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The Runge Approximation Theorem on Compact Riemann Surfaces

by

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Contents:

0.1. Introduction

The purpose of this report is to give a simple and direct proof of a " Runge approximation theorem" for compact Riemann surfaces:

If Ω is an open subset of a compact Riemann surface W, then the meromorphic functions on W with poles off Ω are dense in the space of holomorphic functions on Ω (in the topology of uniform convergence on compact subset of Ω).

Although this is a most natural generalization of the classical Runge theorem ([R1, Theorem 13.9] for example), I have not been able to find theorems of this kind (for compact Riemann surfaces) in any of the standard textbooks on Riemann surfaces.

Even in the journal literature it seems very difficult to find simple and self-contained proofs of the Runge theorem. The most relevant articles I have found are the following ones:

[G1] : Here S. Ya Gusman proves a Mergelyan theorem for compact Riemann surfaces, using the Runge theorem as a starting point. The Runge theorem is quoted from [S1].

[S1] : In this paper some Runge-type theorems are stated, but I must admit that I have not been able to understand the proofs. In any case the proofs are quite different from the proof given in this report.

[T1] : Here the Runge theorem is proved in the special case that $W-\Omega$ is simply connected. The approximation of a given function is achieved by developing it along a complete system of "elementary functions", analogously to the power series expansion of a function in the complex plane. The system of elementary functions is obtained from a Cauchy kernel, similar to the Cauchy kernel costructed in this report (see the Remark on p. 6.).

[C1] : In the Cartan seminars 1951/52 more general and deeper approximation theorems on complex analytic varietes are developed. John Wermer has, in private communication with Prof. Harold S. Shapiro, outlined a proof of the Runge theorem from these theorems.

Such a proof is included as an Appendix in this report.

Thus there seems to be a gap to fill, namely that of a simple and self-contained proof of Runge's theorem, using only elementary properties of Riemann surfaces. This motivates the present report.

In this connection, it should be mentioned that Runge theorems for non-compact Riemann surfaces are included in many textbooks, for example [BS1, Kap. VI. §6] and [HC1, Anhang 2 § 7].

The method of proof used in this paper essentially agrees with that "functional analysis proof" of the Runge theorem for $\mathbb{P}$ (the Riemann sphere) which, more or less explicitly, builds upon the duality between the holomorphic functions on $\Omega \subset \mathbb{P}$ and the holomorphic differentials on $\mathbb{P} - \Omega$ (see [RT1] for these matters and for further references) . To carry this out for a compact Riemann surface $W \neq \mathbb{P}$ one has to construct a substitute for the "Cauchy kernel" , $\frac{d\xi}{\xi - z}$, and this construction is the main part of the paper.

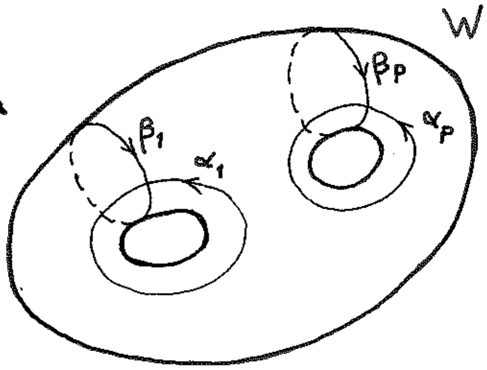
0.2. List of Notations

W will always denote a compact Riemann surface,

$p = \text{genus}(W)$,

$\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_p$ a canonical homology basis on W

as in the figure i.e. $\alpha_1,\dots,\beta_p$ are a basis for the cycles on W such that β_k crosses α_k once from right to left $(k=1,\dots,p)$, and such that $\alpha_1,\dots,\beta_p$ are otherwise disjoint.



$H(\Omega)$, for $\Omega \subset W$ open : the space of holomorphic functions on Ω , provided with topology of uniform convergence on compacts.

$M(\Omega)$: the meromorphic functions on Ω .

$H_{diff}(\Omega)$: the space of holomorphic differentials on Ω with the topology of uniform convergence on compacts , i.e. the topology defined by the semi-norms

$$||df||_{(U,z)} = \left|\left|\frac{df}{dz}\, dz\right|\right|_{(U,z)} = \sup_{\bar U}\left|\frac{df}{dz}\right|$$

for all coordinate-systems (U,z) , $z:U \to \mathbb{C}$ with $\bar U \subset \Omega$ ($\bar U$ compact) .

$M_{diff}(\Omega)$: the meromorphic differentials on Ω .

$H(E)$, for an arbitrary subset $E \subset W$: the space of functions holomorphic in some neighbourhood of E , two functions being identified if they agree on some neighbourhood of E. Likewise for $M(E)$, $H_{diff}(E)$ and $M_{diff}(E)$.

df : meromorphic differentials will often be denoted df, du, ... even if they are not exact ; i.e. the integrals f, u, ... may be additively multiple-valued .

Z, ξ, ... : such letters will often stand for both local variables and points on a Riemann surface .

A^* : the dual space of a topological vector space A .

$S^\perp = \{L \in A^* : L(f) = 0$ for all $f \in S\}$ if S is a subset of A .

$\underset{\bullet}{z}$, $\underset{\bullet}{\xi}$, ... : a dot under a variable indicates that it is "bound" in some sense (for example an integration variable when there are several variables involved) .

$\mathbb{P}$: the Riemann sphere .

I 1 The generalized Cauchy kernel :

Let W be a compact Riemann surface of genus p , z_o, $\xi_o \in W$ two fixed points .

Then there exists a kernel $\varphi(z,\xi)d\xi$ on W with the following properties:

① For fixed $\xi \neq z_o$: $\varphi(z,\xi) \in M(W) \cap H(W-\{\xi,\xi_o\})$,
where the pole at $z = \xi$ is of order = 1 ,
the pole at $z = \xi_o$ is of order $\leq 2p-1$.

If ξ_o is not a Weierstrass point the pole at $z = \xi_o$ may even be taken to be of order $\leq$ p .

② For fixed $z \neq \xi_o$: $\varphi(z,\xi)d\xi \in M_{diff}(W) \cap H_{diff}(W-\{z,z_o\})$,

where the poles are simple poles with $\begin{cases} \text{res} = +1 & \text{at } \xi = z \\ \text{res} = -1 & \text{at } \xi = z_o \end{cases}$

③ For $f \in H(\bar{U})$, $dg \in H_{diff}(W-U)$, where $U \subset W$ is an open set with rectifiable boundary ∂U $(z_o,\xi_o \notin \partial U)$:

$$\begin{cases} (1) & \dfrac{1}{2\pi i}\displaystyle\int_{\partial U} f(\xi)\varphi(z,\xi)d\xi = f(z) \pmod{H(W)} \text{ for } z\in U , \\ (2) & \dfrac{1}{2\pi i}\displaystyle\int_{\partial U} dg(z)\varphi(z,\xi)d\xi = dg(\xi) \pmod{H_{diff}(W)}, \ \xi\in W-\bar{U} . \end{cases}$$

(H(W) is just the constants, and $H_{diff}(W)$ is the p -dimensional space of abelian differentials of the first kind .)

<u>Remark</u> : $\varphi(z,\xi)d\xi$ is not uniquely determined by the properties ① - ③. Infact, if the pole for $\varphi(z,\xi)$ at ξ_o is required to be of minimal order , one finds that any two Cauchy kernels differ by some differential (with respect to ξ) in $H_{diff}(W)$.

There are several ways to normalize $\varphi(z,\xi)d\xi$ so that it becomes uniquely determined . The kernel constructed in [T1] is an example of such a normalization . Namely, z_o is taken to be a non-Weierstrass point , $\xi_o = z_o$, the pole of $\varphi(z,\xi)$ at $z = \xi_o$ is taken to be of order p , and for ξ at z_o it is required that :

$$\varphi(z,\xi)d\xi = - \frac{d\xi}{\xi - z_o} + O((\xi - z_o)^p) .$$

I2 Construction of the Cauchy kernel

In the case $W = \mathbb{P}$, the kernel will be :

$$\varphi(z,\xi)d\xi = \frac{d\xi}{\xi-z} - \frac{d\xi}{\xi-z_o} = \int_{z_o}^{z} \frac{dt d\xi}{(\xi-t)^2}$$

In case genus(W) = p > 0 there is a natural substitute , $h(z,\xi)dzd\xi$, for the differential $\frac{dzd\xi}{(\xi-z)^2}$, but the integral on the right-hand side above will not be single-valued with $h(t,\xi)dtd\xi$ in place of $\frac{dtd\xi}{(\xi-t)^2}$. This trouble is solved by subtractning off the periods of $h(z,\xi)dzd\xi$, and in order to do that , one has to introduce another singularity in the z variable , $z = \xi_o$.

The building blocks in the construction of $\varphi(z,\xi)d\xi$ will be the following standard differentials on W (see for example [W1, §§ 15-16] or [SS1, Ch 3] for this material) :

① The p normalized abelian differentials of the first kind , i.e. the basis $du_1, \ldots, du_p$ of $H_{diff}(W)$ with

(3) $$\int_{\alpha_k} du_j = \delta_{kj} = \begin{cases} 1 & \text{for } k=j \\ 0 & k \neq j \end{cases} , \quad k, j = 1,\ldots,p .$$

② The normalized abelian differentials of the second kind with a single pole of order 2 , i.e. the differentials :

(4) $$h(z,\xi)dz = \frac{dz}{(\xi-z)^2} + \text{regular terms} \in M_{diff}(W) \cap H_{diff}(W-\{\xi\})$$

with periods :

(5) $$\begin{cases} \int_{\alpha_k} h(z,\xi)dz = 0 , & k=1,\ldots,p \\ \int_{\beta_k} h(z,\xi)dz = 2\pi i \frac{du_k(\xi)}{d\xi} , & k=1,\ldots,p . \end{cases}$$

The dependence on ξ is such that

$$h(z,\xi)dzd\xi = \frac{dzd\xi}{(\xi-z)^2} + \text{regular terms}$$

has an invariant meaning and is a meromorphic differential in ξ for fixed z. Infact $h(z,\xi)dzd\xi = h(\xi,z)d\xi dz$.

Now we want $\varphi(z,\xi)d\xi$ essentially to be the integral of $h(z,\xi)dzd\xi$ with respect to z . Hence we must subtract off the β_k-periods , $2\pi i\cdot du_k(\xi)$. Take any fixed point $\xi_o \in W$. Suppose we can find meromorphic differentials $dv_1,\ldots,dv_p$ with poles only at ξ_o and with periods :

$$\begin{cases} \int_{\alpha_k} dv_j = 0 & \text{for all } k,j = 1,\ldots,p \\ \int_{\beta_k} dv_j = 2\pi i\cdot\delta_{kj} . \end{cases}$$

Then the differential

$g(z,\xi)dzd\xi = h(z,\xi)dzd\xi - \sum_{k=1}^{p} dv_k(z)\cdot du_k(\xi)$ is readily seen to be exact in z for each ξ . (Observe that $g(z,\xi)dzd\xi$ has no residues in z (nor in ξ) , since both $h(z,\xi)dz$ and $dv_k(z)$ have only one pole each , the residue of which therefore must be = 0) .

Hence we can define :

$$\varphi(z,\xi)d\xi = \int_{z_o}^{z} g(t,\xi)dtd\xi$$

where $z_o \in W$ is an arbitrary fixed point *.

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* When $z_o = \xi_o$ a slight modification is needed , since the integrand then has a pole at the lower bound $t = z_o$ (namely the pole of dv_k) . For $z_o \neq \xi_o$ we have :

$$\varphi(z,\xi)d\xi = \int_{z_o}^{z} h(t,\xi)dtd\xi - \sum_{k=1}^{p} (v_k(z) - v_k(z_o))du_k(\xi) .$$

For $z_o = \xi_o$: simply omit the term

$$\sum_{k=1}^{p} v_k(z_o)du_k(\xi) .$$

The properties of $\varphi(z,\xi)d\xi$ are most easily exhibited by writing

$$\varphi(z,\xi)d\xi = \frac{d\xi}{\xi - z} - \frac{d\xi}{\xi - z_o} - \sum_{k=1}^{p} v_k(z)\cdot du_k(\xi) + \text{reg} ,$$

where the individual terms are multiple-valued (as functions of z) but the periods cancel between the terms, the $v_k(z)$ have poles at $z = \xi_o$ and are regular elsewhere and the $du_k(\xi)$ are everywhere regular. Apart from the statements about the order of the pole at $z = \xi_o$, the asserted properties (1) and (2) of $\varphi(z,\xi)d\xi$ are now checked by mere inspection .

To check (3) suppose

$$\begin{cases} f \in H(\bar{U}) \\ dg \in H_{diff}(W-U) \end{cases} ,$$

$U \subset W$ open with rectifiable boundary ∂U .

Then the residue theorem gives , for $z \in U$:

$$\frac{1}{2\pi i}\int_{\partial U} f(\xi)\varphi(z,\xi)d\xi = \sum_{\xi \in U} \text{res}\, f(\xi)\varphi(z,\xi)d\xi =$$

$$= \begin{cases} f(z) - f(z_o) & \text{if} \quad z_o \in U \\ f(z) & \phantom{\text{if}} \quad z_o \in W-\bar{U} \end{cases} = f(z) \pmod{H(W)} ,$$

and , for $\xi \in W-\bar{U}$:

$$\frac{1}{2\pi i}\int_{\partial U} dg(z)\varphi(z,\xi)d\xi = -\frac{1}{2\pi i}\int_{\partial(W-\bar{U})} dg(z)\varphi(z,\varphi)d\xi =$$

$$= -\sum_{z \in W-\bar{U}} \text{res}\, dg(z)\varphi(z,\xi)d\xi =$$

$$= \begin{cases} -\underset{z=\xi}{\text{res}} \dfrac{dg(z)}{\xi - z} d\xi + \displaystyle\sum_{k=1}^{p} \underset{z=\xi_o}{\text{res}} (v_k(z)dg(z))\, du_k(\xi) & \text{if} \quad \xi_o \in W-\bar{U} \\ -\underset{z=\xi}{\text{res}} \dfrac{dg(z)}{\xi - z} d\xi & \phantom{\text{if}} \quad \xi_o \in U \end{cases}$$

$$= \begin{cases} dg(\xi) + \displaystyle\sum_{k=1}^{p} \underset{\xi_o}{\text{res}}(v_k dg)\, du_k(\xi) & \text{if} \quad \xi_o \in W-\bar{U} \\ dg(\xi) & \phantom{\text{if}} \quad \xi_o \in U \end{cases}$$

$$= dg(\xi) \pmod{H_{diff}(W)}$$

(the fact that the $v_k(z)$ are not single-valued does not affect the residues $\mathrm{res}(v_k dg)$ since dg is regular in $W-\bar{U}$) .
Thus , to finish the construction we have to settle the existence of the differentials dv_k ,that is to prove the following lemma .

Lemma : Given any point $\xi_o \in W$, there are meromorphic differentials $dv_1,\dots,dv_p$ with poles only at ξ_o and with the periods

$$\begin{cases} \int_{\alpha_k} dv_j = 0 \\ \int_{\beta_k} dv_j = 2\pi i\cdot\delta_{kj} \end{cases} \qquad k,j = 1,\dots,p$$

Moreover, the orders of the poles at ξ_o can always be taken to be $\leq 2p$, and if ξ_o is not a Weierstrass point the orders can even be taken to be $\leq p+1$.
(Since $\varphi(z,\xi)d\xi$ contains the differentials dv_k in integrated form, the order of the pole at ξ_o in $\varphi(z,\xi)d\xi$ will be $\leq 2p-1$ resp. $\leq p$.)

Proof : $dv_1,\dots,dv_p$ are going to be suitable linear combinations of the differentials (in z) :

$$\frac{\partial^n}{\partial\xi^n} h(z,\xi_o)dz = \frac{\partial^n}{\partial\xi^n}\bigg|_{\xi_o} h(z,\xi)dz \quad , \quad n = 0,\ 1,\ 2,\dots \quad .$$

Here ξ is a fixed local parameter at the point ξ_o .
By (4) we have :

$$\frac{\partial^n}{\partial\xi^n} h(z,\xi_o)dz = (-1)^n(n+1)!\,\frac{dz}{(\xi_o-z)^{n+2}} \quad + \text{ regular terms}$$

and :

$$\int_{\alpha_k} \frac{\partial^n}{\partial\xi^n} h(z,\xi_o)dz = \frac{\partial^n}{\partial\xi^n}\bigg|_{\xi_o} \int_{\alpha_k} h(z,\xi)dz = 0 \quad .$$

Hence the linear combinations

$$dv_k(z) = \sum_{n=0}^{m} a_{kn} \frac{\partial^n}{\partial \xi^n} h(z,\xi_o)dz$$

will do if and only if

$$\int_{\beta_j} dv_k(z) = \sum_{n=0}^{m} a_{kn} \frac{\partial^n}{\partial \xi^n}\Big|_{\xi_o} \int_{\beta_j} h(z,\xi_o)dz = 2\pi i \cdot \delta_{kj}$$

hence , by (5) , if and only if

$$\sum_{n=0}^{m} a_{kn} \frac{d^{n+1}u_j}{d\xi^{n+1}} (\xi_o) = \delta_{kj} \qquad , k,j = 1,\ldots,p$$

For fixed m this system of equations can be solved if and only if the matrix

$$M_m = \begin{bmatrix} \frac{du_1}{d\xi}(\xi_o) & \frac{d^2u_1}{d\xi^2}(\xi_o) & \cdots\cdots & \frac{d^{m+1}u_1}{d\xi^{m+1}}(\xi_o) \\ \vdots & \vdots & & \vdots \\ \frac{du_p}{d\xi}(\xi_o) & \frac{d^2u_p}{d\xi^2}(\xi_o) & \cdots\cdots & \frac{d^{m+1}u_p}{d\xi^{m+1}}(\xi_o) \end{bmatrix}$$

has rank = p (i.e. the columus span $\mathbb{C}^p$) .

Consider an m for which the columus of M_m do <u>not</u> span $\mathbb{C}^p$. Then there is a non-zero vector $(b_1,\ldots,b_p)$ such that

$$b_1\frac{d^n u_1}{d\xi^n}(\xi_o) + \ldots + b_p\frac{d^n u_p}{d\xi^n}(\xi_o) = 0$$

for $n = 1, \ldots, m+1$

This means that the everywhere regular differential

$du = b_1 du_1 + \ldots + b_p du_p$

is non-trivial and has a zero of order $\geq m + 1$ at ξ_o .

But it is a classical fact that an everywhere regular differential on a compact Riemann surface of genus = p has exactly 2p-2 zeroes .

Hence $m + 1 \leq 2p - 2$ (if M_m has rank $< p$) .
Therefore M_m must have rank $= p$ for every $m \geq 2p - 2$.
This means that the system of equations can always be solved with $m = 2p - 2$, and the solution will then yield differentials

$$dv_k(z) = \sum_{n=0}^{2p-2} a_{kn} \frac{\partial^n}{\partial \xi^n} h(z,\xi_o)dz$$

with the poles at ξ_o of order $\leq 2p$ as claimed .
Finally , the Weierstrass points on W are (by definition) the $\leq (p-1)p(p+1)$ points $\xi_o \in W$ at which M_{p-1} has rank $< p$.
Hence , if ξ_o is not a Weierstrass point m can be taken $= p - 1$ so that the poles of dv_k will be of order $\leq p+1$.
This proves the lemma and also finishes the construction of the Cauchy kernel .

I.3. Examples of Cauchy kernels

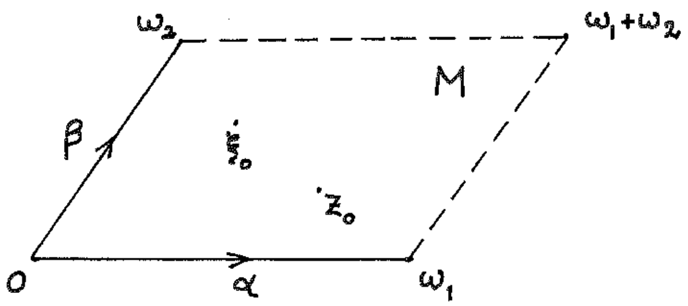
① As already remarked , the simpliest choice of $\varphi(z,\xi)d\xi$ when $p = 0$ ($W = \mathbb{P}$) is :

$$\varphi(z,\xi)d\xi = \frac{d\xi}{\xi - z} - \frac{d\xi}{\xi - z_0}$$

or , with $z_0 = \infty$:

$$\varphi(z,\xi)d\xi = \frac{d\xi}{\xi - z}$$

② When $p = 1$, W is a torus and can be represented as a period parallellogram M with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$, say (reference : [A1 , Ch 7]) .
The meromorphic functions on W correspond to the doubly periodic functions (elliptic functions) with respect to the basis (ω_1,ω_2) . The canonical homology basis (α,β) on W may be taken to be the one represented by the segments $[0,\omega_1]$, $[0,\omega_2]$ on M (see figure) .



Let :

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega}\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right] ,$$

where the sum ranges over $(\omega_1\mathbf{Z} + \omega_2\mathbf{Z}) - \{0\}$,

be the Weierstrass $\wp$-function with respect to (ω_1,ω_2) and put :

$$\eta_j = - \int_0^{\omega_j} \wp(z)dz , \quad j = 1,2.$$

(η_1 and η_2 are related by $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$,Legendre's relation .) .

Let z_o , $\xi_o \in M$ be the two fixed points in the Cauchy kernel $\varphi(z,\xi)d\xi$ for W . Then one finds that the simplest choice of $\varphi(z,\xi)d\xi$ becomes (when $z_o \neq \xi_o$) :

$$\varphi(z,\xi)d\xi = \int_{z_o}^{z} [\wp(t-\xi) - \wp(t-\xi_o)]dt d\xi \quad .$$

The other differentials occurring in the construction of $\varphi(z,\xi)d\xi$ will be :

$$\begin{cases} h(z,\xi)dzd\xi = (\wp(z-\xi) + \dfrac{\eta_1}{\omega_1})dzd\xi , \\ du(\xi) = \dfrac{d\xi}{\omega_1} \\ dv(z) = (\omega_1\wp(z-\xi_o) + \eta_1)dz . \end{cases}$$

Traditionally , the antiderivative of $\wp(z)$ is denoted $-\zeta(z)$, and is normalized so that it is odd . In terms of $\zeta(z)$ the Cauchy kernel thus becomes :

$$\varphi(z,\eta)d\eta = [-\zeta(z-\eta) + \zeta(z_o-\eta) + \zeta(z-\eta_o) - \zeta(z_o-\eta_o)]d\eta_1$$

where we have replaced the variable ξ by η and ξ_o by η_o . With an arbitrary constant A in place of the constant $\zeta(z_o-\eta_o)$, we get the most general Cauchy kernel with the pole at η_o of minimal order (= 1) , at the same time as we account for the case $z_o = \xi_o$:

$$\varphi(z,\eta)d\eta = [-\zeta(z-\eta) + \zeta(z_o-\eta) + \zeta(z-\eta_o) - A]d\eta \quad .$$

II 1. The Runge approximation theorem :

Let W be a compact Riemann surface ,

$\Omega \subset W$ an open subset ,

$E \subset W-\Omega$ a set which intersects each component of $W-\Omega$.

Then : $M(W) \cap H(W-E)$ is dense in $H(\Omega)$.
In particular : $M(W) \cap H(\Omega)$ is dense in $H(\Omega)$.

Also : $M_{diff}(W) \cap H_{diff}(W-E)$ is dense in $H_{diff}(\Omega)$,

and in particular : $M_{diff}(W) \cap H_{diff}(\Omega)$ is dense in $H_{diff}(\Omega)$.

II 2. Proof of the Runge theorem :

① In order to prove that a certain subspace A is dense in $H(\Omega)$ it suffices to prove that for $L \in H(\Omega)^*$,

$$L \in A^{\perp} \Longrightarrow L = 0 \quad .$$

Hence, take a functional $L \in H(\Omega)^*$. To begin with we only assume that $L \in H(W)^{\perp}$ (i.e. that L annihilates the constants).

② By the definition of the topology on $H(\Omega)$ there is a compact $K \subset \Omega$ and a constant M such that

$$|L(f)| \leq M \sup_K |f| \qquad \text{for all} \quad f \in H(\Omega)$$

Hence, by the Hahn-Banach theorem L extends to a functional L' on $H(K)$ (the functions holomorphic in some neighbourhood of K) such that the same estimate holds for all $f \in H(K)$.

③ Choose a Cauchy kernel $\varphi(z,\xi)d\xi$ on W with $z_o \in K \subset \Omega$, $\xi_o \in E \subset W-\Omega$.

For each $\xi \in W-K$, $\varphi(\cdot,\xi) \in H(K)$. Hence $L'(\varphi(\cdot,\xi))$ is meaningful and it has the transformation properties of a differential in ξ, so that :

$$dg(\xi) = L'(\varphi(\cdot,\xi))d\xi$$

has an invariant meaning. Since the difference quotients

$$\frac{\varphi(z,\xi') - \varphi(z,\xi)}{\xi' - \xi} \quad , \qquad \xi \in W-K$$

converge to $\frac{\partial}{\partial \xi}\varphi(z,\xi)$ as $\xi' \to \xi$ <u>uniformly for $z \in K$</u>, it follows that

$\frac{\partial}{\partial\xi} L'(\varphi(\cdot,\xi))$ exists and equals $L'(\frac{\partial}{\partial\xi}\varphi(\cdot,\xi))$.

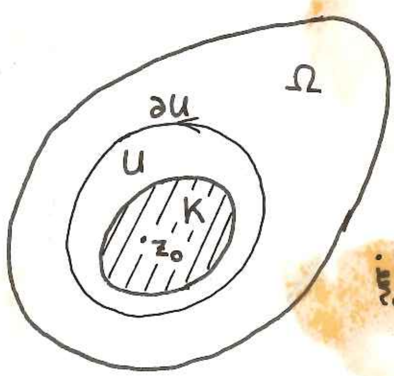
Hence $dg(\xi)$ is a holomorphic differential for $\xi \in W-K$. Repeated use of this argument also shows that

$$\frac{d^{n+1}g(\xi)}{d\xi^{n+1}} = L'\left(\frac{\partial^n}{\partial\xi^n}\varphi(\cdot,\xi)\right) \quad , \quad n = 0,1,2,\ldots, \quad \xi \in W-K .$$

④ For each open U , $K \subset U \subset \bar{U} \subset \Omega$, ∂U rectifiable , we have the formula :

$$\frac{1}{2\pi i}\int_{\partial U} f(\xi)\varphi(z,\xi)d\xi = f(z) \qquad (\text{mod } H(W))$$

for $z \in U$, $f \in H(\bar{U})$.



The Riemann sums of this integral converge to the integral uniformly for $z \in K$. It follows that for $f \in H(\Omega)$:

$$L(f) = L'(f) = \frac{1}{2\pi i} L'\left(\int_{\partial U} f(\xi)\varphi(\cdot,\xi)d\xi\right) =$$

$$= \frac{1}{2\pi i}\int_{\partial U} f(\xi)L'(\varphi(\cdot,\xi))d\xi = \frac{1}{2\pi i}\int_{\partial U} f(\xi)dg(\xi) .$$

⑤ Having this representation formula for the functionals $L \in H(W)^{\perp} \subset H(\Omega)^*$ it is clear , that in order to prove that $L = 0$, it suffices to prove that $dg(\xi) = 0$ in some neighbourhood of $W-\Omega$ (since then we choose U so that $W-U$, and hence ∂U , is included in that neighbourhood) .

In other words , it is enough to prove that the Taylor expansion

$$\frac{dg}{d\xi}(\xi) = \frac{dg}{d\xi}(\xi_1) + \frac{d^2g}{d\xi^2}(\xi_1)\cdot(\xi-\xi_1) + \frac{1}{2!}\frac{d^3g}{d\xi^3}(\xi_1)\cdot(\xi-\xi_1)^2 + \ldots$$

vanishes for each $\xi_1 \in W-\Omega$, or even that it vanishes for one ξ_1 in each component of $W-\Omega$.

⑥ Hence , if $L \in (M(W) \cap H(W-E))^{\perp}$, the choice

$$\frac{\partial^n}{\partial\xi^n}\varphi(\cdot,\xi_1) \in M(W) \cap H(W-E) \quad , \quad \xi_1 \in E \quad , n \geq 0$$

gives :

$$0 = L\left(\frac{\partial^n}{\partial\xi^n}\varphi(\cdot,\xi_1)\right) = L'\left(\frac{\partial^n}{\partial\xi^n}\varphi(\cdot,\xi_1)\right) = \frac{d^{n+1}g}{d\xi^{n+1}}(\xi_1)$$

so that the Taylor expansion in ⑤ vanishes for each $\xi_1 \in E$, and therefore $L = 0$.
This proves the part of the theorem which is about the space $H(\Omega)$.
The proof of the part cocerning $H_{diff}(\Omega)$ is quite similar . The modifications needed are :

① One considers functionals $L \in H_{diff}(W)^{\perp} \subset H_{diff}(\Omega)^*$.

② The estimate becomes :

$$|L(dg)| \leq M\cdot\max_{1\leq j\leq m} \cdot \sup_{V_j} \left|\frac{dg}{dz_j}\right| \quad , \quad dg \in H_{diff}(\Omega)$$

where $(V_1,z_1),\ldots,(V_m,z_m)$ are a finite number of coordinate systems (i.e. z_j conformal $V_j \to \mathbb{C}$) with $K = \bar{V}_1 \cup \ldots \cup \bar{V}_m \subset \Omega$ compact .
Now L extends to a functional L' on $H_{diff}(K)$ such that the estimate holds for all $dg \in H_{diff}(K)$.

③ The Cauchy kernel is chosen with $\xi_o \in K \subset \Omega$, $z_o \in E \subset W-\Omega$. One defines :

$$f(z) = L'(\varphi(z,\xi)d\xi)$$

and f becomes holomorphic in $W-K$ with

$$\frac{d^n f(z)}{dz^n} = L'\left(\frac{\partial^n}{\partial z^n} \varphi(z,\xi)d\xi \right) \quad , \quad n = 0, 1, \ldots, \quad z \in W-K$$

④ The representation formula becomes :

$$L(dg) = -\frac{1}{2\pi i} \int_{\partial U} dg(z)f(z) \quad , \quad dg \in H_{diff}(\Omega) \quad .$$

⑤ To conclude that $L = 0$ it is enough that the Taylor expansion :

$$f(z) = f(z_1) + \frac{df}{dz}(z_1)(z-z_1) + \frac{1}{2!}\frac{d^2f}{dz^2}(z_1)\cdot(z-z_1)^2 + \ldots$$

vanishes for each $z_1 \in E$.

⑥ Since $\frac{\partial^n}{\partial z^n} \varphi(z_1,\xi)d\xi \in M_{diff}(W) \cap H_{diff}(W-E)$

for $z_1 \in E$, $n \geq 0$, this is the case if

$L \in (M_{diff}(W) \cap H_{diff}(W-E))^{\perp}$ by the formula in ③ .

This finishes the proof .

II. 3 Comments

① The Runge theorem can be generalized in several directions. One can for instance allow $\Omega \subset W$ to be any subset (not necessarily open) of W, and introduce on $H(\Omega)$ (resp. $H_{diff}(\Omega)$) the so called hull topology ([K 1 § 19, 27]), and the Runge theorem remains true word for word.

Another generalization is along the lines in [RT1]: Replace the set E in the theorem by a sequence $Z=\{z_1, z_2, \ldots\}$ of points in W, where the same point may occur more than once (finitely or infinitely many times). Define $M_Z(W)$ to be the set of meromorphic functions on W, allowed to have a pole of order n at the point $z \in W$ if and only if z occurs in the sequence Z at least n times.
Then, a minor modification of the argument used in parts ⑤ - ⑥ of the proof of the Runge theorem gives:
If the sequence Z has at least one limit point in each component of $W-\Omega$, then $M_Z(W)$ is dense in $H(\Omega)$ (and $M_{diff,Z}(W)$ dense in $H_{diff}(\Omega)$).

② The Runge theorem can be put in a another light by means of the duality between $H(U)$ and $H_{diff}(W-U)$ mentioned in the introduction. This duality is defined by the bilinear pairing

$$B(f, dg) = \frac{1}{2\pi i} \int_{\partial U} f(z)dg(z)$$

for $f \in H(U)$, $dg \in H_{diff}(W-U)$.

For simplicity, assume that U is open. This means that dg is holomorphic in some neighbourhood of $W-U$, and the integral $\int_{\partial U}$ is understood to be along a path moved a little into U (for details about this bilinear pairing in the case $W=\mathbb{P}$ and $U \subset W$ is an arbitrary subset, see [G 2].

It is easily seen that if $dg \in H_{diff}(W)$ then $B(f,dg) = 0$ for all $f \in H(U)$, and if $f \in H(W)$ (i.e. f is constant) then $B(f,dg) = 0$ for all $dg \in H_{diff}(W-U)$.
Now , the point is that the converses of these statements are true , or even more :

if $B(f,dg)= 0$ for all $f \in H(U) \cap M(W)$

then $dg \in H_{diff}(W)$ resp.

if $B(f,dg) = 0$ for all $dg \in H_{diff}(W-U) \cap M_{diff}(W)$
then $f \in H(W)$.

In fact , this is essentially parts ⑤ - ⑥ in the proof of the Runge theorem .

In particular this means :

if $B(f,dg) = 0$ for all $f \in H(U) \cap M(W)$,

then $B(f,dg) = 0$ for all $f \in H(U)$

(and similarly for the other half) .
Therefore , in view of the Hahn-Banach theorem :

$H(U) \cap M(W)$ is dense in $H(U)$ for any topology on $H(U)$ compatible with the duality between $H(U)$ and $H_{diff}(W-U)$ defined by B , i.e. for any topology on $H(U)$ for which $H_{diff}(W-U)$ represents the dual space $H(U)^*$ via B .
Now , (and finally) , the essence of parts ① - ④ in the proof of the Runge theorem is that the usual topology on $H(U)$ <u>is</u> such a topology (in fact the strongest one) .

III. APPENDIX

As John Wermer has pointed out , the Runge theorem also follows from more general approximation theorems on complex analytic varietes . A proof along those lines will be given below .

For simplicity , we prove the part of the Runge theorem stating that $M(W) \cap H(\Omega)$ is dense in $H(\Omega)$, where W is a compact Riemann surface and $\Omega \subset W$ is open .
Our starting-point will be the following theorem , proved in the Cartan seminars 1951/52 (Théorèm 4 in seminar no.9) :

Let E be a complex analytic variety , and suppose $\mathfrak{F}$ is an algebra of holomorphic functions on E satisfying the following condititons :

1) Every compact $K \subset E$ has a neighbourhood V such that $K_{\mathfrak{F}} \cap V$ is compact . Here

$$K_{\mathfrak{F}} = \{x \in E : |f(x)| \leq \sup_{y \in K} |f(y)| \quad \text{for all} \quad f \in \mathfrak{F}\} .$$

2) $\mathfrak{F}$ separates points on E .
3) $\mathfrak{F}$ furnishes local coordinates at each point on E .

Then $\mathfrak{F}$ is dense in the algebra of holomorphic functions on E (in the topology of uniform convergence on compact subsets on E) .

We want to apply this theorem with $E = \Omega$ (which certainly is a complex analytic variety) and $\mathfrak{F} = M(W) \cap H(\Omega)$. The conclusion of the theorem is then just the Runge theorem we want to prove .

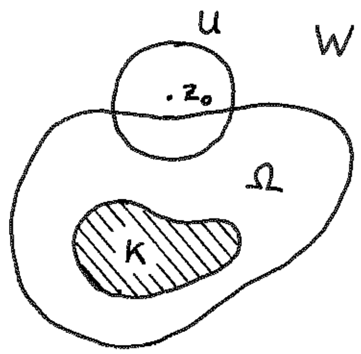
Thus it only remains to check the properties 1) - 3) for .

1) : To check 1), let $K \subset \Omega$ be compact.
For each point $z_o \in W-\Omega$ there is a function $f \in M(W) \cap H(\Omega)$ with a pole at z_o (by the Riemann-Roch theorem).

Let U be a neighbourhood of z_o such that

$$|f(z)| > \sup_{\xi \in K} |f(\xi)| + 1$$

for $z \in U \cap \Omega$.



Then $W-\Omega$ can be covered by finitely many such neighbourhoods $U_1, \ldots, U_m$, and it follows that
$V = \Omega - (\bar{U}_1 \cup \ldots \cup \bar{U}_m)$
is a neighbourhood of K which also contains $K_{\mathcal{F}}$.
Hence $K_{\mathcal{F}} \cap V = K_{\mathcal{F}}$ is compact.

2) : Let $z_1, z_2 \in \Omega$, $z_1 \neq z_2$. We must prove that $f(z_1) \neq f(z_2)$ for some $f \in M(W) \cap H(\Omega)$.
Choose a function $g \in M(W) \cap H(\Omega)$ with a zero of order ≥ 1 at z_1 and a zero of order $N \geq 2p$ at z_2 (p=genus of W).
Such a function exists according to the Riemann-Roch theorem (unless $\Omega = W$, in which case the Runge theorem is trivial).
It is well-known ([AS1 , V 28 C]) that there also exists a function $h \in M(W)$ with a pole of order N (exactly) at z_2 and having no other poles (here we made use of $N \geq 2p$).
The function $f = g \cdot h$ then belongs to $M(W) \cap H(\Omega)$, equals zero at z_1 and is non-zero at z_2.
This proves 2).

3) Let $z_o \in \Omega$. Proceeding similarly as in 2), we choose a function $g \in M(W) \cap H(\Omega)$ with a zero of order $N \geq 2p+1$ at z_o and a function $h \in M(W)$ with a pole of order $N-1$ (exactly) at z_o and having no other poles. Then $f = g\,h \in M(W) \cap H(\Omega)$ has a zero of order 1 (exactly) at z_o. Thus $f'(z_o) \neq 0$, so that f serves as a local coordinate at z_o. This finishes the verifications of 1)-3), and the Runge theorem follows.

IV. REFERENCES

A 1 : Ahlfors : Complex Analysis, Mc Graw-Hill Company, 1966.

AS 1: Ahlfors-Sario: Riemann Surfaces, Princ. Univ. Press 1960.

BS 1: Behnke-Sommer: Theorie der Analytischen Funktionen einer komplexen Veränderlichen, 3:e Auflage, Springer Verlag, 1965.

C 1 : Cartan seminar 1951-52 (nr 9) .

G 1 : Gusman S.Y.: Uniform Approximation of Continuous Functions on Riemann Surfaces, Soviet Math. Dokl 1 (1960), 105-107

G 2 : Grothendieck, A. Sur certains espace de fonctions holomorphes I, Journal fur die reine und angewandte Mathematik, Band 192 (1953), p. 35-64 .

HC 1: Hurwitz-Courant: Funktionentheorie, 4:e Auflage, Springer Verlag, 1964 .

K 1: Köthe : Topological Vector Spaces I, Springer Verlag, 1969.

R 1: Rudin, M. : Real and Complex Analysis, Mc Graw-Hill, 1966 .

RT 1: Rubel-Taylor: Functional Analysis Proofs of some Theorems in Function Theory, Amer. Math. Monthly Vol. 76, no 5, May 1969 .

S 1: Sakakihara, K.: Meromorphic approximations on Riemann surfaces, Journal of the Institutte of Polytechnics, Osaka City University , Vol 5, No 1, Series A, (1954) p. 63-70 .

SS 1 : Schiffer-Spencer : Functionals of Finite Riemann Surfaces, Princ. Univ. Press 1954 .

T 1 : Tietz, H. : Fabersche Entwicklungen auf geschlossenen Riemannschen Flächen, Journal für die reine und angewandte Mathematik, Band 190 (1952), s. 22-33 .

W 1 : Weyl, H. : The Concept of a Riemann Surface (English translation) 3rd edition Addison-Wesley, Inc 1955 .