SIAM J. MATH. ANAL.
Vol. 16, No. 2, March 1985

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APPLICATIONS OF VARIATIONAL INEQUALITIES TO A MOVING BOUNDARY PROBLEM FOR HELE SHAW FLOWS*

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Abstract. We consider a class of two-dimensional moving boundary problems originating from a Hele Shaw flow problem. Concepts of classical and weak solutions are introduced. We show that a classical solution also is a weak solution and, by using variational inequalities, that given arbitrary initial ($t=0$) data there exists a unique weak solution defined on the time interval $0\leq t<\infty$. We also prove some monotonicity properties of weak solutions and that, under reasonable hypotheses, the moving boundaries consist of analytic curves for $t>0$.

Key words. Hele Shaw flow, moving boundary problem, variational inequalities

Introduction. The aim of the present paper is to prove a global existence and uniqueness theorem for a kind of weak solution to a moving boundary problem arising in two-dimensional Hele Shaw flows. The method used is that of transforming the problem into a series of elliptic variational inequalities.

The problem we shall treat is a slight generalization of the following problem. Let, for D any bounded region in $\mathbb{R}^2$ containing the origin, g_D be the Green's function for D with respect to the origin:

$$g_D(z)=\begin{cases}-\log|z|+\text{harmonic} & \text{in } D,\\ 0 & \text{on } \partial D\end{cases}$$

$$\left(z=x+iy,\ \mathbb{R}^2 \text{ being identified with } \mathbb{C}\right).$$

Then, given an initial domain $D=D_0$, we want to find a family of domains $\{D_t\}$ for $t\geq 0$ ($t=$time) such that ∂D_t moves with the velocity $-(\nabla g_{D_t})|_{\partial D_t}$. (It is assumed here that $\nabla g_{D_t}=$the gradient of g_{D_t} has a continuous extension to ∂D_t.)

This problem (essentially) was introduced by S. Richardson in [12]. The physical interpretation for it as described in [12] is, very briefly, that D_t is the two-dimensional picture of the region of flow in a Hele Shaw flow with a (time-dependent) free boundary and a source point. This means more precisely that an incompressible viscous Newtonian fluid occupies part of the space between two parallel, narrowly separated, infinitely extended surfaces and that more fluid is injected at a constant and moderate rate through a hole in one of the surfaces. The region occupied by fluid then will grow as time increases and, since the gap between the two surfaces is very small, that region can be very well described by its projection D_t onto and $\mathbb{R}^2$-plane lying parallel to the surfaces. The origin of that $\mathbb{R}^2$-plane is taken to correspond to the injection point. For more details and for a derivation of the moving boundary condition above, see [12] and [8]. An incompressible viscous flow in the narrow space between two parallel surfaces is called a Hele Shaw flow. See e.g. [10, p. 581ff.].

The approach in [12] is that of formulating the problem as a differential equation for the Riemann mapping function from the unit disc onto D_t, identifying $\mathbb{R}^2$ with $\mathbb{C}$ and assuming that the D_t are simply connected. No proof of existence or uniqueness of solutions of this differential equation is given in [12]. However, a local (for t in a small,

* Received by the editors June 8, 1983.
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two-sided interval about zero) existence and a partial uniqueness proof for the same differential equation have been given in [19]. See also [8].

Since 1972 Richardson's moving boundary problem has been taken up by J. R. Ockendon [11], C. M. Elliott–V. Janovský [6], S. Richardson himself [13], [14], M. Sakai [16], [17], and me [7], [8]. The present paper is, to a large extent, a summary of [7]. It also has much in common with [17] and more detailed references to that paper will be given at relevant places in the text.

The paper is organized as follows. In §1 we define in a precise way what is meant by being a (local) solution of the problem stated above, by introducing a concept of "classical solution." In §2 we also introduce a concept of "weak solution" and prove that a classical solution is a weak solution. In §3 and 4 we prove that being a weak solution is equivalent to satisfying a series of variational inequalities. From this our main result, the existence and uniqueness of weak solutions for arbitrary given initial domains, follows immediately. Section 5 is devoted to proving that a weak solution is equivalently characterized as the solution of what we call "the moment inequality." Finally, in §6, we summarize part of our results in a kind of main theorem (Theorem 8) and also obtain some partial results on the regularity of the boundaries of the domains of a solution.

List of some notation frequently used.

$\mathbb{R}^2$ is identified with $\mathbb{C}$ whenever convenient (by $(x,y)\leftrightarrow z=x+iy$).

$\mathbb{D}(a;r)=\{z\in\mathbb{C}:|z-a|<r\}$.

$\mathbb{D}(r)=\mathbb{D}(0;r)$.

$\mathbb{D}=\mathbb{D}(0;1)$.

$\mathbb{Z}=$ the set of integers.

$(a,b)=\{x\in\mathbb{R}:a<x<b\}$.

$[a,b]=\{x\in\mathbb{R}:a\leq x\leq b\}$.

$D\subset\subset\Omega$ means: $\bar{D}\subset\Omega$ (if Ω is open and bounded).

$|D|=$ area of D (if $D\subset\mathbb{R}^2$).

$\nabla u=\operatorname{grad} u=(\partial u/\partial x,\partial u/\partial y)$ (if $u=u(x,y)$).

$\Delta u=$ the Laplacian of $u=\partial^2u/\partial x^2+\partial^2u/\partial y^2$.

$$\int_D u=\int_D u(x,y)\,dx\,dy.$$

$\chi_D=$ the characteristic function of $D\subset\mathbb{R}^2=\begin{cases}1 & \text{in } D,\\ 0 & \text{in } \mathbb{R}^2\setminus D.\end{cases}$

$C_c^\infty(\Omega)$: the space of infinitely differentiable functions in Ω with compact support.

$H^{m,p}(\Omega), H^m(\Omega)=H^{m,2}(\Omega), H_0^1(\Omega)$: Sobolev spaces as defined in [18]. $H_0^1(\Omega)$ will always be equipped with the inner product

$$(u,v)=\int_\Omega \nabla u\cdot\nabla v=\int_\Omega(\partial u/\partial x\,\partial v/\partial x+\partial u/\partial y\,\partial v/\partial y)\,dx\,dy$$

(Ω will always be bounded) and norm $\|u\|=\sqrt{(u,u)}$.

$\langle u,v\rangle$: the pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ ($u\in H_0^1(\Omega)$, $v\in H^{-1}(\Omega)$) when $H^{-1}(\Omega)$ is regarded as the dual space of $H_0^1(\Omega)$ in the usual way. This is consistent with

$$\langle u,v\rangle=\int_\Omega u\cdot v \text{ for } u\in L^p(\Omega),\ v\in L^q(\Omega),\ 1/p+1/q=1,\ 1\leq p<\infty.$$

We will often use the fact that the Laplacian Δ is an isomorphism of $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ with the property that [18, Thm. 23.1]

$$(u,v)=-\langle u,\Delta v\rangle \quad \text{for } u,v\in H_0^1(\Omega). \tag{0.1}$$

$\mathfrak{S}_\omega$ is defined in §1 (before Definition 1).
$\mathfrak{R}_{\omega,\Omega}$ is defined in §2 (before Definition 2).

1. A classical formulation of the problem. The purpose of this section is to give an example of how to formulate the moving boundary condition for the problem described in the introduction in a rigorous way. This is done by our concept of a classical solution (Definition 1 below). Actually, we have generalized the problem a little by replacing the Green's function g_D by a domain function p_D depending on a positive measure μ. Our concept of classical solution is to be thought of as just formalizing the rule according to which the moving boundary moves, and we have not tried to make any initial value problem out of it.

Let $\mu\neq 0$ be a finite positive measure with compact support in $\mathbb{R}^2$. For domains $D\subset\mathbb{R}^2$ with $\operatorname{supp}\mu\subset D$ let p_D be the (superharmonic) function in D defined by

$$-\Delta p_D=\mu \quad \text{in } D, \tag{1.1}$$

$$p_D=0 \quad \text{on } \partial D. \tag{1.2}$$

Here (1.2) should be interpreted as follows: for each $\varepsilon>0$ there is a compact set $K\subset D$ such that $|p_D|<\varepsilon$ outside K. For all domains D considered in this section, the problem (1.1)–(1.2) has a unique solution. This solution, moreover, satisfies $p_D\geq 0$ by the minimum principle for superharmonic functions. We are going to consider the problem mentioned in the introduction but with the Green's function g_D there replaced by the more general function p_D. The special case $p_D=g_D$ is obtained by choosing $\mu=2\pi\delta$ in (1.1)–(1.2) (where $\delta=$ the Dirac measure at the origin).

Let ω be a fixed open neighborhood of $\operatorname{supp}\mu$ and set

$\mathfrak{S}_\omega=$ the class of all simply connected domains $D\subset\mathbb{R}^2$ with $\omega\subset\subset D$ and such that ∂D is a Jordan curve of class C^2 (i.e. such that there exists a twice continuously differentiable map from the unit circle to ∂D which is bijective and whose derivative never vanishes; such a map will be called a diffeomorphism of class C^2).

For $D\in\mathfrak{S}_\omega$ p_D exists and is unique and moreover both p_D and ∇p_D have continuous extensions to ∂D. Let $I\subset\mathbb{R}$ be an open interval.

DEFINITION 1. A map $I\ni t\to D_t\in\mathfrak{S}_\omega$ is a *classical solution* of our moving boundary problem if there exists a map ζ: $\mathbb{R}/\mathbb{Z}\times I\to\mathbb{R}^2$ of class C^2 (i.e. twice continuously differentiable) such that

(i) $\zeta(s,t)\in\partial D_t$ for all s,t,
(ii) $\zeta(\cdot,t):\mathbb{R}/\mathbb{Z}\to\partial D_t$ is a diffeomorphism (of class C^2) for each $t\in I$, and
(iii)

$$\frac{\partial\zeta(s,t)}{\partial t}=-\nabla p_{D_t}(\zeta(s,t)) \quad \text{for all } s,t. \tag{1.3}$$

Comment. (i) and (ii) say that for each fixed $t,\zeta(\cdot,t)$ parametrizes ∂D_t. The parameter s (in which ζ has period 1) numbers the points on ∂D_t and (iii) says that each such point moves with the velocity $-\nabla p_{D_t}(\zeta(s,t))$. Here ∇p_{D_t} is the continuous extension of the gradient of p_{D_t} to ∂D_t.

For simplicity we have preferred to let the domains of definition of classical solutions be *open* time intervals. For that reason the concept of a classical solution is not immediately well suited to formalize initial value problems of the kind stated in the introduction. Consider e.g. the following attempt in that direction (where μ and ω are given):

Given $D \in \mathbb{S}_\omega$ find, for some $\varepsilon > 0$, a classical solution $(-\varepsilon, +\infty) \ni t \to D_t \in \mathbb{S}_\omega$ such that $D_0 = D$. This formulation has the drawback that it requires the existence of the solution for $-\varepsilon < t < 0$ and, as it turns out (see below), this can only occur if ∂D is analytic.

A perhaps better attempt would therefore be:

Given $D \in \mathbb{S}_\omega$ find a classical solution $(0, +\infty) \ni t \to D_t \in \mathbb{S}_\omega$ such that $\lim_{t \to 0} D_t = D$ in some specified sense. This formulation does away with the problem of the former formulation, but nevertheless one cannot expect a solution to exist for arbitrary $D \in \mathbb{S}_\omega$. This is because we have not, in our definition of a classical solution, built in any possibility for D_t to change connectivity and it is easy to see that without such a possibility global solutions cannot exist in general.

The above remark shows that the concept of a classical solution has to be fairly complicated in order to be well suited for a formulation of a global initial value problem. We have not thought it to be worth the effort to make such a formulation, since our concept of classical solution is introduced mostly in order to motivate our concepts of a weak solution (and for this purpose we think that Definition 1 is good enough). A global concept of a classical solution (allowing connectivity changes) has, however, been given by Sakai [16].

Definition 1 is, however, well suited for formulating a local problem:

(1.4) Given $D \in \mathbb{S}_\omega$ find, for some $\varepsilon > 0$, a classical solution $(-\varepsilon, \varepsilon) \ni t \to D_t \in \mathbb{S}_\omega$ such that $D_0 = D$.

The task of proving existence and unicity for solutions of this problem is seemingly hard. In fact, we will prove here (Theorem 10) that a necessary condition for a solution of (1.4) to exist is that ∂D is an analytic curve. Probably this condition is also sufficient. In any case, in the special case that $\mu = 2\pi\delta$, Vinogradov and Kufarev [19] have proved local existence of solutions when the problem is formulated as a differential equation for the Riemann mapping function (as in [12]), under the condition that ∂D is analytic (see also [8]). It is, however, not quite easy to prove rigorously that a solution in their sense is also a classical solution in our sense. (The converse is easier.) Vinogradov and Kufarev also prove uniqueness for solutions depending analytically on t (of their problem). Here we prove at least that a solution of (1.4) is unique for $t > 0$ (Theorem 10).

Let us next make a remark about the measure μ; namely, as far as classical solutions are concerned, we can always assume that μ is a smooth function. The reason is that nothing but the behaviour of p_D near ∂D comes into Definition 1 and that therefore p_D can be smoothed out in a neighbourhood of $\operatorname{supp} \mu$. To be precise, let h be a smooth (C^∞), positive, radially symmetric (i.e. a function of radius only) function ("mollifier") on $\mathbb{R}^2$ with total mass one ($\int h = 1$) and with compact support in the open unit disk $\mathbb{D}$. Define

$$h_\varepsilon(z) = \frac{1}{\varepsilon^2} h\left(\frac{z}{\varepsilon}\right) \qquad (z \in \mathbb{R}^2). \tag{1.5}$$

(Thus $\operatorname{supp} h_\varepsilon \subset \mathbb{D}(0;\varepsilon)$, $\int h_\varepsilon = 1$, $h_\varepsilon \geq 0$.) Then we have

PROPOSITION 1. *Let μ, ω and I be as before Definition 1 and choose $\varepsilon > 0$ such that $2\varepsilon < \operatorname{dist}(\operatorname{supp}\mu, \partial\omega)$. Then $I \ni t \to D_t \in \mathcal{S}_\omega$ is a classical solution for μ if and only if it is a classical solution for $\mu * h_\varepsilon$ (which is a smooth function). ($*$ denotes convolution.)*

Proof. The proof consists of the observation that the function p_D defined by (1.1)–(1.2) only changes inside ω when μ is replaced by $\mu * h_\varepsilon$ (for $D \in \mathcal{S}_\omega$). In fact, define

$$q_D = \begin{cases} p_D * h & \text{in } \{z \in D : \operatorname{dist}(z, \partial D) > \varepsilon\}, \\ p_D & \text{in } \{z \in D : \operatorname{dist}(z, \operatorname{supp}\mu) > \varepsilon\}. \end{cases}$$

Then q_D is well defined because p_D is harmonic in a whole ε-neighborhood of any point in the overlap between the two domains above, and therefore $p_D * h_\varepsilon = p_D$ in that overlap by the mean-value property for harmonic functions. Since $\Delta(p_D * h_\varepsilon) = \Delta p_D * h_\varepsilon = -\mu * h_\varepsilon$, it is immediately seen that q_D is the solution of (1.1)–(1.2) with $\mu * h_\varepsilon$ in place of μ. Since $q_D = p_D$ near ∂D, the conclusion of the proposition follows immediately.

2. The weak solution. We now introduce the concept of a weak solution and prove that a classical solution is a weak solution.

The concept of a weak solution is much more flexible than that of a classical solution (e.g. one does not have to bother about boundary regularity or connectivity of the domains), it is much easier to show existence of solutions for, and it is also more apt for numerical treatment (because it is closely related to variational inequalities). These are the main reasons for introducing the concept of a weak solution.

Let $\mu \neq 0$ be a finite positive measure with compact support in $\mathbb{R}^2$, and choose bounded open sets ω and Ω in $\mathbb{R}^2$ such that $\operatorname{supp}\mu \subset \omega \subset\subset \Omega$ and with $\partial\Omega$ smooth, and let $T > 0$. Set

$$\mathfrak{R}_{\omega,\Omega} = \text{the class of all open sets } D \subset R^2 \text{ with } \omega \subset\subset D \subset\subset \Omega.$$

In order for the definition below to make sense, we have to assume that μ belongs to the Sobolev space $H^{-1}(\Omega)$. This is an assumption of purely technical nature and does not mean any restriction of the class of problems considered (in view of Proposition 1 above).

DEFINITION 2. A map of $[0, T] \ni t \to D_t \in \mathfrak{R}_{\omega,\Omega}$ is a *weak solution* of our moving problem if, for each $t \in [0, T]$, the function $u_t \in H_0^1(\Omega)$ defined by

$$\chi_{D_t} - \chi_{D_0} = \Delta u_t + t \cdot \mu \tag{2.1}$$

satisfies

$$u_t \geq 0, \tag{2.2}$$

$$\langle u_t, 1 - \chi_{D_t} \rangle = 0. \tag{2.3}$$

Comments. 1) *Notation.* The subscript t in u_t just indicates that u_t depends on t. We never use subscripts for partial derivatives. (2.1) and (2.2), like all other equalities and inequalities on open sets in this paper, are to be interpreted in the sense of distributions. In (2.3) $1 - \chi_{D_t}$ is regarded as an element of $H^{-1}(\Omega) \cong$ the dual space of $H_0^1(\Omega)$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Since, in the present case, $1-\chi_{D_t}\in L^2(\Omega)$ (and $u_t\in H_0^1(\Omega)\subset L^2(\Omega)$), the left member of (2.3) reduces to the Lebesgue integral

$$\langle u_t, 1-\chi_{D_t}\rangle=\int_\Omega u_t\cdot(1-\chi_{D_t})=\int_{\Omega\setminus D_t} u_t,$$

and since $1-\chi_{D_t}\geq 0$, (2.2) and (2.3) together express that $u_t\geq 0$ in Ω and $u_t=0$ a.e. outside D_t.

2) Since $\chi_{D_t}-\chi_{D_0}-t\cdot\mu\in H^{-1}(\Omega)$ and the Laplacian Δ is an isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$, (2.1) really defines u_t uniquely.

3) It is clear from the definition that given $D_0\in\mathfrak{R}_{\omega,R}$, the domains D_t of a weak solution can be unique at most up to two-dimensional Lebesgue measure zero, since (2.1)–(2.3) are not affected if D_t is replaced by another $D_t'\in\mathfrak{R}_{\omega,\Omega}$ such that $\chi_{D_t'}=\chi_{D_t}$ a.e..

4) In order to motivate the concept of a weak solution, we now sketch a proof that a classical solution is a weak solution.

So suppose $t\to D_t$ is a classical solution. Condition (iii) in Definition 1 can be written

$$\frac{\delta n}{\delta t}=-\frac{\partial p_{D_t}}{\partial n}\quad\text{on } \partial D_t, \tag{2.4}$$

where $\delta n/\delta t$ denotes the normal velocity of ∂D_t (in the direction out from D_t), and $\partial/\partial n$ denotes outward normal derivative. (2.4) is equivalent to

$$\int_{\partial D_t}\frac{\delta n}{\delta t}\cdot\varphi\, ds=-\int_{\partial D_t}\frac{\partial p_{D_t}}{\partial n}\cdot\varphi\, ds\quad\text{for all } \varphi\in C_c^\infty(\mathbb{R}^2) \tag{2.5}$$

(ds denotes arc-length measure).

It is not hard to see that

$$\int_{\partial D_t}\frac{\delta n}{\delta t}\varphi\, ds=\frac{d}{dt}\int_{D_t}\varphi\, dx\, dy=\frac{d}{dt}\langle\chi_{D_t},\varphi\rangle.$$

Extend p_{D_t} to all $\mathbb{R}^2$ by setting $p_{D_t}=0$ outside D_t. Then

$$-\Delta p_{D_t}=\mu+\frac{\partial p_{D_t}}{\partial n}ds,$$

where $(\partial p_{D_t}/\partial n)\,ds$ denotes the distribution

$$\varphi\to\int_{\partial D_t}\varphi\frac{\partial p_{D_t}}{\partial n}ds\qquad(\varphi\in C_c^\infty(\mathbb{R}^2)),$$

a nonpositive measure supported by ∂D_t. Hence

$$-\int_{\partial D_t}\frac{\partial p_{D_t}}{\partial n}\cdot\varphi\, ds=\langle\Delta p_{D_t}+\mu,\varphi\rangle,$$

and (2.5) can be written

$$\frac{d}{dt}\langle\chi_{D_t},\varphi\rangle=\langle\Delta p_{D_t}+\mu,\varphi\rangle\quad\text{for all } \varphi\in C_c^\infty(\mathbb{R}^2). \tag{2.6}$$

Now integrate (2.6) with respect to t. With

$$u_t=\int_0^t p_{D_\tau}\,d\tau \tag{2.7}$$

this gives $\chi_{D_t}-\chi_{D_0}=\Delta u_t+t\cdot\mu$, to hold in the sense of distributions. Thus u_t defined by (2.7) satisfies (2.1) of Definition 2. Since $p_{D_\tau}\geq 0$ in D_τ, $p_{D_\tau}=0$ outside D_τ, and because $D_{\tau_1}\subset D_{\tau_2}$ for $\tau_1<\tau_2$ (if $t\to D_t$ is a classical solution) u_t, defined by (2.7), also satisfies

$$\begin{aligned} &u_t\geq 0 \quad \text{in all } \mathbb{R}^2 \quad \text{and}\\ &u_t=0 \quad \text{outside } D_t, \end{aligned}$$

that is, (2.2) and (2.3) of Definition 2.

This was a sketch of a proof that a classical solution is a weak solution. A formal proof of this will be given later (Theorem 1).

5) A nice feature of the concept of weak solution is that the time variable only occurs as a parameter in it: (2.1)–(2.3) is just a series of uncoupled problems (actually free boundary problems), one for each $t\in[0,T]$. The transformation (2.7) plays a crucial role in this respect. The efficiency of transformations similar to (2.7) on certain kinds of free and moving boundary problems is now well known and has been demonstrated in works by Baiocchi, Duvaut, Elliott and others. (See e.g. [2], [4] and [5].)

Just as for classical solutions, there is no loss of generality in assuming that the measure μ in Definition 1 is a smooth function.

PROPOSITION 2. *Let μ,ω,Ω and T be as in Definition 2, choose $\varepsilon>0$ with $2\varepsilon<$ dist$(\operatorname{supp}\mu,\partial\omega)$ and let h_ε be as before Proposition 1. Then $[0,T]\ni t\to D_t\in\mathfrak{R}_{\omega,\Omega}$ is a weak solution for μ if and only if it is a weak solution for $\mu * h_\varepsilon$. In case they are solutions we have*

$$v_t=\begin{cases} u_t * h_\varepsilon & \text{in } \operatorname{supp}\mu+\mathbb{D}(0;\varepsilon),\\ u_t & \text{elsewhere in } \Omega, \end{cases} \tag{2.8}$$

where u_t (v_t) is the function occurring in Definition 2 for μ ($\mu * h_\varepsilon$). ($\operatorname{supp}\mu+\mathbb{D}(0;\varepsilon)=\{z+w\in\mathbb{R}^2 : z\in\operatorname{supp}\mu$ *and* $w\in\mathbb{D}(0;\varepsilon)\}$.)

Proof (*sketch*). Let $t\to D_t$ be a weak solution for μ, let $u_t\in H_0^1(\Omega)$ be defined by (2.1) and define v_t by (2.8). Then u_t is harmonic in $\omega\setminus\operatorname{supp}\mu$ (by (2.1)) and the mean-value property for harmonic functions, together with the properties of h_ε, show that $u_t * h_\varepsilon=u_t$ a distance ε away from $\partial(\omega\setminus\operatorname{supp}\mu)$ in $\omega\setminus\operatorname{supp}\mu$. It follows that v_t is smooth in the join between the two ranges of definition and in particular that $v_t\in H_0^1(\Omega)$. Now it is easy to check that v_t satisfies (2.1)–(2.3) of Definition 2 with $\mu * h_\varepsilon$ in place of μ. This proves the proposition in one direction.

Next, let $t\to D_t$ be a weak solution for $\mu * h_\varepsilon$ and $u_t\in H_0^1(\Omega)$ be defined by (2.1) for μ. In order to prove the proposition in the other direction, we have to prove that u_t also satisfies (2.2) and (2.3).

Define v_t by (2.8). As before we have $v_t\in H_0^1(\Omega)$, and v_t satisfies (2.1) with $\mu * h_\varepsilon$ in place of μ. Since the solution of (2.1) is unique and $t\to D_t$ is a weak solution for $\mu * h_\varepsilon$, v_t also satisfies (2.2) and (2.3). From this it follows immediately that u_t satisfies (2.3). Moreover, $u_t\geq 0$ clearly holds outside $\operatorname{supp}\mu+\mathbb{D}(0;\varepsilon)$ ($u_t=v_t$ there), and in fact also in $\operatorname{supp}\mu+\mathbb{D}(0;\varepsilon)$ because $u_t\geq u_t * h$ $(=v_t\geq 0)$ there due to the fact that u_t (by (2.1)) is superharmonic in ω $(\supset\operatorname{supp}\mu+2\mathbb{D}(0;\varepsilon))$. Thus u_t also satisfies (2.2), and the proposition is proved.

Now we shall prove that a classical solution is a weak solution. Let $\mu \neq 0$ be a finite positive measure with compact support in $\mathbb{R}^2$ such that $\mu \in H^{-1}(\mathbb{R}^2)$, let $\operatorname{supp} \mu \subset \omega$ and let $a<0<T<b$. Then

THEOREM 1. *Suppose $(a,b) \ni t \to D_t \in \mathfrak{S}_\omega$ is a classical solution. Then $[0,T] \ni t \to D_t \in \mathfrak{R}_{\omega,\Omega}$ is a weak solution if Ω is chosen such that $D_T \subset\subset \Omega$. Moreover, the functions p_{D_t} and u_t occurring in the classical and weak solutions respectively, are related by*

$$u_t = \int_0^t p_{D_\tau} \, d\tau \tag{2.9}$$

(a $H_0^1(\Omega)$-valued integral), where p_{D_τ} is extended to all Ω by setting it equal to zero outside D_τ.

Proof. Let $(a,b) \ni t \to D_t \in \mathfrak{S}_\omega$ be the classical solution, which we shall prove to be weak. We shall first prove that

$$\frac{d}{dt} \int_{D_t} \varphi \, dx \, dy = -\int_{\partial D_t} \varphi \cdot \frac{\partial p_{D_t}}{\partial n} ds \tag{2.10}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2)$ and that the right member of (2.10) is a continuous function of t.

Let x, y be the coordinate variables in $\mathbb{R}^2$ and let

$$\xi = \xi(s,t), \qquad \eta = \eta(s,t)$$

denote the components of $\zeta(s,t) \in \mathbb{R}^2$ (see Definition 1). Then (iii) of Definition 1 becomes

$$\frac{\partial \xi(s,t)}{\partial t} = -\frac{\partial p_{D_t}}{\partial x}(\zeta(s,t)), \qquad \frac{\partial \eta(s,t)}{\partial t} = -\frac{\partial p_{D_t}}{\partial y}(\zeta(s,t)).$$

Thus the right member of (2.10) becomes

$$-\int_{\partial D_t} \varphi \frac{\partial p_{D_t}}{\partial n} ds = -\int_{\partial D_t} \varphi \cdot \left(\frac{\partial p_{D_t}}{\partial x} dy - \frac{\partial p_{D_t}}{\partial y} dx \right) = \int_{\partial D_t} \varphi \cdot \left(\frac{\partial \xi}{\partial t} dy - \frac{\partial \eta}{\partial t} dx \right). \tag{2.11}$$

Next we rewrite the left member of (2.10). Choose smooth functions, $a(x,y)$ and $b(x,y)$, on $\mathbb{R}^2$ such that $\partial b/\partial x - \partial a/\partial y = \varphi$ (e.g. $a(x,y)=0$ and $b(x,y) = \int_0^x \varphi(u,y)\, du$). Let $\mathbb{T} = \mathbb{R}/\mathbf{Z}$ (the range of the variable s). Then, using Stokes' formula at the first step, we get

$$\begin{aligned} \frac{d}{dt} \int_{D_t} \varphi \, dx \, dy &= \frac{d}{dt} \int_{\partial D_t} a \, dx + b \, dy \\ &= \frac{d}{dt} \int_{\mathbb{T}} \left[a(\zeta(s,t)) \frac{\partial \xi(s,t)}{\partial s} + b(\zeta(s,t)) \frac{\partial \eta(s,t)}{\partial s} \right] ds \\ &= \int_{\mathbb{T}} \left[a \cdot \frac{\partial^2 \xi}{\partial s \partial t} + b \cdot \frac{\partial^2 n}{\partial s \partial t} \right] ds \\ &\quad + \int_{\mathbb{T}} \left[\left(\frac{\partial a}{\partial x} \frac{\partial \xi}{\partial t} + \frac{\partial a}{\partial y} \frac{\partial \eta}{\partial t} \right) \frac{\partial \xi}{\partial s} + \left(\frac{\partial b}{\partial x} \frac{\partial \xi}{\partial t} + \frac{\partial b}{\partial y} \frac{\partial \eta}{\partial t} \right) \frac{\partial \eta}{\partial s} \right] ds \end{aligned} \tag{2.12}$$

$$=\int_{\mathbb{T}}\left[a\cdot\frac{\partial^2\xi}{\partial s\partial t}+b\cdot\frac{\partial^2\eta}{\partial s\partial t}\right]ds+\int_{\mathbb{T}}\left(\frac{\partial b}{\partial x}-\frac{\partial a}{\partial y}\right)\left(\frac{\partial\xi}{\partial t}\frac{\partial\eta}{\partial s}-\frac{\partial\eta}{\partial t}\frac{\partial\xi}{\partial s}\right)ds$$

$$+\int_{\mathbb{T}}\left[\frac{\partial a}{\partial x}\frac{\partial\xi}{\partial t}\frac{\partial\xi}{\partial s}+\frac{\partial b}{\partial y}\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial s}+\frac{\partial a}{\partial y}\frac{\partial\xi}{\partial t}\frac{\partial\eta}{\partial s}+\frac{\partial b}{\partial x}\frac{\partial\eta}{\partial t}\frac{\partial\xi}{\partial s}\right]ds$$

$$=\int_{\mathbb{T}}\left(\frac{\partial b}{\partial x}-\frac{\partial a}{\partial y}\right)\left(\frac{\partial\xi}{\partial t}\frac{\partial\eta}{\partial s}-\frac{\partial\eta}{\partial t}\frac{\partial\xi}{\partial s}\right)ds+\int_{\mathbb{T}}\frac{\partial}{\partial s}\left(a\frac{\partial\xi}{\partial t}+b\frac{\partial\eta}{\partial t}\right)ds$$

$$=\int_{\mathbb{T}}\varphi\cdot\left(\frac{\partial\xi}{\partial t}\frac{\partial\eta}{\partial s}-\frac{\partial\eta}{\partial t}\frac{\partial\xi}{\partial s}\right)ds=\int_{\partial D_t}\varphi\cdot\left(\frac{\partial\xi}{\partial t}dy-\frac{\partial\eta}{\partial t}\right)dx.$$

By (2.11) and (2.12) we have proven (2.10). It is seen from (2.12) also that the right member of (2.10) is a continuous function of t and that hence $\int_{D_t}\varphi\,dx\,dy$ is continuously differentiable with respect to t.

We next prove that $D_t\in\mathfrak{R}_{\omega,\Omega}$ for $t\in[0,T]$ if Ω is chosen such that $D_t\subset\subset\Omega$. Since $\omega\subset\subset D_0$ ($D_0\in\mathbb{S}_\omega$), it suffices to prove that $D_\tau\subset D_t$ for $\tau<t$. Since $p_{D_t}\geq 0$ in D_t we have $-\partial p_{D_t}/\partial\eta\geq 0$ on ∂D_t. Therefore (2.10) shows that $\int_{D_t}\varphi\,dx\,dy$ is a nondecreasing function of t for all $\varphi\in C_c^\infty(\mathbb{R}^2)$ such that $\varphi\geq 0$. This easily implies that $D_\tau\subset D_t$ for $\tau<t$. Thus we have proved that $D_t\in\mathfrak{R}_{\omega,\Omega}$ for $t\in[0,T]$.

Now let $u_t\in H_0^1(\Omega)$ (for $t\in[0,T]$) be defined by (2.1) and extend p_{D_t} to all Ω by setting it equal to zero outside D_t. Then it is easy to see that $p_{D_t}\in H_0^1(\Omega)$. We now want to prove that

$$u_t=\int_0^t p_{D_\tau}\,d\tau. \tag{2.13}$$

Equation (2.13) means by definition (we are using the "weak" definition of vector-valued integrals as exposed e.g. in [18, p. 73]) that

$$\langle u_t,\rho\rangle=\int_0^t\langle p_{D_\tau},\rho\rangle\,d\tau \tag{2.14}$$

for all $\rho\in H^{-1}(\Omega)$. Since $\Delta: H_0^1(\Omega)\to H^{-1}(\Omega)$ is an isomorphism, (2.14) can be written

$$\langle u_t,\Delta\varphi\rangle=\int_0^t\langle p_{D_\tau},\Delta\varphi\rangle\,d\tau$$

for all $\varphi\in H_0^1(\Omega)$, i.e., using the fact that $\langle u,\Delta\varphi\rangle=\langle\varphi,\Delta u\rangle$ for $u,\varphi\in H_0^1(\Omega)$,

$$\langle\varphi,\chi_{D_t}-\chi_{D_0}-t\mu\rangle=\int_0^t\langle p_{D_\tau},\Delta\varphi\rangle\,d\tau. \tag{2.15}$$

We first prove (2.15) for $\varphi\in C_c^\infty(\Omega)$. Green's (second) formula gives (for $\varphi\in C_c^\infty(\Omega)$)

$$-\int_{\partial D_t}\varphi\cdot\frac{\partial p_{D_t}}{\partial n}ds=-\int_{D_t}\varphi\cdot\Delta p_{D_t}+\int_{D_t}\Delta\varphi\cdot p_{D_t}$$

$$=\int_{D_t}\varphi\cdot\mu+\int_{\Omega}\Delta\varphi\cdot p_{D_t}=\langle\varphi,\mu\rangle+\langle p_{D_t},\Delta\varphi\rangle.$$

Combining this with (2.10) gives

$$\langle \varphi, \chi_{D_t} - \chi_{D_0} \rangle = \int_{D_t} \varphi - \int_{D_0} \varphi = \int_0^t \left(- \int_{\partial D_\tau} \varphi \cdot \frac{\partial p_{D_\tau}}{\partial n} \, ds \right) d\tau$$

$$= \langle \varphi, t \cdot \mu \rangle + \int_0^t \langle p_{D_\tau}, \Delta \varphi \rangle \, d\tau.$$

This proves (2.15) for $\varphi \in C_c^\infty(\Omega)$.

To extend (2.15) to all $\varphi \in H_0^1(\Omega)$ it is enough to prove that the right member of (2.15) depends continuously with respect to the $H_0^1(\Omega)$-norm on φ; for $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ and the left member of (2.15) obviously depends continuously on φ.

We have

$$\left| \int_0^t \langle p_{D_\tau}, \Delta \varphi \rangle \, d\tau \right| = \left| \int_0^t (p_{D_\tau}, \varphi) \, d\tau \right| \le \int_0^t \| p_{D_\tau} \| \cdot \| \varphi \| \, d\tau = \| \varphi \| \cdot \int_0^t \| p_{D_\tau} \| \, d\tau,$$

where $(u, v) = \int_\Omega \nabla u \cdot \nabla v$ is the inner product in $H_0^1(\Omega)$ and $\|u\| = \sqrt{(u,u)}$, and it is not hard to prove that $\int_0^t \| p_{D_\tau} \| d\tau < \infty$. (Details are found in [7, p. 40].) Thus the right-hand side of (2.15) depends continuously on φ and so (2.15) is proven.

Now it only remains to prove that u_t satisfies (2.2) and (2.3). (2.2) follows immediately from (2.13) and $p_{D_\tau} \ge 0$, and (2.3) follows from

$$\langle u_t, 1 - \chi_{D_t} \rangle = \int_0^t \langle p_{D_\tau}, 1 - \chi_{D_t} \rangle \, d\tau$$

by choosing $\rho = 1 - \chi_{D_t}$ in (2.14) and

$$\langle p_{D_\tau}, 1 - \chi_{D_t} \rangle = \int_\Omega p_{D_\tau} (1 - \chi_{D_t}) = 0$$

for $\tau \in [0, t]$, a consequence of $D_\tau \subset D_t$ (already proved) and $p_{D_\tau} = 0$ outside D_τ. This completes the proof of Theorem 1.

3. Complementarity problems and variational equalities. By weakening still more the concept of a solution, we arrive at a series of linear complementarity problems. These are equivalent to a series of variational inequalities which are shown to have unique solutions.

Let μ, ω, Ω and T be as before Definition 2 ($\mu \in H^{-1}(\mathbb{R}^2)$, $\mu \ge 0$, $\mu \ne 0$, ω, Ω open, $\partial\Omega$ nice, $\operatorname{supp} \mu \subset \omega \subset \subset \Omega$ and $T > 0$). Let also $D_0 \in \mathfrak{R}_{\omega,\Omega}$ be given and define $\rho_t = \rho_{t, D_0} \in H^{-1}(\Omega)$ by

$$\rho_t = 1 - \chi_{D_0} - t \cdot \mu. \tag{3.1}$$

Then (2.1)–(2.3) in Definition 2 can be written

$$\Delta u_t - \rho_t = \chi_{D_t} - 1, \quad u_t \ge 0, \quad \langle u_t, 1 - \chi_{D_t} \rangle = 0$$

or

$$\Delta u_t - \rho_t = \chi_{D_t} - 1, \quad u_t \ge 0, \quad \langle u_t, \Delta u_t - \rho_t \rangle = 0.$$

Since $\chi_{D_t} - 1 \leq 0$, this immediately shows

THEOREM 2. *Suppose* $[0,T] \ni t \to D_t \in \mathfrak{R}_{\omega,\Omega}$ *is a weak solution. Then the functions* $u_t \in H_0^1(\Omega)$ *defined by* (2.1) *also solve the linear complementarity problems*

$$\Delta u_t \leq \rho_t, \tag{3.2}$$

$$u_t \geq 0, \tag{3.3}$$

$$\langle u_t, \Delta u_t - \rho_t \rangle = 0 \tag{3.4}$$

($t \in [0,T]$), *where* ρ_t *are defined by* (3.1).

Remark. Clearly (3.4) (in the presence of (3.2) and (3.3)) expresses that at (almost) every point in Ω equality holds in at least one of the inequalities (3.2) and (3.3).

The complementarity problem (3.2)–(3.4) is equivalent to a variational inequality:

THEOREM 3. *Let* $\rho_t \in H^{-1}(\Omega)$ (*e.g. given by* (3.1)). *Then* $u_t \in H_0^1(\Omega)$ *satisfies* (3.2)–(3.4) *if and only if it satisfies*

$$\Delta u_t \leq \rho_t \text{ and } (u - u_t, u_t) \geq 0 \text{ for all } u \in H_0^1(\Omega) \text{ with } \Delta u \leq \rho_t. \tag{3.5}$$

Remark. Theorems 2 and 3 together are similar to [17, Prop. 2]. However, [17] deals with the equivalent variational inequality (3.11) below in place of (3.5).

Proof. If u_t satisfies (3.2)–(3.4), then $\langle u_t, \Delta u - \rho_t \rangle \leq 0 = \langle u_t, \Delta u_t - \rho_t \rangle$ for all $u \in H_0^1(\Omega)$ with $\Delta u \leq \rho_t$. By subtracting $\langle u_t, \rho_t \rangle$ and using (0.1) the theorem immediately follows in one direction.

Conversely, suppose (3.5) holds. Then (by (0.1))

$$\langle u_t, \Delta u - \rho_t \rangle \leq \langle u_t, \Delta u_t - \rho_t \rangle \quad \text{for all } u \in H_0^1(\Omega) \text{ with } \Delta u \leq \rho_t. \tag{3.6}$$

Since $\Delta: H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, we can choose $u \in H_0^1(\Omega)$ in (3.6) such that $\Delta u = \rho_t$ or such that $\Delta u = 2\Delta u_t - \rho_t$ ($\Delta u \leq \rho_t$ is fulfilled in both cases). The first choice shows that the right member of (3.6) is ≥ 0, while the second shows that it is ≤ 0. Thus

$$\langle u_t, \Delta u_t - \rho_t \rangle = 0. \tag{3.7}$$

By (3.7), and writing $\varphi = \Delta u - \rho_t$, (3.6) becomes $\langle u_t, \varphi \rangle \leq 0$ for all $\varphi \in H^{-1}(\Omega)$ with $\varphi \leq 0$. This shows that $u_t \geq 0$. Thus (3.2)–(3.4) hold for u_t and so the theorem is proven in the other direction too.

Remark. The variational inequality (3.5) differs somewhat from those variational inequalities most often met with in the literature in that the condition $\Delta u \leq \rho_t$ is of an unusual kind, but it is equivalent to a variational inequality of "obstacle-type." To be precise, define $\psi_t \in H_0^1(\Omega)$ by

$$-\Delta \psi_t = \rho_t. \tag{3.8}$$

Then, in terms of the function

$$v_t = u_t + \psi_t \tag{3.9}$$

the complementarity problem (3.2)–(3.4) can be written

$$\Delta v_t \leq 0, \quad v_t \geq \psi_t, \quad \langle v_t - \psi_t, \Delta v_t \rangle = 0. \tag{3.10}$$

By an argument very similar to the proof of Theorem 3 (see [7, pp. 43–45] for details), (3.10) can be shown to be equivalent to the variational inequality

(3.11) find $v_t \in H_0^1(\Omega)$ such that $v_t \geq \psi_t$ and $(v - v_t, v_t) \geq 0$ for all $v \in H_0^1(\Omega)$ with $v \geq \psi_t$.

This variational inequality is of obstacle-type (ψ_t describing the obstacle). See [9, Chap. II, §6].

THEOREM 4. *Let* $\rho_t \in H^{-1}(\Omega)$. *Then the variational inequality* (3.5) *has a unique solution* $u_t \in H_0^1(\Omega)$. *If, moreover,* $\rho_t \in L^p(\Omega)$ *for some* $1<p<\infty$, *then* $u_t \in H^{2,p}(\Omega)$, *in particular* u_t *is continuous* (*if* $p>2$ *even continuously differentiable*).

Proof. The existence and unicity of a solution is immediate from the general theory of variational inequalities or in fact just from ordinary Hilbert space theory since (3.5) just expresses that u_t is the unique element of minimum norm in the closed and convex set $K=\{u \in H_0^1(\Omega)\colon \Delta u \le \rho_t\}$. The regularity of the solution also follows from the general theory of variational inequalities, e.g. by first rewriting the problem into the form (3.11) as indicated in the Remark above, and then invoking [3, Théorème I.1]. There is, however, also a direct and rather nice proof of the regularity. This goes as follows.

Let $\rho_t \in L^p(\Omega)$ with $1<p<\infty$. To prove that $u_t \in H^{2,p}(\Omega)$, we shall consider a new variational inequality, namely

find $w_t \in H_0^1(\Omega)$ such that

$$\min(0,\rho_t) \le \Delta w_t \le \rho_t \quad \text{and} \tag{3.12}$$

$$(w-w_t, w_t) \ge 0 \text{ for all } w \in H_0^1(\Omega) \text{ with } \min(0,\rho_t) \le \Delta w \le \rho_t. \tag{3.13}$$

This variational inequality has a unique solution $w_t \in H_0^1(\Omega)$ for the same reason as for (3.5). Moreover, this solution a priori belongs to $H^{2,p}(\Omega)$ since (3.12) shows that $\Delta w_t \in L^p(\Omega)$ (and $\Delta w_t \in L^p(\Omega)$ implies $w_t \in H^{2,p}(\Omega)$). Thus, to prove that $u_t \in H^{2,p}(\Omega)$, it is enough to prove that $u_t = w_t$. For that purpose we only have to show that w_t satisfies (3.2)–(3.4) (by Theorem 3).

To prove (3.2)–(3.4) for w_t, first rewrite (3.13) as

$$\langle w_t, \Delta w - \rho_t \rangle \le \langle w_t, \Delta w_t - \rho_t \rangle$$

for all $w \in H_0^1(\Omega)$ with $\min(0,\rho_t) \le \Delta w \le \rho_t$. Setting $\varphi = \Delta w - \rho_t$ and using that the bracket $\langle \cdot, \cdot \rangle$ in the present case reduces to a Lebesgue integral, we get

$$\int_\Omega w_t \cdot \varphi \le \int_\Omega w_t \cdot (\Delta w_t - \rho_t) \tag{3.14}$$

for all $\varphi \in H^{-1}(\Omega)$ with $\min(0, -\rho_t) \le \varphi \le 0$. First choose $\varphi = 0$ in (3.14). This gives

$$\int_\Omega w_t \cdot (\Delta w_t - \rho_t) \ge 0. \tag{3.15}$$

Then choose

$$\varphi = \begin{cases} \min(0, -\rho_t) & \text{in } N, \\ \Delta w_t - \rho_t & \text{in } \Omega \setminus N, \end{cases}$$

where $N = \{z \in \Omega : w_t(z) < 0\}$. (Since $w_t \in H^{2,p}(\Omega)$, w_t is a continuous function and so $w_t(z) < 0$ has a natural meaning and N is a well-defined open set.) We get

$$\int_N w_t \cdot \min(0, -\rho_t) \le \int_N w_t \cdot (\Delta w_t - \rho_t),$$

or

$$\int_N w_t \cdot (\Delta w_t - \rho_t - \min(0, -\rho_t)) \geq 0. \tag{3.16}$$

Since $\Delta w_t - \rho_t - \min(0, -\rho_t) = \Delta w_t - \min(0, \rho_t) \geq 0$ and $w_t < 0$ in N (3.16) shows that

$$\Delta w_t - \min(0, \rho_t) = 0 \quad \text{in } N.$$

In particular,

$$\Delta w_t \leq 0 \quad \text{in } N. \tag{3.17}$$

But now $w_t = 0$ on $\partial N \cup \partial \Omega$ (by the definition of N, and since w_t is continuous and belongs to $H_0^1(\Omega)$). Therefore (3.17) implies $w_t \geq 0$ in N (minimum principle for superharmonic functions). Comparing with the definition of N, we conclude that N is the empty set. Hence

$$w_t \geq 0 \quad (\text{in } \Omega).$$

Thus (3.3) (for w_t) is proven. (3.2) is part of (3.12), and (3.4) follows by combining (3.15) with (3.2) and (3.3). Hence w_t satisfies the complementarity conditions (3.2)–(3.4) which characterize u_t, hence $w_t = u_t$ as we wanted to prove.

The statements about continuity and continuous differentiability of u_t follow from Sobolev's inequalities. See e.g. [18, Thm. 24.2]. This completes the proof.

4. From variational inequalities to a weak solution. Up to now we have performed a series of weakenings of the concept of solution,

$$\begin{matrix}\text{classical} \\ \text{solution}\end{matrix} \Rightarrow \begin{matrix}\text{weak} \\ \text{solution}\end{matrix} \Rightarrow \begin{matrix}\text{solution of} \\ \text{complementarity} \\ \text{problems}\end{matrix} \Leftrightarrow \begin{matrix}\text{solution of} \\ \text{variational} \\ \text{inequalities}\end{matrix}$$

and we have proved existence and uniqueness of solutions at the right end point of this series. On the way from weak solution to complementarity problems we have also lost the domain D_t from the problem.

Now we want to perform the step

$$\begin{matrix}\text{solution of} \\ \text{complementarity} \\ \text{problems}\end{matrix} \Rightarrow \begin{matrix}\text{weak} \\ \text{solution}\end{matrix}$$

thereby also proving existence of weak solutions with a given initial domain (uniqueness is already proven, by uniqueness of solutions of the variational inequalities). This step involves among other things recovering the regions D_t from the functions u_t (constituting the solution of the complementarity problems). We need two lemmas.

LEMMA 1. *Let $\rho_t \in H^{-1}(\Omega)$ and let $u_t \in H_0^1(\Omega)$ be the solution of the complementarity problem* (3.2)–(3.4). *Then $u_t \leq u$ for all $u \in H_0^1(\Omega)$ which satisfy $\Delta u \leq \rho_t$ and $u \geq 0$.*

Comment. Lemma 1 says that among the functions that satisfy the inequalities (3.2) and (3.3), there is a smallest function, namely that function for which these inequalities hold complementarily.

Lemma 1 is closely related to [9, Thm. 6.4, Chap. II]. In fact, that theorem says that if $v_t \in H_0^1(\Omega)$ is the solution of the variational inequality (3.11) or, equivalently, to the problem (3.10), then $v_t \leq v$ for all $v \in H_0^1(\Omega)$ which satisfy $v \geq \psi_t$ and $\Delta v \leq 0$ ($\psi_t \in H_0^1(\Omega)$). In view of the remark after Theorem 3, this gives a proof of our lemma by setting $u_t = v_t - \psi_t$, $u = v - \psi_t$ and by defining $\psi_t \in H_0^1(\Omega)$ by (3.8).

Let us, however, give an independent proof of Lemma 1. For simplicity we restrict ourselves to the case that $\rho_t \in L^p(\Omega)$ for some $p > 1$ (which suffices for our purposes).

Proof of Lemma 1 in case $\rho_t \in L^p(\Omega)$, $p > 1$. Suppose $\rho_t \in L^p(\Omega)$ where $1 < p < \infty$. Then $u_t \in H^{2,p}(\Omega)$, in particular u_t is continuous, by Theorem 4. Thus

$$I = \{z \in \Omega : u_t(z) = 0\} \quad \text{and} \quad D = \Omega \setminus I = \{z \in \Omega : u_t(z) > 0\}$$

are well-defined (relatively) closed and open sets in Ω respectively.

Put $w = u - u_t$. Thus we want to prove that $w \geq 0$. In view of (3.2) and (3.3) it follows from (3.4), which can be interpreted as a Lebesgue integral in this case, that $\Delta u_t = \rho_t$ in D. Thus

$$\Delta w = \Delta u - \rho_t \leq 0 \quad \text{in } D. \tag{4.1}$$

Take an $\varepsilon > 0$ and define $N = \{z \in \Omega : u_t(z) < \varepsilon\}$. Then N is an open neighborhood of $I \cup \partial\Omega$ in Ω. Now, from $u \geq 0$ and (4.1) we have

$$w + \varepsilon \geq 0 \quad \text{in } N \quad \text{and} \tag{4.2}$$

$$\Delta(w + \varepsilon) \leq 0 \quad \text{in } D. \tag{4.3}$$

Since $\partial D \subset I \cup \partial\Omega$, it essentially follows from (4.2), (4.3) and the minimum principle for superharmonic functions that $w + \varepsilon \geq 0$ in D and hence

$$w + \varepsilon \geq 0 \quad \text{in } \Omega = D \cup N.$$

The only problem is that w is not (necessarily) a nice function but just an element of $H_0^1(\Omega)$, so that some care is needed in applying the minimum principle.

To this end, choose $r > 0$ with $2r < \operatorname{dist}(\partial N, \partial D)$ and let h_r be as before Proposition 1. Then an application of the ordinary minimum principle to $(w + \varepsilon) * h_r$ in $\{z \in D : \operatorname{dist}(z, \partial D) > r\}$ shows that

$$(w + \varepsilon) * h_r \geq 0 \quad \text{in } \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > r\}. \tag{4.4}$$

Letting first $r \to 0$ and then $\varepsilon \to 0$, (4.4) yields $w \geq 0$ in Ω, as we wanted to prove.

COROLLARY 1. *Let $\rho, \rho' \in H^{-1}(\Omega)$ and let u and u' be the solutions of* (3.2)–(3.4) *for $\rho_t = \rho$ and ρ' respectively. Suppose that $\rho' \leq \rho$. Then $u \leq u'$.*

Proof. This follows from the lemma with $\rho_t = \rho$, $u_t = u$ since $\Delta u' \leq \rho' \leq \rho$ and $u' \geq 0$.

Remark. There is also an inequality in the other direction. Namely, let $\psi, \psi' \in H_0^1(\Omega)$ be defined by $-\Delta\psi = \rho$ and $-\Delta\psi' = \rho'$ respectively. Then, if $\rho' \leq \rho$, we have $u' \leq u + (\psi - \psi')$. This follows by applying the lemma with $u_t = u'$ and "u in the lemma" $= u + (\psi - \psi')$.

COROLLARY 2. *The solution u_t of* (3.2)–(3.4) *is monotonically increasing* ($=$*non-decreasing*) *as a function of each of μ, D_0 and t* (*more generally, as a function of $\chi_{D_0} + t \cdot \mu$*) *when ρ_t is given by* (3.1), *i.e. if $D_0 \subset D_0'$, $\mu \leq \mu'$ and $t \leq \tau$ then $u_t \leq u_\tau'$* (*self-explanatory notations*).

Proof. This is just a special case of Corollary 1.

LEMMA 2. *Suppose $\mu \in L^p(\Omega)$ for some $1 < p < \infty$ and let $u_t \in H_0^1(\Omega)$ be the solution of* (3.2)–(3.4) *with ρ_t given by* (3.1). *Define*

$$D_t = D_0 \cup \{z \in \Omega : u_t(z) > 0\}. \tag{4.5}$$

Then

$$\Delta u_t=\chi_{D_t}-\chi_{D_0}-t\cdot\mu=\chi_{D_t}-1+\rho_t. \tag{4.6}$$

Proof. By Theorems 3 and 4 $u_t\in H^{2,p}(\Omega)$, in particular u_t is continuous. Thus D_t is a well-defined open set. Observe also that the definition (4.5) is consistent for $t=0$ since $u_t=0$ is the solution of (3.2)–(3.4) for $t=0$ (in view of $\rho_0\geq 0$).

Define $I_t=\{z\in\Omega: u_t(z)=0\}$. Because $u_t\in H^{2,p}(\Omega)$, all partial derivatives of u_t of order ≤ 2 vanish almost everywhere on I_t. (This follows e.g. from [9, Lemma A.4, p. 53].) In particular

$$\Delta u_t=0 \quad \text{a.e. on } I_t. \tag{4.7}$$

Hence (3.2) shows that $\rho_t\geq 0$ a.e. on I_t. By (3.1), using that $\operatorname{supp}\mu\subset D_0$ and $\mu\geq 0$, this gives $\rho_t=1-\chi_{D_0}$ a.e. on I_t, or, using (4.7),

$$\Delta u_t-\rho_t=\chi_{D_0}-1 \quad \text{a.e. on } I_t. \tag{4.8}$$

Now, since $\Delta u_t-\rho_t\in L^p(\Omega)$ and u_t is continuous, the left member of (3.4) can be interpreted as a Lebesgue integral, and it follows from (3.2)–(3.4) and the fact that $u_t>0$ in $\Omega\setminus I_t$ that

$$\Delta u_t-\rho_t=0 \quad \text{in } \Omega\setminus I_t. \tag{4.9}$$

Equation (4.8) together with (4.9) gives

$$\Delta u_t-\rho_t=(\chi_{D_0}-1)\cdot\chi_{I_t}=\chi_{(\Omega\setminus D_0)\cap I_t}$$

(a.e. or in the sense of distributions). Since $(\Omega\setminus D_0)\cap I_t=\Omega\setminus D_t$, this shows that $\Delta u_t=\chi_{D_t}-1+\rho_t$, which is the desired result.

THEOREM 5. *Let μ,ω,Ω and T be as before Definition 2 with $\mu\in L^\infty$ (for simplicity), let $D_0\in\mathfrak{R}_{\omega,\Omega}$ and let ρ_t be defined by* (3.1). *Suppose $u_t\in H_0^1(\Omega)$ and solve* (3.2)–(3.4) *for $t\in[0,T]$. Then, if Ω is large enough and D_t is defined by $D_t=D_0\cup\{z\in\Omega: u_t(z)>0\}$, the map*

$$[0,T]\ni t\to D_t\in\mathfrak{R}_{\omega,\Omega} \tag{4.10}$$

is well defined and is a weak solution. Further, the function "u_t" appearing in the definition of a weak solution is identical with the u_t above.

Remark. This theorem, showing that solutions of the complementarity problems give rise to weak solutions, is similar to [17, Prop. 3].

Proof. We first show that the map (4.10) is well defined. , i.e. that $\Omega\subset\subset D_t\subset\subset\Omega$ for all $t\in[0,T]$.

$\omega\subset\subset D_t$ is evident since $\omega\subset\subset D_0$ and $D_0\subset D_t$. Next, choose $M>0$ and $0<r<R$ such that $\mu\leq M\cdot\chi_{\mathbb{D}(r)}$ and $D_0\subset\mathbb{D}(R)$, and define $R_t>0$ for $t\in[0,T]$ by

$$|\mathbb{D}(R_t)|=tM|\mathbb{D}(r)|+|\mathbb{D}(R)|.$$

Then I claim that $D_t\subset\mathbb{D}(R_t)$ for all t, and hence that it suffices to choose Ω such that $\mathbb{D}(R_T)\subset\subset\Omega$.

To see this, put $\rho_t'=1-\chi_{\mathbb{D}(R)}-tM\chi_{\mathbb{D}(r)}$ and define $u_t'\in H_0^1(\Omega)$ by $\Delta u_t'=\chi_{\mathbb{D}(R_t)}-\chi_{\mathbb{D}(R)}-tM\chi_{\mathbb{D}(r)}$. Then it can be checked that $u_t'>0$ in $\mathbb{D}(R_t)$, $u_t'=0$ outside $\mathbb{D}(R_t)$. This shows that u_t' is the solution of (3.2)–(3.4) corresponding to ρ_t'.

But now $\rho_t'\leq\rho_t$. Therefore $u_t\leq u_t'$ by Corollary 1 of Lemma 1. Hence $u_t=0$ outside $\mathbb{D}(R_t)$, showing that $D_t\subset\mathbb{D}(R_t)$.

It remains to prove that u_t and D_t satisfy (2.1)–(2.3) of Definition 2. (2.1) follows from Lemma 2, (2.2) is (3.3) and (2.3) is (3.4) combined with (4.6). This completes the proof of Theorem 5.

THEOREM 6. (corollary of Theorem 5). *Let μ, ω, Ω and T be as before Definition 2 and let $D \in \mathfrak{R}_{\omega,\Omega}$ be given. Then, if merely Ω is large enough, there exists a weak solution*

$$[0,T] \ni t \to D_t \in \mathfrak{R}_{\omega,\Omega} \tag{4.11}$$

with $D_0 = D$. This solution is unique up to sets of two-dimensional Lebesgue measure zero. Moreover, let $u_t \in H_0^1(\Omega)$ be the function appearing in the definition of a weak solution. Then u_t is unique (as an element of $H_0^1(\Omega)$) and D_t above can be chosen to be

$$D_t = D_0 \cup \{z \in \Omega : u_t(z) > 0\}. \tag{4.12}$$

(Equation (2.1) *shows that u_t is continuous outside* supp μ, *in particular outside D_0, so that the right-hand side of* (4.12) *is a well-defined open set.) Further, the weak solution* (4.11) *is monotonically increasing (=nondecreasing) as a function of each of μ, D_0 and t (more generally, as a function of $\chi_{D_0} + t \cdot \mu$) i.e. if $\mu \le \mu'$, $D_0 \subset D_0'$ and $t \le \tau$ then $D_t \subset D_\tau'$ up to null sets.*

Proof. Suppose first that $\mu \in L^\infty(\Omega)$. Since the problem (3.2)–(3.4) has a unique solution (Theorems 3 and 4), it follows immediately from Theorem 5 that there exists a weak solution (4.11) such that (4.12) holds. Since $u_0 = 0$ for the solution u_t of (3.2)–(3.4), we also have $D_0 = D$. As to the unicity, suppose we have two solutions, $t \to D_t$ and $t \to D_t'$, with $D_0 = D_0' = D$. Then we get, by Theorem 2, two solutions, u_t and u_t', of (3.2)–(3.4) for the same ρ_t. Thus $u_t = u_t'$ and (2.1) shows that $\chi_{D_t} = \chi_{D_t'}$. This is what the unicity statement of the theorem amounts to. The last sentence of the theorem follows immediately from Corollary 2 of Lemma 1. Thus the theorem is proved in case $\mu \in L^\infty(\Omega)$.

If $\mu \notin L^\infty(\Omega)$, we merely apply Proposition 2. Then we are back in the previous case and the theorem follows as before, noting only that the function u_t and v_t in Proposition 2 differ only inside ω, in particular inside D_0, so that (4.12) is not affected by the smoothing process. Note also that the application of Proposition 2 does not destroy the validity of the last sentence in the theorem since the smoothing process is order-preserving (we used positive mollifiers in Proposition 2). This proves the theorem.

5. The moment inequality. We have hitherto shown the equivalence between three concepts of solution for our moving boundary problem, namely the concept of a weak solution, the solution of the linear complementarity problems and the solution of the variational inequalities. There is another equivalent concept of solution which we now want to discuss.

With $\mu \neq 0$ a finite positive measure with compact support, with ω, Ω and T as before Definition 2, and with $D_0 \in \mathfrak{R}_{\omega,\Omega}$ given, let us say that a map $[0,T] \ni t \to D_t \in \mathfrak{R}_{\omega,\Omega}$ satisfies *the moment inequality* if for each $t \in [0,T]$

$$\int_{D_t} \varphi - \int_{D_0} \varphi \ge t \cdot \int \varphi \, d\mu \tag{5.1}$$

for every function $\varphi \in H^2(\mathbb{R}^2)$ which is subharmonic in D_t.

The reason for calling this property the moment inequality is that with $\mu = \delta$ (the Dirac measure at the origin) and by choosing $\varphi = \pm \mathrm{Re}\, z^n$ and $\pm \mathrm{Im}\, z^n$ ($n \ge 0$) in a neighborhood of $\overline{D}_0 \cup \overline{D}_t$, (5.1) yields

$$|D_t| = |D_0| + t \ (n=0) \quad \text{and} \quad \int_{D_t} z^n = \int_{D_0} z^n \ (n \ge 1).$$

The quantities $\int_D z^n$ are called the complex (or analytic) moments of the region D. Thus, if (5.1) holds for $t \to D_t$, all complex moments of order ≥ 1 of D_t are preserved under the map $t \to D_t$, while the zeroth order moment (=the area of the domain) increases linearly.

The fact that solutions of the Hele Shaw problem have this moment preserving property was discovered by Richardson ([12]). The idea to consider relations such as (5.1) for subharmonic functions φ is due to Sakai ([16] and [17]).

We shall now prove that satisfying the moment inequality is equivalent to being a weak solution. More complete results in the same direction are given in [17] (our Theorem 7 corresponds to [17, Props. 1 and 4]).

THEOREM 7. *Let μ, ω, Ω and T be as above with $\mu \in L^\infty(\Omega)$ (for simplicity). Then a map $[0,T] \ni t \to D_t \in \mathcal{R}_{\omega,\Omega}$ is a weak solution if and only if it satisfies the moment inequality.*

Proof. Suppose $[0,T] \ni t \to D_t \in \mathcal{R}_{\omega,\Omega}$ is a weak solution. Since $D_t \subset\subset \Omega$, it is enough to check (5.1) for all $\varphi \in H^2(\mathbb{R}^2)$ which are subharmonic in D_t and vanish on $\partial\Omega$. Thus we assume $\varphi \in H^2(\mathbb{R}^2) \cap H_0^1(\Omega)$.

Let $u_t \in H_0^1(\Omega)$ be the function defined by (2.1). Then $u_t \geq 0$ and $u_t = 0$ a.e. on $\Omega \setminus D_t$ (by (2.2) and (2.3)). Moreover, u_t is continuous and bounded.

Now let $\varphi \in H_0^1(\Omega) \cap H^2(\mathbb{R}^2)$ be subharmonic in D_t. Then $\Delta\varphi \geq 0$ in D_t in the sense of distributions. Moreover, since $\Delta\varphi \in L^2(\Omega)$, the above properties of u_t show that $u_t \cdot \Delta\varphi \in L^2(\Omega)$, $u_t \cdot \Delta\varphi = 0$ a.e. on $\Omega \setminus D_t$ and hence $\int_{\Omega \setminus D_t} u_t \cdot \Delta\varphi = 0$. Using these facts and (2.1), we get

$$\int_{D_t} \varphi - \int_{D_0} \varphi = \langle \chi_{D_t} - \chi_{D_0}, \varphi \rangle = \langle \Delta u_t + t\mu, \varphi \rangle = \langle u_t, \Delta\varphi \rangle + t\langle \mu, \varphi \rangle$$

$$= \int_\Omega u_t \Delta\varphi + t\int \varphi\, d\mu = \int_{D_t} u_t \Delta\varphi + t\int \varphi\, d\mu \geq t\int \varphi\, d\mu.$$

Thus the moment inequality holds.

Conversely, suppose that the moment inequality is satisfied for $[0,T] \ni t \to D_t \in \mathcal{R}_{\omega,\Omega}$. Again define $u_t \in H_0^1(\Omega)$ by (2.1). In terms of u_t(5.1) takes the form

$$\langle \varphi, \Delta u_t \rangle \geq 0 \tag{5.2}$$

for all $\varphi \in H^2(\mathbb{R}^2)$ subharmonic in D_t. Since the restriction mapping $H^2(\mathbb{R}^2) \to H^2(\Omega)$ is onto [18, Thm. 26.7] the test class, $H^2(\mathbb{R}^2)$ for φ in (5.2) can be replaced by $H^2(\Omega)$. In particular (5.2) holds for all $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ which are subharmonic in D_t. For $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ we have $\langle \varphi, \Delta u_t \rangle = \langle u_t, \Delta\varphi \rangle$. Therefore, and since Δ maps $H^2(\Omega) \cap H_0^1(\Omega)$ onto $L^2(\Omega)$, (5.2) can be written

$$\langle u_t, \rho \rangle \geq 0 \tag{5.3}$$

for all $\rho \in L^2(\Omega)$ with $\rho \geq 0$ in D_t. (5.3) shows that

$$u_t \geq 0 \qquad (\text{in } \Omega) \tag{5.4}$$

(because all nonnegative $\rho \in L^2(\Omega)$ are allowed in (5.3)). The choice $\rho = \chi_{D_t} - 1$ is also allowed in (5.3). This gives $\langle u_t, \chi_{D_t} - 1 \rangle \geq 0$ and therefore, since $u_t \geq 0$, $\chi_{D_t} - 1 \leq 0$,

$$\langle u_t, \chi_{D_t} - 1 \rangle = 0. \tag{5.5}$$

The fact that (5.4) and (5.5) hold for u_t defined by (2.1) shows that our map $t \to D_t$ is a weak solution. Thus the proof of Theorem 7 is completed.

6. Summarizing results and further properties of weak solutions. In this final section we shall first reformulate the main result, Theorem 6, so that it becomes more self-contained and simple, and then we shall prove some modest results on properties of weak solutions.

Theorem 6 and the definition of a weak solution suffer from being a bit complicated because of our desire to work in the Sobolev spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. The following theorem is just Theorem 6 liberated from these complications, and it defines implicitly a more simple concept of a weak solution.

THEOREM 8. *Given a finite positive measure μ with compact support in $\mathbb{R}^2$ and a bounded open set D_0 in $\mathbb{R}^2$ with* $\operatorname{supp}\mu\subset D_0$ *there exist, for each $t>0$, a unique open bounded set D_t containing* $\operatorname{supp}\mu$ *and a unique distribution u_t in $\mathbb{R}^2$ such that*

$$\chi_{D_t}-\chi_{D_0}=\Delta u_t+t\cdot\mu, \tag{6.1}$$

$$u_t\geq 0 \quad \textit{and} \tag{6.2}$$

$$D_t=D_0\cup\{z\in\mathbb{R}^2: u_t(z)>0\}, \tag{6.3}$$

where (6.1) *shows that u_t has a representation in form of a function continuous outside* $\operatorname{supp}\mu$ *(in particular outside D_0) and* (6.3) *refers to any such representative.*

Further, D_t is monotonically increasing as a function of each of μ, D_0 and t (more generally, as a function of $\chi_{D_0}+t\cdot\mu$), i.e. if $\mu\leq\mu'$, $D_0\subset D_0'$ and $t\leq\tau$ then $D_t\subset D_\tau'$. Finally

$$\int_{D_t}\varphi-\int_{D_0}\varphi\geq t\cdot\int\varphi\,d\mu \tag{6.4}$$

holds for every function $\varphi\in H^2(\mathbb{R}^2)$ which is subharmonic in D_t.

Proof (sketch). If μ is not sufficiently smooth ($\mu\notin H^{-1}(\mathbb{R}^2)$), we first smooth it out by convolving it with some radially symmetric mollifier (as in Proposition 2) so that $\operatorname{supp}\mu$ is still contained in D_0. Then by choosing appropriate ω,Ω and T and by applying Theorem 6, we obtain functions u_t and domains D_t, related by (4.12) for arbitrary large t. It is easily seen that if we extend the u_t by putting them equal to zero outside Ω, both the u_t and the D_t become independent of the choices of ω,Ω and T, and they satisfy (6.1)–(6.3) (for the smoothed out μ).

Now the D_t will actually provide a solution also for the original μ and the u_t will satisfy (6.1)–(6.3) after a change inside D_0. The details of this are completely similar to the application of Proposition 2 at the end of the proof of Theorem 6 and are therefore omitted. (The details in the case that $\mu=2\pi\delta$ are given in [7, §IVa].)

The unicity and monotonicity properties of D_t also follow easily from Theorem 6. As to the moment inequality (6.4) it follows from Theorem 7 that

$$\int_{D_t}\varphi-\int_{D_0}\varphi\geq t\cdot\int\varphi\,d(\mu*h_\varepsilon)$$

for all $\varphi\in H^2(\mathbb{R}^2)$ subharmonic in D_t, where h_ε is that mollifier, defined by (1.5) for some suitable $\varepsilon>0$, used in the beginning of this proof. But, since $\varphi\leq\varphi*h_\varepsilon$ in a neighborhood of $\operatorname{supp}\mu$ by the sub-mean value property of subharmonic functions, we have

$$\int\varphi\,d(\mu*h_\varepsilon)=\int(\varphi*h_\varepsilon)\,d\mu\geq\int\varphi\,d\mu$$

and so

$$\int_{D_t}\varphi-\int_{D_0}\varphi\geq t\cdot\int\varphi\, d\mu$$

for $\varphi\in H^2(\mathbb{R}^2)$ subharmonic in D_t. This ends the proof of Theorem 8.

Now consider a weak solution $t\to D_t$ in the sense of Theorem 8. There are strong reasons to believe that D_t in some sense becomes nicer as t increases. One for example expects that for any $t>0$, D_t is bounded by analytic curves even if D_0 is not. This we cannot prove (and it is not true for completely arbitary initial domains D_0, as we shall see in a moment). What we can prove is the following.

THEOREM 9. *Let $t\to D_t$ be as in Theorem* 8 *and suppose that, for some fixed t, D_t is connected and finitely connected, and $D_0\subset\subset D_t$. Then ∂D_t is a finite disjoint union of analytic curves and isolated points.*

By an "analytic curve" we mean precisely the following: a subset of $\mathbb{C}$ is an analytic curve if it is the image of $\partial\mathbb{D}$ under some nonconstant function holomorphic in a neighbourhood of $\partial\mathbb{D}$. Thus an analytic curve is allowed to have singular points. For the proof of Theorem 9 we need the following lemma which shows to what extent the term D_0 in (6.3) is necessary.

LEMMA 3. *Let, in the situation of Theorem* 8,

$$U_t=\{z\in\mathbb{C}:u_t(z)>0\}$$

where u_t, being superharmonic in D_0 by (6.1), *is normalized to be lower semicontinuous in D_0 (and continuous outside* $\operatorname{supp}\mu$). *Then, for $t>0$,*

(i) *If N is a component of D_0, then either $N\subset U_t$ or $N\cap U_t=\varnothing$, and the latter case can occur only if $N\cap\operatorname{supp}\mu=\varnothing$.*

(ii) *D_t is the union of U_t and those components of D_0 which do not meet U_t.*

(iii) *If D_0 is connected, then $D_t=U_t$ and D_t is connected.*

Proof of the lemma.

(i) In D_0, and in particular in N, u_t is a superharmonic function. Therefore, since N is connected and $u_t\geq 0$, if u_t attains the value 0 in N, it must be constantly equal to 0 in N. Thus either $u_t>0$ in N or $u_t\equiv 0$ in N. Moreover, it is obvious (from (6.1)) that the latter case can occur only if N does not meet $\operatorname{supp}\mu$. This proves (i).

(ii) is an immediate consequence of (i) and the definition (6.3) of D_t.

(iii) Since $\operatorname{supp}\mu\subset D_0$, it follows from (i) (with $N=D_0$) that $D_0\subset U_t$. Thus, $D_t=U_t$. It remains to show that U_t is connected. But, since D_0 is connected and $D_0\subset U_t$, if U_t were not connected, there would be some component V of U_t such that $V\cap D_0=\varnothing$. And this is impossible because u_t is subharmonic outside $\operatorname{supp}\mu$, in particular outside D_0, $u_t>0$ in V and $=0$ on ∂V. This completes the proof.

Proof of Theorem 9. Let $U_t=\{z\in\mathbb{C}:u_t(z)>0\}$. Then $D_0\subset\subset D_t=D_0\cup U_t$ shows that $\partial D_0\subset U_t$. This implies that each component of D_0 intersects U_t, and so, by (ii) of Lemma 3, $D_t=U_t$.

Now let γ be a component of $\partial D_t=\partial U_t$ and we shall show that γ is an analytic curve or a point. Let $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ denote the Riemann sphere. Since U_t is connected, there is exactly one component of $\hat{\mathbb{C}}\setminus\gamma$ which contains U_t. Let V denote that component. Then it is easy to see that $\partial V=\gamma$. Since γ is connected, this also shows that V is simply connected.

Put $W=D_t\setminus\overline{D}_0=U_t\setminus\overline{D}_0\subset V$, and define

$$S(z)=\bar{z}-4\frac{\partial u_t}{\partial z} \tag{6.5}$$

for $z \in W \cup \gamma$ $(\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y))$. Due to the assumption that D_t is finitely connected, $\overline{D_t}$ is a neighbourhood of γ in V (the other components of ∂D_t cannot cluster at γ). Since $\overline{D_0}$ is a compact subset of D_t (also by assumption), it follows that also W is a full neighborhood of γ in V.

It follows from (6.1) that u_t is continuously differentiable outside μ. Hence $S(z)$ is a continuous function on $W \cup \gamma$. On $\gamma \subset \mathbb{C} \setminus U_t$, u_t attains its minimum $(u_t = 0)$. Therefore $\partial u_t/\partial z = 0$ on γ, so

$$S(z) = \bar{z} \quad \text{on } \gamma. \tag{6.6}$$

In W $S(z)$ is holomorphic since, by (6.5) and (6.1),

$$\frac{\partial S}{\partial \bar{z}} = 1 - \Delta u_t = 1 - \chi_{D_t} + \chi_{D_0} + t \cdot \mu = 0 \quad \text{in } W.$$

Now it is known (cf. [1, Lemma 6.1] or [9, p. 152]) that the existence of a function with the properties of $S(z)$ above gives the desired conclusion for γ. To be precise, if γ just consists of one point, we are done. Otherwise (since V is simply connected and $\partial V = \gamma$) V can be mapped conformally onto $\mathbb{D}$. Let $f\colon \mathbb{D} \to V$ be the inverse map.

Then $S(f(\zeta))$ is holomorphic in the neighborhood $f^{-1}(W)$ of $\partial\mathbb{D}$ in $\mathbb{D}$ and (6.6) shows that

$$S(f(\zeta)) - \overline{f(\zeta)} \to 0 \quad \text{as } \zeta \to \partial\mathbb{D} \quad (\zeta \in \mathbb{D}).$$

It can be seen that this implies that $f(\zeta)$ extends analytically across $\partial\mathbb{D}$ by defining $f(\zeta) = \overline{S(f(1/\bar{\zeta}))}$ for ζ in a neighborhood of $\partial\mathbb{D}$ in $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Moreover, it is seen that $f(\partial\mathbb{D}) = \gamma$. This shows that γ is an analytic curve and the theorem is proven.

Remark. Theorem 9 is not quite satisfactory because of its three assumptions a priori on D_t. The first of these, that D_t is connected, is however harmless and is automatically fulfilled if D_0 is connected (by (iii) of Lemma 3).

The second of the assumptions, that D_t is finitely connected, I do not know how to get rid of although I suspect that it is also automatically fulfilled (possibly some weak assumption on D_0 is needed).

The third assumption, that $D_0 \subset\subset D_t$, can be replaced by either one of the following two assumptions.

(i) t is sufficiently large.

(ii) D_0 is connected and is bounded by finitely many disjoint analytic curves.

As to (i), we actually have $D_0 \subset\subset D_t$ for t sufficiently large. This is seen by comparing our solution $t \to D_t$ (corresponding to the measure μ and initial domain D_0) with some suitable known solution $t \to D_t'$ defined by some measure μ' and initial domain D_0'. As indicated by Proposition 2 we can assume that μ is a continuous function. Since $\mu \neq 0$, μ must be strictly positive somewhere, say at the origin. Then we can choose μ' to be a radially symmetric function such that $\mu' \leq \mu$ and D_0' to be a disc, centered at the origin, such that $D_0' \subset D_0$. Then, by the monotonicity properties of D_t as a function of μ and D_0, we have $D_t' \subset D_t$ for all $t > 0$. On the other hand D_t' is a disc (this follows from radial symmetry and uniqueness of solutions of (6.1)–(6.3)) which grows beyond all bounds as t increases (it follows e.g. by integrating both sides of (6.1) over $\mathbb{R}^2$ that $|D_t'| = |D_0'| + t \cdot \int d\mu'$). Therefore $D_0 \subset\subset D_t'$, in particular $D_0 \subset\subset D_t$, for all sufficiently large t, as we wanted to prove.

To prove that (ii) can replace $D_0 \subset\subset D_t$ we first note that, in the proof of Theorem 9, we still have $D_t = U_t$ by (iii) of Lemma 3. The assumption that ∂D_0 is analytic implies that there exists a function $S_0(z)$, defined and continuous in $(D_0 \setminus K) \cup \partial D_0$, where K is some compact subset of D_0, and holomorphic in $D_0 \setminus K$ such that

$$S_0(z)=\bar{z} \quad \text{on } \partial D_0.$$

Now in the proof of Theorem 8 we change the definitions of W and $S(z)$ to $W=D_t \setminus (K \cup \operatorname{supp} \mu)$ and

$$S(z)=\bar{z}-4\frac{\partial u_t}{\partial z}+\chi_{D_0}(z)\cdot(S_0(z)-\bar{z}) \quad \text{for } z \in W \cup \gamma.$$

Here it is assumed that $S_0(z)$ is extended to $W \cup \gamma$ in some way, e.g. by $S_0(z)=\bar{z}$ for $z \in (W \cup \gamma) \setminus \overline{D}_0$. Then $S(z)$ is continuous on $W \cup \gamma$, holomorphic in W (since (6.1) shows that $\partial S/\partial \bar{z}=0$ in $W \setminus \partial D_0$ and ∂D_0 is a nice curve) and $S(z)=\bar{z}$ on γ. The rest of the proof of Theorem 8 works as before and so (ii) is proved.

I am sure that the assumption $D_0 \subset\subset D_t$ in Theorem 9 can be replaced by some much weaker assumption on D_0 than (ii). However some assumption is needed as the following example shows. Choose D_0 such that ∂D_0 has positive two-dimensional Lebesgue measure. D_0 could e.g. be a square with a lot of slits (of constant length) along one side, spaced as a Cantor set of positive length. Then it will take a positive time for D_t to move through ∂D_0 (since $|D_t|-|D_0|=t\cdot\int d\mu$) and therefore $D_0 \subset\subset D_t$ cannot hold for small $t>0$. Moreover ∂D_t cannot be analytic for these t. This shows that the conclusion of Theorem 9 is not valid if the hypothesis $D_0 \subset\subset D_t$ is completely omitted.

Despite its weaknesses Theorem 9 is strong enough to ensure that classical solutions always are bounded by analytic curves.

THEOREM 10. *Let μ, ω and I be as before Definition* 1 *and let $I \ni t \to D_t \in \mathbb{S}_\omega$ be a classical solution. Then*

(i) *If $I \ni t \to D'_t \in \mathbb{S}_\omega$ is another classical solution and $D'_\tau = D_\tau$ for some $\tau \in I$ then $D'_t = D_t$ for all $t \in I$ with $t > \tau$.*

(ii) *∂D_t is an analytic curve for every $t \in I$.*

Proof. We can assume that μ is a nice function, by Proposition 1. To prove (i) assume, without loss of generality since the concept of a classical solution is invariant with respect to time translations, that $\tau=0 \in I$ and then apply Theorem 1 with $T \geq t$. Combined with the unicity statement of Theorem 6 this gives that $D'_t = D_t$ up to null sets. But it is easy to see that, in view of the regularity assumptions on ∂D_t and $\partial D'_t$, this implies that $D'_t = D_t$ everywhere. This proves (i).

To prove (ii) take $\tau \in I$, $\tau < t$. We may assume that $\tau=0$. Now we apply Theorem 1 with $T \geq t$. This shows that $t \to D_t$ is a weak solution in the sense of Definition 2. According to Theorem 6 (uniqueness part)

$$D_t = D_0 \cup \{z \in \Omega : u_t(z) > 0\} \tag{6.7}$$

up to null sets, where u_t is the function defined by (2.9).

Now using the regularity of ∂D_t it is not very hard to see that (6.7) actually holds everywhere. In fact we have $D_t = \{z \in \Omega : p_{D_t}(z) > 0\}$ for all $0 < t \leq T$, and the function $p_{D_{t'}}(z)$ increases with t', and from this it follows that $D_t = \{z \in \Omega : u_t(z) > 0\}$ (for $0 < t \leq T$). Moreover, the regularity of ∂D_t also implies that $-\partial p_{D_t}/\partial n > 0$ on ∂D_t for all t. In view of the continuity of $\partial \zeta/\partial t = -\nabla p_{D_t}$ (Definition 1) this easily implies that $D_{t_1} \subset\subset D_{t_2}$ for $t_1 < t_2$, in particular that $D_0 \subset\subset D_t$ for $t > 0$. Now it follows from Theorem 9 that ∂D_t is an analytic curve.

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