Acta Applicandae Mathematicae **1**, 209–240. 0167–8019/83/0013–0209$04.80

Quadrature Identities and the Schottky Double

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(Received: 1 March, 1983; accepted: 12 September, 1983)

Abstract. By using Riemann surface theory we obtain results on quadrature domains and identities for analytic functions, e.g., existence of multiply-connected quadrature domains, descriptions of their algebraic boundaries and results on the multitude of quadrature domains associated to a fixed quadrature identity. The main idea is to characterize quadrature domains in terms of meromorphic functions and differentials on Riemann surfaces conformally equivalent to the Schottky doubles of the domains.

AMS (MOS) subject classifications (1980). 30E20, 30C99, 30F30.

Key words. Quadrature identity, quadrature domain, Schottky double, symmetric Riemann surface.

1. Introduction

The present paper, which is a slightly revised version of unpublished paper [11], deals with so-called quadrature domains and quadrature identities for analytic functions. Let us make the following (preliminary) definition.

Let Ω be a domain in $\mathbb{C}$, $L_a^p(\Omega)$ $(1 \leqslant p \leqslant \infty)$ the subspace of the Lebesgue space $L^p(\Omega, \mathrm{d}x\,\mathrm{d}y)$ ($\mathrm{d}x\,\mathrm{d}y$ denotes two-dimensional Lebesgue measure) consisting of analytic functions in Ω and let $\Lambda(\Omega)$ be a subset of $L_a^1(\Omega)$. Then Ω is a *quadrature domain* for the test class Λ if there exist points $z_1, \ldots, z_m$ in Ω and complex numbers a_{kj} for $1 \leqslant k \leqslant m$ and $0 \leqslant j \leqslant n_k - 1$, say, such that the *quadrature identity*

$$\int_\Omega f\,\mathrm{d}x\,\mathrm{d}y = \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} a_{kj} f^{(j)}(z_k) \tag{1.1}$$

holds for $f \in \Lambda(\Omega)$. The integer $n = \Sigma_{k=1}^m n_k$ is called the order of the quadrature identity (provided $a_{k, n_k - 1} \neq 0$ for all k and the z_k are distinct).

Quadrature identities (1.1) are only of interest if the test class Λ is sufficiently large in some sense. Usually $\Lambda(\Omega)$ will be dense in $L_a^1(\Omega)$ or, possibly, dense in $L_{as}^1(\Omega)$, where $L_{as}^p(\Omega)$ $(1 \leqslant p \leqslant \infty)$ is the subspace of $L_a^p(\Omega)$ consisting of those functions which have a single-valued integral in Ω. Until otherwise stated in this introduction, Λ will be L_a^1.

The principal example of a quadrature domain is any disc, in which case $m = 1, n = 1$, z_1 is the center and $a_{1,0}$ the area of the disc. This particular quadrature identity (sometimes with harmonic functions as test functions) has drawn some attention for a couple of decades and different authors have shown, under varying *a priori* assumptions, that discs are the only domains which satisfy such a simple quadrature identity. See [1] or [6] for short accounts of this and other references.

More general quadrature identities for analytic functions seem to have been first studied by Davis ([8] and papers referred to there) and, in more depth, by Aharonov and Shapiro [1]. Later came [6] and [11] and, more recently, [18] and [19]. Beside these works of a general character, there are works dealing with various particular quadrature identities (other than discs): [7, 14, 17, 21, 22].

The present work originates from [1] in the sense that it is inspired by [1] and that the kinds of problems we consider come from [1]. On the other hand, our work is technically independent of [1] and uses other methods.

Let us briefly review some of the results in [1].

(1) Ω is a quadrature domain if and only if there exists a meromorphic function $H(z)$ in Ω such that

$$H(z) = \bar{z} \quad \text{on } \partial\Omega. \tag{1.2}$$

([1], Lemma 2.3, see also [8], Theorem p. 154.)

(2) A simply-connected domain is a quadrature domain if and only if it is the conformal image of the open unit disc $\mathbb{D}$ under a rational function (with the poles off $\mathbb{D}^-$). In particular, there exist plenty of simply-connected quadrature domains ([1], Theorem 1 and [8], Theorem p. 158.)

(3) If Ω is a quadrature domain, $\partial\Omega$ is part of an algebraic curve. ([1], Theorem 3).

In all three cases, the test class of functions is $L^1_a(\Omega)$ and the domains Ω are assumed *a priori* to fulfill

$$\int_\Omega \frac{\mathrm{d}x\,\mathrm{d}y}{|z|} < \infty \quad (z = x + iy). \tag{1.3}$$

(When Ω is unbounded, (1.2) has to be accompanied by a growth condition at infinity.)

Among the questions left open in [1] are:

(4) Do there exist multiply-connected quadrature domains?

(5) Is it possible for two different domains to satisfy the same quadrature identity (1.1)?

In the present paper we do, among other things, the following:

(a) We settle question (4): for any bounded domain W, bounded by finitely many disjoint analytic curves, there exist quadrature domains Ω arbitrarily close to W and conformally equivalent to W (Theorem 4). This result, which was new when [11] appeared, is proved by Riemann surface technique. Today, similar results can also be proved by other methods, due essentially to Sakai [18, 19]. We have included such a result here (Theorem 9 with corollary).

(b) As to uniqueness question (5), we have no answer in the simply connected case (i.e., we do not know whether two simply connected domains can satisfy the same quadrature identity) but we prove that, in general, there exist continuous families of multiply connected domains satisfying one and the same quadrature identity (Theorems 11 and 12).

(c) Further, point (3) above is elaborated a little: We show that the boundary of a quadrature domain must be a whole algebraic curve (Theorem 5) and explicit relations between the coefficients of the polynomial function of that curve and the data of the quadrature identity are obtained (Theorem 10).

The general idea underlying most results in this paper is that of completing a plane domain Ω with a 'back side' $\tilde{\Omega}$ so that a compact Riemann surface

$$\hat{\Omega} = \Omega \cup \partial\Omega \cup \tilde{\Omega},$$

the *Schottky double* of Ω, is obtained (see [20], Ch. 2.2).

From this point of view (1.2) simply means that the pair $(z, H(z))$ defines a meromorphic function on $\hat{\Omega}$, namely that function which equals z on Ω, equals $\overline{H(z)}$ on $\tilde{\Omega}$ and extends continuously over $\partial\Omega$ by (1.2). This gives rise to a generalization of (2) to the multiply connected case:

(d) Let W be a standard domain representing a certain conformal type. Then all quadrature domains conformally equivalent to W are conformal images of W under functions meromorphic on the Schottky double $\hat{W} = W \cup \partial W \cup \tilde{W}$ of W (Theorem 3). (Another generalization of (2) has been given by Avci [6].)

It will be convenient at certain stages to also consider more general types of quadrature identities than (1.1), namely quadrature identities also involving curve integrals:

$$\int_\Omega f\,dx\,dy = \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} a_{kj} f^{(j)}(z_k) + \sum_{k=1}^{r} b_k \int_{\gamma_k} f\,dz + \sum_{k=1}^{p} c_k \int_{\alpha_k} f\,dz, \qquad (1.4)$$

to hold for all $f \in A(\Omega)$. Here $\gamma_1, \ldots, \gamma_r$ are arcs in Ω, $\alpha_1, \ldots, \alpha_p$ are closed curves in Ω and b_k, c_k are complex numbers. Identities of the kind (1.4) will be called quadrature identities and the domains Ω appearing there, quadrature domains. While a quadrature identity (1.1) holds iff the function z on Ω extends to a meromorphic function on $\hat{\Omega}$, as indicated above, a quadrature identity (1.4) holds iff the differential dz on Ω extends to a meromorphic differential on $\hat{\Omega}$ (Theorem 3 with corollary).

A limitation with our methods of working with the Schottky double is that we have to require our domains to have finite area and to be bounded by finitely many continua (Lemma 1). In [1] they are able to work with the weaker assumption (1.3). However, as far as the largeness of Ω is concerned, the difference in assumptions is insignificant because Sakai [18], Theorem 11.2, has shown that any quadrature domain Ω which *a priori* only fulfills

$$\int_{\Omega \setminus \mathbb{D}} \frac{dx\,dy}{|z|^2} < \infty \qquad (1.5)$$

actually has finite area (provided the test class is $L_a^1(\Omega)$).

Another difference is that our methods make the test class $L_a^2(\Omega)$ more natural than $L_a^1(\Omega)$. However, the assumption of finite area implies that $L_a^2(\Omega) \subset L_a^1(\Omega)$ and it can be shown (according to [1], Section 1.3) that the assumption that Ω is bounded by finitely many continua implies that $L_a^2(\Omega)$ is dense in $L_a^1(\Omega)$, so also this difference is insignificant.

1.1. NOTATIONS AND TERMINOLOGY

$\mathbb{D}(a; r) = \{z \in \mathbb{C}: |z - a| < r\}$.
$\mathbb{D} = \mathbb{D}(0; 1)$.
$\mathbb{P} = \mathbb{C} \cup \{\infty\}$ = the Riemann sphere.

domain:	An open, connected and non-empty subset of a Riemann surface.
analytic:	Holomorphic (about functions and differentials).
conformal map:	A map between two Riemann surfaces which is analytic, one-to-one and onto.
continuum:	A closed connected set consisting of more than one point.

analytic curve, regular analytic curve: see Remark 2 after Lemma 1.

$\Omega^- = \overline{\Omega} = \Omega \cup \partial\Omega$:	The closure of the point set Ω in the Riemann surface it is regarded as a subset of. (Also: $\bar{z}$ = the complex conjugate of $z \in \mathbb{C}$.)

j, $\tilde{z}$, $\tilde{f}$, $d\tilde{f}$ and other notations for symmetric Riemann surfaces are defined at the beginning of Section 2.

X':	The dual space of a Banach space X.
$H(\Omega)$:	The space of holomorphic functions on Ω provided with the topology of uniform convergence on compact subsets (Ω may be any Riemann surface).
$M(\Omega)$:	The space of meromorphic functions on Ω (any Riemann surface).

$L_a^p(\Omega)$ and $L_{as}^p(\Omega)$ were defined above for $\Omega \subset \mathbb{C}$, $1 \leqslant p \leqslant \infty$.
$L_a^2(\Omega)$ and $L_{as}^2(\Omega)$ are complex Hilbert spaces with the inner product

$$(f, g) = \int_\Omega f\bar{g}\, dx\, dy = -\frac{1}{2i}\int_\Omega f\bar{g}\, dz\, d\bar{z}.$$

When Ω is a Riemann surface there is no natural substitute for the Lebesgue measure $dx\, dy$, so $L_a^2(\Omega)$ and $L_{as}^2(\Omega)$ then have to be replaced by spaces of differentials:

$\Gamma_a(\Omega)$: the Hilbert of square-integrable holomorphic one-forms (differentials) on Ω with inner product

$$(\omega_1, \omega_2) = -\frac{1}{2i}\int_\Omega \omega_1 \wedge \overline{\omega}_2$$

$\Gamma_{ae}(\Omega)$: the subspace of $\Gamma_a(\Omega)$ consisting of exact forms, i.e., forms of the kind $\omega = df$ with f a single-valued holomorphic function on Ω.

(We are following the notation of [3] here.)

In case Ω is a domain in $\mathbb{C}$ $\Gamma_a(\Omega)$ and $\Gamma_{ae}(\Omega)$ are isometrically isomorphic with $L_a^2(\Omega)$ and $L_{as}^2(\Omega)$ respectively, namely via the map

$$L_a^2(\Omega) \ni f \mapsto f\,dz \in \Gamma_a(\Omega). \tag{1.6}$$

Since a holomorphic differential is always locally exact, we find it convenient to denote holomorphic and meromorphic differentials by symbols such as $df, dg, \ldots$ even when they are not exact. One then has to keep in mind that the symbols $f, g, \ldots$ may stand for functions which are additively multiple-valued. (The work 'function' without further attributes will, however, always mean 'single-valued function'.)

It will be convenient to have names for the kinds of functionals appearing in the right members of (1.1) and (1.4). A functional on $L_a^2(\Omega)$ or $L_{as}^2(\Omega)$, which can be written in the form of the right member of (1.4), will be called a *finite functional*. If all b_k and c_k can be chosen equal to zero, i.e., if it can be written in the form of the right member of (1.1), it will be called a *point functional*. The number $n = \Sigma_{k=1}^m n_k$ in (1.1) is called the order of the point functional, provided $a_{k,n_k-1} \neq 0$ for all k and the z_k are distinct. It is easy to see that every finite functional is continuous. In Remark 4 after Theorem 1, we define finite functionals and point functionals on $\Gamma_a(\Omega)$ in a way which is consistent with the above definition under the identification (1.6).

2. A Preparatory Result on Symmetric Riemann Surfaces

The usual construction of the Schottky double applies only to domains bounded by analytic curves (or, more generally, to bordered Riemann surfaces). This is too restrictive for our purposes. Therefore, we introduce the concept of a symmetric Riemann surface which allows us to take the double (in a slightly more abstract sense) of a larger class of domains, namely those which are conformally equivalent to 'one half of' a (compact) symmetric Riemann surface.

By a symmetric Riemann surface we mean a pair (S, j) consisting of a Riemann surface S and an anticonformal involution j on S (cf. [4, 20]). The latter means that j: $S \to S$ is an anti-analytic map with $j \cdot j = \text{id}$ (the identity map). The principal example for us is the symmetric Riemann surface obtained by taking the Schottky double of a plane domain. The construction of this is briefly as follows. (See [20], Section 2.2 or [3], II.3E for details.)

Let W be a domain in $\mathbb{C}$ with $\Gamma = \partial W$ consisting of finitely many disjoint regular analytic curves. Take copy $\tilde{W}$ of W and weld W and $\tilde{W}$ together along Γ so that a compact surface $\hat{W} = W \cup \Gamma \cup \tilde{W}$ is obtained. If $z \in W$ let $\tilde{z}$ denote the corresponding point on $\tilde{W}$. Then an involution j on $\hat{W}$ is defined by

$$j(z) = \tilde{z} \quad \text{and}$$

$$j(\tilde{z}) = z \quad \text{for } z \in W,$$

$$j(z) = z \quad \text{for } z \in \Gamma.$$

The conformal structure on W, inherited from $\mathbb{C}$, extends in a natural way across Γ to a conformal structure on all of $\hat{W}$. This makes $\hat{W}$ into a Riemann surface. The conformal structure on $\tilde{W}$ will be the opposite to that on W, which means that the function $\tilde{z} \mapsto \bar{z}$ serves as a local variable on $\tilde{W}$, and j becomes anti-analytic. Thus $(\hat{W}, j)$ is a symmetric Riemann surface; in fact, even a compact symmetric Riemann surface (i.e., $\hat{W}$ is compact as a topological space).

Let (S, j) be the compact symmetric Riemann surface obtained by taking the double of a plane domain W, so that $S = \hat{W} = W \cup \Gamma \cup \tilde{W}$. Then it is easy to recover W, $\tilde{W}$ and Γ from (S, j): Γ is the set of fixed points of j, and W and $\tilde{W}$ are the two components of $S \setminus \Gamma$. The surfaces W and $\tilde{W}$ are, however, indistinguishable in the sense that when looking only at (S, j), it is impossible to decide whether (S, j) was constructed as the double of W or as the double of a domain conformally equivalent to $\tilde{W}$. (Notice that W and $\tilde{W}$ need not be conformally equivalent. $\tilde{W}$ is, e.g., conformally equivalent to $\{\bar{z} \in \mathbb{C} : z \in W\}$ if $W \subset \mathbb{C}$.)

Let (S, j) be a compact symmetric Riemann surface in general and let Γ be the set of fixed points of j. Then $S \setminus \Gamma$ consists of either one or two components. That the first case can occur is shown by the example $S = \mathbb{P}$, $j(z) = -1/\bar{z}$, in which case Γ is empty (although $S \setminus \Gamma$ may consist of only one component, even if Γ is nonempty). In the second case, let W be one component of $S \setminus \Gamma$. Then W is a Riemann surface, though not necessarily (conformally equivalent to) a plane domain, and (S, j) can be considered as the double of W, the 'back side', and remaining component of $S \setminus \Gamma$, being $\tilde{W} = j(W)$.

In the rest of this paper we shall only consider symmetric Riemann surfaces of the second kind stated above, i.e., with $S \setminus \Gamma$ consisting of two components. We will then often express ourselves in a somewhat abbreviated way and write, e.g., 'let $\hat{W} = W \cup \Gamma \cup \tilde{W}$ be a compact symmetric Riemann surface' to mean 'let $(\hat{W}, j)$ be a compact symmetric Riemann surface such that $\hat{W} \setminus \Gamma$ consists of two components, W and $\tilde{W}$, Γ being the set of fixed points of j'.

We will always denote the involution on a symmetric Riemann surface by the letter j or by putting a ~ over the argument. Thus, $j(z) = \tilde{z}$. z and $\tilde{z}$ are called conjugate points. For meromorphic functions f and differentials $\mathrm{d}f$ we define

$$\tilde{f}(z) = \bar{f}(\tilde{z}), \qquad \mathrm{d}\tilde{f} = \mathrm{d}(\tilde{f}). \tag{2.1}$$

Thus, $\tilde{f}$ and $\mathrm{d}\tilde{f}$ are also meromorphic. The set of fixed points of j will always be denoted Γ, and is also called the symmetry line of the symmetric Riemann surface.

The following lemma shows, among other things, which plane domains can be doubled in the above sense of being identifiable with one half of a compact symmetric Riemann surface.

LEMMA 1. *Let Ω be a domain in $\mathbb{P}$. Then a necessary and sufficient condition that there exists a compact symmetric Riemann surface (S, j) such that, with Γ the set of fixed points of j, $S \setminus \Gamma$ consists of two components and such that Ω is conformally equivalent to one of them, W, is that*

(a) *the number of components of* $\mathbb{P}\setminus\Omega$ *is finite and at least one, and*
(b) *no component of* $\mathbb{P}\setminus\Omega$ *consists of a single point.*

Suppose (a) *and* (b) *hold and let* $f\colon W\to\Omega$ *be a conformal map. Then*

(i) Ω *has a finite area if and only if* $\mathrm{d}f\in\Gamma_{ae}(W)$.
(ii) *If* f *extends to a meromorphic function in a neighbourhood of* $W\cup\Gamma$ *then* $\partial\Omega$ *is a finite union of analytic curves.*
(iii) *If* f *extends to a meromorphic function on* S *then* $\partial\Omega\cap\mathbb{C}$ *is part of an algebraic curve. If, moreover,* $\Omega\cup\partial\Omega\neq\mathbb{P}$ *then* $\partial\Omega\cap\mathbb{C}$ *is a whole algebraic curve, possibly minus a finite number of points.*

REMARKS. (1) The conditions (a) and (b) can be summarized as '$\partial\Omega$ (or $\mathbb{P}\setminus\Omega$) consist of finitely many, and at least one, continua'.

(2) (ii) requires a definition of 'analytic curve'. We shall use the following terminology. A connected subset Γ of a Riemann surface W is a *regular analytic curve* if for each $\zeta\in\Gamma$ there exists a conformal map f from $\mathbb{D}$ onto a neighbourhood U of ζ such that f maps $\mathbb{D}\cap\mathbb{R}$ onto $U\cap\Gamma$. This is the same as saying that Γ is a one-dimensional real analytic submanifold of W (with the subset topology on Γ and regarding W as a two-dimensional real analytic manifold). Note that we do not require Γ to be closed so that Γ may be an 'arc' only. It is easy to see that, e.g., the symmetry line of a compact symmetric Riemann surface is a finite disjoint union of regular analytic curves.

A subset of a Riemann surface is an *analytic curve* if it is the image of a regular analytic curve under some nonconstant analytic map defined in some neighbourhood of that regular analytic curve.

(3) An analytic curve, as defined above, may have various kinds of singular points. It is, however, easy to see that the singularities on $\partial\Omega$ that can appear in (ii) of the theorem, are of very restricted types due to the fact that the map f in the lemma (which parametrizes $\partial\Omega$) is univalent in W. In fact, the only possible singularities are cusps pointing inwards to Ω and different parts of $\partial\Omega$ having points or segments in common (this includes, e.g., rectilinear slits).

(4) By an algebraic curve (in $\mathbb{C}$) we mean a subset of $\mathbb{C}$ of the kind $\{x+iy\in\mathbb{C}: Q(x,y)=0\}$, where Q is a nonconstant polynomial with real coefficients and irreducible over the complex numbers.

(5) The hypothesis that $\Omega\cup\partial\Omega\neq\mathbb{P}$ in (iii) is necessary for the last conclusion to hold. An example which shows this is

$$S=\mathbb{P},\qquad j(z)=\bar{z},\qquad W=\{z\in\mathbb{C}:\operatorname{Im}z>0\},\qquad f(z)=z^2.$$

Here f is univalent on W and meromorphic on S, but $\Omega=f(W)=\mathbb{C}\setminus[0,\infty)$ so that $\partial\Omega\cap\mathbb{C}=[0,\infty)$, which is not a whole algebraic curve ($[0,\infty)=\{x\in\mathbb{R}:x\geqslant 0\}$).

Proof of Lemma 1. If Ω satisfies (a) and (b), then Ω can be mapped conformally onto a domain $W\subset\mathbb{P}$ bounded by regular analytic curves by repeated use of the Riemann mapping theorem in a well-known manner, and this domain can be doubled in the usual way.

Conversely, if Ω is conformally to W with W as in the lemma, then $\partial W = \Gamma$, and Γ has only finitely-many components, and at least one, and therefore the same must be true for $\partial\Omega$. Therefore (a) holds. (Clearly, connectivity (Ω) = genus $(S) + 1$; cf. [20], Section 2.2.) Moreover, Γ consists of regular analytic curves, in particular no component of Γ consists of a single point. This proves (b), since the property of having an isolated boundary component which consists of a single point is a conformally-invariant property for subdomains of compact Riemann surfaces.

(i) Follows from

$$\text{area }(\Omega) = -\frac{1}{2i}\int_\Omega \mathrm{d}z\,\mathrm{d}\bar{z} = -\frac{1}{2i}\int_W \mathrm{d}f \wedge \mathrm{d}\tilde{f} = \|\mathrm{d}f\|^2.$$

(ii) If f extends to a meromorphic function in a neighbourhood of $W \cup \Gamma$ $f(\Gamma)$ is by definition a finite union of analytic curves (since Γ is a finite union of regular analytic curves). Since $f(\Gamma) = \partial\Omega$, as is easily checked, this proves (ii)..

(iii) Suppose f extends to a meromorphic function on S and put $g = \tilde{f}$. Then f and g are two nonconstant meromorphic functions on the compact Riemann surface S. Hence, by the classical theory [9], Proposition IV.11.6, there exists a non-trivial, irreducible polynomial

$$P(z, w) = \sum a_{kj} z^k w^j \tag{2.2}$$

such that $P(f, g) = 0$ on S. Since $g = \bar{f}$ on Γ this shows that

$$P(z, \bar{z}) = 0, \quad \text{for } z \in f(\Gamma) \cap \mathbb{C}. \tag{2.3}$$

(If $\infty \in f(\Gamma)$ (2.3) has no obvious interpretation at $z = \infty$, so we exclude that point from consideration.)

The polynomial P with the above properties is uniquely determined up to a constant factor and it is not hard to see that this factor can be chosen so that

$$a_{kj} = \bar{a}_{jk}. \tag{2.4}$$

We shall call polynomials P with the property (2.4) *self-conjugate*. Equation (2.4) is equivalent to the polynomial $Q(x, y) = P(x + iy, x - iy)$ having real coefficients.

Thus, choosing P in (2.3) to be self-conjugate, the first conclusion of (iii) follows from (2.3) since $\partial\Omega = f(\Gamma)$.

To prove the second statement of (iii) a more detailed investigation is necessary. Let m be the order of f, that is the number of times it takes almost every value. Then g also has order m and the polynomial (2.2) is of degree at most m in each of z and w separately. This follows from the classical construction of P (see [3], Ch. V.25 or [9], Ch. IV.11). Actually, the degree of P in each of z and w is exactly m as we shall see in a moment.

Put $V = \{z \in \mathbb{C} : P(z, \bar{z}) = 0\}$, $V_0 = \{\text{isolated points in } V\}$, $V_1 = V \setminus V_0$. Then it is easy to see that V_0 is a finite set. (More generally, V has only finitely-many components.) Now, under the assumption $\Omega \cup \partial\Omega \neq \mathbb{P}$, we shall prove that $\partial\Omega \cap \mathbb{C} = V_1$, which will complete the proof of (iii).

We have already seen that $\partial\Omega \cap \mathbb{C} \subset V$, which, of course, implies $\partial\Omega \cap \mathbb{C} \subset V_1$. Therefore, and since $\partial\Omega = f(\Gamma)$, it only remains to prove that $f(\Gamma) \cap \mathbb{C} \supset V_1$, or simply

$$f(\Gamma) \supset V_1. \tag{2.5}$$

Our first step is to prove that f and g generate the field of meromorphic functions on S (that f and g are a primitive pair, in the terminology of [9]). By assumption, $\mathbb{P}\setminus(\Omega \cup \partial\Omega) = \mathbb{P}\setminus f(W \cup \Gamma)$ is a nonempty open set. Therefore there exists a point $z \in \mathbb{P}\setminus f(W \cup \Gamma)$ whose pre-image under f, $f^{-1}(\{z\})$, consists of m distinct points, $\zeta_1, \ldots, \zeta_m$ (because, quite generally when f has order m, there are only finitely-many z for which there are less than m points in the pre-image of z). Clearly $\zeta_1, \ldots, \zeta_m \in \tilde{W} = j(W)$ (since $z \notin f(W \cup \Gamma)$). Now, since f is univalent on W, g is univalent on $\tilde{W}$. Therefore, $g(\zeta_1), \ldots, g(\zeta_m)$ are distinct and it is well-known ([3], Ch. V, 25D and 25F) that the existence of a point z such that g takes m (= order of f) distinct values on $f^{-1}(\{z\})$ is sufficient (and necessary) for f and g to generate the function-field.

In passing we remark that the fact that f and g generate the function field on S implies that the polynomial (2.2) is of a degree exactly m in z and w separately ([9], Proposition IV.11.9).

Now take a point $z_0 \in V_1$. In order to prove (2.5) we have to find a point $\zeta_0 \in \Gamma$ with $f(\zeta_0) = z_0$.

By the definition of V_1, there exists a sequence $\{z_n\}$ $(n = 1, 2, \ldots)$ of distinct points in V converging to z_0. For each n $(n = 1, 2, \ldots)$ there is a point $\zeta_n \in S$ with

$$f(\zeta_n) = z_n, \qquad g(\zeta_n) = \bar{z}_n. \tag{2.6}$$

In fact, because P is irreducible there is for any pair $(z, w) \in \mathbb{C}^2$ with $P(z, w) = 0$ a point $\zeta \in S$ with $f(\zeta) = z, g(\zeta) = w$ (see e.g. [9], proof of Theorem IV.11.4), and $(z_n, \bar{z}_n)$ is such a pair. Since S is compact, we may assume that $\{\zeta_n\}$ converges to some point $\zeta_0 \in S$ and, by continuity, (2.6) then will also hold for $n = 0$.

From (2.6) we have

$$f(\tilde{\zeta}_n) = \overline{g(\zeta_n)} = f(\zeta_n), \qquad g(\tilde{\zeta}_n) = \overline{f(\zeta_n)} = g(\zeta_n) \tag{2.7}$$

for all n, that is both f and g take the same value at $\tilde{\zeta}_n$ as at ζ_n $(n = 0, 1, 2, \ldots)$. We aim at showing that this implies $\tilde{\zeta}_0 = \zeta_0$. Since f and g generate the function field on S, every meromorphic function h on S can be written in the form

$$h(\zeta) = R_0(f(\zeta)) + R_1(f(\zeta))g(\zeta) + \cdots + R_{m-1}(f(\zeta))g(\zeta)^{m-1}, \tag{2.8}$$

where $R_0, \ldots, R_{m-1}$ are rational functions ([9], Proposition IV.11.10). Equation (2.8) holds in the point-wise sense for all $\zeta \in S$ for which the right member makes sense, the exceptional set being those finitely-many ζ for which there appear expressions such as $0/0$, ∞/∞, $\infty \pm \infty$, $0 \cdot \infty$ in the right member. In particular (2.8) holds for almost all $\zeta = \zeta_n$. Therefore, by (2.7), $h(\tilde{\zeta}_n) = h(\zeta_n)$ for almost all n and hence, by continuity, $h(\tilde{\zeta}_0) = h(\zeta_0)$.

Thus, every meromorphic function on S takes the same value at $\tilde{\zeta}_0$ at ζ_0. But this clearly implies $\tilde{\zeta}_0 = \zeta_0$ since, e.g., the Riemann–Roch theorem implies that the meromorphic functions on S separate points on S. Therefore, $\zeta_0 \in \Gamma$ and since $f(\zeta_0) = z_0$ and $z_0 \in V_1$ was arbitrary, we have established (2.5). This completes the proof of (iii) (and Lemma 1).

3. Some Basic Results on Quadrature Identities

We now begin the study of quadrature identities. This section contains some basic results on quadrature identities on compact symmetric Riemann surfaces.

THEOREM 1. *Let $\hat{W} = W \cup \Gamma \cup \tilde{W}$ be a compact symmetric Riemann surface of genus p with W conformally equivalent to a plane domain, and let* $\mathrm{d}g$ *be a meromorphic differential on $\hat{W}$ with no singularities on $W \cup \Gamma$ (in particular* $\mathrm{d}g \in \Gamma_a(W)$*). Then, for all* $\mathrm{d}f \in \Gamma_a(W)$

$$\int_W \mathrm{d}f \wedge \overline{\mathrm{d}g} = 2\pi i \sum_W \operatorname{res}(f\,\mathrm{d}\tilde{g}) - \sum_{k=1}^{p} \int_{\alpha_k} \mathrm{d}f \int_{\beta_k} \mathrm{d}\tilde{g}, \tag{3.1}$$

where $\alpha_1, \beta_1, \ldots, \alpha_p, \beta_p$ is a canonical homology basis for $\hat{W}$ such that each α_k lies in W and such that each β_k avoids the poles of $\mathrm{d}\tilde{g}$. *Here, f denotes any single-valued integral of* $\mathrm{d}f$ *in $W \setminus \bigcup_{k=1}^{p} \beta_k$.*

REMARKS. (1) That $\alpha_1, \beta_1, \ldots, \alpha_p, \beta_p$ is a canonical homology basis for $\hat{W}$ means that it is a system of closed oriented curves in $\hat{W}$ which defines a basis for the first homology group of $\hat{W}$, and having the further property that, for each $k = 1, \ldots, p$, α_k intersects β_k once from the left to the right and that no other curves intersect each other. See [9], Ch. III.1, for details. In our applications we will always choose the basis $\alpha_1, \ldots, \beta_p$ for $\hat{W}$ such that each α_k lies in W. The easiest way to construct such a basis is as follows. Let $\Gamma_1, \ldots, \Gamma_{p+1}$ be the components of Γ, oriented so that W lies to the left. For each $k = 1. \ldots, p$ take α_k to be Γ_k moved slightly into W and take β_k such that $\beta_k \cap W$ goes from Γ_{p+1} to Γ_k, such that $\tilde{\beta}_k = \beta_k$ (as a point set) and such that β_k intersects only α_k. It is obvious that with $\alpha_1, \ldots, \beta_p$ constructed this way, the surface $W \setminus \bigcup_{k=1}^{p} \beta_k$ will be connected and simply connected. Actually, this holds independently of the construction of $\alpha_1, \ldots, \beta_p$, as a mere consequence of $\alpha_k \subset W$, $k = 1, \ldots, p$. Of course, the curves $\alpha_1, \ldots, \beta_p$ can always be chosen such that they avoid any given finite set (e.g., the poles of $\mathrm{d}\tilde{g}$ in the theorem).

(2) Since $W \setminus \bigcup_{k=1}^{p} \beta_k$ is connected and simply connected there exists a single-valued integral f of $\mathrm{d}f$ in $W \setminus \bigcup_{k=1}^{p} \beta_k$. Here f is uniquely determined up to an additive constant, and because $\Sigma_W \operatorname{res} \mathrm{d}\tilde{g} = \Sigma_{\hat{W}} \operatorname{res} \mathrm{d}\tilde{g} = 0$, the value of $\Sigma_W \operatorname{res}(f\,\mathrm{d}\tilde{g})$ does not depend upon this constant.

(3) Suppose the singular parts of $\mathrm{d}\tilde{g}$ are

$$\mathrm{d}\tilde{g}(\zeta) = \sum_{j=0}^{n_k} \frac{a_{kj}\,\mathrm{d}\zeta}{(\zeta - \zeta_k)^{j+1}} + \text{regular}, \tag{3.2}$$

expressed in terms of suitable local variables ζ at the singular points $\zeta_1, \ldots, \zeta_m$ of $\mathrm{d}\tilde{g}$, and put

$$c_k = -\int_{\beta_k} \mathrm{d}\tilde{g}. \tag{3.3}$$

Then (3.1) takes the form

$$\int_W \mathrm{d}f \wedge \mathrm{d}\bar{g} = 2\pi i \sum_{k=1}^{m} \sum_{j=0}^{n_k} \frac{a_{kj}}{j!} f^{(j)}(\zeta_k) + \sum_{k=1}^{p} c_k \int_{\alpha_k} \mathrm{d}f, \tag{3.4}$$

where $f^{(j)}(\zeta_k)$ denotes the jth derivative of f at ζ_k with respect to the local variable chosen there. Observe that $\Sigma_{k=1}^{m} a_{k0} = \Sigma \operatorname{res} \mathrm{d}\tilde{g} = 0$ in (3.2) and (3.4).

(4) Functionals on $\Gamma_a(W)$ (with W as in the theorem) of the kind appearing in the right member of (3.4) (with $\Sigma_{k=1}^{m} a_{k0} = 0$) will be called *finite functionals*. Those which moreover satisfy $a_{k0} = 0$ ($k = 1, \dots, m$) and $c_k = 0$ ($k = 1, \dots, p$) will be called *point functionals*. If W happens to be a subdomain of $\mathbb{C}$ these notions agree with the corresponding notions for $L_a^2(W)$ (see Section 1) when $L_a^2(W)$ is identified with $\Gamma_a(W)$ via (1.6). (Observe that $\Sigma_{k=1}^{m} a_{k0} = 0$ implies that there exist arcs $\gamma_1, \dots, \gamma_r$ in $W \setminus \bigcup_{k=1}^{p} \beta_k$ with end points among $\zeta_1, \dots, \zeta_m$ such that, for suitable coefficients $b_1, \dots, b_r$, $\Sigma_{k=1}^{m} a_{k0} f(\zeta_k) = \Sigma_{k=1}^{r} b_k \int_{\gamma_k} \mathrm{d}f$ for every $\mathrm{d}f \in \Gamma_a(W)$, with f a single-valued integral of $\mathrm{d}f$ in $W \setminus \bigcup_{k=1}^{p} \beta_k$.) It is easy to see that every finite functional is continuous.

Proof of Theorem 1. It suffices to prove (3.1) for $\mathrm{d}f$ which extend continuously to $W \cup \Gamma$ since such $\mathrm{d}f$ are dense in $\Gamma_a(W)$ and both members of (3.1) depend continuously on $\mathrm{d}f$. Let $W' = W \setminus \bigcup_{k=1}^{p} \beta_k$. Then W' is a simply connected subdomain of W and $\mathrm{d}f$ has a single-valued integral f in W'. Let us split each oriented curve $\beta_k \cap W$ ($k = 1, \dots, p$) into two identical copies, β_k^+ and β_k^-, in such a way that the oriented boundary of W' becomes

$$\partial W' = \Gamma + \sum_{k=1}^{p} \beta_k^+ - \sum_{k=1}^{p} \beta_k^-,$$

where Γ is oriented so that W lies to the left of it. Thus β_k^+ is $\beta_k \cap W$ regarded as bounding the part of W' to the left of $\beta_k \cap W$, and similarly for β_k^-.

If $\zeta \in \beta_k \cap W$, let $\zeta^+ \in \beta_k^+$ and $\zeta^- \in \beta_k^-$ denote the two boundary points of W' arising from ζ. Then

$$f(\zeta^+) - f(\zeta^-) = \int_{\alpha_k} \mathrm{d}f. \tag{3.5}$$

Moreover, we have

$$\int_{\beta_k} \mathrm{d}\tilde{g} = \int_{\beta_k \cap W} \mathrm{d}\tilde{g} + \int_{\beta_k \cap \tilde{W}} \mathrm{d}\tilde{g} = \int_{\beta_k \cap W} \mathrm{d}\tilde{g} - \int_{\beta_k \cap W} \mathrm{d}\bar{g} = \int_{\beta_k^-} \mathrm{d}\tilde{g} - \int_{\beta_k^-} \mathrm{d}\bar{g}. \tag{3.6}$$

Now, using (3.5), (3.6) and the fact that $\mathrm{d}\bar{g} = \mathrm{d}\tilde{g}$ along Γ we get

$$\int_W \mathrm{d}f \wedge \mathrm{d}\bar{g} = \int_{W'} \mathrm{d}f \wedge \mathrm{d}\bar{g} = \int_{\partial W'} f\, \mathrm{d}\bar{g}$$

$$= \int_\Gamma f\, \mathrm{d}\bar{g} + \sum_{k=1}^{p} \left(\int_{\beta_k^+} f\, \mathrm{d}\bar{g} - \int_{\beta_k^-} f\, \mathrm{d}\bar{g} \right) = \int_\Gamma f\, \mathrm{d}\tilde{g} + \sum_{k=1}^{p} \int_{\alpha_k} \mathrm{d}f \cdot \int_{\beta_k^-} \mathrm{d}\bar{g}$$

$$= \int_{\partial W'} f\,\mathrm{d}\tilde{g} - \sum_{k=1}^{p} \left(\int_{\beta_k^+} f\,\mathrm{d}\tilde{g} - \int_{\beta_k^-} f\,\mathrm{d}\tilde{g} \right) + \sum_{k=1}^{p} \int_{\alpha_k} \mathrm{d}f \cdot \int_{\beta_k^-} \mathrm{d}\bar{g}$$
$$= \int_{\partial W'} f\,\mathrm{d}\tilde{g} - \sum_{k=1}^{p} \int_{\alpha_k} \mathrm{d}f \cdot \int_{\beta_k^-} \mathrm{d}\tilde{g} + \sum_{k=1}^{p} \int_{\alpha_k} \mathrm{d}f \cdot \int_{\beta_k^-} \mathrm{d}\bar{g}$$
$$= 2\pi i \sum_{W} \operatorname{res} f\,\mathrm{d}\tilde{g} - \sum_{k=1}^{p} \int_{\alpha_k} \mathrm{d}f \cdot \int_{\beta_k} \mathrm{d}\bar{g}.$$

This proves the theorem.

THEOREM 2. *With* $\hat{W} = W \cup \Gamma \cup \tilde{W}$ *as in Theorem* 1, *let* $\mathrm{d}g \in \Gamma_a(W)$. *Then* $\mathrm{d}g$ *extends to a meromorphic differential on* $\hat{W}$ *if and only if the functional* $L \in \Gamma_a(W)'$ *defined by*

$$L(\mathrm{d}f) = \int_W \mathrm{d}f \wedge \mathrm{d}\bar{g} \quad (\mathrm{d}f \in \Gamma_a(W))$$

is a finite functional. When that is the case the singular parts and periods of $\mathrm{d}\tilde{g}$ *and the data of L are related according to* (3.2)–(3.4).

Proof. The 'only if' part of theorem follows immediately from Theorem 1. To prove the 'if' part, suppose L in the theorem is a finite functional, say is given by

$$L(\mathrm{d}f) = 2\pi i \sum_{k=1}^{m} \sum_{j=0}^{n_k} \frac{a_{kj}}{j!} f^{(j)}(\zeta_k) + \sum_{k=1}^{p} c_k \int_{\alpha_k} \mathrm{d}f \quad (\mathrm{d}f \in \Gamma_a(W)),$$

where $\alpha_1, \beta_1, \ldots, \alpha_p, \beta_p$ is a canonical homology basis for $\hat{W}$ as in Theorem 1, and which avoids all the ζ_k. It is a classical result in the theory of compact Riemann surfaces [9], Ch. III.2–3, that on $\hat{W}$ (or any compact Riemann surface) there exists a unique meromorphic differential $\mathrm{d}G$ with

(i) poles, located outside the curves $\alpha_1, \ldots, \beta_p$, with arbitrarily prescribed singular parts, subject to the only condition that the sum of the residues be zero, and

(ii) the periods $\int_{\beta_k} \mathrm{d}G$ ($k = 1, \ldots, p$) (alternatively, the periods $\int_{\alpha_k} \mathrm{d}G$) arbitrarily prescribed.

In particular, there exists a meromorphic differential $\mathrm{d}G$ on $\hat{W}$ such that the singular parts and the periods of $\mathrm{d}\tilde{G}$ are given by

$$\mathrm{d}\tilde{G}(\zeta) = \sum_{j=0}^{n_k} \frac{a_{kj}\,\mathrm{d}\zeta}{(\zeta - \zeta_k)^{j+1}} + \text{regular}$$

at ζ_k ($k = 1, \ldots, m$) and

$$\int_{\beta_k} \mathrm{d}\tilde{G} = -c_k \quad (k = 1, \ldots, p). \tag{3.7}$$

(It is more convenient to apply (i) and (ii) on $\mathrm{d}\tilde{G}$ than on $\mathrm{d}G$ itself.)

Now, Theorem 1 together with Remark 3 following it shows that this gives

$$\int_W \mathrm{d}f \wedge \mathrm{d}\bar{G} = 2\pi i \sum_{k=1}^{m} \sum_{j=0}^{n_k} \frac{a_{kj}}{j!} f^{(j)}(\zeta_k) + \sum_{k=1}^{p} c_k \int_{\alpha_k} \mathrm{d}f = L(\mathrm{d}f) \quad \text{for all} \quad \mathrm{d}f \in \Gamma_a(W)$$

Thus $\int_W \mathrm{d}f \wedge \mathrm{d}\bar{G} = \int_W \mathrm{d}f \wedge \mathrm{d}\bar{g}$ for all $\mathrm{d}f \in \Gamma_a(W)$. Since $\mathrm{d}g$ and $\mathrm{d}G|_W$ both belong to $\Gamma_a(W)$ this shows that $\mathrm{d}G|_W = \mathrm{d}g$. Hence, $\mathrm{d}G$ provides an extension of $\mathrm{d}g$ to $\hat{W}$ and the theorem is proven.

REMARK. Given any continuous linear functional L on $\Gamma_a(W)$ there clearly exists a $\mathrm{d}g \in \Gamma_a(W)$ with $L(\mathrm{d}f) = \int_W \mathrm{d}f \wedge \mathrm{d}\bar{g}$ $(\mathrm{d}f \in \Gamma_a(W))$ since the form $-(1/2i)\int_W \mathrm{d}f \wedge \mathrm{d}\bar{g}$ is an inner product on $\Gamma_a(W)$. Theorem 2 says that this $\mathrm{d}g$ extends meromorphically to $\hat{W}$ if and only if L happens to be a finite functional.

4. Main Theorems

By means of conformal mappings the results of Section 3 lead quickly to our principal theorems.

Let $g\colon W \to \Omega$ be a conformal map between two Riemann surfaces W and Ω. Then g gives rise to an isometric isomorphism

$$g^*\colon \Gamma_a(\Omega) \to \Gamma_a(W)$$

by pull-back of differentials, i.e.,

$$g^*(\mathrm{d}f) = \mathrm{d}(f \circ g) \quad \text{for } \mathrm{d}f \in \Gamma_a(\Omega).$$

Further, g^* has an adjoint

$$g^{**}\colon \Gamma_a(W)' \to \Gamma_a(\Omega)'$$

defined by

$$g^{**}(L) = L \circ g^* \quad \text{for } L \in \Gamma_a(W)'.$$

Now suppose $L \in \Gamma_a(W)'$ is a finite functional. Then the same holds for $g^{**}(L)$. Indeed, if L is given by

$$L(\mathrm{d}f) = \sum_{k=1}^{m} \sum_{j=0}^{n_k} a_{kj} f^{(j)}(\zeta_k) + \sum_{k=1}^{p} c_k \int_{\alpha_k} \mathrm{d}f \quad (\mathrm{d}f \in \Gamma_a(W)), \tag{4.1}$$

then, for $\mathrm{d}F \in \Gamma_a(\Omega)$,

$$\begin{aligned} g^{**}(L)(\mathrm{d}F) &= L(g^*(\mathrm{d}F)) = L(\mathrm{d}(F \circ g)) \\ &= \sum_{k=1}^{m} \sum_{j=0}^{n_k} a_{kj} (F \circ g)^{(j)}(\zeta_k) + \sum_{k=1}^{p} c_k \int_{\alpha_k} \mathrm{d}(F \circ g) \\ &= \sum_{k=1}^{m} \sum_{j=0}^{n_k} b_{kj} F^{(j)}(z_k) + \sum_{k=1}^{p} c_k \int_{g(\alpha_k)} \mathrm{d}F, \end{aligned} \tag{4.2}$$

where $z_k = g(\zeta_k)$ and the b_{kj} are linear combinations of the a_{kj} with coefficients involving the derivatives of g at the ζ_k.

After these preliminaries, we now state one of our main theorems.

THEOREM 3. *Suppose Ω is a domain in $\mathbb{C}$ bounded by finitely many continua and of finite area, $\hat{W} = W \cup \Gamma \cup \tilde{W}$ is a compact symmetric Riemann surface and $g\colon W \to \Omega$ is a* *conformal map. Then* dg *extends to a meromorphic differential on* $\hat{W}$ *if and only if the functional*

$$L(f) = \int_{\Omega} f \, \mathrm{d}x \, \mathrm{d}y \tag{4.3}$$

on $L_a^2(\Omega)$ *is a finite functional. Moreover, the integral* g *of* dg *is single-valued on* $\hat{W}$ *if and only if* L *is a point functional.*

Proof. Regarding L as a functional on $\Gamma_a(\Omega)$ instead of $L_a^2(\Omega)$ via (1.6) it takes the form $L(f \, \mathrm{d}z) = -(1/2i) \int_{\Omega} f \, \mathrm{d}z \wedge \mathrm{d}\bar{z}$ ($f \, \mathrm{d}z \in \Gamma_a(\Omega)$), or

$$L(\mathrm{d}f) = -\frac{1}{2i} \int_{\Omega} \mathrm{d}f \wedge \mathrm{d}\bar{z} \quad (\mathrm{d}f \in \Gamma_a(\Omega)). \tag{4.4}$$

Let $\phi\colon \Omega \to W$ be the inverse map of f. Then

$$\begin{aligned} \phi^{**}(L)(\mathrm{d}f) = L(\mathrm{d}(f \circ \phi)) &= -\frac{1}{2i} \int_{\Omega} \mathrm{d}(f \circ \phi) \wedge \mathrm{d}\bar{z} \\ &= -\frac{1}{2i} \int_{\Omega} \mathrm{d}(f \circ \phi) \wedge d(\overline{g \circ \phi}) \\ &= -\frac{1}{2i} \int_{W} \mathrm{d}f \wedge \mathrm{d}\bar{g} \quad (\mathrm{d}f \in \Gamma_a(W)). \end{aligned} \tag{4.5}$$

Now L is a finite functional if and only if $\phi^{**}(L)$ is a finite functional as we saw above and, by (4.5) and Theorem 2, this occurs if and only if dg extends to a meromorphic differential on $\hat{W}$. This proves the first statement in the theorem. The second (and last) statement follows from the relations between the singularities and periods of dg and the data in the functional L (see Remark 4, below).

REMARKS. (1) Notice that when Ω, as a subdomain of $\mathbb{C}$, is bounded by finitely-many continua and has finite area, then it is bounded by finitely many, and at least one, continua regarded as subdomain of $\mathbb{P}$. Therefore it follows from Lemma 1 that when Ω satisfies the hypotheses in the theorem the Riemann surface $\hat{W}$ and the map g always exist. Therefore Theorem 3 shows how to generate all quadrature domains of finite area and bounded by finitely many continua.

(2) Disregarding differences in *a priori* assumptions on the domains, Theorem 3 generalizes the result ([1], Theorem 1; [8], Theorem p. 158) that the simply connected quadrature domains (for point functionals) are obtained as images of the unit disc $\mathbb{D}$ under rational functions with all poles outside $\mathbb{D}^-$. In fact, when Ω is simply connected, W in the theorem can be taken to be $\mathbb{D}$ and $\hat{W}$ can be identified with $\mathbb{P}$, the involution being $\phi(z) = 1/\bar{z}$. Since the meromorphic functions on $\hat{W}$ are exactly the rational functions, this identifies our result with that of [1] and [8] in the simply connected case.

(3) Our technique of considering extensions of functions to the Schottky double can be regarded as a kind of reflection method. Another type of reflection method has been

used by Avci [6], Theorems 13 and 14, to obtain results analogous to our Theorem 3.

(4) The relations between the singular parts and periods of dg on the one hand, and the data of L on the other hand, are obtained by combining (3.2)–(3.4) with (4.1)–(4.2) and (4.4)–(4.5). This yields that if dg is given by (3.2)–(3.3) the quadrature identity of $\Omega = g(W)$ is

$$-\frac{1}{2i}\int_\Omega df \wedge d\bar{z} = \sum_{k=1}^{m}\sum_{j=0}^{n_k} b_{kj} f^{(j)}(z_k) + \sum_{k=1}^{p} c_k \int_{g(\alpha_k)} df \quad (df \in \Gamma_a(\Omega)) \tag{4.6}$$

for suitable b_{kj} and c_j, or (with $a_{kj} = b_{k,j+1}$)

$$\int_\Omega f\,dx\,dy = \sum_{k=1}^{m}\sum_{j=0}^{n_k-1} a_{kj} f^{(j)}(z_k) + \sum_{k=1}^{r} b_k \int_{\gamma_k} f\,dz + \\ + \sum_{k=1}^{p} c_k \int_{g(\alpha_k)} f\,dz \quad (f \in L_a^2(\Omega)). \tag{4.7}$$

Here $\alpha_1, \beta_1, \ldots, \alpha_p, \beta_p$ is a canonical homology basis for $\hat{W}$ as in Theorem 1, $m, n_1, \ldots, n_m$ and p are the same integers as in (3.2)–(3.3), $z_k = g(\zeta_k)$, a_{kj} are linear combinations of the a_{kj} in (3.2) and the c_k in (4.7) are $-(1/2i)$ times the c_k in (3.3). Further, $\gamma_1, \ldots, \gamma_r$ are arcs in $\Omega \setminus \bigcup_{k=1}^{p} g(\beta_k)$ with end points among $z_1, \ldots, z_m$ and $b_1, \ldots, b_r$ coefficients such that (with the b_{k0} in (4.6))

$$\sum_{k=1}^{m} b_{k0} f(z_k) = \sum_{k=1}^{r} b_k \int_{\gamma_k} df \tag{4.8}$$

for all $df \in \Gamma_a(\Omega)$. In (4.6) and (4.8) f denotes any single-valued integral of df in $\Omega \setminus \bigcup_{k=1}^{p} g(\beta_k)$. Equation (4.8) is possible to satisfy because $\Sigma_{k=1}^{m} b_{k0} = -\pi \Sigma \operatorname{res} d\tilde{g} = 0$ (it is easy to see that $b_{k0} = -\pi \operatorname{res}_{\zeta_k} d\tilde{g}$).

It follows from these relations between (3.2)–(3.3) and (4.7) that the second term in the right member of (4.7) vanishes if and only if dg has no residues and that the last term vanishes if and only if dg has no 'β-periods'. Since dg, moreover, has no 'α-periods' (because g is single-valued on W), it follows in particular that the right member of (4.7) is a point functional if and only if the integral g of dg is single-valued on all of $\hat{W}$.

By restating Theorem 3 in another form, we obtain the following result similar to [1], Lemma 2.3.

COROLLARY 3.1. *Suppose Ω is a domain in $\mathbb{C}$ bounded by finitely many continua and of finite area. Then the functional* (4.3) *on $L_a^2(\Omega)$ is a finite functional if and only if there exists a meromorphic differential $h(z)\,dz$ in Ω such that*

$$dz = \overline{h(z)\,dz} \quad \text{along } \partial\Omega. \tag{4.9}$$

Moreover, (4.3) *is a point functional if and only if there exists a meromorphic function $H(z)$ in Ω, extending continuously to $\Omega \cup \partial\Omega$ such that*

$$z = \overline{H(z)} \quad \text{on } \partial\Omega. \tag{4.10}$$

REMARKS. (1) (4.9) is to be interpreted as follows. There exists a single-valued branch $H(z)$ of the integral of $h(z)\,\mathrm{d}z$ in some neighbourhood of $\partial\Omega$, extending continuously to $\partial\Omega$ such that $z = \overline{H(z)}$ + const. on each component of $\partial\Omega$.

(2) Equation (4.10) shows that $H(z)$ is the so-called Schwarz function for $\partial\Omega$ (see [8]).

Proof of Corollary 3.1. For simplicity we prove only the second half of the corollary. The proof of the first half is similar, the only difference being that one has to handle a multiple-valued meromorphic function $H(z)$ instead of a single-valued one.

Take a compact symmetric Riemann surface $\hat{W} = W \cup \Gamma \cup \tilde{W}$ and a conformal map $g: W \to \Omega$. This is possible by Lemma 1. Let ϕ be the inverse map of g.

Suppose that (4.3) is a point functional. Then g extends meromorphically to $\hat{W}$ by Theorem 3 and the function $H = \tilde{g} \circ \phi$ is seen to have the required properties.

Conversely, suppose that H exists such that (4.10) holds and put $f = H \circ g$. Then f is a meromorphic function on W which extends continuously to Γ and, by (4.10), satisfies $f = \bar{g}$ there. Hence, f yields an extension of g to $\hat{W}$, namely by setting $g(\tilde{z}) = \overline{f(z)}$ for $\tilde{z} \in \tilde{W} \cup \Gamma$. By applying Theorem 3 again the desired conclusion follows.

Now, let us return to Theorem 3. It gives a method of producing quadrature domains for the class $L^2_a(\Omega)$ of arbitrary conformal types (subject to the usual restrictions given by Lemma 1): take a bounded domain $W \subset \mathbb{C}$ bounded by regular analytic curves and of the desired conformal type. Let $\hat{W} = W \cup \Gamma \cup \tilde{W}$ be the Schottky double of W. If we can find a meromorphic differential $\mathrm{d}g$ on $\hat{W}$ such that $\mathrm{d}g|_W \in \Gamma_{ae}(W)$ and such that its integral g is univalent on W then $\Omega = g(W)$ will be a quadrature domain by Theorem 3.

The condition $\mathrm{d}g|_W \in \Gamma_{ae}(W)$ means exactly that $\mathrm{d}g$ shall have all its singularities on the back side $\tilde{W}$ and that g shall be single-valued on W. These conditions are easily satisfied. In fact, the poles and singular parts (with sum of redidues equal to zero) of $\mathrm{d}g$ as well as the periods $\int_{\alpha_k} \mathrm{d}g$ can be arbitrarily prescribed. The problem is to get g univalent on W.

This problem can be solved by an approximation argument as follows. There certainly exist functions which are defined and are univalent in some neighbourhood of $W \cup \Gamma$ in $\hat{W}$. One can, for example, take the identity function $z: W \to \mathbb{C}$ which embeds W in $\mathbb{C}$. Since Γ consists of regular analytic curves, both regarded as a subset of $\mathbb{C}$ and regarded as a subset of $\hat{W}$, z extends by reflection in Γ to an analytic function f from a neighbourhood of $W \cup \Gamma$ in $\hat{W}$ to $\mathbb{C}$. It is easy to see that f will also be univalent in some neighbourhood of $W \cup \Gamma$.

Now there is a Runge approximation theorem for compact Riemann surfaces, stating that if U is any open subset of a compact Riemann surface S then $M(S) \cap H(U)$ is dense in $H(U)$ in the toplogy of uniform convergence on compact subsets of U ([5], [10], [13], Satz 1). Applying this theorem with $S = \hat{W}$ and U = (a neighbourhood of $W \cup \Gamma$ in which f is defined) we get f approximated uniformly on some neighbourhood of $W \cup \Gamma$ by functions in $M(\hat{W})$. By making the approximation sufficiently fine we also achieve that the approximating function g is univalent in some neighbourhood of $W \cup \Gamma$ (in particular in W).

This function g (or rather the differential dg) has all the properties we require. Even more, since g is single-valued on $\hat{W}$, the resulting quadrature identity on $\Omega = g(W)$ will be of the kind (1.1). Notice also that Ω is close to W, since g approximates the identity function on W. Therefore, we have proved the following.

THEOREM 4. *Suppose W is a bounded domain in $\mathbb{C}$, bounded by finitely many disjoint regular analytic curves. Then there are domains arbitrarily close to W and conformally equivalent to W, which admit quadrature identities of the kind* (1.1) *for the test class L_a^2.*

REMARKS. (1) It is easy to modify the above construction to obtain quadrature domains admitting identities of the kind (1.4) with some b_k and/or some c_k different from zero. In fact, if we want to have $c_1 \neq 0$, then we only have to take meromorphic differential dh on $\hat{W}$ such that $dh|_W \in \Gamma_{ae}(W)$ and such that $\int_{\beta_1} dh \neq 0$. (Such differentials exist, since we allow for arbitrarily many poles on $\tilde{W}$). Then, with g as above and with $\varepsilon \neq 0$ sufficiently small, $g + \varepsilon h$ will be univalent in W and map W onto a domain with the required property. Similarly, if we want to have some b_k different from zero, we apply the same procedure but with the condition $\int_{\beta_1} dh \neq 0$ replaced by the condition that dh shall have some pole (on $\tilde{W}$) with nonzero residue.

(2) Using slightly stronger variants of the Runge approximation theorem ([5], cf. also [15]) one can prescribe, e.g., the numbers n_k or m in (1.1). With U and S as above, it is true that not only $M(S) \cap H(U)$ but even $M(S) \cap H(S \setminus E)$ in dense in $H(U)$, where E is any subset of $S \setminus U$ which intersects each component of $S \setminus U$. In our case $S \setminus U$ can be assumed to be connected so that E can be taken to consist of a single point. This shows that we can prescribe $m = 1$ in (1.1). By using another variant of the approximation theorem, one can prescribe all n_k to be equal to one (but then one, of course, loses the control over m).

Another consequence of Theorem 3 is that a quadrature domain must have a nice boundary.

THEOREM 5. *Suppose Ω is a domain in $\mathbb{C}$ of finite area and bounded by finitely many continua. Then, if Ω admits a quadrature identity of the kind* (1.4) *for the test class $L_a^2(\Omega)$, $\partial\Omega$ is a finite union of analytic curves* (*possibly containing singular points of the kinds described in Remark* 3 *after Lemma* 1). *If, moreover, all b_k and c_k in* (1.4) *vanish, i.e., the identity is of the kind* (1.1), *then $\partial\Omega$ is an algebraic curve, possibly minus a finite number of points.*

Proof. Combine Theorem 3 with Lemma 1.

REMARK. It is not true that every domain whose boundary is an algebraic curve is a quadrature domain (in our sense). For example, the interior of an ellipse is not (but satisfies other kinds of quadrature identities [8], pp. 132–133).

In the case that $\partial\Omega$ is an algebraic curve ($b_k, c_k = 0$) the relations between the coefficients in the polynomial equation of $\partial\Omega$ and the data in the quadrature identity can be stated explicitly to a certain extent. This is the topic of Section 6.

The fact that $\partial\Omega$ is part of an algebraic curve when $b_k, c_k = 0$ was first proven in [1], Theorem 3, where also results about $\partial\Omega$ when only $b_k = 0$ can be found (Theorems 8–10).

5. Some Special Results

By Theorem 3, we have set up a relationship between quadrature domains and certain meromorphic functions and differentials on compact symmetric Riemann surfaces. A consequence of this is that we may get results about quadrature domains as corollaries of (old or new) results about functions and differentials on compact Riemann surfaces. Theorems 4 and 5 can be said to be results obtained this way. In this section we shall give a few other such results, but of a more special character.

The first of these, Corollary 6.1, is a rather classical kind of converse of the mean-value property for analytic functions. Different variants of it have been proved by many authors (see [1], Sections 1.1 and 1.3, for a survey of this and for references) and it has been included here only for the purpose of illustrating the technique.

Corollary 7.1 is, apart from differences in *a priori* assumptions, a generalization of [1], Theorem 4. It was first proved in [11], Corollary 4.5. Another proof is given in [6], Theorem 6, and a special case of Corollary 7.1 is in [21], Theorem 2.

Corollary 8.1 seems to be a new result.

In Corollaries 6.1–8.1 the domains Ω are assumed *a priori* to be of finite area and bounded by finitely many continua.

THEOREM 6. *Suppose g is a meromorphic function of order one on a compact symmetric Riemann surface $\hat{W} = W \cup \Gamma \cup \tilde{W}$. Then g maps $\hat{W}$ conformally onto $\mathbb{P}$ and takes W onto the interior or exterior of a disc or onto a half-plane.*

Proof. The first statement is well-known and the second statement follows by considering the symmetric-Riemann-surface structure that g induces on $\mathbb{P}$: the involution must be an anti-möbius transformation having a nonempty fixed point set C and $g(W)$ must be one of the components of $\mathbb{P}\setminus C$. Since C is easily seen to be a circle or a straight line the theorem follows.

COROLLARY 6.1. *Suppose*

$$\int_\Omega f \,\mathrm{d}x\,\mathrm{d}y = af(z_0) \quad \text{for all } f \in L_a^2(\Omega). \tag{5.1}$$

Then Ω is a disc and z_0 its center.

Proof. Take a compact symmetric Riemann surface $\hat{W} = W \cup \Gamma \cup \tilde{W}$ and a conformal map $g\colon W \to \Omega$ as in Lemma 1. Then Theorem 3 shows that g is as in Theorem 6 (observe that necessarily $a \neq 0$ in (5.1), by choosing $f = 1$). Hence, $\Omega = g(W)$ is a disc (since it has finite area). Moreover, it follows from Remark 4 after Theorem 3 that the pre-image of z_0 under g and the pole of g are conjugate on $\hat{W}$ with respect to the involution on $\hat{W}$, hence, z_0 and ∞ are conjugate points on $\mathbb{P}$ with respect to the induced involution on $\mathbb{P}$, and so are mirror points with respect to the circle $\partial\Omega$. Hence, z_0 is the center of the disc Ω.

REMARKS. (1) With the test class $L^1_a(\Omega)$ the conclusion of Corollary 6.1 holds true with the single assumption (1.5) on Ω ([18], Example 1.1 together with Theorem 11.2). On the other hand, some assumption on Ω is necessary in Corollary 6.1 because there exist domains of infinite area other than the entire complex plane, which satisfy (5.1) for the test class L^1_a (see [18], Theorem 11.5).

(2) As is shown in [1], Theorem 7 (and [11], Corollary 4.3) the conclusion of Corollary 6.1 still holds if the test class is shrunk to $L^2_{as}(\Omega)$, provided Ω is bounded by finitely many continua and has finite area.

THEOREM 7. *Suppose $\hat{W} = W \cup \Gamma \cup \tilde{W}$ is a compact symmetric Riemann surface and g a meromorphic function on $\hat{W}$ such that*

(i) *g has order two,*

(ii) *g is univalent on W,*

(iii) $\Omega^- \nsubseteq \mathbb{P}$, *where $\Omega = g(W)$.*

Then $\hat{W}$ has genus zero, hence is conformally equivalent to the Riemann sphere.

Proof. Suppose g is as in the theorem. Being of order two g gives rise to an automorphism $\sigma_g\colon \hat{W} \to \hat{W}$ of order two (i.e., $\sigma_g \circ \sigma_g = \mathrm{id}$), defined by $\sigma_g(\zeta_1) = (\zeta_2)$ whenever $\{\zeta_1, \zeta_2\} = g^{-1}(z)$ for some $z \in \mathbb{P}$. Put $U_1 = \sigma_g(W)$, $U_2 = \tilde{W} \backslash U_1^-$. Then $U_1 \subset \tilde{W}$ (because g is univalent on W), $W \cup U_1 = g^{-1}(\Omega)$, $U_2 = g^{-1}(\mathbb{P} \backslash \Omega^-) = \sigma_g(U_2)$ and $U_2 \neq \phi$ (since $\mathbb{P}\backslash\Omega^- \neq \phi$). We shall, however, see that all this is impossible if the genus p of W is greater than zero.

The case $p > 1$. In this case $\hat{W}$ is a hyperelliptic Riemann surface and it is known ([9], Proposition III.7.9 with corollaries) that σ_g does not depend upon g (σ_g is *the* hyperelliptic automorphism). In particular $\sigma_g = \sigma_{\bar{g}}$, which implies that $g(\zeta_1) = g(\zeta_2)$ if and only if $g(\tilde{\zeta}_1) = g(\tilde{\zeta}_2)$ ($\zeta_1, \zeta_2 \in \hat{W}$). But take $\zeta_1 \in U_2$, $\zeta_2 = \sigma_g(\zeta_1) \in U_2$. Then $g(\zeta_1) = g(\zeta_2)$ and, therefore, $g(\tilde{\zeta}_1) = g(\tilde{\zeta}_2)$. Since, however, $\tilde{\zeta}_1, \tilde{\zeta}_2 \in W$ and we could have chosen $\zeta_1 \in U_2$ such that $\zeta_2 \neq \zeta_1$ (σ_g has only finitely many fixed points and U_2 is open) this contradicts the univalency of g on W.

The case $p = 1$. In this case a different argument is needed since σ_g now depends on g. We may represent $\hat{W}$ as $\hat{W} = \mathbb{C}/G$, where G is a discrete group of the kind $G = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with $\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$.

Let $z_1, z_2 \in \mathbb{C}$ be (represent) the poles of g. Then $\sigma_g(z_1) = z_2 \pmod G$. Hence, Abel's theorem ([2], Ch. 7, Theorem 6) shows that $\zeta_1 + \zeta_2 = z_1 + z_2 \pmod G$ whenever $\zeta_2 = \sigma_g(\zeta_1) \pmod G$. This shows that σ_g is of the form

$$\sigma_g(\zeta) = -\zeta + a \pmod G \quad (\zeta \in \mathbb{C}), \tag{5.2}$$

where a is a constant ($a = z_1 + z_2$). But (5.2) is in conflict with the fact that σ_g maps W (exactly half of $\hat{W}$) onto U_1 (strictly included in the other half of $\hat{W}$) since, e.g., (5.2) shows that σ_g preserves area in $\mathbb{C}$ whereas area $(U_1) <$ area (W).

This concludes the proof of the theorem.

COROLLARY 7.1. *Suppose Ω admits a quadrature identity* (1.1) *of order two for the test class $L^2_a(\Omega)$. Then Ω is simply connected.*

Proof. Take a compact symmetric Riemann surface $\hat{W} = W \cup \Gamma \cup \tilde{W}$ and a conformal map $g\colon W \to \Omega$ as in Lemma 1. Then Theorem 3 shows that g has the properties (i)–(iii) of Theorem 7. Hence, $\hat{W}$ has genus zero and W and Ω have connectivity one.

THEOREM 8. *Let S be a compact Riemann surface,* $\mathrm{d}g$ *a meromorphic differential on S with two simple poles as its only singularities, and suppose that* $\mathrm{d}g$ *has vanishing periods with respect to all curves in a canonical homology basis for S. Then S must have genus zero.*

Proof. Let ζ_0 and ζ be the poles of $\mathrm{d}g$, with residues $-a$ and $+a$ say ($a \neq 0$). If all the periods of $\mathrm{d}g$ with respect to a canonical homology basis for S vanish, then g is multiple-valued only to the extent of additive multiples of $2\pi i a$. It follows that $f = \exp(g/a)$ is a single-valued meromorphic function on S whose divisor consists of a simple pole at ζ_0 and a simple zero at ζ. Hence, f is a meromorphic function of order one on S, and since such a function exists only if S has genus zero, the theorem is proved.

COROLLARY 8.1. *Suppose $\int_\Omega f\,\mathrm{d}x\,\mathrm{d}y = b\int_\gamma f\,\mathrm{d}z$ for all $f \in L^2_a(\Omega)$, where $b \in \mathbb{C}$ and γ is an arc in Ω. Then Ω is simply connected.*

Proof. Similar to the proof of Corollary 7.1 (with $S = \hat{W}$).

REMARK. There do exist plenty of simply-connected domains satisfying quadrature identities of the above kind. One example is given in [8], p. 162 ff. See also [18], Example 9.6, where a uniqueness result is proved.

The above results can be summarized by saying that no multiply-connected domain Ω admits a quadrature identity of any of the following four types for the test class $L^2_a(\Omega)$:

$$\int_\Omega f\,\mathrm{d}x\,\mathrm{d}y = a_1 f(z_1) + a_2 f(z_2),$$

$$\int_\Omega f\,\mathrm{d}x\,\mathrm{d}y = a_1 f(z_1) + a_2 f'(z_1), \tag{5.3}$$

$$\int_\Omega f\,\mathrm{d}x\,\mathrm{d}y = a_1 f(z_1) + \sum_{k=1}^{p} c_k \int_{\alpha_k} f\,\mathrm{d}z, \tag{5.4}$$

$$\int_\Omega f\,\mathrm{d}x\,\mathrm{d}y = b\int_\gamma f\,\mathrm{d}z.$$

Here $z_1, z_2 \in \Omega$, $a_1, a_2, b, c_k \in \mathbb{C}$, $\alpha_1, \ldots, \alpha_k$ are closed curves in Ω and γ is an arc in Ω. In (5.3) and (5.4) a_1 is necessarily nonzero (choose $f = 1$). (5.4) arises by reformulating the assertion of Remark 2 after Corollary 6.1 for the test class $L^2_a(\Omega)$.

One might therefore ask what the simplest quadrature identity for a multiply connected domain is. One answer is given in [14] where Levin has constructed a doubly-connected domain Ω for which an identity (5.3) holds for all $f \in L^2_{as}(\Omega)$, or, equivalently $\int_\Omega f\,\mathrm{d}x\,\mathrm{d}y = a_1 f(z_1) + a_2 f'(z_1) + c_1 \int_{\alpha_1} f\,\mathrm{d}z$ for all $f \in L^2_a(\Omega)$. Another answer will be given by Corollary 9.1 below, which, as a special case, shows that there exists a domain Ω of connectivity of at least two for which an identity

$$\int_{\Omega} f\,\mathrm{d}x\,\mathrm{d}y = a_1 f(z_1) + a_2 f(z_2) + a_3 f(z_3)$$

holds for all $f \in L_a^2(\Omega)$.

Theorem 9 below and its corollary are the only results in this paper which are not proved by using the Riemann surface theory. It is a drawback of using a Runge approximation theorem in the proof of Theorem 4 that it does not yield any information about the orders of the quadrature identities obtained. Theorem 9 is really a corollary of a result by Sakai about quadrature domains for subharmonic functions ([19], Theorem together with Lemma 5 and its corollary or [18], Theorem 3.7; compare also [12], Corollary 16.1).

Theorem 9 and its corollary have been formulated for the test class $L_a^1(\Omega)$ instead of for the usual one, $L_a^2(\Omega)$, because Sakai's results are stated for L^1-functions. Since, however, our theorem and corollary claim the existence of certain quadrature domains of finite area, this only makes the results stronger and they, *a fortiori*, also hold for $L_a^2(\Omega)$.

THEOREM 9. *Suppose Δ_k, $k = 1, \dots, m$ are disjoint open discs in $\mathbb{C}$ such that $K = \bigcup_{k=1}^m \overline{\Delta}_k$ is connected and such that $\mathbb{C}\setminus K$ has n components. Then there is a domain Ω containing K and of connectivity at least n admitting a quadrature identity of the kind*

$$\int_{\Omega} f\,\mathrm{d}x\,\mathrm{d}y = \sum_{k=1}^{m} a_k f(z_k) \tag{5.5}$$

for all $f \in L_a^1(\Omega)$. Here z_k is the center of Δ_k and $a_k > 0$ $(k = 1, \dots, m)$.

Proof. The above-mentioned result by Sakai states, among other things, that given a domain D in $\mathbb{C}$ of finite area and a bounded measurable function μ on D with $\mu \geqslant 1$, there exists a domain Ω of finite area and containing D such that

$$\int_{D} s\mu\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{\Omega} s\,\mathrm{d}x\,\mathrm{d}y \tag{5.6}$$

for every integrable subharmonic function s on Ω. Clearly (5.6) implies that

$$\int_{D} f\mu\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega} f\,\mathrm{d}x\,\mathrm{d}y \tag{5.7}$$

for all $f \in L_a^1(\Omega)$.

To apply (5.7), let A be the minimum of the areas of the components of $\mathbb{C}\setminus K$. Clearly $A > 0$. Let D_k $(k = 1, \dots, m)$ be new discs with the same centers z_k as Δ_k but slightly (and strictly) larger, so that

$$\sum_{k=1}^{m} |D_k \setminus \Delta_k| < A \tag{5.8}$$

holds, where $|..|$ denotes area. Now take

$$D = \bigcup_{k=1}^{m} D_k,$$

$\mu(z)$ = the number of discs D_k containing z.

Then D is connected and $\mu \geqslant 1$ in D. Applying (5.7) shows that there exists a domain Ω containing D such that

$$\int_\Omega f\,\mathrm{d}x\,\mathrm{d}y = \int_D f\mu\,\mathrm{d}x\,\mathrm{d}y = \sum_{k=1}^{m} \int_{D_k} f\,\mathrm{d}x\,\mathrm{d}y = \sum_{k=1}^{m} a_k f(z_k) \tag{5.9}$$

for all $f \in L^1_a(\Omega)$, where $a_k = |D_k|$.

Now it only remains to prove that Ω has connectivity of at least n. But taking $f = 1$ in (5.9) and using (5.8) gives $|\Omega| = \Sigma_{k=1}^{m} |D_k| < \Sigma_{k=1}^{m} |\Delta_k| + A = |K| + A$. Therefore, and since $K \subset \Omega$, Ω has not enough area to cover any of the n components of $\mathbb{C}\backslash K$. Thus also $\mathbb{C}\backslash\Omega$ has at least n components, which proves the theorem.

COROLLARY 9.1. *For each $m \geqslant 3$ there is a domain Ω of connectivity at least* $2(m-2)$ *admitting a quadrature identity* (5.5) (*of order m*) *for the test class* $L^1_a(\Omega)$.

Proof. In Theorem 9, take first $\Delta_1, \Delta_2, \Delta_3$ to be three mutually tangent discs. Then $\overline{\Delta}_1 \cup \overline{\Delta}_2 \cup \overline{\Delta}_3$ is connected and $\mathbb{C}\backslash(\overline{\Delta}_1 \cup \overline{\Delta}_2 \cup \overline{\Delta}_3)$ has two components, which proves the corollary for $m = 3$. Proceeding inductively, suppose we have chosen $\Delta_1, \ldots, \Delta_m$ ($m \geqslant 3$) such that $\overline{\Delta}_1 \cup \cdots \cup \overline{\Delta}_m$ is connected and such that $\mathbb{C}\backslash(\overline{\Delta}_1 \cup \cdots \cup \overline{\Delta}_m)$ has at least $2(m-2)$ components. Let E be one of these components which is bounded and choose Δ_{k+1} to be a disc in E with largest possible radius. Then Δ_{k+1} must be tangent to at least three of the other discs and, therefore, $E\backslash\overline{\Delta}_{k+1}$ must have at least three components. Thus $\overline{\Delta}_1 \cup \cdots \cup \overline{\Delta}_{k+1}$ is connected and $\mathbb{C}\backslash(\overline{\Delta}_1 \cup \cdots \cup \overline{\Delta}_{k+1})$ has two more components than $\mathbb{C}\backslash(\overline{\Delta}_1 \cup \cdots \cup \overline{\Delta}_k)$, that is, at least $2((m+1)-2)$ components. By induction and Theorem 9, this proves the corollary.

6. Quadrature Domains Bounded by Algebraic Curves

In this section we shall study domains Ω admitting quadrature identities of the kind

$$\int_\Omega f\,\mathrm{d}x\,\mathrm{d}y = \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} c_{kj} f^{(j)}(z_k) \quad (f \in L^2_a(\Omega)) \tag{6.1}$$

in more detail, namely the relationship between the coefficients c_{kj} in (6.1) and the polynomial equation for the boundary curve $\partial\Omega$. All domains Ω considered will be assumed *a priori* to be bounded by finitely many continua and to have finite area. Moreover, we will always assume that $c_{k,n_k-1} \neq 0$ ($k = 1, \ldots, m$) and that the z_k are distinct in (6.1). Let us first summarize what we know about domains Ω satisfying an identity (6.1) for the test class $L^2_a(\Omega)$.

(1) They are all produced in the following way. Take a compact symmetric Riemann surface $\hat{W} = W \cup \Gamma \cup \tilde{W}$ and a meromorphic function g on $\hat{W}$ such that all poles of g are in $\tilde{W}$ and such that g is univalent on W. Then $\Omega = g(W)$ is such a quadrature domain,

and if $\zeta_1, \ldots, \zeta_m \in \tilde{W}$ are the poles of g, then (referring to (6.1)) $z_k = g(\zeta_k)$ and n_k = the order of the pole at ζ_k. All this follows from Theorem 3 together with Remark 4 which follows the theorem.

(2) With $\hat{W}$, g and Ω as in (1) g and $\tilde{g}$ generate the field of meromorphic functions on $\hat{W}$ and they satisfy an equation $P(g, \tilde{g}) = 0$, where

$$P(z, w) = \sum_{k,j=1}^{n} a_{kj} z^k w^j \tag{6.2}$$

is a self-conjugate (i.e., $a_{jk} = \bar{a}_{kj}$) irreducible polynomial. The degree n of P in each of z and w equals the order of g and also equals the order of the quadrature identity. Thus, $n = \Sigma_{k=1}^{m} n_k$. Moreover, $\partial\Omega = \{z \in \mathbb{C} : P(z, \bar{z}) = 0\} \backslash V_0$, where V_0 is a finite set. All this follows from Lemma 1 and its proof.

(3) Working in the domain Ω itself rather than on $\hat{W}$, one can define a function S on Ω by

$$S(g(\zeta)) = \tilde{g}(\zeta) \quad \text{for } \zeta \in W \tag{6.3}$$

(i.e., $S = \tilde{g} \circ (g|_W)^{-1}$). Then $S(z)$ is meromorphic in Ω with poles of order n_k at z_k ($k = 1, \ldots, m$), $S(z) = \bar{z}$ on $\partial\Omega$ and $P(z, S(z)) \equiv 0$. Conversely, if on a given domain Ω there exists a meromorphic function $S(z)$ such that $S(z) = \bar{z}$ on $\partial\Omega$, then Ω is a quadrature domain of the kind (6.1). These things follow from (1) and (2) above and Corollary 3.1 where the function S is denoted H.

Now suppose Ω is a quadrature domain such that (6.1) holds and let $S(z)$ be the meromorphic function (6.3). The relations between the coefficients c_{kj} and the singular parts of $S(z)$ are most easily obtained directly as follows. If

$$S(z) = \sum_{k=1}^{m} \sum_{j=1}^{n_k} \frac{b_{kj}}{(z - z_k)^j} + \text{holomorphic function}, \tag{6.4}$$

then Stokes' formula gives, for $f \in L_a^2(\Omega)$,

$$\int_\Omega f \, dx \, dy = \frac{1}{2i} \int_\Omega f \, d\bar{z} \, dz = \frac{1}{2i} \int_{\partial\Omega} f(z)\bar{z} \, dz = \frac{1}{2i} \int_{\partial\Omega} f(z)S(z) \, dz$$

$$= \pi \sum_{k=1}^{m} \operatorname*{res}_{z=z_k} f(z)S(z) = \pi \sum_{k=1}^{m} \sum_{j=1}^{n_k} b_{kj} \frac{f^{(j-1)}(z_k)}{(j-1)!}.$$

Hence,

$$c_{kj} = \frac{\pi}{j!} b_{k,j+1} \tag{6.5}$$

for $j = 0, \ldots, n_k - 1$, $k = 1, \ldots, m$.

Next, let $P_j(z) = \Sigma_{k=0}^{n} a_{kj} z^k$ so that $P(z, w) = \Sigma_{j=0}^{n} P_j(z) w^j$. Since $P(z, S(z)) \equiv 0$ we have

$$0 \equiv \frac{P(z, S(z))}{S(z)^{n-1}} = \frac{P_0(z)}{S(z)^{n-1}} + \cdots + \frac{P_{n-2}(z)}{S(z)} + P_{n-1}(z) + P_n(z)S(z)$$

$$= R(z) + P_{n-1}(z) + P_n(z)S(z),$$

where

$$R(z) = \frac{P_0(z)}{S(z)^{n-1}} + \cdots + \frac{P_{n-2}(z)}{S(z)}.$$

Clearly, $R(z) = \mathcal{O}((z - z_k)^{n_k})$ as $z \to z_k$. In particular $P_n(z)S(z) = -R(z) - P_{n-1}(z)$ is bounded as $z \to z_k$. This implies that $P_n(z)$ contains the factor $(z - z_k)^{n_k}$. Therefore, and since $P_n(z)$ has degree $n = \Sigma_{k=1}^{m} n_k$, $P_n(z)$ must be

$$P_n(z) = a_{nn}(z - z_1)^{n_1} \cdot \ldots \cdot (z - z_m)^{n_m}. \tag{6.6}$$

As a consequence, $a_{nn} \neq 0$, for otherwise $P(z, w)$ would have a degree less than n in w, which it does not have (by 2) above.

Another consequence of (6.6) (and $R(z) = \mathcal{O}((z - z_k)^{n_k})$) is that $R(z)/P_n(z)$ is bounded as $z \to z_k$. Hence,

$$S(z) = -\frac{R(z)}{P_n(z)} - \frac{P_{n-1}(z)}{P_n(z)} = -\frac{P_{n-1}(z)}{P_n(z)} + \mathcal{O}(1) \quad \text{as } z \to z_k,$$

showing that the principal parts of $S(z)$ agree with those of $-(P_{n-1}(z)/P_n(z))$. Thus,

$$\sum_{k=1}^{m} \sum_{j=1}^{n_k} \frac{b_{kj}}{(z - z_k)^j} \equiv -\frac{P_{n-1}(z)}{P_n(z)} + C, \tag{6.7}$$

where

$$C = \lim_{z \to \infty} \frac{P_{n-1}(z)}{P_n(z)} = \frac{a_{n,n-1}}{a_{nn}}. \tag{6.8}$$

Now, if the polynomial $P(z, w)$ is known, (6.7) and (6.8) together with (6.5) give the coefficients c_{kj} in (6.1). Conversely, suppose that (6.1), and, hence by (6.5), Equation (6.4) is known. The polynomial $P(z, w)$, subject to $a_{kj} = \bar{a}_{jk}$, is still only determined up to a nonzero real factor. Since we know that $a_{nn} \neq 0$, it is natural to normalize $P(z, w)$ by requiring $a_{nn} = 1$. This will be assumed in the following. Then the last column in the coefficient matrix $A = (a_{kj})$ of $P(z, w)$ is obtained from the identity (6.6), that is

$$P_n(z) = z^n + a_{n-1,n} z^{n-1} + \cdots + a_{0,n} \equiv (z - z_1)^{n_1} \cdot \ldots \cdot (z - z_m)^{n_m}.$$

The condition $a_{kj} = \bar{a}_{jk}$ further gives $a_{n,n-1} = \bar{a}_{n-1,n}$.

The remaining coefficients in the $(n - 1)$th column of $A = (a_{kj})$ are now uniquely obtained from (6.7) and (6.8), i.e.,

$$\sum_{k=1}^{m} \sum_{j=1}^{n_k} \frac{b_{kj}}{(z-z_k)^j} \equiv -\frac{P_{n-1}(z)}{P_n(z)} + a_{n,n-1}.$$

It turns out that the coefficient $a_{n-1,n-1}$ automatically becomes real, as the condition $a_{kj} = \bar{a}_{jk}$ requires it to be. In fact, a manipulation (carried out in [11], p. 5.6f) shows that

$$a_{n-1,n-1} = |a_{n,n-1}|^2 - \frac{1}{\pi} \sum_{k=1}^{m} c_{k0} = |a_{n,n-1}|^2 - \frac{|\Omega|}{\pi}.$$

Using (6.5), the discussion can be summarized as follows:

THEOREM 10. *The identity*

$$\frac{1}{\pi} \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} \frac{j!\, c_{kj}}{(z-z_k)^{j+1}} \equiv a_{n,n-1} - \frac{P_{n-1}(z)}{P_n(z)},$$

where

$$P_{n-1}(z) = a_{n,n-1} z^n + a_{n-1,n-1} z^{n-1} + \cdots + a_{0,n-1},$$

$$P_n(z) = z^n + a_{n-1,n} z^{n-1} + \cdots + a_{0,n},$$

sets up a one-to-one correspondence between, on the one hand, the last two columns (and rows) of coefficient matrices $A = (a_{kj})$ $(0 \leqslant k, j \leqslant n)$ of normalized $(a_{nn} = 1)$ self-conjugate polynomials (6.2) *and, on the other hand, quadrature data $z_1, \ldots, z_m$, $n_1, \ldots, n_m$, c_{kj} $(0 \leqslant j \leqslant n_k - 1,\ 1 \leqslant k \leqslant m)$ with $\Sigma_{k=1}^m n_k = n$ and $\Sigma_{k=1}^m c_{k,0}$ real such that whenever Ω is a quadrature domain (for a point functional) the quadrature identity* (6.1) *and the normalized self-conjugate polynomial equation $P(z, \bar{z}) = 0$ for the boundary of Ω are related according to this correspondence.*

Thus, in the coefficient matrix $A = (a_{kj})$ of P only the last two columns and rows are directly related to the data in the quadrature identity, as above. The remaining coefficients, (a_{kj}) with $0 \leqslant k, j \leqslant n-2$, make up a vector space of dimension $(n-1)^2$ over the reals. Hence, given the quadrature data, this vector space can be thought of as a parameter space for all algebraic curves, which are candidates for being boundary curves for domains Ω admitting a quadrature identity with the given data. Just how many of these curves really correspond to the quadrature domains seems to be hard to decide in general (cf. Corollary 10.1 below). However, we at least get an upper bound for the dimensionality of the space of quadrature domains of the kind (6.1) admitting one and the same quadrature identity, namely $(n-1)^2$.

From Theorem 10 we may easily deduce a uniqueness result which generalizes Corollary 6.1 and results by Aharonov and Shapiro [1], Theorem 4 together with the statement after its proof, and Ullemar [21], Theorem 3:

COROLLARY 10.1. *If* (6.1) *holds and the right member there, $L(f)$, is of order one or two, then Ω is uniquely determined by L.*

Proof. Let us first show that

$$\Omega = \{z \in \mathbb{C} : P(z, \bar{z}) < 0\} \cup \text{(a finite set)}, \tag{6.9}$$

where P is as in the theorem. Since $P(z, w)$ is irreducible there are, at most, finitely many z for which the equations $P(z, w) = 0$ and $\partial P(z, w)/\partial w = 0$ have a common solution in w (namely the zeroes of the discriminant of $P(z, w)$ with respect to w). Therefore, $\partial P(z, \bar{z})/\partial \bar{z} \neq 0$ on $\partial\Omega = \{z \in \mathbb{C} : P(z, \bar{z}) = 0\} \backslash$(a finite set) except possibly at a finite set. This shows that $P(z, \bar{z})$ always changes sign over $\partial\Omega$. From this (6.9) follows easily, using that Ω is connected and that $P(z, \bar{z}) > 0$ in a neighbourhood of infinity.

If L is of order one, (6.9) immediately implies the corollary since Theorem 10 shows that P is uniquely determined by L in this case ($n = 1$). Note that the 'finite set' in (6.9) is uniquely determined by the fact that Ω shall be bounded by continua.

If L is of order two, we may write (6.9) in the form

$$\Omega = \{z \in \mathbb{C} : Q(z, \bar{z}) < c\} \cup \text{(a finite set)}, \tag{6.10}$$

where $Q(z, \bar{z}) = P(z, \bar{z}) - a_{00}$ and $c = -a_{00}$. Then c is real and Q is uniquely determined by L ($n = 2$ in Theorem 10). Moreover, the area of Ω is determined by L, namely equal to $L(1)$. But the area of the right member of (6.10) strictly increases with c. Therefore, c also is uniquely determined by L, and the corollary follows.

7. Questions of Uniqueness

In this section we shall study the folowing question: given a point functional

$$L(f) = \sum_{k=1}^{m} \sum_{j=0}^{n_k - 1} a_{kj} f^{(j)}(z_k) \tag{7.1}$$

on $L^2_a(\Omega)$ or $L^2_{as}(\Omega)$, how many different domains Ω can there be for which

$$\int_\Omega f \, dx \, dy = L(f) \tag{7.2}$$

holds for all $f \in L^2_a(\Omega)$ or $f \in L^2_{as}(\Omega)$? As usual, we only consider domains of finite area and bounded by finitely many continua.

Results such as Corollary 10.1 may raise the conjecture that Ω is always uniquely determined by L. We shall see in this section that this is definitely false if multiply-connected domains are allowed. In fact, Theorem 12, below, shows that it is typical for multiply-connected quadrature domains for a fixed functional to appear in continuous families. However, if we only allow simply-connected domains, the question seems to be open: no one has constructed two different simply-connected domains (of finite area) which admit one and the same quadrature identity for the test class L^2_a (or L^1_a). More generally, 'simply connected' in the last sentence can be replaced by 'conformally equivalent'.

On the other hand, Sakai [16] has constructed two different simply-connected Jordan domains Ω_1 and Ω_2 such that $\int_{\Omega_1} f \, dx \, dy = \int_{\Omega_2} f \, dx \, dy$ holds for all polynomials f, hence for a set of functions which is dense in both $L_a^1(\Omega_1)$ and $L_a^1(\Omega_2)$. This result indicates that the answer to the uniqueness question is negative, even in the simply-connected case.

Let us also mention here that without the assumption of the finite area of the domain or some similar assumption (e.g., (1.5)), uniqueness is known to fail, even in the simply connected case. There are, e.g., a lot of simply-connected domains Ω of infinite area such that $\int_\Omega f\,dx\,dy = 0$ for all $f \in L_a^1(\Omega)$. (See [17]. Compare also Remark after Corollary 7.1 in the present paper.)

For more results on the uniqueness question, se [18], in particular Section 9.

THEOREM 11. *Given any plane domain W bounded by $p + 1$ disjoint analytic curves ($p \geqslant 0$) there exist functionals* (7.1) *such that* (7.2) *holds for all $f \in L_{as}^2(\Omega)$ for a family of domains Ω, conformally equivalent to W and depending on at least q real parameters, where*

$$q = \begin{cases} 0 & \text{if } p = 0 \\ 2 & \text{if } p = 1 \\ 3 & \text{if } p > 1. \end{cases}$$

CORRECTION. In [11], Theorem 7.1, it is erroneously stated that $q = 1$ in the case $p = 1$.

THEOREM 12. *For every $p \geqslant 0$ there exist functionals* (7.1) *such that* (7.2) *holds for a family of domains Ω of connectivity $p + 1$ depending on at least*

(i) $3p$ *real parameters if the test class is $L_{as}^2(\Omega)$,*

(ii) p *real parameters if the test class is $L_a^2(\Omega)$.*

Proofs. We begin with Theorem 11. Let $\hat{W} = W \cup \Gamma \cup \tilde{W}$ be the Schottky double of W. By Theorem 3, together with Remark 4 following it, any quadrature domain Ω conformally equivalent to W and satisfying an identity (7.1) and (7.2) for the test class $L_{as}^2(\Omega)$ is obtained as $\Omega = g(W)$ for a meromorphic differential $\mathrm{d}g$ on $\hat{W}$, all of whose poles are residue-free and lie in $\tilde{W}$ and which has a single-valued and univalent integral g on W. The proof of Theorem 11, roughly speaking, consists of relating the parameters necessary to describe $\mathrm{d}g$ to the parameters of L in (7.1) and count the overflow.

A differential $\mathrm{d}g$ on $\hat{W}$ of the above kind is uniquely described by specifying the singular parts of $\mathrm{d}\tilde{g}$ (it is more convenient to specify $\mathrm{d}\tilde{g}$ than $\mathrm{d}g$), say in the form

$$\mathrm{d}\tilde{g}(t) = \sum_{k=1}^{m} \sum_{j=1}^{n_k} b_{kj} \frac{\mathrm{d}t}{(t - t_k)^{j+1}} + \text{holomorphic differential} \tag{7.3}$$

for $t \in W$ (cf. the proof of Theorem 2; observe that half of the periods of $\mathrm{d}\tilde{g}$, namely the 'α-periods', are prescribed to be zero by the requirement that $\mathrm{d}g$ shall have a single-valued integral on W).

Let $\mathrm{d}g = \mathrm{d}g^{(0)}$ be a fixed differential of the above kind which moreover has the property that its integral $g = g^{(0)}$ on W is univalent in a neighbourhood of $W \cup \Gamma$. We know by Theorem 4 that such differentials exist. Let $(t_k^{(0)}, b_{kj}^{(0)})$ $(1 \leqslant j \leqslant n_k,\ 1 \leqslant k \leqslant m)$ be the parameters describing $\mathrm{d}g^{(0)}$ by (7.3). We may assume that the $t_k^{(0)}$ are distinct.

Keeping m and $n_1, \ldots, n_m$ fixed we now vary (t_k, b_{kj}) in a neighbourhood U of $(t_k^{(0)}, b_{kj}^{(0)})$ in $W^m \times \mathbb{C}^{n_1+\cdots+n_m}$. It is fairly obvious that if U is sufficiently small the corresponding differentials $\mathrm{d}g$ defined by (7.3) will also have integrals which are univalent in a neighbourhood of $W \cup \Gamma$. Thus, we have a map $(t_k, b_{kj}) \mapsto \mathrm{d}g$ from U into the set of differentials we are interested in.

The transition from $\mathrm{d}g$ to an integral g of $\mathrm{d}g$ on W requires the specification of an integration constant. This we do by specifying $g(t_1)$. Thus, we have a map $(t_k, b_{kj}, c) \mapsto \mathrm{d}g \mapsto g$, defined on $U \times \mathbb{C}$ by (7.3) and

$$g(t) = c + \int_{t_1}^{t} \mathrm{d}g. \tag{7.4}$$

Now, each g, as above, determines a quadrature domain Ω by $\Omega = g(W)$, Ω determines a point functional L by (7.2), and L determines (z_k, a_{kj}) by (7.1). (To be precise, the z_k are determined only up to a permutation, but since they are the images of the t_k under g, we may naturally order them so that $z_k = g(t_k)$.)

It follows from Remark 4 after Theorem 3 that the integers $m, n_1, \ldots, n_k$ appearing in (7.3) and (7.1) are the same under this correspondence. Thus, we have described a map

$$\tau\colon (t_k, b_{kj}, c) \mapsto (z_k, a_{kj})$$

from $U \times \mathbb{C} \subset W^m \times \mathbb{C}^n \times \mathbb{C}$ into $\mathbb{C}^m \times \mathbb{C}^n$, where $n = \Sigma_{k=1}^m n_k$. Considered as a map between the underlying real spaces, τ goes from an open subset V of $\mathbb{R}^{2(m+n+1)}$ into $\mathbb{R}^{2(m+n)}$. It is not hard to see that τ is smooth and in fact, is even real analytic.

Actually, the image of τ is contained in a linear subspace of $\mathbb{R}^{2(m+n)}$ of (real) codimension one, since $\Sigma_{k=1}^m a_{k0}$ is always real in (7.1)–(7.2), namely equal to the area of Ω. Therefore the maximum value r of the rank of the Jacobian matrix of τ in V is, at most, $2(m+n) - 1$. On the other hand, this maximum is attained on some open subset V_r of V, and it now follows from the implicit function theorem that in V_r, τ takes constant values on submanifolds of dimension $2(m + n + 1) - r \geqslant 3$.

We have proved that there is an open set V_r on which the composed map

$$(t_k, b_{kj}, c) \mapsto g \mapsto \Omega \mapsto L \mapsto (z_k, a_{kj}) \tag{7.5}$$

takes constant values on manifolds of real dimension of at least three. What we are really interested in is, however, the map $\Omega \mapsto (z_k, a_{kj})$ and we must account for the possibility that the map $(t_k, b_{kj}, c) \mapsto \Omega$ might not be one-to-one. We have already remarked that $(t_k, b_{kj}, c) \mapsto g$ is one-to-one (locally). However, $g \mapsto \Omega$, defined by $\Omega = g(W)$, is not necessarily so.

Indeed, $g_1(W) = g_2(W)$ if and only if $\varphi = g_1^{-1} \circ g_2$ is an automorphism on W. Keeping g_1 fixed, it follows that those g_2 for which $g_2(W) = g_1(W)$ stand in one-to-one correspondence with the automorphisms φ on W. Since we are only considering g_1 and g_2 which are close to each other, we are only interested in the case when there are automorphisms arbitrarily close to the identity. It is well-known ([9], Ch. V.4) that this is the case only when $p = 0$ and $p = 1$ and that the automorphy-group $\mathrm{aut}(W)$ in these cases are Lie groups of real dimensions three and one respectively.

Let M be a typical submanifold of V_r of dimension $2(m + n + 1) - r$, on which τ is constant. Then the above Lie groups generate orbits in M so that (t_k, b_{kj}, c) and (t'_k, b'_{kj}, c') lie on the same orbit if and only if they are mapped onto the same Ω in (7.5). These orbits are submanifolds of dimension three and one in the cases $p = 0$ and $p = 1$ respectively. (When $p > 1$ the orbits consist of isolated points only.) Now we can take a submanifold N of M which is transversal to these orbits and such that

$$\dim N = \begin{cases} (2(m+n+1) - r - 3 & \text{if } p = 0, \\ (2(m+n+1) - r - 1 & \text{if } p = 1, \\ (2(m+n+1) - r & \text{if } p > 1. \end{cases}$$

Then the map $(t_k, b_{kj}, c) \mapsto \Omega$ restricted to N is one-to-one and so parametrizes, locally and in a bijective way, domains which satisfy the same quadrature identity for the class L^2_{as}. Since $\dim N \geqslant q$ with q as in the statement of the theorem, this proves Theorem 11.

To prove Theorem 12 we also have to let the conformal type of the domain W vary, keeping only its connectivity fixed. Such variations are easily performed by taking W to be a horizontal slit domain, i.e., of the kind

$$W = \mathbb{P}\setminus\bigcup_{k=0}^{p} \{w_k + tr_k : -1 \leqslant t \leqslant 1\}. \tag{7.6}$$

Here $w_k \in \mathbb{C}$, $r_k > 0$ $(k = 0, 1, \dots, p)$ are parameters, assumed chosen so that the slits $\{w_k + tr_k : -1 \leqslant t \leqslant 1\}$ are disjoint.

On each such domain W we have functions g defined by the parameters (t_k, b_{kj}, c) (for fixed $m, n_1, \dots, n_m$) as before. We shall consider pairs (W, g) of horizontal slit domains and functions in a sufficiently small neighbourhood of some fixed pair $(W^{(0)}, g^{(0)})$, chosen in such a way that each g is univalent in a neighbourhood of $W \cup \Gamma$ in $\hat{W}$. This yields a map

$$(w_k, r_k, t_k, b_{kj}, c) \mapsto (W, g) \tag{7.7}$$

from an open set V in $\mathbb{C}^{p+1} \times \mathbb{R}^{p+1} \times \mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}$ into the set of domains and functions we are interested in. To be precise, (7.7) is defined by (7.6), (7.3) and (7.4). Clearly, (7.7) is one-to-one if V is small enough.

Proceeding as in the proof of Theorem 11, we compose (7.7) with the maps $(W, g) \mapsto \Omega \mapsto L \mapsto (z_k, a_{kj})$, where $\Omega = g(W)$, L is defined by (7.2) and (z_k, a_{kj}) by (7.1). The composed map τ: $(w_k, r_k, t_k, b_{kj}, c) \mapsto (z_k, a_{kj})$, regarded as a map between the underlying real spaces, then goes from an open subset (still called V) of $\mathbb{R}^{3(p+1)+2(m+n+1)}$ into $\mathbb{R}^{2(m+n)}$, or actually into a subspace of $\mathbb{R}^{2(m+n)}$ of dimension $2(m + n) - 1$. Just as before, this shows that there must be submanifolds of V of dimension of at least $3p + 6$ on which τ is constant.

The map (7.7) is one-to-one as we have remarked, but not $(W, g) \mapsto \Omega$. In fact, $g_1(W_1) = g_2(W_2)$ if and only if $\varphi = g_2^{-1} \circ g_1$ maps W_1 conformally onto W_2. Keeping (W_1, g_1) fixed, it follows that pairs (W_2, g_2) mapped onto the same $\Omega = g_1(W_1)$ stand in bijective correspondence to conformal mappings φ on W_1 such that $\varphi(W_1)$ is also a horizontal slit domain. It is well-known that such maps φ depend on six real parameters.

To be precise, if W is a horizontal slit domain, then, given $t_0 \in W$ and $a, b \in \mathbb{C}$ with $a \neq 0$, there is a unique univalent function φ on W with

$$\varphi(t) = \frac{a}{t - t_0} + b + \mathcal{O}(t - t_0) \quad \text{as } t - t_0 \tag{7.8}$$

such that $\varphi(W)$ is also a horizontal slit region (cf. [2], Ch. 6, Section 5.3). (For $t_0 = \infty$ (7.8) takes the form $\varphi(t) = at + b + \mathcal{O}(t^{-1})$ as $t \to \infty$.)

In just the same way as in the proof of Theorem 11, the existence of these maps reduces the number of parameters from $3p + 6$ to $3p$, from which (i) of Theorem 12 follows.

To prove (ii) of Theorem 12, recall (Remark 4 after Theorem 3) that (7.1) and (7.2) holds for all $f \in L_a^2(\Omega)$ if and only if it holds for all $f \in L_{as}^2(\Omega)$ and the differential $d\tilde{g}$ in (7.3) has the further property that

$$\int_{\beta_k} d\tilde{g} = 0, \quad \text{for } k = 1, \ldots, p \tag{7.9}$$

(so that g is single-valued on all of $\hat{W}$). Thus, the requirement that (7.1) and (7.2) shall hold for all $f \in L_a^2(\Omega)$ imposes $2p$ real conditions on g. These conditions can be satisfied as we know (Theorem 4). We just have to check that they are linearly independent (to rule out the possibility that things get pathological so that (7.9) defines a set of greater codimension than $2p$).

Consider the map

$$(w_k, r_k, t_k, b_{kj}, c) \mapsto g \mapsto \left(\int_{\beta_1} dg, \ldots, \int_{\beta_p} dg \right) \tag{7.10}$$

(the first step defined by (7.7)). It can be seen, without too much trouble, that the rank of the Jacobian of this map (regarded as a map into $\mathbb{R}^{2p}$) can be less than $2p$ only at points $(w_k, r_k, t_k, b_{kj}, c)$ for which the index of speciality of the divisor $\Sigma_{k=1}^m n_k \cdot (t_k)$ (see [9], Ch. III.4) is greater than zero. (We have assumed here that $b_{kn_k} \neq 0$ for $k = 1, \ldots, m$.) However, the degree of this divisor is equal to $n = \Sigma_{k=1}^m n_k$ and the index of speciality is equal to zero for all divisors of a degree greater than $2p - 2$. Therefore, the Jacobian of (7.10) has rank $2p$ at all points with $b_{kn_k} \neq 0$ $(k = 1, \ldots, m)$ whenever $n > 2p - 2$.

From this it follows that if n is large enough, the conditions (7.9) define submanifolds in the $(w_k, r_k, t_k, b_{kj}, c)$-space of real codimension $2p$. Proceeding now as in the proof of (i) but working in such a manifold instead, we reach the conclusion of (ii). This proves Theorem 12.

REMARKS. (1) Actually Theorems 11 and 12 describe the typical cases, i.e., quadrature domains of connectivity p 'in general' occur in families depending on exactly the number of parameters given by the theorems.

(2) Since, by Corollary 9.1, there exist multiply-connected quadrature domains admitting quadrature identities of order three for the test class L^2_a, Theorem 12(ii) indicates that the uniqueness question already has a negative answer for certain finite functionals (7.1) of order three. A more detailed analysis, not carried out here, shows that this actually is the case.

(3) The results in Theorems 11 and 12 can be made believable in the following way. First note that the difference between the number of parameters in Theorem 12(i) and Theorem 11, namely $3p - q$, equals the real dimensionality of the space of conformal equivalence classes for connectivity $p + 1$ ([20], Section 2.11) as one might expect. Second, the difference between the number of parameters in (i) and (ii) of Theorem 12 is just the real codimension of $L^2_{as}(\Omega)$ in $L^2_a(\Omega)$. Thus, it only remains to motivate (ii) of Theorem 12.

For this purpose we shall consider a number of moving boundary problems for $(p + 1)$-connected domains. Let $\Omega^{(0)}$ be a quadrature domain (for L^2_a) of connectivity $p + 1$ such that $\partial\Omega^{(0)}$ consists of disjoint regular analytic curves $\Gamma_1^{(0)}, \ldots, \Gamma_{p+1}^{(0)}$. Let $\mathcal{N}$ be a neighbourhood of $\Omega^{(0)}$ in the space of all domains bounded by $p + 1$ disjoint regular analytic curves. This neighbourhood is assumed to be so small so that for each $\Omega \in \mathcal{N}$, the boundary components $\Gamma_1, \ldots, \Gamma_{p+1}$ of Ω are unambiguously ordered by the requirement that Γ_k shall be close to $\Gamma_k^{(0)}$ $(1 \leqslant k \leqslant p + 1)$.

For each $(p + 1)$-tuple of real numbers $c = (c_1, \ldots, c_{p+1})$ and each $\Omega \in \mathcal{N}$ there is a (unique) harmonic function $u = u_\Omega$ on Ω having the boundary values c_k on Γ_k $(k = 1, \ldots, p + 1)$. Let $\partial u/\partial n$ denote the outward normal derivative of u on $\partial\Omega$. Then, for $c \in \mathbb{R}^{p+1}$ fixed, we consider the following moving boundary problem:

> Find a map $(-\varepsilon, \varepsilon) \ni t \mapsto \Omega_t \in \mathcal{N}$ for some $\varepsilon > 0$ such that $\partial\Omega_t$ propagates with velocity $-(\partial u_{\Omega_t}/\partial n)$, measured in the direction of the outward normal of $\partial\Omega_t$, and such that $\Omega_0 = \Omega^{(0)}$. (7.11)

There are reasons to believe that this problem has a unique solution for $\varepsilon > 0$ sufficiently small. Suppose $t \mapsto \Omega_t$ is a solution. Then

$$\int_{\Omega_t} f \, \mathrm{d}x \, \mathrm{d}y = \text{const} = \int_{\Omega_0} f \, \mathrm{d}x \, \mathrm{d}y$$

for each $f \in L^2_a(\cup_{|t| < \varepsilon} \Omega_t)$. In fact, denoting an arc-length parameter along the positively oriented boundary of Ω_t by s, using Green's formula and that $\partial f/\partial n = -i(\partial f/\partial s)$ (Cauchy–Riemann), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} f \, \mathrm{d}x \, \mathrm{d}y = \int_{\partial\Omega_t} f \cdot \left(-\frac{\partial u_{\Omega_t}}{\partial n} \right) \mathrm{d}s = -\int_{\partial\Omega_t} u_{\Omega_t} \cdot \frac{\partial f}{\partial n} \, \mathrm{d}s$$

$$= i \cdot \int_{\partial\Omega_t} u_{\Omega_t} \cdot \frac{\partial f}{\partial s} \, \mathrm{d}s = i \cdot \sum_{k=1}^{p+1} c_k \int_{(\Gamma_k)_t} \frac{\partial f}{\partial s} \, \mathrm{d}s = 0.$$

Thus, for each $c \in \mathbb{R}^{p+1}$ the moving boundary problem (7.11) yields a one-parameter family of domains in $\mathcal{N}$ admitting the same quadrature identity for the test class L^2_a as

$\Omega^{(0)}$. Observe that any two $c \in \mathbb{R}^{p+1}$ that differ by an additive constant (i.e., by $(\alpha, \alpha, \ldots, \alpha)$ for some $\alpha \in \mathbb{R}$) yield the same problem (7.11). Therefore, by varying $c \in \mathbb{R}^{p+1}$ there arises exactly a p-parameter family of problems (7.11) and so we get a p-parameter family of domains in $\mathcal{N}$ satisfying the same quadrature identity as $\Omega^{(0)}$ as claimed in (ii) of Theorem 12.

One may also notice that the numbers $3p$ and p in Theorem 12 equal the (real) codimensions of $\operatorname{Re} L^2_{as}(\Omega)$ and $\operatorname{Re} L^2_a(\Omega)$ respectively in $L^2_h(\Omega)$, where $\operatorname{Re} L^2_{a(s)}(\Omega)$ denotes the set of real parts of functions in $L^2_{a(s)}(\Omega)$ and $L^2_h(\Omega)$ is the space of real-valued square-integrable harmonic functions. This gives another explanation of the numbers $3p$ and p, because quadrature domains for harmonic functions can be expected to be more or less uniquely determined by their quadrature functionals (cf. [18], Theorem 4.7 with corollaries).

Acknowledgements

I wish to express my great indebtedness to Professor Harold S. Shapiro for having introduced me to the topic of this paper and for many inspiring discussions on this and related topics throughout almost a decade now. Without his interest and encouragement this work would perhaps never have been carried out.

References

1. Aharonov, D. and Shapiro, H. S.: *J. Anal. Math.* **30**, (1976), 39–73.
2. Ahlfors, L. V.: *Complex Analysis*, McGraw-Hill, 1966.
3. Ahlfors, L. V. and Sario, L.: *Riemann Surfaces*, Princeton Universty Press, 1960.
4. Alling, N. L. and Greenleaf, N.: *Foundations of the Theory of Klein Surfaces*, Lecture Notes in Mathematics, Vol. 219, Springer Verlag, 1971.
5. Auderset, C.: 'Sur le Théorème d'approximation de Runge', in *L'enseignement Mathématique*, Vol. 26, parts 3–4, 1980.
6. Avci, Y.: 'Quadrature Identities and the Schwartz Function', Doctoral dissertation, Stanford University, 1977.
7. Avci, Y.: *J. London Math. Soc.* **23** (1980), 123–128.
8. Davis, P. J.: *The Schwarz Function and its Applications*, The Carus Mathematical Monographs 17, The Mathematical Association of America, 1974.
9. Farkas, H. M. and Kra, I.: *Riemann Surfaces*, GTM 71, Springer Verlag, 1980.
10. Gustafsson, B.: 'The Runge Approximation Theorem on Compact Riemann Surfaces', TRITA-MAT-1977-2, Mathematics, Royal Institute of Technology, Stockholm.
11. Gustafsson, B.: 'Quadrature Identities and the Schottky Double', TRITA-MAT-1977-3, Mathematics, Royal Institute of Technology, Stockholm.
12. Gustafsson, B.: 'Applications of Variational Inequalities to a Moving Boundary Problem for Hele Shaw Flows', TRITA-MAT-1981-9, Mathematics, Royal Institute of Technology, Stockholm.
13. Kõditz, H. and Timmann, S.: *Mathematische Annalen* **217** (1975) 157–159.
14. Levin, A. L.: *Proc. Amer. Math. Soc.* **60** (1976), 163–168.
15. Rubel, L. A. and Taylor, B. A.: *Amer. Math. Monthly* **76** (1969), 483–489.
16. Sakai, M.: *Proc. Amer. Math. Soc.* **70** (1978), 35–38.
17. Sakai, M.: *J. Anal. Math.* **40** (1981), 144–154.
18. Sakai, M.: *Quadrature Domains*, Lecture Notes in Mathematics, Vol. 934, Springer Verlag, 1982.
19. Sakai, M.: *Trans. Amer. Math. Soc.* **276** (1983), 267–279.
20. Schiffer, M. and Spencer, D. C.: *Functionals of Finite Riemann Surfaces*, Princeton University Press, 1954.
21. Ullemar, C.: 'Symmetric Plane Domains Satisfying Two-point Quadrature Identities for Analytic Functions', TRITA-MAT-1977-24, Mathematics, Royal Institute of Technology, Stockholm.
22. Ullemar, C.: 'A Uniqueness Theorem for Domains Satisfying a Quadrature Identity for Analytic Functions', TRITA-MAT-1980-37, Mathematics, Royal Institute of Technology, Stockholm.