ANALYTIC CONTINUATION OF CAUCHY AND EXPONENTIAL TRANSFORMS

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Abstract. We review some recent results concerning analytic continuation properties of the Cauchy transform of a domain in the complex plane, of a corresponding exponential transform and of the resolvent of a hyponormal operator associated with the domain.

The main result states the equivalence between the mentioned analytic continuations. As a corollary we obtain apriori regularity of boundaries admitting analytic continuation of the Cauchy transform.

1. Introduction

In this note we review some recent results [12], [13], [14], [8] concerning analytic continuation properties of different objects related to the Cauchy kernel.

The first object is the Cauchy transform of the characteristic function of a domain in the complex plane. This can be viewed as a kind of one variable trace (more precisely, a derivative or residue at infinity) of a function of two complex variables, known as the exponential transform of the domain. This is our second object of study.

The exponential transform can be traced back to operator theoretic studies by R.W. Carey, J.D Pincus [3] and K. Clancey [4], [5] (see also [11]). In these works it appears in a formula involving the resolvent of a hyponormal operator associated to a bounded positive function (called the principal function of the operator), which in our case will be the characteristic function of the domain.

This resolvent is our third object of study, but rather than taking an abstract operator theoretic point of view, we work within an explicit function theoretic model of the Hilbert space on which the operator acts. The inner product in the Hilbert space is defined via an adjoint exponential transform, and in the so obtained model the resolvent of the hyponormal operator, specialized at a certain vector, simply becomes the Cauchy kernel itself.

The main result [8] is that the above three objects have analytic continuations from outside the domain across its boundary under exactly the same conditions. In fact, we give (or, in this note, outline) a direct proof that if one of them has an analytic continuation then so has the others. As an application we get apriori regularity of boundaries which admit analytic continuation of the Cauchy transform. This gives an alternative approach to part of M. Sakai's complete solution [15], [16] of that regularity problem.

Details and complete proofs of most matters discussed here can be found in [12], [13], [14], [8]. Minor parts of the contents are a slightly tentative or inprecise, and for these we intend to provide full details in [9] and other forthcoming papers.

2. Definition and elementary properties of the exponential transform

We first recall the definition of the Cauchy transform of a function (or distribution) ρ with compact support:

$$\hat{\rho}(z) = -\frac{1}{\pi}\int \frac{\rho(\zeta)dA(\zeta)}{\zeta - z},$$

where dA denotes two-dimensional Lebesgue measure. Then $\frac{\partial\hat{\rho}}{\partial\bar{z}} = \rho$ in the sense of distributions. The corresponding *exponential transform* is defined to be

$$E_\rho(z,w) = \exp[-\frac{1}{\pi}\int \frac{\rho(\zeta)dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}],$$

at least when $\rho \geq 0$. We shall mainly be concerned with the case that $\rho = \chi_\Omega$ for some an open subset Ω of $\mathbb{C}$ and we then write

$$E_\Omega(z,w) = \exp[-\frac{1}{\pi}\int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}].$$

If the domain Ω is clear from the context we shall sometimes delete it from notation.

Below we list a few elementary properties of the exponential transform.

- $E_\Omega(z,w)$ is analytic in z for $z \notin \overline{\Omega}$, antianalytic in w for $w \notin \overline{\Omega}$.
- $E_{a\Omega+b}(az+b, aw+b) = E_\Omega(z,w)$ for any $a, b \in \mathbb{C}$, $a \neq 0$.
- $|E_\Omega(z,w)| \leq 2$. Equality is attained only if Ω is a disc (up to nullsets) and z, w diametrically opposite points on the boundary.
- $E_\rho(z,w) = 1 - \frac{1}{\pi z\bar{w}}\int \rho dA +$ smaller terms, as $|z|, |w| \to \infty$.
- $E_\rho(z,w) = 1 - \frac{\hat{\rho}(z)}{\bar{w}} +$ smaller terms, as $|w| \to \infty$ for fixed z.

The latter property says that $\hat{\rho}(z)$ is a derivative, or residue, at infinity with respect to w of $E_\rho(z,w)$. For example we have, for large enough values of R and with counterclockwise integration,

$$\hat{\rho}(z) = \frac{1}{2\pi i}\int_{|w|=R} E_\rho(z,w)d\bar{w}.$$

This can also be written

$$\hat{\rho}(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial E_\rho(z,w)}{\partial w}dA(w).$$

Taking the z-bar derivative gives

$$\rho(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial^2 E_\rho(z,w)}{\partial\bar{z}\partial w}dA(w). \tag{2.1}$$

In the definition of the exponential transform we understand that $\exp[-\infty] = 0$. Then $E_\rho(z,z)\rho(z) = 0$. Using this we can compute some distributional derivatives of E_ρ to be

$$\frac{\partial E_\rho(z,w)}{\partial\bar{z}} = \frac{E_\rho(z,w)}{\bar{z} - \bar{w}}\rho(z), \tag{2.2}$$

$$\frac{\partial E_\rho(z,w)}{\partial w} = -\frac{E_\rho(z,w)}{z - w}\rho(w),$$

$$\frac{\partial^2 E_\rho(z,w)}{\partial\bar{z}\partial w} = -\frac{E_\rho(z,w)}{|z - w|^2}\rho(z)\rho(w). \tag{2.3}$$

In the sequel we stick to the case $\rho = \chi_\Omega$.

Example: For the unit disc $\mathbb{D} = \mathbb{D}(0,1)$ we have

$$E(z,w) = \begin{cases} 1 - \frac{1}{z\bar{w}} & \text{for } z, w \notin \overline{\mathbb{D}}, \\ 1 - \frac{\bar{z}}{\bar{w}} & \text{for } z \in \mathbb{D}, w \notin \overline{\mathbb{D}}, \\ 1 - \frac{w}{z} & \text{for } z \notin \overline{\mathbb{D}}, w \in \mathbb{D}, \\ \frac{|z-w|^2}{1-z\bar{w}} & \text{for } z, w \in \mathbb{D}. \end{cases} \tag{2.4}$$

Investigation of the above expressions shows that $E(z,w)$, for the unit disc, is continuous everywhere. This is almost true in general: for any bounded open set Ω, $E(z,w)$ is

- continuous in each variable separately,
- jointly continuous except at points (z,z) with z in

$$Z = \left\{ z \in \partial\Omega : \int_\Omega \frac{dA(\zeta)}{|\zeta - z|^2} < \infty \right\}. \tag{2.5}$$

The set Z consists of those points (if any) on $\partial\Omega$ at which Ω is "thin", e.g. has a sharp outward cusp. It enters into the description of the behaviour of E on the diagonal:

$$E(z,z) = \begin{cases} > 0 & \text{for} \quad z \in (\mathbb{C} \setminus \overline{\Omega}) \cup Z, \\ = 0 & \text{for} \quad z \in \Omega \cup (\partial\Omega \setminus Z). \end{cases} \tag{2.6}$$

The general structure of $E(z,w)$ as to analyticity/antianalyticity also agrees with that for the disc, namely

$$E(z,w) = \begin{cases} \text{analytic/antianalytic} & \text{in} \quad \overline{\Omega}^c \times \overline{\Omega}^c, \\ (\bar{z} - \bar{w}) \cdot \text{analytic/antianalytic} & \text{in} \quad \Omega \times \overline{\Omega}^c, \\ (z - w) \cdot \text{analytic/antianalytic} & \text{in} \quad \overline{\Omega}^c \times \Omega, \\ |z-w|^2 \cdot \text{analytic/antianalytic} & \text{in} \quad \Omega \times \Omega. \end{cases} \tag{2.7}$$

3. A functional model.

The exponential transform originally appeared in the theory of hyponormal operators, see [3], [4], [5], [12], [13], [14], [11], Ch.XI. Below we give a functional model in which the hyponormal operator acts on a Hilbert space of (equivalence classes of) functions and distributions on the given domain. The model agrees with the "standard" model in this context (see [11], Ch.XI.3) except that we have switched the roles between the operator and its adjoint.

So let $\Omega \subset \mathbb{C}$ be a bounded domain. In addition to the exponential transform itself, we shall need what we call the *adjoint exponential transform* of Ω. It is defined as

$$H(z,w) = H_\Omega(z,w) = -\frac{\partial^2 E_\Omega(z,w)}{\partial\bar{z}\partial w} = \frac{E_\Omega(z,w)}{|z-w|^2} \qquad (z, w \in \Omega).$$

It is immediate from (2.7) that $H(z,w)$ is analytic in z, antianalytic in w.

Example: From (2.4) we see that, for the unit disc $\Omega = \mathbb{D}$,

$$H(z,w) = \frac{1}{1 - z\overline{w}} \quad (z, w \in \mathbb{D}). \tag{3.1}$$

Although we shall not need it, we mention that H can be viewed as a renormalized version of one over the exponential transform of the complementary domain (see [9]). Precisely:

$$H_\Omega(z,w) = \lim_{R\to\infty} \frac{1}{R^2 E_{\mathbb{D}(0,R)\setminus\Omega}(z,w)}.$$

Now, for $\phi, \psi \in \mathcal{D}(\Omega) = C_0^\infty(\Omega)$ we define

$$\langle \phi, \psi \rangle = \frac{1}{\pi^2} \int_\Omega \int_\Omega H(z,w)\phi(z)\overline{\psi(w)}dA(z)dA(w), \tag{3.2}$$
$$\|\phi\| = \sqrt{\langle \phi, \phi \rangle}.$$

We cite, from e.g. [11], the following nontrivial fact.

Lemma 3.1. *The adjoint exponential transform $H(z,w)$ is nonnegative definite, e.g. in the sense that $\langle \phi, \phi \rangle \geq 0$ for all $\phi \in \mathcal{D}(\Omega)$.*

Thus $\langle \cdot, \cdot \rangle$ is an inner product on a quotient space of $\mathcal{D}(\Omega)$. Let $\mathcal{H}(\Omega)$ denote the completion of that quotient space. Then $\mathcal{H}(\Omega)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and we have natural linear map (given by the construction)

$$\mathcal{D}(\Omega) \to \mathcal{H}(\Omega) \tag{3.3}$$

with dense range.

Much of $\mathcal{D}(\Omega)$ collapses under the above map, i.e., it has a big kernel. For example, since $H(z,w)$ is analytic/antianalytic, partial integration shows that, for any $\varphi \in \mathcal{D}(\Omega)$, $\frac{\partial \varphi}{\partial \bar{z}} = 0$ as an element of $\mathcal{H}(\Omega)$. Thus all elements of $\mathcal{D}(\Omega)$ can be said to be "analytic", in some very weak sense.

Looking at the expression (3.1) we easily find that

$$\int_\Omega \int_\Omega |H(z,w)| dA(z)dA(w) < \infty \tag{3.4}$$

in the case of the unit disc, and this persists to hold at least for smoothly bounded domains. (Whether (3.4) holds for all domains we do not know.)

Assuming that (3.4) holds, the inner product (3.2) makes sense for all $\phi, \psi \in L^\infty(\Omega)$, and it is easy to see that $L^\infty(\Omega)$ is in the closure of $\mathcal{D}(\Omega)$ with respect to the norm of $\mathcal{H}(\Omega)$. Thus the map (3.3) extends to $L^\infty(\Omega) \to \mathcal{H}(\Omega)$. In particular, we have $1 \in \mathcal{H}(\Omega)$, and also $k_z \in \mathcal{H}(\Omega)$ for $z \notin \overline{\Omega}$, where

$$k_z(\zeta) = \frac{1}{\zeta - z}$$

denotes the Cauchy kernel, regarded as a function of $\zeta \in \Omega$ with $z \in \mathbb{C}$ a parameter.

Similarly, any distribution (or even analytic functional) with compact support in Ω can be considered as an element of $\mathcal{H}(\Omega)$. Indeed, if $\mu \in \mathcal{E}'(\Omega)$ (the distributions with compact support in Ω) then there is a sequence $\varphi_n \in \mathcal{D}(\Omega)$ tending weakly as distributions towards μ. Since $H(z,w)$ is smooth in both variables the tensor product $\mu \otimes \overline{\mu}$ acts on H, hence $\|\mu\|^2 = \langle \mu, \mu \rangle = \frac{1}{\pi^2}(\mu \otimes \overline{\mu})(H)$ has a natural meaning. The same is true for $\|\mu - \varphi_n\|^2$, and it follows from the weak convergence that $\|\mu - \varphi_n\|^2 \to 0$ as $n \to \infty$. We conclude that $\{\varphi_n\} \subset \mathcal{D}(\Omega)$ is a Cauchy sequence with respect to $\|\cdot\|$ and that μ in a natural sense represents the limit of this Cauchy sequence in $\mathcal{H}(\Omega)$. In other words, the map (3.3) lifts to

$$\mathcal{E}'(\Omega) \to \mathcal{H}(\Omega).$$

Thus $k_z \in \mathcal{H}(\Omega)$ also for $z \in \Omega$. It is possible to show that $k_z \in \mathcal{H}(\Omega)$ even for $z \in \partial\Omega$ and that the map $\mathbb{C} \to \mathcal{H}(\Omega)$ given by $z \mapsto k_z$ is weakly continuous (i.e., $z \mapsto \langle k_z, \phi \rangle$ is continuous for each $\phi \in \mathcal{H}(\Omega)$). It need not be strongly continuous, though. See [4] or [11], Ch.XI.

We denote by $\mathcal{A}(\Omega)$ the subspace of $\mathcal{H}(\Omega)$ consisting of analytic functions in the ordinary sense. More precisely, we may define it as:

$$\mathcal{A}(\Omega) = \text{clos}_{\mathcal{H}(\Omega)}\text{span}\{k_z : z \notin \Omega\}.$$

Example. In the case of the unit disc we have, by (3.1),

$$\langle \phi, \psi \rangle = \frac{1}{\pi^2} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{\phi(z)\overline{\psi(w)}}{1 - z\overline{w}} dA(z) dA(w).$$

If ϕ and ψ are analytic this reduces, by the meanvalue property of analytic functions, to

$$\langle \phi, \psi \rangle = \phi(0)\overline{\psi(0)}.$$

Thus the holomorphic subspace $\mathcal{A}(\Omega)$ of $\mathcal{H}(\Omega)$ collapses to a one-dimensional space in the case of the unit disc.

From (2.1) (with $\rho = \chi_\Omega$) and the definition of $H(z,w)$ we conclude that

$$\langle \phi, 1 \rangle = \frac{1}{\pi} \int_\Omega \phi dA \tag{3.5}$$

for any $\phi \in \mathcal{H}(\Omega)$. In particular,

$$\langle k_z, 1 \rangle = -\hat{\chi}_\Omega(z). \tag{3.6}$$

More generally,

$$\langle \rho k_z, 1 \rangle = -\hat{\rho}(z)$$

for (say) $\rho \in \mathcal{E}'(\Omega)$.

Since $1 - E(z,w)$ vanishes at infinity in each of the variables we may represent it as a double Cauchy-integral of its second mixed derivatives:

$$1 - E(z,w) = \frac{1}{\pi^2} \int \int \frac{\partial^2}{\partial \overline{u} \partial v}(1 - E(u,v)) \frac{dA(u)}{u - z} \frac{dA(v)}{\overline{v} - \overline{w}}.$$

Thus we get a representation also of the exponential transform in terms of k_z:

$$\langle k_z, k_w \rangle = 1 - E(z,w). \tag{3.7}$$

Both (3.6) and (3.7) are valid for all $z, w \in \mathbb{C}$.

From (3.7) combined with (2.6) we see that

$$\|k_z\| = \begin{cases} < 1 & \text{for } z \in (\mathbb{C} \setminus \overline{\Omega}) \cup Z, \\ = 1 & \text{for } z \in \Omega \cup (\partial\Omega \setminus Z). \end{cases} \tag{3.8}$$

Also, by (3.5), $\|1\| = \sqrt{\frac{|\Omega|}{\pi}}$.

By the definition (3.2) of the inner product, the adjoint exponential transform can be gotten by using δ_z, the Dirac measure at z, instead of k_z above:

$$\langle \delta_z, \delta_w \rangle = \frac{1}{\pi^2} H(z,w).$$

This could also have been obtained by differentiating (3.7) and using the fact that

$$\frac{\partial}{\partial \bar{z}} k_z = -\pi \delta_z. \tag{3.9}$$

Next we define a bounded linear operator

$$T : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$$

by setting, for $\phi \in \mathcal{D}(\Omega)$,

$$(T\phi)(z) = z\phi(z).$$

The adjoint of T is given by

$$(T^*\phi)(z) = \bar{z}\phi(z) + \hat{\phi}(z).$$

In fact, using partial integration and the fact that $\frac{\partial}{\partial w}[(z-w)H(z,w)] = -H(z,w)$ we have, for $\phi, \psi \in \mathcal{D}(\Omega)$,

$$\begin{aligned} &\langle z\phi(z), \psi(z)\rangle - \langle \phi(z), \bar{z}\psi(z)\rangle \\ &= \frac{1}{\pi^2}\int_\Omega\int_\Omega H(z,w)(z-w)\phi(z)\overline{\psi(w)}dA(z)dA(w) \\ &= \frac{1}{\pi^2}\int_\Omega\int_\Omega H(z,w)(z-w)\phi(z)\overline{\frac{\partial\hat{\psi}(w)}{\partial \bar{w}}}dA(z)dA(w) \\ &= \frac{1}{\pi^2}\int_\Omega\int_\Omega H(z,w)\phi(z)\overline{\hat{\psi}(w)}dA(z)dA(w) \\ &= \langle \phi, \hat{\psi}\rangle, \end{aligned}$$

proving the assertion.

We proceed to compute the commutator of T and T^*. Multiplication by z certainly commutes with multiplication by $\bar{z}$, so we just get

$$[T,T^*]\phi(z) = z(-\frac{1}{\pi}\int_\Omega \frac{\phi(\zeta)dA(\zeta)}{\zeta - z}) - (-\frac{1}{\pi}\int_\Omega \frac{\zeta\phi(\zeta)dA(\zeta)}{\zeta - z}) = \frac{1}{\pi}\int_\Omega \phi dA$$

Thus, using (3.5),

$$[T,T^*] = 1 \otimes 1,$$

where the right member is understood as the operator

$$(1\otimes 1)(\phi) = \langle \phi, 1\rangle 1.$$

This is a positive multiple of the orthogonal projection onto the subspace spanned by $1 \in \mathcal{H}(\Omega)$. In particular, $[T,T^*]$ is a positive operator, in other words T^* is a hyponormal operator.

Using ζ as the running variable (argument) for the "functions" in $\mathcal{H}(\Omega)$ and regarding z as a parameter we have

$$(T-z)k_z(\zeta) = (\zeta - z)\cdot \frac{1}{\zeta - z} = 1.$$

It follows that, as elements in $\mathcal{H}(\Omega)$,

$$(T-z)^{-1}1 = k_z,$$

at least for values of z for which the inverse $(T-z)^{-1}$ exists, namely for $z \notin \overline{\Omega}$. Thus, in view of (3.6) and (3.7), we can also express the Cauchy and exponential transforms in terms of T:

$$\langle (T-z)^{-1}1, 1\rangle = -\hat{\chi}_\Omega(z),$$

$$\langle (T-z)^{-1}1, (T-w)^{-1}1\rangle = 1 - E(z,w).$$

4. Analytic continuation properties

In this section we study analytic continuation properties of functions related to the Cauchy kernel, for example the Cauchy transform $\hat{\chi}_\Omega$ of a domain Ω. Recall that $\hat{\chi}_\Omega$ is analytic in the exterior of Ω, but not inside Ω. A basic question is: for which domains does this exterior part of the Cauchy transform have an analytic continuation across $\partial\Omega$ into Ω?

For the unit disc $\mathbb{D}$ we have $\hat{\chi}_\mathbb{D}(z) = \frac{1}{z}$ for $z \notin \mathbb{D}$, which continues analytically to $\mathbb{C} \setminus \{0\}$, so this is one example of analytic continuation. In general, it is not hard to show that any domain with an analytic boundary has the continuation property, whereas a rough boundary, having for example a corner, never admits analytic continuation.

A complete characterization of boundaries admitting analytic continuation of the Cauchy transform has been given by M. Sakai [15], [16]. A different approach was given in [8], based on previous investigations in [13], [14]. Before stating our main result (namely the main result of [8]), let us illustrate the ideas by an example.

Example ("Classical quadrature domains").

A classical quadrature domain [1], [6], [7], [17], [12] is a domain $\Omega \subset \mathbb{C}$ such that, like in the case of the unit disc, the Cauchy transform $\hat{\chi}_\Omega$ admits analytic continuation from the exterior of the domain down to finitely many points in Ω, with only polar singularities at these points. In other words, it is a domain for which the exterior Cauchy transform is a rational function. An equivalent statement (cf. the equivalence between (4.4) and (4.6) below) is that there exists a distribution ρ with support in a finite number of points in Ω (namely the same points as those above) such that

$$\int_\Omega \varphi dA = \rho(\varphi) \tag{4.1}$$

for all integrable analytic functions φ in Ω. (The right member of (4.1) is to be interpreted as the action of ρ on the test function φ.)

Suppose that Ω is quadrature domain as above. Then we can write

$$\hat{\chi}_\Omega(z) = \frac{r(z)}{p(z)} \quad \text{for } z \notin \Omega,$$

where $p(z)$ and $r(z)$ are polynomials without common factors. The distribution ρ in (4.1) is supported at the zeros of $p(z)$, with a multiple zero corresponding to ρ being of higher order at the point. This implies that $\rho(p\varphi) = 0$ for every analytic function φ. The degree of $p(z)$,

$$n = \deg p,$$

is called the order of the quadrature domain. Then $r(z)$ will have degree $n-1$ because of the behaviour of the Cauchy transform at infinity.

By (4.1) and the discussion above,

$$\int_\Omega p\varphi dA = 0$$

for every $\varphi \in \mathcal{A}(\Omega)$. Since the kernel $H(z,w)$ itself is analytic/antianalytic this leads to

$$\|p\varphi\|^2_{\mathcal{H}(\Omega)} = \frac{1}{\pi^2}\int_\Omega\int_\Omega H(z,w)p(z)\varphi(z)\overline{p(w)\varphi(w)}dA(z)dA(w) = 0$$

for $\varphi \in \mathcal{A}(\Omega)$. In short:

$$p \cdot \mathcal{A}(\Omega) = 0. \tag{4.2}$$

The polynomial $p(z)$ is minimal with this property, so it follows that

$$\dim \mathcal{A}(\Omega) = n$$

if Ω is a quadrature domain of order n. Conversely it can be shown (cf. [12], [13]) that $\mathcal{A}(\Omega)$ being of finite dimension n implies that Ω is a quadrature domain of order n.

Continuing the example, we show that not only does $\langle k_z, 1\rangle = -\hat{\chi}_\Omega(z)$ have an analytic continuation up to the zeros of $p(z)$ when Ω is a quadrature domain, but also k_z itself has such a continuation, as an element of $\mathcal{H}(\Omega)$. Indeed, using (4.2) we have, for $z \notin \Omega$ and regarding functions of ζ as elements in $\mathcal{H}(\Omega)$,

$$k_z(\zeta) = \frac{1}{\zeta - z} = \frac{1}{\zeta - z} - p(\zeta)\cdot\frac{1}{p(z)(\zeta - z)} = \frac{q(\zeta, z)}{p(z)},$$

where

$$q(\zeta, z) = -\frac{p(\zeta) - p(z)}{\zeta - z},$$

a polynomial of degree $n-1$ in each of ζ and z. For us it is appropriate to regard $q(\cdot, z)$ as a polynomial in z with coefficients in $\mathcal{H}(\Omega)$. Then we see from the expression above that k_z, for $z \notin \Omega$, is a rational function in z with coefficients in $\mathcal{H}(\Omega)$ and that this rational function has singularities only at the zeros of $p(z)$. This proves the continuability of k_z.

In addition, using (3.8) and the fact that the set Z (see (2.5)) easily can be shown to be empty in the present case, we find that the boundary of Ω satisfies

$$\partial\Omega \subset \{z \in \mathbb{C} : \|\frac{q(\cdot, z)}{p(z)}\|_{\mathcal{H}(\Omega)} = 1\} = \{z \in \mathbb{C} : \|q(\cdot, z)\|^2 = |p(z)|^2\}.$$

Since $\|q(\cdot, z)\|^2 = \langle q(\cdot, z), q(\cdot, z)\rangle$ is a polynomial in z and $\bar{z}$ it follows that $\partial\Omega$ is a subset of an algebraic curve. This result was first obtained by H.S. Shapiro and D. Aharonov [1]. One can prove that $\partial\Omega$ is actually the whole algebraic curve in the right member above, minus at most finitely many points [7].

This finishes the example on (classical) quadrature domains. Our main result (below) can be viewed as a generalization of the example to a more general class of quadrature domains (sometimes called "quadrature domains in the wide sense", see [17]). It amounts to replacing the finite set in the example by an arbitrary compact subset of Ω.

Theorem 4.1. ([8]) *Let $\Omega \subset \mathbb{C}$ be open and bounded and let $K \subset \Omega$ be compact. Then the following statements are equivalent.*

- *(i) The map $\mathbb{C} \setminus \Omega \to \mathbb{C}$ given by*

$$z \mapsto \langle k_z, 1\rangle$$

extends analytically to $\mathbb{C} \setminus K \to \mathbb{C}$.

- *(ii) The map* $(\mathbb{C}\setminus\Omega)^2\to\mathbb{C}$ *given by*
$$(z,w)\mapsto\langle k_z,k_w\rangle$$
extends analytically/antianalytically to $(\mathbb{C}\setminus K)^2\to\mathbb{C}$.
- *(iii) The Hilbert space-valued map* $\mathbb{C}\setminus\Omega\to\mathcal{H}(\Omega)$ *given by*
$$z\mapsto k_z$$
extends analytically to $\mathbb{C}\setminus K\to\mathcal{H}(\Omega)$.

More precise forms of the statements are that there exist, in the three cases, analytic functions (or maps)

- $f:\mathbb{C}\setminus K\to\mathbb{C}$,
- $F:(\mathbb{C}\setminus K)^2\to\mathbb{C}$ (analytic/antianalytic),
- $\Phi:\mathbb{C}\setminus K\to\mathcal{H}(\Omega)$,

such that, respectively,

- $\langle k_z,1\rangle=f(z)$ for $z\in\mathbb{C}\setminus\Omega$,
- $\langle k_z,k_w\rangle=F(z,w)$ for $z,w\in\mathbb{C}\setminus\Omega$,
- $k_z=\Phi(z)$ for $z\in\mathbb{C}\setminus\Omega$.

As for analyticity of Hilbert space-valued maps we adopt a weak definition: $\Phi:\mathbb{C}\setminus K\to\mathcal{H}(\Omega)$ is analytic iff the function $z\mapsto\langle\Phi(z),\phi\rangle$ is analytic for each $\phi\in\mathcal{H}(\Omega)$. Since a function of two complex variables (e.g., $\langle k_z,k_w\rangle$) is analytic iff it is analytic in each variable separately, the essence of the theorem therefore is the implication
$$z\mapsto\langle k_z,1\rangle\quad\text{extends analytically}\Rightarrow$$
$$z\mapsto\langle k_z,\phi\rangle\quad\text{extends analytically for every }\phi\in\mathcal{H}(\Omega).$$

Proof. We shall prove the theorem under the simplifying assumptions that (3.4) holds and that $\Omega=\operatorname{int}\overline{\Omega}$. The general case is treated in [8]. The approach we present here is however different from that in [8].

As indicated above, it is enough to prove (i) $\Rightarrow$ (iii). So assume (i), let $f(z)$ denote the analytic continuation of $\langle k_z,1\rangle=-\hat{\chi}_\Omega(z)$ to $\mathbb{C}\setminus K$ and we shall find an analytic continuation $\Phi(z)$ of k_z itself. It is convenient to continue f further, in an arbitrary fashion over K to all of $\mathbb{C}$, and then think of f as the Cauchy transform of
$$\rho=\frac{\partial f}{\partial\bar{z}}.$$
In this process we may have to modify f near ∂K and let the support of ρ go slightly outside K, but then we are on the other hand free to take ρ as smooth as we like, which simplifies the interpretation of the forthcoming formulas. This enlargement of the support does not change anything in principle, so for simplicity of notation we still assume that
$$\operatorname{supp}\rho\subset K.\tag{4.3}$$

Thus assumption (i) now takes the form
$$\hat{\chi}_\Omega=\hat{\rho}\quad\text{on}\quad\mathbb{C}\setminus\Omega\tag{4.4}$$
or, equivalently,
$$\langle k_z,1\rangle=\langle\rho k_z,1\rangle\quad\text{for}\quad z\in\mathbb{C}\setminus\Omega.$$

We claim that the extension of k_z itself is given by ρk_z, i.e., that, in the notation introduced after the statement of the theorem,

$$\Phi(z) = \rho k_z. \tag{4.5}$$

Similarly we will have that the continuation of $\langle k_z, k_w\rangle$ in (ii) of the theorem is given by

$$F(z,w) = \langle \rho k_z, \rho k_w\rangle = \langle \Phi(z), \Phi(w)\rangle.$$

That Φ defined by (4.5) is, in fact, analytic $\mathbb{C}\setminus K \to \mathcal{H}(\Omega)$ is immediate from the definition of analyticity and from (4.3): in the expression for $\langle \Phi(z), \phi\rangle$ (where $\phi \in \mathcal{H}(\Omega)$), z will not meet any singularity until it reaches K. Alternatively, we may compute, using (3.9):

$$\frac{\partial}{\partial \bar z}\Phi(z) = -\pi\rho\delta_z = -\pi\rho(z)\delta_z,$$

which is the zero element in $\mathcal{H}(\Omega)$ when $z \notin \operatorname{supp}\rho$.

It remains to verify that $\Phi(z) = k_z$ as elements in $\mathcal{H}(\Omega)$ for $z \notin \Omega$. By an approximation argument [2], (4.4) can be shown to imply that

$$\int_\Omega \varphi dA = \int_\Omega \varphi\rho dA \tag{4.6}$$

for every integrable analytic function φ in Ω. Note that (4.4) says exactly that (4.6) holds for all $\varphi = k_z$, $z \notin \Omega$, so the opposite implication (that (4.6) implies (4.4)) is trivially true.

Choosing in (4.6) the integrable analytic function

$$\varphi(\zeta) = \int_\Omega H(\zeta,w)k_z(\zeta)\overline{\phi(w)}dA(w)$$

for $z \notin \overline{\Omega}$, $\phi \in \mathcal{D}(\Omega)$ gives

$$\langle k_z, \phi\rangle = \langle \rho k_z, \phi\rangle.$$

Thus $k_z = \rho k_z$ as elements in $\mathcal{H}(\Omega)$ for $z \notin \overline{\Omega}$, and by continuity also for $z \notin \Omega$. This proves the theorem.

□

Taking instead

$$\varphi(\zeta) = \int_\Omega H(\zeta,w)\overline{\phi(w)}dA(w)$$

in (4.6) gives similarly $\langle 1, \phi\rangle = \langle \rho, \phi\rangle$ $(\phi \in \mathcal{D}(\Omega))$. Hence

$$1 = \rho \quad \text{as elements in } \mathcal{H}(\Omega). \tag{4.7}$$

Conversely, (4.7) implies that $k_z = \rho k_z$ for $z \notin \Omega$. Thus an additional equivalent statement to those in Theorem 4.1 is that $1 \in \mathcal{H}(\Omega)$ is "carried" by K in the sense that it has a representative (such as ρ above) with support in K.

The above analytic continuation result may look innocent, but it is indeed powerful. Our main application is the apriori regularity of some free boundaries in two dimensions:

Corollary 4.2. ([15], [8]) *If $\hat\chi_\Omega$ has an analytic continuation from the exterior of Ω across $\partial\Omega$, then $\partial\Omega$ is contained in a real analytic variety.*

There are examples showing that $\partial\Omega$ may have singular points, of certain specific types. These can be completely classified [16].

Proof. By direct arguments, as in [10] (proof of Lemma 2.11) or [8] (Lemma 4.4), or by using a newly invented theory of "quasi balayage" [18], one shows that the set Z defined by (2.5) is empty under the assumtions of the corollary. Then it follows from (2.6) or (3.8) that, in the notation of the theorem,

$$\partial\Omega \subset \{z \in \mathbb{C} \setminus K : F(z, z) = 0\} = \{z \in \mathbb{C} \setminus K : \|\Phi(z)\| = 1\},$$

which proves the corollary. $\square$

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