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Restriction operators, balayage and doubly orthogonal systems of analytic functions☆

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Abstract

Systems of analytic functions which are simultaneously orthogonal over each of two domains were apparently first studied in particular cases by Walsh and Szegö, and in full generality by Bergman. In principle, these are very interesting objects, allowing application to analytic continuation that is not restricted (as Weierstrassian continuation via power series) either by circular geometry or considerations of locality. However, few explicit examples are known, and in general one does not know even gross qualitative features of such systems. The main contribution of the present paper is to prove qualitative results in a quite general situation.

It is by now very well known that the phenomenon of "double orthogonality" is not restricted to Bergman spaces of analytic functions, nor even indeed has it any intrinsic relation to analyticity; its essence is an eigenvalue problem arising whenever one considers the operator of restriction on a Hilbert space of functions on some set, to a subset thereof, provided this restriction is injective and compact. However, in this paper only Hilbert spaces of analytic functions are considered, especially Bergman spaces. In the case of the Hardy spaces Fisher and Micchelli discovered remarkable qualitative features of doubly orthogonal systems, and we have shown how, based on the classical potential-theoretic notion of balayage, and its modern generalizations, one can deduce analogous results in the Bergman space set-up, but with restrictions imposed on the geometry of the considered domains and measures; these were not needed in the Fisher–Micchelli analysis, but are necessary here as shown by examples.

From a more constructive point of view we study the Bergman restriction operator between the unit disk and a compactly contained quadrature domain and show that the representing kernel of this operator is rational and it is expressible (as an inversion followed by a

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logarithmic derivative) in terms of the polynomial equation of the boundary of the inner domain.

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1. Introduction

Let $\Omega_0 \subset \overline{\Omega_0} \subset \Omega_1$ be a pair of bounded planar domains. The main object of study is the modulus $|R| = (R^*R)^{1/2}$ of the restriction operator:

$$R : AL^2(\Omega_1) \to AL^2(\Omega_0).$$

A list of notations can be found at the end of the introduction. The operator R is nuclear and injective, therefore its modulus square R^*R is an injective, nuclear, non-negative self-adjoint contraction of $AL^2(\Omega_1)$. We will denote by

$$\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots,$$

its eigenvalues. Actually, it is not hard to prove that the eigenvalues λ_n decay exponentially, in the sense that $\limsup \lambda_n^{1/n} < 1$, see [23] or [20,22].

The corresponding eigenfunctions $f_n \in AL^2(\Omega_1)$, $n \geqslant 0$, satisfy the identity

$$\int_{\Omega_0} f_n \bar{g}\, dA = \lambda_n \int_{\Omega_1} f_n \bar{g}\, dA, \quad g \in AL^2(\Omega_1). \tag{1}$$

An equivalent statement is that the functions f_n, $n \geqslant 0$, form an orthogonal system with respect to both domains:

$$\langle f_n, f_m \rangle_{\Omega_1} = \langle f_n, f_m \rangle_{\Omega_0} = 0, \quad m \neq n.$$

This double orthogonality property seems to have initially been investigated by Bergman [4]. A related study of double orthogonality with respect to a system of two Jordan curves was undertaken around the same time by Walsh [41,43] and Szegö [36,37]. Some of its further applications and ramifications have appeared in the work of the second author [31,32]. See also [8,20,26,30,35,38,42].

Let $K_\Omega(z, \bar{w})$ denote the Bergman kernel of the domain Ω, cf. [4]. A direct consequence of Eq. (1) is the following integral equation:

$$\int_{\Omega_0} f_n(z) K_{\Omega_1}(w, \bar{z})\, dA(z) = \lambda_n f_n(w), \quad w \in \Omega_1. \tag{2}$$

Due to the general properties of the Bergman kernel, this shows for instance that the function f_n analytically extends across any analytic arc in the boundary of Ω_1.

By replacing g by gf_n in Eq. (1) we obtain

$$\int_{\Omega_0} |f_n|^2 \bar{g}\, dA = \lambda_n \int_{\Omega_1} |f_n|^2 \bar{g}\, dA, \quad g \in H^\infty(\Omega_1).$$

The latter identity can be regarded as a quadrature formula for analytic functions:

$$\int g\, d\mu_n = \int g\, d\nu_n, \quad g \in H^\infty(\Omega_1),$$

where $d\mu_n = \chi_{\Omega_0} |f_n|^2\, dA$ and $d\nu_n = \lambda_n |f_n|^2\, dA$ are two positive measures, absolutely continuous with respect to the area measure. As usual, χ_S denotes the characteristic function of the set S.

If we assume in addition that the domain Ω_1 is simply connected, then the above equality added to its complex conjugate implies

$$\int_{\Omega_0} |f_n|^2 h\, dA = \lambda_n \int_{\Omega_1} |f_n|^2 h\, dA, \tag{3}$$

where h is any harmonic function sufficiently smooth up to the boundary, for example defined in a neighbourhood of $\overline{\Omega_1}$. In this case we write $h \in H(\overline{\Omega_0})$. This relation can be interpreted as a balayage formula and will have far reaching consequences. In particular it will impose, under additional assumptions, strong restrictions on the eigenfunctions f_n.

For instance, knowing that the constant function **1** is an eigenfunction of the operator R^*R is a non-trivial piece of information. In that case

$$\int_{\Omega_0} h\, dA = \lambda \int_{\Omega_1} h\, dA, \quad h \in H(\overline{\Omega_1}).$$

This implies that the logarithmic potential of the domain Ω_1 can be harmonically continued, up to the exterior boundary of Ω_0. Then the boundary of Ω_1 turns out to be real analytic with inner cusps or double tangency points as the only possible singularities. This phenomenon was well studied in the context of quadrature domains for harmonic or subharmonic functions, for references see [27,14,34]. We will return to these topics in Section 5 and the following ones.

At the same general level, let us also remark that the Courant–Fischer minimax principle implies the inequality

$$\int_{\Omega_0} |g|^2\, dA \leqslant \lambda_n \int_{\Omega_1} |g|^2\, dA, \quad g \perp \{f_0, f_1, \ldots, f_{n-1}\},\ n \geqslant 1, \tag{4}$$

while for an arbitrary subspace V of codimension n the smallest choice of λ_V satisfying

$$\int_{\Omega_0} |g|^2\, dA \leqslant \lambda_V \int_{\Omega_1} |g|^2\, dA, \quad g \in V,$$

is never less than λ_n and equals λ_n for at least one choice of V.

We illustrate, by means of a simplified situation, how we will use the minimax principle in finding properties of the eigenfunctions f_n. Assume that the spectrum of R^*R is simple, the boundary of Ω_1 is C^1 smooth, the eigenfunction f_n is continuous on $\overline{\Omega_1}$ and it does not vanish on $\partial\Omega_1$. If the inequality

$$\int_{\Omega_0} |g|^2 |f_n|^2 \, dA \leqslant \lambda_n \int_{\Omega_1} |g|^2 |f_n|^2 \, dA, \quad g \in AL^2(\Omega_1),$$

holds (and we shall see that this is the case under quite general conditions), then the closed subspace V of $AL^2(\Omega_1)$ spanned by the multiples gf_n, $g \in H^\infty(\Omega_1)$, of f_n has codimension at least n, hence because $|f_n|$ is bounded from below by a positive constant on $\partial\Omega_1$, the space V coincides with the subspace of functions vanishing on the finite set of zeros of f_n. Thus f_n has at least n zeros in the domain Ω_1.

Two well-known examples are serving as an optimal scenario throughout the rest of the note. We briefly discuss them below.

Example 1.1 (Two concentric disks). Let $\Omega_1 = \mathbf{D}$ be the unit disk and $\Omega_0 = r\mathbf{D}$ be a concentric disk of radius $r < 1$. Then the monomials $f_n(z) = z^n$, $n \geqslant 0$, are doubly orthogonal with respect to the area measures of the two domains. Hence they are the eigenfunctions of the operator R^*R. The corresponding eigenvalues are then $\lambda_n = r^{2n+2}$, $n \geqslant 0$.

Example 1.2 (Two confocal ellipses). Let $w = \frac{1}{2}(z + \frac{1}{z})$ be the Joukowski map. It conformally transforms the annulus $A(1,r) = \{z \in \mathbf{C}; 1 < |z| < r\}$ onto the ellipse E_r minus the straight line segment joining its two foci ± 1. Any ellipse with foci at ± 1 can be obtained in this way by varying the parameter $r > 1$.

Let $U_n(\cos\zeta) = \frac{\sin(n+1)\zeta}{\sin\zeta}$, $n \geqslant 0$, be the Chebyshev polynomials of the second kind. By analytically extending them to the complex domain we have

$$U_n(w(z)) = \frac{z^{n+1} - \frac{1}{z^{n+1}}}{z - \frac{1}{z}}.$$

Thus, by changing variables in the domain of integration we obtain for every $m \neq n$:

$$\begin{aligned}
\int_{E_r} U_n \overline{U_m} \, dA &= \int_{E_r \setminus [-1,1]} U_n \overline{U_m} \, dA \\
&= \int_{A(1,r)} \frac{z^{n+1} - \frac{1}{z^{n+1}}}{z - \frac{1}{z}} \left(\frac{z^{m+1} - \frac{1}{z^{m+1}}}{z - \frac{1}{z}} \right)^* \left(1 - \frac{1}{z^2}\right) \left(1 - \frac{1}{z^2}\right)^* \frac{dA(z)}{4} \\
&= 1/4 \int_{A(1,r)} \left(z^n - \frac{1}{z^{n+2}}\right) \left(z^m - \frac{1}{z^{m+2}}\right)^* dA(z) = 0.
\end{aligned}$$

The complex conjugation in the above formulas was denoted by $*$.

This shows that the Chebyshev polynomials U_n are the eigenfunctions of $R_{ts}^* R_{ts}$, where R_{ts} is the restriction operator between the Bergman spaces corresponding to *any* pair of confocal ellipses:

$$R_{ts} : AL^2(E_s) \to AL^2(E_t), \quad s > t.$$

A simple computations yields

$$||U_n||_{2,E_r}^2 = \frac{\pi}{4(n+1)} (r^{n+1} - r^{-(n+1)})^2,$$

therefore the eigenvalues of $|R_{ts}|$ are

$$\lambda_n = \frac{t^{n+1} - t^{-(n+1)}}{s^{n+1} - s^{-(n+1)}}.$$

In the limiting case, $r \to 1$, we obtain the original orthogonality property of the Chebyshev polynomials

$$\int_{-1}^{1} U_n(t) U_m(t) \sqrt{1-t^2}\, dt = 0, \quad m \neq n.$$

It is important to remark that in both examples above a continuous family of domains, filling the whole complex plane, shares the same eigenfunctions for the associated Bergman restriction operator between any pair of them.

In the early 1930s Szegö [36,37] has classified all continuous, nested systems of Jordan curves which admit a common system of orthogonal polynomials. They are essentially reducible to the above two examples.

The contents is the following. Section 2 reviews, with some modified arguments, the known facts, mainly due to Fisher and Micchelli, about the eigenfunctions of the restriction operator defined on the Hardy space of the unit disk. In Section 3 we present an analytic continuation criterion derived from the Fourier series of a function with respect to the eigenfunctions of the Bergman restriction operator. Bergman himself seems to have seen applications to analytic continuation as one of the main uses of doubly orthogonal systems, and this Section can be regarded as a step in this direction, even though much remains to be done.

Section 4 deals with continuous families of domains sharing the same eigenfunctions for the corresponding restriction operators. The main result, Theorem 4.1, states the non-vanishing of these functions on the region swept out by the moving boundaries. In Section 5 a different, free boundary type, approach to the eigenfunctions of the Bergman restriction operator is discussed and this is applied to give a new proof of an "inverse" theorem of Andersson.

Section 6 is rather technical, so let us try to explain briefly its significance. The key concept underlying the Fischer–Micchelli work, is the notion of *balayage* from classical potential theory (even though they do not use this term; however their use of the Poisson kernels amounts essentially to this). A good description of classical balayage, in the sense we use the term here, is in Chapter IV of [21].

It is a remarkable result of theirs that eigenfunctions of restriction operators from the Hardy space H^2 of the unit disk to $L^2(\mu)$, regardless of the measure μ, never vanish on the unit circle, and this is the key to a series of interesting deductions. This is no longer true if we take the Bergman space in place of H^2. The deeper reason for this is that the characterization of eigenfunctions in the Bergman space scenario is not in terms of classical balayage (i.e. "sweeping out" of measures onto boundaries) but a generalized kind of balayage involving "sweeping out" of measures throughout the whole domain. Interestingly, the latter kind of balayage has been developed in recent times in connection with hydrodynamics (especially Hele Shaw flows) and the so-called inverse problem of Newtonian gravitation. In pure mathematics, the distinction turns up e.g. in the current notion (due to Hedenmalm, Korenblyum, etc.) of "inner function" in the Bergman space, which is not simply describable in terms of boundary behaviour as with classical Hardy space inner functions. Since this generalized balayage may be unfamiliar to most readers, what is needed for our purposes is developed in Section 6. One consequence of this is the result that any function holomorphic on a neighbourhood of the closed unit disk (in particular it may have zeros on the boundary) will arise as one of the doubly orthogonal eigenfunctions associated to the restriction operator from the Bergman space of the unit disk, and that of a suitable subdomain. Another significant feature distinguishing classical from generalized balayage, and having important repercussions for our eigenvalue problem, is that in the former the swept-out measure always has a potential majorizing the original, whereas in the latter this is not always so, and has to be studied case by case. This too is analyzed in the context of our eigenvalue problem in Section 6.

The generalized balayage gets a new twist in Section 7, and leads to one of our main results: if the support of μ is sufficiently concentrated around the centre of the disk, then also in the Bergman space scenario eigenfunctions cannot vanish on the boundary, with the consequence that the further results "of Fisher–Micchelli type" hold.

Section 8 studies the representing kernel for the restriction operator between Bergman spaces, *qua* integral operator. In the special case where the pair of domains consists of the unit disk, and a "quadrature domain" in its interior (see below for definition and references) there are simple formulae relating the representing kernel of the integral operator and the equation of the boundary of the quadrature domain.

Finally, Section 9 deals with the case of finite rank restriction operators acting on a fixed Bergman space.

Notation. For a given domain Ω and $1 \leqslant p \leqslant \infty$ we denote by $L^p(\Omega, d\mu)$ the Lebesgue spaces with respect to the positive measure μ. If no measure is mentioned it will be understood to be the area measure dA. The prefixes $AL^p(\Omega)$, $HL^p(\Omega)$, $SL^p(\Omega)$ mean the spaces of analytic, harmonic, respectively, subharmonic elements of $L^p(\Omega, dA)$. Similarly $HC(\bar{\Omega}), SC(\bar{\Omega})$ stand for the spaces of harmonic, respectively, subharmonic functions in Ω which are continuous on the closure $\bar{\Omega}$.

The Hardy spaces of a domain with smooth boundary will be denoted by $H^p(\Omega), 1 \leqslant p < \infty$. By $H^\infty(\Omega)$ we mean the algebra of bounded analytic functions in Ω.

Complex conjugation of a number is denoted by $z \mapsto \bar{z}$, or sometimes $z^* = \bar{z}$. However, $\bar{\Omega}$ will be the closure of Ω and not its image under complex conjugation. The open unit disk is denoted by $\mathbf{D}$. The Bergman kernel of a domain Ω will be denoted by $K_\Omega(z, \bar{w})$.

2. Restriction from the Hardy space

Motivated by some questions of approximation theory (Kolmogorov widths in spaces of analytic functions) Fisher and Micchelli [12,13] have studied the restriction operator from the Hardy space $H^2(\mathbf{D})$ of the unit disk to $L^2(\mu)$, where μ is a positive measure compactly supported by $\mathbf{D}$. Their remarkable results serve as a model and partial aim in the Bergman space framework of the present paper. We briefly recall, with some modifications imposed by later developments in our paper, the main ideas of [13]. See also [24].

Let μ be a positive Borel measure compactly supported by the unit disk $\mathbf{D}$ and let

$$R : H^2(\mathbf{D}) \to L^2(\mu)$$

be the restriction operator. Then the operator R is injective, and its modulus $|R|$ is nuclear and non-negative. Let λ_n be the eigenvalues of R^*R arranged in decreasing order, and let $\{f_n\} \subset H^2$ be the corresponding eigenfunctions.

Analogously as in our earlier discussion the functions f_n satisfy the integral equation:

$$\lambda_n f_n(z) = \frac{1}{\pi} \int_{\mathbf{D}} \frac{f_n(w)\, d\mu(w)}{1 - \bar{w}z}, \quad z \in \mathbf{D}. \tag{5}$$

This shows that each eigenfunction f_n analytically extends from the disk to the domain bounded by the Schwarz reflection in the unit circle $\mathbf{T}$ of the exterior boundary of the closed support of μ.

The main results of [13] assert that

(a) for all n and all $z \in \mathbf{T}, f_n(z) \neq 0$;
(b) the operator $|R|$ has simple spectrum;
(c) f_n has exactly n zeros in $\mathbf{D}$.

We take the liberty to present (with some modifications) some of the arguments of [13] which also will be applicable to our Bergman space situation.

Assume that assertion (a) holds. To prove (b) we assume by contradiction that f and g are two eigenfunctions corresponding to the same eigenvalue. Then any linear combination $f + \alpha g$ will still be an eigenfunction, and by choosing α appropriately $f + \alpha g$ would vanish at any given point of the boundary, a contradiction.

Assuming that (a) and (b) hold, it is a standard matter of perturbation theory to check that each normalized eigenfunction f_n has exactly n zeros in the unit disk. To be more specific, let ν be the arc length measure carried by a circle $r\mathbf{T}$ of radius $r<1$ and center at 0. We will consider the family of measures $\mu_t = t\mu + (1-t)\nu,\ t\in[0,1]$, and associated with them the integral operators:

$$(A_t f)(z) = \frac{1}{\pi}\int_{\mathbf{D}} \frac{f(w)\,d\mu_t}{1 - z\bar{w}}, \quad f\in H^2.$$

Since there exists a constant C and a compact set K of the unit disk with the property

$$|\langle (A_t - A_s)f, f\rangle| \leqslant C|t-s|\,\|f\|^2_{\infty,K}, \quad s,t\in[0,1],$$

we infer that, for each fixed n the function $t\mapsto\lambda_{t,n}$ is Lipschitz continuous (see [12, p. 254] for full details). Then Dunford's integral formula for the spectral projection associated to the n-th eigenvalue shows that $f_{t,n}$ depends continuously on t. Since $f_{0,n}(z^n) = c_n z^n$, with an appropriate constant c_n, and each $f_{t,n},\ t\in[0,1]$, does not vanish on $\mathbf{T}$, we conclude as in [13] that $f_n = f_{1,n}$ has exactly n zeros in the disk.

So, the crucial point is the non-vanishing statement (a). We can establish this fact as follows. From the variational condition fulfilled by the eigenfunction f_n we find that

$$\int |f_n|^2 u\,d\mu = \lambda_n \int_{\mathbf{T}} |f_n|^2 u\,ds, \tag{6}$$

for all harmonic functions u in $\mathbf{D}$ which are continuous on $\bar{\mathbf{D}}$. At this point Fisher and Micchelli choose u to be the Poisson kernel of a fixed point z on the unit disk, and use simple estimates to obtain a positive lower bound for $|f(z)|$ when $|z|$ is close to 1. We prefer an equivalent, but conceptually perhaps more suggestive procedure, based on the *balayage* concept and which guided us in the search for analogous results involving the Bergman eigenfunctions. In these terms (6) expresses that the measure $\lambda_n|f_n|^2\,ds$ is the balayage onto $\mathbf{T}$ of the compactly supported positive measure $|f_n|^2\,d\mu$ in $\mathbf{D}$. Now, it is well known that on a smooth portion of the boundary the balayage measure is absolutely continuous with respect to arc length measure and the density (Radon–Nikodym derivative) is continuous and strictly positive. This follows easily from the special case where we do balayage of a point mass, the swept-out measure then being harmonic measure, whose density with respect to arc length is the Poisson kernel.

Because $|f_n|^2\,ds$ is a swept-out measure, we obtain the following stronger version of the variational condition (6). If λ_n, f_n are as above, then

$$\int |f_n|^2 v\,d\mu \leqslant \lambda_n \int_{\mathbf{T}} |f_n|^2 v\,ds, \tag{7}$$

for every continuous function $v\in C(\bar{\mathbf{D}})$ which is subharmonic in $\mathbf{D}$.

Indeed, let u be the harmonic function equal to v on $\mathbf{T}$. Then $v \leqslant u$ everywhere in the disk, therefore

$$\int |f_n|^2 v \, d\mu \leqslant \int |f_n|^2 u \, d\mu = \lambda_n \int_{\mathbf{T}} |f_n|^2 u \, ds = \lambda_n \int_{\mathbf{T}} |f_n|^2 v \, ds.$$

In the case of Bergman eigenfunctions the analogous extension of the variational condition will be established assuming the positivity of the biharmonic Green's function of the larger domain.

To illustrate the strength of the subharmonic estimate (7) we offer a different proof of one the main results of [13]:

Theorem 2.1. *For every positive integer n there are points $a_j \in \mathbf{D}$ and functions $h_j \in H^2(\mathbf{D}), 1 \leqslant j \leqslant n$, with the property that*

$$\left\| f - \sum_{j=1}^{n} h_j f(a_j) \right\|_{2,\mu} \leqslant \lambda_n \|f\|_{2,\mathbf{T}},$$

and the constant λ_n is the smallest possible among any other choices of a_j and h_j's.

Proof. Let V_n be the subspace of functions in $H^2(\mathbf{D})$ which vanish at the n zeros of f_n. Then V_n is a codimension n subspace and since f_n analytically extends across the boundary of the disk we remark that $V_n = f_n H^2(\mathbf{D})$.

Let $g \in V_n$. Then $g = f_n h$ with $h \in H^2(\mathbf{D})$ and according to inequality (7) we have

$$\int |g|^2 \, d\mu \leqslant \lambda_n \int_{\mathbf{T}} |g|^2 \, ds.$$

This means that the subspace V_n is optimal in the min–max computation of λ_n:

$$\max_{g \in V_n} \frac{\int |g|^2 \, d\mu}{\int_{\mathbf{T}} |g|^2 \, ds} = \min_{codim\ V=n} \ \max_{g \in V} \frac{\int |g|^2 \, d\mu}{\int_{\mathbf{T}} |g|^2 \, ds} = \lambda_n.$$

But the orthogonal complement of V_n is spanned by the evaluation functionals $\frac{1}{1-\overline{a_j}z}$ at the zeros a_j of f_n. (In case of multiple roots derivatives of these fractions should be considered.) Therefore the orthogonal projection P_n onto $V_n^{\perp}$ is a finite rank operator of the form

$$P_n = \sum_{j=1}^{n} h_j \left\langle *, \frac{1}{1-\overline{a_j}z} \right\rangle,$$

where $h_j \in H^2(\mathbf{D}), \ 1 \leqslant j \leqslant n$.

In conclusion, for every function $f \in H^2(\mathbf{D})$, we obtain the estimate in the statement and the proof is complete.

We note that, even if the eigenvalues of a positive definite matrix are simple, the optimal subspaces in the min–max criterion may not be unique. This can easily be seen on a 3×3 diagonal matrix with simple spectrum. For more details about non-unique optimal subspaces in the min–max criterion we refer to [39,44].

3. Analytic continuation via eigenfunction expansion

Let Ω_0 be a subdomain of the bounded domain Ω_1, and let us denote as before by $R: AL^2(\Omega_1) \to AL^2(\Omega_0)$ the restriction operator. We are interested in such pairs of domains for which the operator R is compact.

Let $K_{\Omega_1}(z, \bar{w})$ denote the Bergman kernel of Ω_1. Then the representing kernel of the operator R^*R is

$$L(z, \bar{w}) = \chi_{\Omega_0}(z)\chi_{\Omega_0}(w)K_{\Omega_1}(z, \bar{w}),$$

where χ_A is the characteristic function of the set A. Therefore the operator R^*R is compact whenever

$$I(\Omega_0) = \int_{\Omega_0} K_{\Omega_1}(z, \bar{z})\, dA(z) < \infty.$$

In particular, if $\Omega_1 = \mathbf{D}$ is the unit disk, then

$$K_{\mathbf{D}}(z, \bar{z}) = \frac{1}{\pi(1 - |z|^2)^2},$$

and $I(\Omega_0)$ turns out to be the non-Euclidean area of Ω_0. To give an example, the compactness condition $I(\Omega_0) < \infty$ holds if Ω_0 is a fundamental domain for the modular group, that is a non-Euclidean triangle with three zero angles and vertices on the unit circle. The discussion about the representing kernel L of R^*R will be resumed in Section 8.

Assuming that the operator R is compact, the spectrum of R^*R consists then of a positive sequence which decreases to zero:

$$\lambda_0 \geqslant \lambda_1 \geqslant \cdots .$$

The corresponding eigenfunctions will be denoted, as before, by f_n.

From this point on, throughout this section we make the additional assumption that both Ω_0, Ω_1 are smoothly bounded Jordan domains of the complex plane and that Ω_0 is relatively compact in Ω_1. In this case R is obviously a compact operator.

Defining $g_n = f_n/\sqrt{\lambda_n}$ we obtain an orthonormal system for the Bergman space $AL^2(\Omega_0)$. Let

$$f = \sum_{n=0}^{\infty} a_n g_n$$

be the decomposition of an arbitrary element $f \in AL^2(\Omega_0)$ with coefficients

$$a_n = \frac{\int_{\Omega_0} f\overline{f_n}\, dA}{\sqrt{\lambda_n}}.$$

In general, the sequence (a_n) is square summable, and no more. If we assume that

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{\lambda_n} < \infty, \tag{8}$$

then one can define an element $F \in AL^2(\Omega_1)$ by

$$F = \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{\lambda_n}} f_n,$$

so that $RF = f$.

Thus we obtain the well-known result that the necessary and sufficient condition for a function $f \in AL^2(\Omega_0)$ to analytically extend to $AL^2(\Omega_1)$ is the convergence of series (8).

Now, in between the square summability of the sequence (a_n) and hypothesis (8) lie the conditions

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{\lambda_n^t} < \infty, \quad 0 < t < 1. \tag{9}$$

It is natural to ask whether (9), in the case $0 < t < 1$, is necessary and/or sufficient for the analytic continuability of f to some domain Ω_t intermediate between Ω_0 and Ω_1, that can be specified explicitly. Although we have not been able to answer this completely, we have the following result as a first step in this direction.

Theorem 3.1. *Let Ω_0, Ω_1 be smoothly bounded Jordan domains and let $f \in AL^2(\Omega_0)$ satisfy* (9) *for some $t > 0$. Then f is analytically continuable to a neighbourhood of $\overline{\Omega_0}$.*

Remark. (a) Under the stated hypotheses we have $\lambda_n \sim cp^n$, for positive constants c, and $p < 1$, in view of results of Parfenov [23]. Therefore the hypothesis in Theorem 3.1 is equivalent to

$$\sum_{n=0}^{\infty} |a_n|^2 e^{ns} < \infty, \tag{10}$$

for some $s > 0$.

(b) In the case where Ω_i are confocal ellipses, a complete result can be obtained, with Ω_t being a certain ellipse confocal with the given ones; this is completely analogous to a classical theorem of Bernstein, whereby the role played by a_n is replaced by the distance in sup-norm from the polynomials of degree at most n

(cf. [5] for details). Since the analysis and the conclusions for approximation in L^2 norm is very similar to that for sup norms, we do not give further details for this scenario.

As partial steps towards the proof of Theorem 3.1 we state a couple of lemmas.

Lemma 3.2. *Let a be a point of $\Omega_1 \backslash \overline{\Omega_0}$, of distance $d(a)$ from Ω_0. Then, for sufficiently small $d(a)$:*

$$|f_n(a)| \leqslant C\lambda_n^{\frac{1-bd(a)}{2}}, \tag{11}$$

where C, b are positive constants depending only on the geometric configuration $\Omega_0 \subset \Omega_1$ and not on n or a.

For the proof it is convenient first to state

Lemma 3.3. *Let D_1 denote the open unit disk and let D_0 be an interior concentric disk of radius $r < 1$. For a point $w \in D_1 \backslash \overline{D_0}$, the harmonic measure $h(w)$ at w (relative to the domain $D_1 \backslash \overline{D_0}$) of the inner circle satisfies*

$$h(w) = \frac{\log|w|}{\log r} \sim 1 - \frac{1}{r\log(1/r)}(|w| - r), \tag{12}$$

for small values of $|w| - r$.

The proof of Lemma 3.3 is a straightforward verification.

Proof of Lemma 3.2. We assume $d(a)$ small enough so that the set

$$G_0 = \{z \in \Omega_0; \quad dist(z, \partial\Omega_0) > d\}$$

is a non-empty Jordan domain. Because of the condition

$$\int_{\Omega_0} |f_n|^2 \, dA = \lambda_n,$$

the integral of $|f_n|^2$ over a disk centred at any point of ∂G_0, with radius d, is less than λ_n, so we have

$$|f_n(z)|^2 \leqslant \frac{\lambda_n}{\pi d^2}, \quad z \in \partial G_0. \tag{13}$$

In like manner, if G_1 denotes the set of all points $z \in \Omega_1$ satisfying $dist(z, \partial\Omega_1) > d$, we have

$$|f_n(z)|^2 \leqslant \frac{1}{\pi d^2}, \quad z \in \partial G_1. \tag{14}$$

We now estimate $f_n(a)$ at the point a, from the two inequalities (13), (14) using the two-constant theorem, with respect to the domain bounded by ∂G_0 and ∂G_1. For a

near the boundary ∂G_0 the harmonic measure at a is asymptotically $1 - Kd(a)$, where K is a constant depending only on the geometric configuration of Ω_0, Ω_1. This follows easily from Lemma 3.3 by conformal mapping, and the conformal invariance of harmonic measure. The estimate obtained is then precisely (11). $\square$

Proof of Theorem 3.1. It is now easy to complete the proof of Theorem 3.1. For $z \in \Omega_1 \setminus \overline{\Omega_0}$, the terms of the series

$$\sum_{n=0}^{\infty} \frac{a_n}{\sqrt{\lambda_n}} f_n(z) \tag{15}$$

are majorized by

$$\sum_{n=0}^{\infty} |a_n| C \lambda_n^{\frac{-bd(z)}{2}}.$$

For z in a sufficiently small neighbourhood of $\overline{\Omega_0}$ we have $bd(z)/2 < t/4$, so finally series (15) is majorized by a constant multiple of

$$\sum_{n=0}^{\infty} \frac{|a_n| \lambda_n^{t/4}}{\lambda_n^{t/2}},$$

which is finite in view of our assumption and the fact that the sequence $(\lambda_n^{t/4})$ is square summable. $\square$

4. Continuous families of domains

Throughout this section and in Section 7 we study conditions which assure that the Bergman eigenfunctions do not vanish on the boundary of the domain. We start with the following generous hypothesis, fulfilled for instance by the families of concentric disks or confocal ellipses. Let Ω_t, $0 \leqslant t \leqslant 1$, be an increasing family of simply connected, bounded planar domains with C^1-smooth boundaries: $\overline{\Omega_s} \subset \Omega_t$, $s < t$. We can assume in this case that there exists a scalar function ϕ defined on a neighbourhood U of $\overline{\Omega_1}$ such that

$$\Omega_t = \{z \in U; \phi(z) < t\},$$

and possessing non-vanishing gradient along all curves $\phi(z) = t$, $0 \leqslant t \leqslant 1$.

Let

$$R_t : AL^2(\Omega_1) \to AL^2(\Omega_t), \quad t \geqslant 0,$$

be the restriction operators. Then the following result holds.

Theorem 4.1. *Assume that the function $f \in AL^2(\Omega_1)$, $f \neq 0$, is a common eigenfunction for all operators $R_t^* R_t$, $0 \leqslant t \leqslant 1$. Then f does not vanish on $\Omega_1 \backslash \overline{\Omega_0}$.*

Proof. By assumption there exists a real constant $\lambda(t)$ such that for every analytic function $g \in AL^2(\Omega_1)$ one has

$$\int_{\Omega_t} f\bar{g}\, dA = \lambda(t) \int_{\Omega_0} f\bar{g}\, dA.$$

Therefore, we find as in the previous section that

$$\int_{\Omega_t} h|f|^2\, dA = \lambda(t) \int_{\Omega_0} h|f|^2\, dA, \tag{16}$$

for every harmonic function $h \in HC(\overline{\Omega_1})$.

In particular, this relation implies

$$\lambda(t) = \frac{\int_{\Omega_t} |f|^2\, dA}{\int_{\Omega_0} |f|^2\, dA}.$$

By subtracting the equations above for two different values $0 < u < t < 1$ we obtain an identity of averages of the harmonic function h:

$$\frac{\int_{\Omega_t \backslash \Omega_u} h|f|^2\, dA}{\int_{\Omega_t \backslash \Omega_u} |f|^2\, dA} = \frac{\int_{\Omega_0} h|f|^2\, dA}{\int_{\Omega_0} |f|^2\, dA}. \tag{17}$$

Let ds denote the arc length measure along the curve $\partial\Omega_u$. According to the Coarea Theorem (see [11, Proposition 3, Section 3.4.4]) the positive function $g_u = \frac{1}{|\nabla\phi|} \in C(\partial\Omega_u)$ has the property

$$\lim_{t \to u} \frac{\int_{\Omega_t \backslash \Omega_u} \phi\, dA}{\int_{\Omega_t \backslash \Omega_u} dA} = \frac{\int_{\partial\Omega_u} \phi g_u\, ds}{\int_{\partial\Omega_u} g_u\, ds}, \quad \phi \in C(\mathbf{C}).$$

We denote by

$$d\mu_u = \frac{|f|^2 g_u\, ds}{\int_{\partial\Omega_u} |f|^2 g_u\, ds}$$

the associated probability measure supported by $\partial\Omega_u$, and by

$$d\nu = \frac{|f|^2 \chi_{\Omega_0}\, dA}{\int_{\Omega_0} |f|^2\, dA},$$

the fixed probability measure arising from the right-hand side of the above identities.

In conclusion, by passing to the limit in (17) we obtain the balayage identity

$$\int_{\partial\Omega_u} h\, d\mu_u = \int_{\Omega_0} h\, dv, \ \ h \in HC(\overline{\Omega_u}).$$

Actually, we first obtain the identity for all harmonic functions $h \in HC(\overline{\Omega_1})$. Due to the simple connectedness of Ω_u and its boundary smoothness, an approximation argument shows that the identity holds for all harmonic functions in Ω_u which are continuous up to the boundary.

Thus $d\mu_u$ is the balayage on the curve $\partial\Omega_u$ of the probability measure dv. Then it is known that the Radon–Nikodym derivative $d\mu_u/ds = |f|^2 g_u$ is strictly positive on $\partial\Omega_u$. In particular f does not vanish on this curve, and the proof is complete. $\square$

Corollary 4.2. *In the conditions of Theorem* 3.1, *for every continuous subharmonic function s defined on* $\overline{\Omega_1}$, *the function*

$$t \mapsto \frac{\int_{\Omega_t} s|f|^2\, dA}{\int_{\Omega_t} |f|^2\, dA}$$

is increasing.

Proof. A first proof of such an inequality goes back to Sakai's work [27, Theorem 10.13]. In our context the proof uses the maximum principle, as follows. For a fixed value of the parameter t, let h be the harmonic function in Ω_t which has boundary values on $\partial\Omega_t$ equal to s. Then

$$\begin{aligned}\frac{d}{dt}\log\frac{\int_{\Omega_t} s|f|^2\, dA}{\int_{\Omega_t} |f|^2\, dA} &= \frac{\int_{\partial\Omega_t} s|f|^2 g_t\, ds}{\int_{\Omega_t} s|f|^2\, dA} - \frac{\int_{\partial\Omega_t} |f|^2 g_t\, ds}{\int_{\Omega_t} |f|^2\, dA}\\ &\geqslant \frac{\int_{\partial\Omega_t} h|f|^2 g_t\, ds}{\int_{\Omega_t} h|f|^2\, dA} - \frac{\int_{\partial\Omega_t} |f|^2 g_t\, ds}{\int_{\Omega_t} |f|^2\, dA}\\ &= \frac{d}{dt}\log\frac{\int_{\Omega_t} h|f|^2\, dA}{\int_{\Omega_t} |f|^2\, dA} = 0.\end{aligned}$$

By keeping the conditions of Theorem 3.1, assume that the operators $R_t^* R_t$ have the same eigenfunctions. This is the same as stating that they commute: $[R_s^* R_s, R_t^* R_t] = 0, \ s, t \in [0,1]$. Then we know that each common eigenfunction f_n does not vanish in the region $\Omega_1 \backslash \overline{\Omega_0}$.

In addition, let us suppose that $\partial\Omega_1$ is smooth and real analytic. Then, as remarked in the introduction, each eigenfunction f_n extends analytically across $\partial\Omega_1$. Moreover, the proof of Theorem 4.1 shows that f_n does not vanish on $\partial\Omega_1$.

This implies, as shown in Section 2, that the spectrum of the restriction operators $R_t^* R_t$ is simple.

Thus, we can speak without ambiguity of the nth eigenfunction f_n. Actually, the proof of the simplicity of the spectrum uses only the analyticity of a portion of the boundary of Ω_1, hence the next result.

Corollary 4.3. *Assume, in the conditions of Theorem* 4.1, *that the boundary* $\partial\Omega_1$ *contains an analytic arc. Then the spectrum of each self-adjoint operator* $R_t^*R_t$, $0 \leqslant t \leqslant 1$, *is simple.*

Corollary 4.4. *Assume, in the conditions of Theorem* 4.1 *that the boundary* $\partial\Omega_1$ *is real analytic smooth. Then each eigenfunction* f_n *has exactly n zeros, all contained in* $\overline{\Omega_0}$.

Proof. According to Corollary 4.2 applied to the subharmonic function $s = |h|^2$, $h \in AL^2(\Omega_1)$, we find that

$$\frac{\int_{\Omega_0} |h|^2 |f_n|^2 \, dA}{\int_{\Omega_0} |f_n|^2 \, dA} \leqslant \frac{\int_{\Omega_1} |h|^2 |f_n|^2 dA}{\int_{\Omega_1} |f_n|^2 \, dA},$$

or equivalently

$$\int_{\Omega_0} |h|^2 |f_n|^2 \, dA \leqslant \lambda_n \int_{\Omega_1} |h|^2 |f_n|^2 \, dA. \tag{18}$$

Let V_n be the closed subspace of $AL^2(\Omega_1)$ generated by the functions hf_n with $h \in H^\infty(\Omega_1)$. According to the minimax principle, the space V_n has codimension at least equal to n.

On the other hand, since the function f_n is free of zeros in a neighbourhood of $\partial\Omega_1$, the space V_n coincides with the space of all functions $f \in AL^2(\Omega_1)$ which vanish, with the same order, at the (finite) zero set of f_n. In conclusion f_n has at least n zeros, all contained in $\overline{\Omega_0}$.

To prove that the function f_n has exactly n zeros we use the main lines in the proof of the Hardy space case. We start with the identity

$$\int_{\partial\Omega_1} f_n \overline{f_m} g_0 \, ds = c \int_{\Omega_0} f_n \overline{f_m} \, dA = 0, \quad m \neq n,$$

where c is a positive constant, derived as in the proof of Theorem 4.1 (where the function g_0 was defined). Hence f_n are the eigenfunctions of the restriction operator:

$$T : P^2(\partial\Omega_1; g_0 \, ds) \to AL^2(\Omega_0).$$

We have denoted by $P^2(\Gamma, d\tau)$ the closure of polynomials in $L^2(\Gamma, d\tau)$ where Γ is a smooth closed curve and τ is a positive measure supported on Γ.

At this point it is perhaps necessary to recall a couple of simple observations concerning the change of the domain by a conformal map. Note that a conformal map $\phi : U \to V$ induces by pull back a unitary operator:

$$\phi^* : AL^2(V) \to AL^2(U, |\phi'|^2\, dA), \quad \phi^* f = f \circ \phi.$$

Moreover, if we have a positive measure μ supported by a compact subset of U, then the push forward measure $\phi_* \mu$ is supported by V and the map

$$\phi^* : L^2(\phi_* \mu) \to L^2(\mu)$$

is still unitary, and compatible with the restriction operators:

$$R_U : AL^2(U, |\phi'|^2 dA) \to L^2(\mu), \quad R_V : AL^2(V) \to L^2(\phi_* \mu).$$

More exactly $R_V \phi^* = \phi^* R_U$ and consequently the two restriction operators have unitarily equivalent moduli: $(\phi^*)^{-1} R_V^* R_V \phi^* = R_U^* R_U$. A similar unitary equivalence can be established for weighted Hardy spaces.

Via a conformal map (recall that $\partial\Omega_1$ was supposed to be real analytic) the operator T is unitarily equivalent, as remarked before, to the restriction operator:

$$T' : P^2(\mathbf{T}; k ds) \to L^2(v),$$

where k is a strictly positive, continuous function on $\mathbf{T}$ and v is a positive measure supported by a compact subset of $\mathbf{D}$. The eigenfunctions of $T'^* T'$ will be of the form $f_n(\phi(z))\phi'(z)$, where ϕ is the conformal mapping. Then it follows as outlined in Section 2, by deforming the function k to $\mathbf{1}$ and the measure v to $\chi_{r\mathbf{D}}$, $r < 1$, that the eigenfunctions corresponding to the continuous path of simple self-adjoint restriction operators have the same number of zeros in the unit disk.

In conclusion, for each $n \geqslant 0$, the function $f_n(\phi(z))\phi'(z)$ has n zeros in $\mathbf{D}$, and consequently f_n has n zeros, all contained in $\overline{\Omega_0}$.

In view of a result of Szegö [36], if, in the above scenario, each eigenfunction f_n is a polynomial of exact degree n, then the conformal mapping of the unit disk onto the complement on the Riemann sphere of $\overline{\Omega_0}$ pulls back the continuous family $\Omega_t \setminus \overline{\Omega_0}$ onto concentric annuli. Moreover, the geometry of these moving boundaries turns out to be very rigid, essentially reducible to the two examples presented in the introduction, see [36] for details.

5. A free boundary approach

An alternative method for studying the eigenfunctions of the restriction operator between Bergman spaces begins by characterizing them as solutions to a certain free boundary problem. This idea is developed below.

Let us return to the original setting of two bounded domains $\Omega_0 \subset \overline{\Omega_0} \subset \Omega_1$ and assume that they have C^1 boundary. We will also assume that Ω_1 is simply connected.

Let $f \in AL^2(\Omega_1)$ be an eigenfunction of the restriction operator R^*R, with corresponding eigenvalue λ. Then

$$\int_{\mathbf{C}} (\chi_{\Omega_0} - \lambda \chi_{\Omega_1}) f \bar{g} \, dA = 0, \quad g \in AL^2(\Omega_1).$$

According to Havin's Lemma for the Cauchy–Riemann operator [19], there exists a function v in the Sobolev space $W_0^{1,2}(\Omega_1)$ with the property that

$$(\chi_{\Omega_0} - \lambda \chi_{\Omega_1}) f = \frac{\partial v}{\partial z},$$

in $\mathbf{C}$, in the sense of distributions.

Let F be a holomorphic primitive of f : $F' = f$. Then there are analytic functions $a \in AL^2(\Omega_1)$ and $b \in AL^2(\Omega_1 \backslash \Omega_0)$ satisfying

$$v(z) = -\lambda F(z) + \overline{b(z)}, \quad z \in \Omega_1 \backslash \Omega_0 \tag{19}$$

and

$$v(z) = (1 - \lambda) F(z) + \overline{a(z)}, \quad z \in \Omega_0. \tag{20}$$

As a matter of notation we put for an analytic function h, $h^*(z) = \overline{h(\bar{z})}$ whenever these compositions make sense.

By taking into account that the function v is continuous up to the boundary of Ω_1 and it vanishes there, we obtain $\lambda F^*(\bar{z}) = b(z)$, $z \in \partial\Omega_1$, whence the necessary condition:

(i) The function $F^*(\bar{z})$ analytically extends from $\partial\Omega_1$ to $\Omega_1 \backslash \Omega_0$.

In our notation this extension is $\lambda^{-1} b(z)$. The second matching condition, on the boundary of Ω_0 implies that $F^*(\bar{z}) + a(z) = b(z)$, $z \in \partial\Omega_0$. Hence the second necessary condition:

(ii) The function $b(z) - F^*(\bar{z})$ analytically extends from $\partial\Omega_0$ to Ω_0.

Assume in addition that the boundaries of Ω_0 and Ω_1 are smooth and real analytic. Let $S_i(z)$ be the Schwarz function of the domain Ω_i, $i = 0, 1$. That is $S_i(z) = \bar{z}$, $z \in \partial\Omega_i$, and S is analytically continuable, possibly as a multivalued function, inside Ω_i, $i = 0, 1$. For details see [34]. Then the above analytic continuation conditions can be restated as follows:

(i)′ The function $F^*(S_1(z))$ analytically extends from $\partial\Omega_1$ to $\Omega_1 \backslash \overline{\Omega_0}$;
(ii)′ The function $\lambda F^*(S_0(z)) - F^*(S_1(z))$ analytically extends from $\partial\Omega_0$ to Ω_0.

The converse, in the C^1 smoothness case, but still under the simply connectdness assumption imposed on Ω_1, can be proved by reversing the above argument. In conclusion we can state the following proposition.

Proposition 5.1. *Let $\Omega_0 \subset \overline{\Omega_0} \subset \Omega_1$ be domains with C^1 boundaries and assume that Ω_1 is simply connected. Let $f \in AL^2(\Omega_1)$ with analytic primitive F.*

The function f is an eigenfunction for the modulus of the restriction operator

$$R : AL^2(\Omega_1) \to AL^2(\Omega_0)$$

if and only if there exists a constant λ with the property that conditions (i) *and* (ii) *are satisfied. In this case λ is necessarily real and positive.*

As an application we prove, by a different method, the following observation originally due to Andersson [3].

Corollary 5.2. *Assume in the conditions of Proposition* 3.5 *that $\Omega_1 = \mathbf{D}$ and that there exists an integer $n \geqslant 0$ with the property that z^n is an eigenfunction of the modulus of the restriction operator R.*

Then $\Omega_0 = r\mathbf{D}$ is a concentric disk $(r < 1)$.

Proof. Let us assume that there exists $n \geqslant 1$ such that the function $F(z) = z^n$ satisfies the analytic continuation assumptions (i) and (ii).

The Schwarz function S_1 of the unit disk is $S_1(z) = z^{-1}$, therefore, in our preceding notation $b(z) = \frac{\lambda}{z^n}$. Condition (ii) implies then that the function

$$a(z) = \frac{\lambda}{z^n} - \overline{z^n}, \ z \in \partial\Omega_0,$$

analytically extends inside Ω_0.

But this implies that the real function

$$z^n a(z) = \lambda - |z|^{2n}, \ z \in \partial\Omega_0,$$

analytically extends in Ω_0. Hence this function is a constant and $\partial\Omega_0$ is a part of a circle $\{z; |z| = r\}$. The smoothness assumptions imply then that Ω_0 is the disk $r\mathbf{D}$.

The same conclusion can be reached if we assume that the function $z - \alpha$ is the analytic primitive of an eigenfunction of the restriction operator. Indeed, in this case we know that $\frac{\lambda}{z} - \bar{z}$ analytically extends from the boundary of Ω_0 to its interior. But this shows that the Schwarz function of the boundary of Ω_0 is meromorphic, with exactly one simple pole at the origin. Therefore Ω_0 is a disk centered at 0. For more details about quadrature domains we refer to the monograph [34] and Section 8.

A related result, obtained by the same technique, follows.

Proposition 5.3. *Assume that Ω_0 is a quadrature domain, relatively compact in $\Omega_1 = \mathbf{D}$. Then either Ω_0 is a concentric disk, or no eigenfunction of the restriction operator can be a polynomial.*

Proof. Assume that $p(z)$ is a polynomial which is an eigenfunction of the restriction operator. Let $P(z)$ be a polynomial antiderivative of $p(z)$. Then the rational function $P^*(1/z)$ extends from the unit circle to the boundary Γ of Ω_0. Let $S(z)$ be the Schwarz function of Γ, that is $\bar{z} = S(z), z \in \Gamma$ and $S(z)$ is a meromorphic function inside Ω_0. Then, according to condition (ii) above, there exists $\lambda > 0$ such that the function

$$\lambda P^*(1/z) - P^*(S(z)), \quad z \in \Gamma,$$

analytically extends inside Ω_0. But this implies that $S(z)$ cannot have other poles than $z = 0$, and second, by a degree count, that its pole must be of order one. Therefore, the only choice for Ω_0 is to be a disk centred at $z = 0$.

6. On inverse balayage of modulus square of an analytic function

In this section we discuss in some generality possible requirements on measures ν, compactly supported in a domain Ω, which can be considered as inverse balayage measures of a density $|f|^2\chi_\Omega$, f analytic, and also the relationship between these requirements and the condition that f is an eigenfunction for the restriction operator from Ω to a smaller domain.

We assume throughout that $\Omega \subset \mathbf{C}$ is a bounded domain with $\partial\Omega$ consisting of finitely many smooth analytic curves and that f is analytic in a neighbourhood of $\bar{\Omega}$. We allow f to have zeros but not to be identically zero.

By using the Cauchy–Kovalevskaya theorem or, more elementary, by exploiting the Schwarz function as indicated below, one constructs a function u (sometimes called the "modified Schwarz potential") and a distribution ν with compact support in Ω satisfying

$$\Delta u = |f|^2 - \nu \text{ in } \Omega, \quad u = |\nabla u| = 0 \text{ on } \partial\Omega. \tag{21}$$

Here ν is determined by u, while u itself is uniquely determined only in a neighbourhood of $\partial\Omega$.

The distribution ν can always be taken to be a signed measure. In fact, one may even choose u and ν to be a smooth functions. However, we shall usually prefer not to choose them that way.

The reason that the overdetermined boundary value problem (21) is related to inverse balayage is that if we extend u by zero outside Ω then $\Delta u = |f|^2\chi_\Omega - \nu$ in all of $\mathbf{C}$, showing that u is the difference between the logarithmic potentials of the measures $|f|^2\chi_\Omega$ and ν. Thus, since $u = 0$ outside Ω, these potentials agree outside Ω, saying that $|f|^2\chi_\Omega$ can be viewed as a form of balayage of ν.

Another way to put it is to say that (21) is equivalent to the quadrature identity

$$\int_\Omega h|f|^2\,dA = \int h\,dv, \quad h\in HL^1(\Omega). \tag{22}$$

Taking $h(\zeta)=\log|\zeta - z|$ for $z\notin\Omega$ gives the earlier statement concerning logarithmic potentials, while for the other direction we have

$$\int_\Omega h|f|^2\,dA - \int h\,dv = \int_\Omega h\Delta u\,dA = \int_\Omega u\Delta h\,dA = 0,$$

where the partial integration can be justified in various ways.

Let $S(z)$ denote the Schwarz function [9,34] of $\partial\Omega$, i.e., the analytic function defined in a neighbourhood of $\partial\Omega$ and satisfying $S(z)=\bar{z}$ on $\partial\Omega$. In terms of $S(z)$ the solution u of (21) can be constructed directly as follows. Let $F(z)$ be a primitive function of $f(z)$ and $G(z)$ a primitive function of $\overline{F(\overline{S(z)})}f(z)$. Then it is easy to check that

$$u(z) = \tfrac{1}{4}[\overline{F(z)}(F(z)-F(\overline{S(z)})) - G(z) + G(\overline{S(z)})] \tag{23}$$

solves (21) (with $v=0$) in a neighbourhood of $\partial\Omega$, say in $\Omega\backslash K$, where $K\subset\Omega$ is compact with smooth boundary.

Note that u will not be changed if we add constants to F and G, so the above construction works even if F and G exist as single-valued functions only locally.

Next, we may continue u continuously to all of Ω by requiring that it solves

$$\Delta u = |f|^2 \text{ in } \mathit{int}\, K. \tag{24}$$

Then (21) holds, with v a signed measure supported by a nullset:

$$\mathit{supp}\, v\subset\partial K. \tag{25}$$

It is many times desirable that u and v satisfy additional conditions. Examples of such conditions are

$$v\geqslant 0, \tag{26}$$

$$u\geqslant 0, \tag{27}$$

$$\nabla u = 0 \text{ at zeros of } f. \tag{28}$$

In (28) the orders of the zeros should be taken into account. The precise form of the statement is that if f has a zero of order m at a point $z_0\in\Omega$ then u should satisfy

$$\frac{\partial u}{\partial z}(z_0) = \cdots = \frac{\partial^m u}{\partial z^m}(z_0) = 0. \tag{29}$$

When this holds all derivatives of u at z_0 of orders $\leqslant m$ will vanish by virtue of (21) (assuming $z_0 \notin supp\, \nu$).

We remark here that if f has a zero of order m at a point $z_0 \in \partial\Omega$, then it follows directly from (21) (or (23)) that all derivatives of u of order $\leqslant m+1$ automatically vanish there. Thus a condition like (29) need not be stated separately when z_0 is on the boundary.

In place of (25) it may be desirable to have ν of the form

$$d\nu = \frac{1}{\lambda}|f|^2 \chi_{D_\lambda}\, dA \tag{30}$$

for some $0<\lambda<1$ and some open set $D_\lambda \subset \Omega$. As a final requirement we may ask that

$$f \neq 0 \text{ on } supp\, \nu. \tag{31}$$

In case (31) is satisfied we set

$$d\mu = |f|^{-2}\, d\nu. \tag{32}$$

In the above construction we can, by enlarging K if necessary, always assume that (31) holds.

In order to state the relevance of the above conditions in the context of quadrature identities and eigenfunction identities we introduce also a weaker version of (21) allowing u to take different constant values on different boundary components:

$$\Delta u = |f|^2 - \nu \text{ in } \Omega, \quad \nabla u = 0 \text{ on } \partial\Omega. \tag{33}$$

Here we may normalize u by taking it to be zero on the outer boundary component. In case Ω is simply connected (21) and (33) then are the same.

Proposition 6.1. *Let ν be a signed measure with compact support in Ω. Then*

(i) *Eq.* (22) *holds if and only if there exists u satisfying* (21);
(ii) *assuming* (26) *the quadrature inequality*

$$\int_\Omega s|f|^2\, dA \geqslant \int s\, d\nu, \quad s \in SL^1(\Omega) \tag{34}$$

holds if and only if there exists u satisfying (21) *and* (27);
(iii) *the quadrature identity*

$$\int_\Omega g|f|^2\, dA = \int g\, d\nu, \quad g \in AL^1(\Omega) \tag{35}$$

holds if and only if there exists u satisfying (33);

(iv) *assuming that* (31) *holds, the eigenfunction identity*

$$\int_{\Omega} f\bar{g}\,dA = \int f\bar{g}\,d\mu, \quad g \in AL^2(\Omega) \tag{36}$$

holds if and only if there exists u satisfying (33) *and* (28).

Remark. In (ii) assumption (26) has no deeper significance, it is just an easy way to ensure that the right member of (34) makes sense. It can still attain the value $-\infty$. In general, the test classes above are not the optimal ones (it would have been more appropriate to require integrability with respect to $|f|^2\,dA$ in (22), (34), (35), for example), but with our regularity assumptions on $\partial\Omega$ the exact choice of test class is of minor importance.

Proof. Statements (i)–(iii) are in principle well known. See e.g [27,28] (Proposition 6) [29] for the case $f = 1$ but with no regularity assumptions on $\partial\Omega$. Under our regularity assumptions there are no difficulties in extending these results to general f analytic in a neighbourhood of $\bar{\Omega}$.

It remains to prove (iv). If (36) holds, take

$$g(\zeta) = \frac{f(\zeta)}{\zeta - z}, \quad z \notin \bar{\Omega}.$$

In view of (32) this gives that

$$(|f|^2\chi_\Omega)\hat{} = \hat{v} \text{ outside } \bar{\Omega}$$

(and by continuity up to $\partial\Omega$), where "hat" denotes Cauchy-transform, e.g.

$$\hat{v}(z) = -\frac{1}{\pi}\int \frac{dv(\zeta)}{\zeta - z}.$$

But this is nothing else than (33). Indeed, defining u to be the logarithmic potential of $|f|^2\chi_\Omega - v$ we have

$$4\frac{\partial u}{\partial z} = (|f|^2\chi_\Omega - v)\hat{}$$

and it follows that (33) is satisfied.

If f has a zero of order m at a point $z_0 \in \Omega$ we may also take

$$g(\zeta) = \frac{f(\zeta)}{(\zeta - z_0)^k} \quad (1 \leqslant k \leqslant m)$$

in (36), showing that $(|f|^2\chi_\Omega - v)\hat{}$ and its first $m-1$ derivatives vanish at z_0, i.e., that (29) holds.

Conversely, assume that (33), (28) hold and (for simplicity of notation) that f has only one zero, say at $z_0 \in \Omega$. Then for g analytic in a neighbourhood of $\bar{\Omega}$ (and such g are dense in $AL^2(\Omega)$) we get

$$\begin{aligned}
&\int_\Omega \bar{f}g\,dA - \int \bar{f}g\,d\mu \\
&= \int_\Omega \frac{g}{f}(|f|^2\,dA - dv) \\
&= \int_\Omega \frac{g}{f}\Delta u\,dA = -2i \lim_{\varepsilon\to 0} \int_{\Omega\setminus\mathbf{D}(z_0,\varepsilon)} \frac{g}{f}\frac{\partial^2 u}{\partial\bar{z}\partial z}\,d\bar{z}dz \\
&= -2i \lim_{\varepsilon\to 0}\left[\int_{\Omega\setminus\mathbf{D}(z_0,\varepsilon)} \frac{\partial}{\partial\bar{z}}\left(\frac{g}{f}\right)\frac{\partial u}{\partial z}\,d\bar{z}dz - \int_{\partial(\Omega\setminus\mathbf{D}(z_0,\varepsilon))} \frac{g}{f}\frac{\partial u}{\partial z}\,dz\right] \\
&= 2i\int_{\partial\Omega} \frac{g}{f}\frac{\partial u}{\partial z}\,dz - 2i \lim_{\varepsilon\to 0}\int_{\partial\mathbf{D}(z_0,\varepsilon)} \frac{g}{f}\frac{\partial u}{\partial z}\,dz = 0,
\end{aligned}$$

the last equality due to the vanishing of $\frac{\partial u}{\partial z}$ on $\partial\Omega$ and, with appropriate multiplicities (29), at z_0.

Remark. Condition (28) says that the zeros of f should be what is known as "special points" for the quadrature identity (35). This terminology was coined in [33], but the type of points were studied already in [27]. One may define a special point for (35) as a point in $\Omega\setminus supp\ v$ at which g may have a pole and the quadrature identity still holds. It is in fact obvious that (36) is equivalent to (35) holding for all g for which gf is analytic, i.e., for g allowed to have poles up to the orders of the zeros of f. (Note however that both members of (35) make sense for g having poles of higher orders than that, so our special points are not required to be "maximally special".)

Now we return to the inverse balayage and to the question to what extent it is possible to choose u and v in (21) so that (25)–(31) are satisfied. With the construction given initially in this section we saw that conditions (25) and (31) could always be met. It is not entirely obvious that v will be positive (26). In fact, this will not always be the case, but by choosing the compact K carefully, or simply large enough, (26) will hold. See [18, Theorem 3.3] for example.

As for the further properties we summarize everything in a theorem.

Theorem 6.2. *With Ω and f as in the beginning of this section it is always possible to find u, v so that they, in addition to* (21), *satisfy*

$$(25),\ (26),\ (28)\ (\textit{with}\ (29)),\ (31).$$

In this list we may replace (25) *by* (30) *for $\lambda > 0$ sufficiently small. If* (27) *happens to hold for the original v* (*in* (25)) *then* (*and only then*) *we can allow all $0<\lambda<1$ and still have $D_\lambda \subset \Omega$* (*plus the remaining properties in the list*).

If f has no zeros on $\partial\Omega$ then we can (with a slightly different construction compared to above) choose u, v to satisfy

$$(26),\ (27),\ (28)\ (\textit{with}\ (29)),\ (31).$$

In this case v will not be as in (25), *but can still be chosen of the form* (30) *for* $0<\lambda<1$ *sufficiently close to one (and with $D_\lambda \subset \Omega$).*

If f has zeros on $\partial\Omega$ then (27) *can not always be satisfied, not even in a neighbourhood of $\partial\Omega$.*

Various parts of the above theorem can be translated into the setting of eigenfunctions for the restriction operator $R : AL^2(\Omega) \to L^2(\mu)$. We state explicitly only one such example.

Corollary 6.3. *Let Ω be a bounded finitely connected domain with real analytic boundary and let f be an analytic function defined in a neighbourhood of $\bar{\Omega}$ and not identically equal to zero.*

Then there exists a positive measure μ compactly supported in Ω such that f is an eigenfunction for the modulus of the restriction operator $R : AL^2(\Omega) \to L^2(\mu)$.

For the proof it suffices to deduce from Theorem 6.2 the existence of u and v satisfying (21) (in particular (33)), (31) and (28). Then the corollary follows from (iv) of Proposition 6.1.

Proof of Theorem 6.2. For the first statement of the theorem it remains to show that (28) and (30) can be satisfied.

Assume for simplicity that f has only one zero, say at $z_0 \in \Omega$. Set

$$\Omega_0 = \Omega \backslash \{z_0\}$$

and define u in a neigbourhood of $\partial\Omega$ by (23) as before and in a neighbourhood of $\partial\Omega_0 \backslash \partial\Omega = \{z_0\}$ by

$$u(z) = \tfrac{1}{4}|F(z) - F(z_0)|^2. \tag{37}$$

Then u is defined in $\Omega_0 \backslash K$ for some compact $K \subset \Omega_0$. Extending again u by (24) to all of Ω_0 we obtain u, v so that (21), (25) hold for Ω_0.

Next, the main result (Theorem 4.3) in [18] shows that $v \geqslant 0$ if K is chosen large enough. Finally, it is immediate to verify that (21) holds also for Ω itself with (29) holding at z_0.

If we want to have v satisfying (10) in place of (5) we sweep it out to the desired density $\frac{1}{\lambda}|f|^2$, i.e., we construct the quadrature domain (or open set) D_λ satisfying

$$\frac{1}{\lambda}\int_{D_\lambda} s|f|^2\,dA \geqslant \int s\,dv, \quad s \in SL^1(D_\lambda). \tag{38}$$

See [27,28,17] for example. For D_λ to exist v needs to be sufficiently big on its support. When supp v is a nullset it is enough that $v \geqslant 0$, and this is what we have (by (25), (26)).

Simple estimates show that $D_\lambda \subset \Omega$ if λ is small enough. In principle, the construction of D_λ works for all $\lambda > 0$ provided just $|f|^2$ is defined everywhere (e.g., if f is an entire function). In any case, for those λ for which D_λ exists, it is uniquely determined (up to nullsets) by (38), it is monotone increasing in λ and, even though it need not be connected, each component of D_λ contains a component of supp v.

Using (ii) of Proposition 6.1 and the uniqueness of D_λ in (38) we see that $D_1 = \Omega$ if and only if (27) holds. Since the D_λ are monotone in λ it follows that $D_\lambda \subset \Omega$ holds for all $0 < \lambda < 1$ exactly in that case. This proves the first part of the theorem.

If f has no zeros on $\partial\Omega$ then it is not hard to see that u defined by (23) will be positive in a neighbourhood of $\partial\Omega$. In case there are zeros inside Ω we set

$$\Omega_0 = \Omega \setminus \{\text{zeros of } f\}$$

and define u near the zeros by (37) as before. Then for $\varepsilon > 0$ sufficiently small there is a compact $K \subset \Omega_0$ such that $0 < u < \varepsilon$ in $\Omega_0 \setminus K$, $u = \varepsilon$ on ∂K with $\frac{\partial u}{\partial n} \leqslant \text{constant} < 0$ on ∂K.

If we extend u to K by (24) as in the previous case it may happen that we destroy the positivity of u. Therefore we choose another extension (which however has the drawback that it violates (25)): we simply set $u = \varepsilon$ in int K. This yields v of the form

$$v = v_{\partial K} + |f|^2 \chi_K$$

for some $v_{\partial K} \geqslant 0$ with supp $v_{\partial K} \subset \partial K$.

One can actually do a little better by taking u to satisfy, for some $\delta > 0$,

$$\Delta u = -\delta |f|^2 \text{ in } \mathit{int}\ K, \quad u = \varepsilon \text{ on } \partial K.$$

Then (21) and (27) hold with v of the form

$$v = v_{\partial K} + (1 + \delta)|f|^2 \chi_K$$

where *supp* $v_{\partial K} \subset \partial K$ and where $v_{\partial K} \geqslant 0$ still holds if δ is small enough.

This v is big enough on its support to guarantee the existence of a quadrature domain D_λ as in (18) for any λ with $\frac{1}{\lambda} \leqslant 1 + \delta$. By previous remarks this gives the desired monotone family $D_\lambda \subset \Omega$ for $\frac{1}{1+\delta} \leqslant \lambda < 1$.

The proof of the final statement of the theorem is contained in the example below.

Example. This is an example to show that the modified Schwarz potential u may become strictly negative close to a boundary point at which f has a zero. It is closely related, via conformal mapping with F, to the well-known fact that u typically becomes negative inside a cusp in case of constant weight (we will have $f = 1$ in the transformed domain).

Take

$$\Omega = \{z \in \mathbf{C} : \operatorname{Im} z > 0\},$$

$$f(z) = 2z + 3az^2$$

($a \in \mathbf{C}$). The fact that Ω is not bounded does not matter very much. Notice that f has a zero at $0 \in \partial\Omega$.

Constructing u as in (23) we have

$$S(z) = z,$$

$$F(z) = z^2 + az^3,$$

$$G(z) = \frac{z^4}{2} + \frac{3az^5}{5} + \frac{2\bar{a}z^5}{5} + \frac{|a|^2 z^6}{2}.$$

This gives (after some computations)

$$u(z) = \tfrac{1}{4}\left(|z^2 + az^3|^2 - \operatorname{Re} z^4 - |a|^2 \operatorname{Re} z^6 - \tfrac{2}{5} \operatorname{Re}\left[(2\bar{a} + 3a)z^5\right]\right).$$

For $z = iy$ we get

$$u(iy) = \tfrac{1}{2}|a|^2 y^6 - \tfrac{2}{5}(\operatorname{Im} a)y^5$$

showing that $u < 0$ for small $y > 0$ if $\operatorname{Im} a > 0$. This finishes the example.

As remarked in the proof of Theorem 6.2, if f is an entire function then the family D_λ obtained in (38) exists for all $\lambda > 0$. Leaving inverse balayage for a moment and turning to forward balayage we may, for any bounded domain Ω, apply (38) to $v = |f|^2 \chi_\Omega$ to obtain domains $D_\lambda \supset \Omega$ for $\lambda > 1$ which in particular satisfy

$$\frac{1}{\lambda} \int_{D_\lambda} g|f|^2 \, dA = \int_\Omega g|f|^2 \, dA, \quad g \in AL^1(\Omega).$$

As λ increases the boundary ∂D_λ propagates with normal velocity equal to $|f|^{-2}$ times the density of classical balayage of $|f|^2 \chi_\Omega$ onto ∂D_λ (see beginning of Section 4). It thus has infinite speed at zeros of f. Numerically produced pictures of the propagating boundary near zeros of f can be found in [6].

Thus, f having zeros is no obstruction to the development of D_λ in itself, if it is just asked to satisfy the quadrature identities and the boundary ∂D_λ is allowed to propagate in a nonsmooth way. However, the additional condition (28) necessary to pass to the corresponding eigenvalue identities is not so easy to satisfy. Even if it happens to hold for one particular value of λ it will usually be violated as λ changes.

The only cases we know of for which (28) is valid for all λ in an interval are certain symmetrical situations involving changes of topology, for example when a hole is filled in at a zero of f. (Example: start with $\Omega=\{r<|z|<R\}$, $f(z)=z$ and let, as λ increases, r decrease to zero and R increase, so that from a certain point on we have a growing disk.) Going in the other direction ($\lambda<1$) this is the way the D_λ which were constructed by inverse balayage in the proof of Theorem 6.2 behave near zeros of f.

The rest of this section is devoted to constructing a counterexample related to the eigenfunction identity.

Example. We produce two simply connected smoothly bounded domains $\Omega_0\subset\Omega_1$ such that $f=1$ is an eigenfunction for the corresponding restriction operator but such that there exists no continuous monotone chain $\Omega_0\subset\Omega_t\subset\Omega_1$ $(0<t<1)$ of simply connected domains so that $f=1$ is an eigenfunction for all intermediate restriction operators corresponding to the couples Ω_t,Ω_1.

The example is based on Example 1.2 in [27]. Let a_0, a_1 be numbers having dimension "area", satisfying

$$0<a_0<\pi<a_1<e\pi.$$

For any $a>0$, $0<b<\pi$, let

$$D(a,b)=\left\{z\in\mathbf{C}:\sqrt{\frac{b}{\pi}}<|z|<\sqrt{\frac{a+b}{\pi}}\right\}$$

be the annulus of area a for which the "hole" has area b. It is a quadrature domain for analytic functions for the uniform measure on $|z|=1$ with total mass a. For every $0<a<e\pi$ there is a unique choice of $b=b(a)$ $(0<b(a)<\pi)$ such that

$$\int_{D(a,b(a))}\log|z|\,dA(z)=0.$$

Thus $b=b(a)$ guarantees that the annulus is a quadrature domain also for harmonic functions. It is in fact even a subharmonic quadrature domain.

Take, as a preliminary choice of domains,

$$D_0=D(a_0,b(a_0)),\quad D_1=\left\{z\in\mathbf{C}:|z|<\sqrt{\frac{a_1}{\pi}}\right\}.$$

Then $|D_0|=a_0$, $|D_1|=a_1$. We assume that a_0 is taken so small that $D_0\subset D_1$. With $\lambda=\frac{a_1}{a_0}$ we have

$$\int_{D_0}h\,dA=\lambda\int_{D_1}h\,dA,\quad h\in HL^1(D_1),\tag{39}$$

i.e., D_1 is a quadrature domain for harmonic function for the measure $\frac{1}{\lambda}\chi_{D_0}$. There is also a quadrature domain for subharmonic functions for the same measure, but this is not D_1, it is

$$D = D(a_1, b(a_1)).$$

Despite (39) there exists no continuous monotone chain of domains $D_0 \subset D_t \subset D_1$ $(0 < t < 1)$ satisfying

$$\int_{D_0} h\,dA = \lambda(t) \int_{D_t} h dA, \quad h \in HL^1(D_t), \tag{40}$$

where necessarily $\lambda(t) = \frac{|D_0|}{|D_t|}$. In fact, by Sakai [27, Theorem 10.13] or our Corollary 4.2, this would necessarily be a chain of quadrature domains for subharmonic functions for the measures $\frac{1}{\lambda(t)} \chi_{D_0}$, in particular we would have $D_1 = D$, which is not the case.

Remark. If we relax the test class $HL^1(D_t)$ to $AL^1(D_t)$ in (40) then there does exist a monotone chain, due to the fact that we in that case can vary the annuli $D(a, b)$ freely, without the restriction $b = b(a)$. To deform D_0 to D_1 we just let the inner radius of D_0 tend to zero and the outer radius tend to that of D_1 as t goes from zero to one.

Next, we modify the above construction so that both initial domains become simply connected. We may take

$$\Omega_0 = D_0 \backslash \{z \in \mathbf{C} : |arg\, z| < \varepsilon\}$$

for $\varepsilon > 0$ small enough, or an approximating domain with analytic boundary. Keeping the previous notation one easily shows that there exists a quadrature domain Ω for subharmonic functions for the measure $\frac{1}{\lambda}\chi_{\Omega_0}$ and that it has the approximate shape of D, in particular is doubly connected.

Also a simply connected domain Ω_1, approximating D_1 and being a quadrature domain for harmonic functions will exist. As Ω_0 was obtained from D_0 by removing a small sectorial piece, call it P, Ω_1 should be obtained from D_1 by making a corresponding removal, but with factor $\frac{1}{\lambda}$: we will have to remove the mass $\frac{1}{\lambda}\chi_P$ from χ_{D_1} and the result should be of the form χ_{Ω_1}. In concrete terms this means that we look for a domain Ω_1 containing $\bar{P}$ and satisfying

$$\int_{D_1} s\,dA \geqslant \frac{1}{\lambda} \int_P s dA + \int_{\Omega_1} s\,dA, \quad s \in SL^1(D_1).$$

This is a problem of inverse balayage, and the construction of Ω_1 essentially follows the proof of the second statement of Theorem 6.2 above (see also [14] for more details): one first makes a basic step of inverse balayage from D_1 to obtain a sufficiently concentrated measure of compact support, then one sweeps that measure to the desired density $1 + \frac{1}{\lambda}\chi_P$. It is easy to make sure that Ω_1 becomes simply connected. Note that the total mass of $\frac{1}{\lambda}\chi_P$ can be made arbitrarily small by choosing ε above small enough.

So now we have produced two simply connected domains $\Omega_0 \subset \Omega_1$ satisfying (39) (with Ω in place of D) and such that no continuous monotone chain of intermediate domains Ω_t satisfies the corresponding identities (40). Since $f = 1$ has no zeros the corresponding eigenvalue identity is (by Proposition 6.1) the same as (39) holding for analytic functions.

It follows that there is at least no chain of simply connected domains satisfying the eigenvalue identities (since for simply connected domains analytic and harmonic test functions give the same).

There still exists a chain Ω_t satisfying (40) for analytic test functions if we allow multiply connected domains. This fact can easily be deduced from Corollary 4.1 in [15], stating that the class of quadrature domains for analytic functions and a given measure is arcwise connected, i.e., that any two such quadrature domains can be deformed into each other within the class. In the present case we have two different measures (χ_{Ω_0} and $\frac{1}{\lambda}\chi_{\Omega_0}$) but also these can be deformed into each other so that causes no problem. It seems likely that, at least in the present case, the deformation can be chosen to be monotone with respect to inclusion (as in Remark above). End of Example.

7. Harmonic function equigravitational with a given one

Another concept of balayage will provide throughout this section a different approach to proving the non-vanishing on the boundary of the Bergman eigenfunctions. This involves constructing a harmonic function in a domain that, as a mass density, is "equigravitational" with a given one. This idea seems to be due to A.I. Prilepko, in work dating from 1966, and has been used in connection with inverse problems of potential theory by several investigators from the Soviet school; for an account, with references see [7, Chapter 7], especially Theorem 7.1 of that book. This encompasses the first assertion of our Theorem 7.2 (which we found independently). The second assertion in Theorem 7.2 as well as Corollary 7.3 and its application in Theorem 7.5 are believed to be new. Due to its possible independent interest, we will present the potential theoretic part in an n-dimensional setting.

First, we note an example showing that in the Bergman space framework the eigenfunctions of the restriction operator may vanish on the boundary of the larger domain. This is an oversimplified version (including the proof) of Theorem 6.2.

Theorem 7.1. *Let Ω be a bounded simply connected planar domain with real analytic boundary and let f be an analytic function defined in a neighbourhood of $\bar{\Omega}$ and which does not vanish in Ω.*

Then there exists a positive measure μ compactly supported in Ω such that f is an eigenfunction for the modulus of the restriction operator $R: AL^2(\Omega) \to L^2(\mu)$.

Proof. By virtue of results of [18] there exists a positive measure ν compactly supported in Ω and such that

$$\int u\, d\nu = \int_\Omega u|f|^2\, dA, \tag{41}$$

for all harmonic and integrable functions u in Ω. In particular, this is true for $u = \bar{h}$ with $h \in AL^2(\Omega)$:

$$\int \bar{h}\, d\nu = \int_\Omega f\overline{fh}\, dA. \tag{42}$$

Define now the measure $d\mu = |f|^{-2} d\nu$. This is again a positive, compactly supported measure satisfying

$$\int f\overline{fh}\, d\mu = \int_\Omega f\overline{fh}\, dA, \quad h \in AL^2(\Omega). \tag{43}$$

In virtue of the regularity assumption on the boundary of Ω and the non-vanishing of f in Ω, any function $g \in AL^2(\Omega)$ can be approximated in the Bergman metric by functions of the form fh, $h \in AL^2(\Omega)$. But this means that

$$\int f\bar{g}\, d\mu = \int_\Omega f\bar{g}\, dA, \quad g \in AL^2(\Omega), \tag{44}$$

that is f is an eigenfunction of the modulus of the restriction operator R, corresponding to the eigenvalue 1.

Note that in the above statement f can have any finite number of zeros on the boundary.

Let us now focus on conditions under which the Bergman eigenfunctions cannot vanish on the boundary of the domain. Until further notice Ω is a bounded domain with smooth boundary of $\mathbf{R}^n$ (n arbitrary). The volume measure will be denoted by dx.

Theorem 7.2. *Let $f \in C^\infty(\bar{\Omega})$. Then there exists a unique $h \in HC(\bar{\Omega})$ such that*

$$\int_\Omega fu\, dx = \int_\Omega hu\, dx, \quad u \in HL^1(\Omega). \tag{45}$$

Moreover, if the biharmonic Green function of Ω is positive and f is subharmonic, then also:

$$\int_\Omega fs\, dx \geqslant \int_\Omega hs\, dx, \quad s \in SL^1(\bar{\Omega}). \tag{46}$$

The biharmonic Green's function is a function $G_y \in C^\infty(\bar{\Omega}\backslash\{y\})$ defined for each point $y \in \partial\Omega$ and such that $\Delta\Delta G_y = \delta_y$ in the sense of distributions and G_y and ∇G_y vanish on $\Gamma = \partial\Omega$.

Proof. If we can find $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that

$$\Delta v = f - h, \quad v|_\Gamma = 0, \nabla v|_\Gamma = 0, \tag{47}$$

then for $u \in HC(\bar{\Omega})$ we would have

$$\int_\Omega (f - h)u \, dx = \int_\Omega (\Delta v)u \, dx = \int_\Omega v \Delta u \, dx = 0.$$

So, by an approximation argument relation (45) in the statement would hold.

Now remark that (47) implies $\Delta\Delta v = \Delta f$ and together with the boundary conditions (47) this is a correct Dirichlet problem that determines a unique solution v. Then the function $h = f - \Delta v$ will be harmonic, and by the regularity up to the boundary results for elliptic equations will find that $v \in C^2(\bar{\Omega})$ and $h \in HC(\bar{\Omega})$.

The function h is unique, because if (45) holds for h_1 and h_2 then $\int_\Omega (h_1 - h_2)u \, dx = 0$ for all harmonic functions u. In particular, $\int_\Omega (h_1 - h_2)^2 dx = 0$, showing that $h_1 = h_2$.

Assume now that the biharmonic Green function of Ω is positive and that f is a subharmonic function. Let $s \in SC(\bar{\Omega})$. Then

$$\int_\Omega (f - h)s \, dx = \int_\Omega (\Delta v)s \, dx = \int_\Omega v \Delta s \, dx.$$

Since $\Delta f \geqslant 0$, the positivity of the Green function implies $v \geqslant 0$. Therefore inequality (46) is proved.

Thus we have proved that every function f admits a unique *equigravitational* harmonic function h. We will denote in short $h = EGH[f]$. In other terms we remark that $EGH[f]$ is the orthogonal projection of f onto the subspace of square integrable harmonic functions, with respect to the L^2 inner product.

Corollary 7.3. *Under the conditions in the second part of Theorem* 6.2,

$$f(y) \geqslant EGH[f](y), \quad y \in \partial\Omega.$$

The proof of the corollary is a consequence of the inequality (46) and the following general result.

Lemma 7.4. *If for a domain Ω in $\mathbf{R}^n$, y is a boundary point whose neighbourhood $\partial\Omega$ is smooth, then there is, associated to y, a "subharmonic δ-function".*

That is, there exists a sequence $(s_m)_{m=1}^\infty$ satisfying:

(i) $s_m \in SC^\infty(\bar{\Omega})$, $m \geqslant 1$,
(ii) $s_m \geqslant 0$ in Ω and $\int_\Omega s_m \, dx = 1$,
(iii) $s_m \to \delta_y$ in the weak* topology of bounded measures on $\bar{\Omega}$.

Proof. Let $\sigma \in SC^\infty(\mathbf{R}^n \setminus \{0\})$ be a subharmonic function satisfying also $\sigma > 0$ and $\int_{|x|<1} \sigma dx = \infty$. Such a function is $\sigma(x) = |x|^{-r}$ with $r \geqslant n$.

Let $y \in \partial\Omega$ be the fixed boundary point and let y_m be a sequence of points in $\mathbf{R}^n \setminus \bar{\Omega}$ which converges to y. Define

$$s_m(x) = \frac{\sigma(x - y_m)}{\int_\Omega \sigma(x - y_m)\, dx},$$

so that clearly s_m are positive, normalized subharmonic smooth functions in $\bar{\Omega}$. It remains to prove that this is a δ_y sequence. In view of the positivity and the normalization condition, it is enough to prove that

$$\int s_m f \, dx \to f(y),$$

for a dense subspace V of the space $\{g \in C(\bar{\Omega}); g(y) = 0\}$. We will choose the space V to consist of all functions $g \in C(\bar{\Omega})$ which vanish in a neighbourhood of y. So, we need only to verify that, for each $\varepsilon > 0$,

$$\lim_{m\to\infty} \int_{\Omega \cap \{|x-y| \geqslant \varepsilon\}} s_m(x)\, dx = 0.$$

In equivalent terms, we must verify that

$$\lim_{m\to\infty} \frac{\int_{\Omega \cap \{|x-y| \geqslant \varepsilon\}} \sigma(x - y_m)\, dx}{\int_\Omega \sigma(x - y_m)\, dx} = 0$$

holds for each fixed $\varepsilon > 0$.

Clearly the numerator remains bounded, since for large m we integrate over $\Omega \cap \{|x - y| \geqslant \varepsilon/2\}$. But the denominator tends to infinity because, if it remained bounded, an application of Fatou's lemma would yield $\int_\Omega \sigma(x - y)\, dx < \infty$ which would violate the choice of σ and the smoothness of $\partial\Omega$.

Remark. (1) Even for non-smooth boundaries, satisfying for instance at every point an outer cone condition, the choice $\sigma(x) = |x|^{-r}$ with large r would imply the same conclusion.

(2) A corresponding "*harmonic* delta function" cannot exist at a smooth boundary point; this is shown by a technique similar to that employed in proving Theorem 8.2 in [34].

Now we can return to planar Bergman spaces. Let $\Omega = \mathbf{D}$ be the unit disk, and let $H(z, w)$ denote the *harmonic kernel function*, that is the reproducing kernel of the Hilbert space $HL^2(\mathbf{D})$. Thus $z \to H(z, w)$ is real, harmonic, symmetric, square integrable for each $w \in \mathbf{D}$ and it satisfies:

$$\int_{\mathbf{D}} H(z,w)u(w)\,dA(w) = u(z), \quad u \in HL^2(\mathbf{D}).$$

Moreover, in the case of the disk we have an explicit formula in terms of the Bergman kernel:

$$H(z,w) = \frac{1}{\pi}\left[\frac{1}{(1-z\bar{w})^2} + \frac{1}{(1-w\bar{z})^2} - 1\right].$$

For each fixed $w \in \mathbf{D}$ the kernel $H(z,w)$ extends real analytically across $\mathbf{T}$. Since $H(0,\zeta) = 1/\pi$, $\zeta \in \mathbf{T}$, there exists a neighbourhood E of 0, henceforth called a *harmonic kernel positivity set*, with the property that:

$$H(z,w) > 0, \quad z \in \mathbf{T}, \quad w \in E. \tag{48}$$

A simple computation shows that the open disk $D(0, \sqrt{2}-1)$ is a positivity set for the kernel H.

Theorem 7.5. *Let μ be a positive measure supported by a harmonic kernel positivity set of the unit disk and let f_n be an eigenfunction of the modulus of the restriction operator $R : AL^2(\mathbf{D}) \to L^2(\mu)$.*

Then f_n does not vanish on $\mathbf{T}$.

Proof. Let λ_n be the corresponding eigenvalue, and let $h = EGH[\lambda_n |f_n|^2]$. That is h is harmonic in the disk, and for every $u \in HC(\overline{\mathbf{D}})$ we have

$$\int_{\mathbf{D}} uh\,dA = \int_{\mathbf{D}} \lambda_n |f_n|^2 u\,dA = \int |f_n|^2 u\,d\mu.$$

In particular, this identity applies to the harmonic kernel and it yields

$$h(w) = \int_{\mathbf{D}} h(z)H(z,w)\,dA(z) = \int_E |f_n(z)|^2 H(z,w)\,d\mu(z),$$

which is bounded below by a positive constant c_n for all w in $\mathbf{D}$ sufficiently close to $\mathbf{T}$, because of our hypothesis. Therefore, in virtue of Corollary 6.3 one finds:

$$\lambda_n |f_n(z)|^2 \geqslant h(z) > c_n, \quad z \in \mathbf{T}.$$

Once Theorem 7.5 is proved, the deformation argument contained in Section 2 applies and shows that each f_n has exactly n zeros in the unit disk.

As a final remark to this section we include an estimate of the uniform norm of the Bergman space contractive divisors recently discovered by Hedenmalm; for a simplified construction of these functions see [10]. Let Ω be a relatively compact subdomain of the unit disk $\mathbf{D}$ and let E be a finite subset of $\mathbf{D}$. Following [10] there exists a *contractive divisor* $g_E \in AL^2(\mathbf{D})$ satisfying: $||g_E||_{2,\mathbf{D}} = 1$, the zero set of g_E is equal to E, g_E analytically extends to the closure of the disk and one has the estimate:

$$||fg_E||_{2,\mathbf{D}} \geqslant ||f||_{2,\mathbf{D}}, \quad f \in AL^2(\mathbf{D}).$$

Let $\lambda_n = \lambda_n(\Omega), f_n$ be, as before, the eigenvalues and, respectively, the eigenfunctions of the self-adjoint operator R^*R.

Then the subspace $g_E AL^2(\mathbf{D})$ has codimension n, and by Courant–Fischer's min–max principle we find a non-zero function $h \in AL^2(\mathbf{D})$ such that:

$$\begin{aligned} ||g_E||_{\infty,\Omega} \int_\Omega |h|^2 \, dA &\geqslant \int_\Omega |hg_E|^2 \, dA \geqslant \lambda_n \int_{\mathbf{D}} |hg_E|^2 \, dA \\ &\geqslant \lambda_n \int_{\mathbf{D}} |h|^2 \geqslant \frac{\lambda_n}{\lambda_0} \int_\Omega |h|^2 \, dA. \end{aligned}$$

Thus we have proved the following

Proposition 7.6. *Let Ω be a relatively compact subdomain of the unit disk. Then*

$$\min_{\sharp(E)=n} ||g_E||_{\infty,\Omega} \geqslant \frac{\lambda_n(\Omega)}{\lambda_0(\Omega)}.$$

Note the independence of Ω from the zero sets E above. In the case of restriction from the Hardy space, such an inequality is well known in connection to estimates of n-widths; there g_E being replaced by a finite Blaschke product, and the minimum being attained at the distribution of points appearing in Theorem 2.1 above, see [12].

8. The representing kernel

Let Ω be a relatively compact subdomain of the unit disk $\mathbf{D}$ and let $R : AL^2(\mathbf{D}) \to AL^2(\Omega)$ be the restriction operator. In this section study a few properties of the representing kernel L of the (nuclear self-adjoint) integral operator R^*R. This function will turn out to be expressible in compact form, as a function of the exponential transform of Ω, a kernel which has recently appeared in operator theory [25,16]. Thus, by combining the known facts about the exponential transform with some standard computations we will show for instance that, whenever Ω is a quadrature domain, the kernel L is a simple rational function depending only on the defining equation of Ω. This observation implies that a finite section of the Taylor series at a fixed point, of the kernel L, or equivalently, of the matrix attached to R^*R in the standard basis of $AL^2(\mathbf{D})$, determines in a constructive way the underlying quadrature domain Ω.

First a few remarks about a matricial representation of the operator R^*R. Let $e_n = \sqrt{\frac{n+1}{\pi}} z^n,\ n \geqslant 0$, be the standard orthonormal basis of the Bergman space of the unit disk. The corresponding matrix of R^*R has entries:

$$\langle R^*Re_m, e_n \rangle = \int_\Omega e_m(z)\overline{e_n(z)}\, dA(z) = \frac{\sqrt{(m+1)(n+1)}}{\pi} a_{mn}, \quad m, n \geqslant 0,$$

where

$$a_{mn} = \int_\Omega z^m \bar{z}^n dA(z), \quad m, n \geqslant 0$$

are the moments of the set Ω. It is well known that the moment matrix (a_{mn}) determines, even in a canonical way, the domain Ω, see [16,25]. Thus the matrix of R^*R with the standard monomial basis of $AL^2(\mathbf{D})$ determines Ω. We will see later in this section that for special domains even a finite section $\langle R^*Re_m, e_n \rangle_{m,n=0}^N$ of the matrix suffices to recover Ω.

The related question whether the spectrum of R^*R determines Ω (possibly modulo non-Euclidean rigid motions) remains open. We remark however that the spectrum of R^*R contains geometric information, as for instance one finds out by computing the trace

$$\begin{aligned} Tr(R^*R) &= \sum_{n=0}^{\infty} \langle R^*Re_n, e_n \rangle = \sum_{n=0}^{\infty} \|Re_n\|^2 \\ &= \sum_{n=0}^{\infty} \frac{n+1}{\pi} \int_\Omega |z|^{2n}\, dA(z) = \frac{1}{\pi} \int_\Omega \frac{dA(z)}{(1-|z|^2)^2} = I(\Omega), \end{aligned}$$

and in the latter we recognize the non-Euclidean area of Ω.

Let us turn now to $L_\Omega(z, \bar{w}),\ z, w \in \mathbf{D}$, the continuous kernel of the operator R^*R:

$$(R^*Rf)(z) = \int_{\mathbf{D}} L_\Omega(z, \bar{w}) f(w)\, dA(w),\ f \in AL^2(\mathbf{D}).$$

To simplify notations we put $L = L_\Omega$ unless the subscript is necessary in the context.

By imposing on the function $L(z, \bar{w})$ to be analytic in z and antianalytic in $\bar{w}$ we have a single choice:

$$L(z, \bar{w}) = \sum_{n=0}^{\infty} \lambda_n f_n(z)\overline{f_n(w)}, \quad z, w \in \mathbf{D},$$

where $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2 \cdots$ are the eigenvalues of R^*R and f_n are the corresponding normalized eigenfunctions.

On the other hand, for any function $f \in AL^2(\mathbf{D})$ we have

$$\begin{aligned}\langle R^*Rf, f\rangle_{2,\mathbf{D}} &= \langle f, f\rangle_{2,\Omega} = \int_\Omega f(\zeta)\overline{f(\zeta)}\, dA(\zeta)\\ &= \int_\Omega \frac{1}{\pi}\int_{\mathbf{D}} \frac{f(w)\, dA(w)}{(1-\zeta\bar{w})^2}\frac{1}{\pi}\int_{\mathbf{D}}\frac{\overline{f(z)}\, dA(z)}{(1-\bar{\zeta}z)^2}\, dA(\zeta)\\ &= \int_{\mathbf{D}^2}\frac{1}{\pi^2}\int_\Omega \frac{dA(\zeta)}{(1-\zeta\bar{w})^2(1-\bar{\zeta}z)^2} f(w)\overline{f(z)}\, dA(z)\, dA(w).\end{aligned}$$

Therefore we obtain an integral formula for the kernel L:

$$L(z,\bar{w}) = \frac{1}{\pi^2}\int_\Omega \frac{dA(\zeta)}{(1-\zeta\bar{w})^2(1-\bar{\zeta}z)^2}, \quad z,w\in\mathbf{D}. \tag{49}$$

Let

$$\omega = \left\{\frac{1}{\bar{z}};\ z\in(\mathbf{C}\cup\{\infty\})\backslash\bar{\Omega}\right\}$$

be the open subset of the complex plane obtained as the complement on the Riemann sphere of the Schwarz reflection of $\bar{\Omega}$ with respect to the unit circle. Let Ω_* be the connected component of ω which contains the unit disk. Then

$$\Omega\subset\mathbf{D}\subset\Omega_*,$$

and the boundary of Ω_* is the Schwarzian reflection of the exterior boundary of Ω.

Lemma 8.1. *The kernel $L_\Omega(z,\bar{w})$ extends analytically/antianalytically to $z,w\in\Omega_*$. Each eigenfunction f_n extends analytically to Ω_*.*

Proof. The integral formula (49) gives directly the desired analytic extension. Recall that the eigenfunction f_n satisfies the integral equation

$$f_n(z) = \frac{\lambda_n}{\pi}\int_\Omega \frac{f_n(\zeta)\, dA(\zeta)}{(1-\bar{\zeta}z)^2} \tag{50}$$

which gives the analytic extension.

As one can easily see, if the boundary of Ω is real analytic, then the kernel L and each eigenfunction f_n can further be continued analytically across $\partial\Omega_*$.

The *exponential transform* of the domain Ω is the kernel

$$E_\Omega(z,\bar{w}) = \exp\left(-\frac{1}{\pi}\int_\Omega \frac{dA(\zeta)}{(\zeta-z)(\bar{\zeta}-\bar{w})}\right),$$

defined by a convergent integral for all values $z, w \in \mathbf{C} \backslash \bar{\Omega}$ or everywhere where $z \neq w$. In case $z = w \in \bar{\Omega}$ produces a divergent integrand, we take by convention $\exp(-\infty) = 0$. With these assumptions, the function E_Ω is defined on $\mathbf{C}^2$ and it turns out to be separately continuous everywhere.

The restriction to the diagonal function $F(z) = E_\Omega(z, \bar{z})$ is real analytic for $z \in \mathbf{C} \backslash \bar{\Omega}$. If an arc τ of the boundary of Ω is real analytic smooth, then $F(z)$ extends real analytically across τ and the extension vanishes of the first order there.

Suppose that Ω is a *quadrature domain* in the original terminology of Aharonov and Shapiro [1], that is there exists a distribution u of finite support in Ω such that

$$\int_\Omega f dA = u(f), \quad f \in AL^1(\Omega).$$

In this case Ω is a domain defined by a polynomial inequality:

$$\Omega = \{z \in \mathbf{C}; \quad Q(z, \bar{z}) < 0\}.$$

Moreover, one proves that Q is an irreducible polynomial of two variables.

Let $p(z)$ be the minimal monic polynomial (of degree d) which vanishes on the support of the quadrature distribution u. Then the defining equation of Ω has the following special form:

$$Q(z, \bar{w}) = p(z)\overline{p(w)} - \sum_{i=0}^{d-1} q_i(z)\overline{q_i(w)},$$

where q_i are polynomials of degree i, respectively. Moreover, in this case a remarkably simple relation between the exponential transform and these data exists:

$$E_\Omega(z, \bar{w}) = \frac{Q(z, \bar{w})}{p(z)\overline{p(w)}}, \quad |z|, |w| > 1. \tag{51}$$

Note that the kernel

$$1 - E_\Omega(z, \bar{w}) = \sum_{i=0}^{d-1} \frac{q_i(z)}{p(z)} \overline{\frac{q_i(w)}{p(w)}}$$

is positive definite for $z, w \in \mathbf{C} \backslash \bar{\Omega}$, and in addition $|1 - E_\Omega(z, \bar{w})| < 1$ for all these values. Let us also remark that $E_\Omega(\infty, \bar{w}) = E_\Omega(z, \infty) = 0$.

A detailed account, with references to the original papers, of the theory of quadrature domains is contained in [34]. For the results related to the exponential transform see [25,16]. For an early reference, in a different context, to the same exponential transform see [2].

Lemma 8.2. *The kernels L_Ω and E_Ω of a bounded planar domain Ω are related in a simple way*:

$$L_\Omega(z,\bar{w}) = -\frac{1}{\pi}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{w}}\log E_\Omega\left(\frac{1}{\bar{w}},\frac{1}{z}\right), \quad z,w\in\Omega_*. \tag{52}$$

Proof. Indeed, for $z,w\in\Omega_*\setminus\{0\}$ we have

$$\begin{aligned} L_\Omega(z,\bar{w}) &= \frac{1}{\pi^2}\frac{1}{z^2\bar{w}^2}\int_\Omega \frac{dA(\zeta)}{(\zeta-\frac{1}{\bar{w}})^2(\bar{\zeta}-\frac{1}{z})^2} \\ &= \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{w}}\frac{1}{\pi^2}\int_\Omega \frac{dA(\zeta)}{(\zeta-\frac{1}{\bar{w}})(\bar{\zeta}-\frac{1}{z})}. \end{aligned}$$

The logarithm above is well defined because $|1-E_\Omega(z,\bar{w})|<1$ whenever $z,w\in\mathbf{C}\setminus\bar{\Omega}$.

In the case of a quadrature domain a more precise formula, linking directly the defining equation to the kernel of the restriction operator is available.

First we need one more definition. For a polynomial $p(z)$ of degree d we denote

$$p^\sharp(z) = z^d p\left(\frac{1}{z}\right).$$

This involutive operation reverses the order of the coefficients and it is well known in the (Hurwitz–Cohn–Schur) theory of separation of polynomial roots. The defining polynomial $Q(z,\bar{w})$ of a quadrature domain has the leading term equal to $z^d\bar{w}^d$. Henceforth we denote

$$Q^\sharp(z,\bar{w}) = z^d\bar{w}^d Q\left(\frac{1}{z},\frac{1}{\bar{w}}\right).$$

Let us remark that $Q^\sharp(0,0)=1$ and that, for all $z\in\mathbf{D}$, we have

$$Q^\sharp(z,\bar{z}) = |z|^{2d} Q\left(\frac{1}{z},\frac{1}{\bar{z}}\right) > 0.$$

Note also that

$$Q^\sharp(z,0) = p^\sharp(z).$$

As a direct application of the previous lemma and the fact that the mixed logarithmic derivative annihilates the pure terms in the denominator of E_Ω we obtain the following result.

Theorem 8.3. *Let Ω be a quadrature domain of order d, with defining polynomial $Q(z,\bar{z})$. Then*

$$L_\Omega(z,\bar{z}) = -\frac{1}{4\pi}\Delta\log Q^\sharp(\bar{z},z), \quad z\in\mathbf{D}. \tag{53}$$

In other terms, we have the explicit formula:

$$L_\Omega(z,\bar z) = -\frac{Q^\sharp(\bar z,z)\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}Q^\sharp(\bar z,z) - \frac{\partial}{\partial z}Q^\sharp(\bar z,z)\frac{\partial}{\partial \bar z}Q^\sharp(\bar z,z)}{\pi Q^\sharp(\bar z,z)^2}.$$

A direct computation shows that the leading terms of degree d in either z or $\bar z$ cancel in the numerator, so that $L_\Omega(z,\bar z)$ turns out to be a rational function expressible as a quotient of polynomials of degree at most equal to $2d-2$ in each variable in the numerator and exact degree $2d$ in each variable in the denominator.

The last equation of L_Ω can of course be polarized. Note that the polynomial Q is intrinsic to the domain Ω, while the fact that we are restricting from the Bergman space of the unit disk is reflected by the Schwarz inversion $z \mapsto 1/\bar z$ involved in formula (53).

Similar, and even simpler, formula can be obtained by restricting the Hardy space of the disk to the Bergman space of a quadrature domain contained (and relatively compact) in the disk.

To verify these formulae, let us consider the case $\Omega = r\mathbf{D}$, $r<1$, of a concentric disk (Example 2.1). Then

$$E_{r\mathbf{D}}(z,\bar w) = 1 - \frac{r^2}{z\bar w}, \quad |z|,|w| > r,$$

while

$$\begin{aligned} L_{r\mathbf{D}}(z,\bar w) &= \frac{1}{\pi^2}\int_{r\mathbf{D}} \frac{dA(\zeta)}{(1-\zeta\bar w)^2(1-\bar\zeta z)^2} \\ &= \frac{1}{\pi}\frac{r^2}{(1-r^2 z\bar w)^2} = -\frac{1}{\pi}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar w}\log(1-r^2 z\bar w). \end{aligned}$$

In this case L is exactly the Bergman kernel of the disk $\Omega_* = r^{-1}\mathbf{D}$. That this is not an accident results from the following observation.

Proposition 8.4. *Let Ω be a simply connected domain relatively compact in the unit disk and with smooth real analytic boundary. Let $K_{\Omega_*}(z,\bar w)$ be the Bergman kernel of Ω_*.*

Then the function $L_\Omega(z,\bar w) - K_{\Omega_*}(z,\bar w)$, $z,w \in \Omega_*$, extends analytically/antianalytically across $\partial\Omega_*$.

Proof. Let $\phi : \Omega_* \to \mathbf{D}$ be a conformal mapping. Due to the assumptions on the boundary of Ω, the map ϕ extends analytically across Ω_*.

The Bergman kernel of the domain Ω_* is given by the formula:

$$L_{\Omega_*}(z,\bar w) = -\frac{1}{\pi}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar w}\log(1-\phi(z)\overline{\phi(w)}), \quad z,w \in \Omega_*.$$

Therefore, by formula (52) we obtain, for all $z, w \in \Omega_* \backslash \mathbf{D}$:

$$L(z, \bar{w}) - K_{\Omega_*}(z, \bar{w}) = \frac{1}{\pi} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{w}} \log \frac{1 - \phi(z)\overline{\phi(w)}}{E_\Omega(\frac{1}{\bar{w}}, \frac{1}{z})}.$$

Let $S(z)$ be the Schwarz function of the boundary of Ω. Then it is known (see [16]) that the function

$$\frac{E_\Omega(a, \bar{b})}{\bar{b} - S(a)}.$$

extends analytically from the region $a, b \in \mathbf{D} \backslash \bar{\Omega}$ across the boundary $\partial\Omega$, and moreover the extension is free of zeros on $\partial\Omega$.

The conclusion follows then by keeping track of the commutation relation of ϕ and the inversion with the Schwarz reflections in the corresponding curves.

The idea used in the latter proof can be related to the analytic continuation pattern of the eigenfunctions f_n of the operator R^*R. More specifically, choose antiderivatives $F_n \in AL^2(\mathbf{D})$ of f_n such that $F_n(0) = 0$. By integrating twice relation (52) we obtain

$$\log E_\Omega\left(\frac{1}{\bar{w}}, \frac{1}{z}\right) = -\pi \sum_{n=0}^{\infty} \lambda_n F_n(z)\overline{F_n(w)}, \quad z, w \in \Omega_*. \tag{54}$$

On the other hand, the Bergman kernel of the unit disk is

$$\frac{1}{\pi(1 - z\bar{w})^2} = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}, \quad z, w \in \mathbf{D}.$$

By integrating twice this identity we get

$$\log(1 - z\bar{w}) = -\pi \sum_{n=0}^{\infty} F_n(z)\overline{F_n(w)}. \tag{55}$$

By subtracting Eq. (55) from (54) and changing variables we obtain

$$\log \frac{E_\Omega(z, \bar{w})}{1 - \frac{\overline{S(z)}}{w}} = \pi \sum_{n=0}^{\infty} \left[F(\overline{S(z)}) - \lambda_n F_n\left(\frac{1}{\bar{z}}\right)\right] F_n\left(\frac{1}{\bar{w}}\right), \quad z, w \in \mathbf{D} \backslash \Omega.$$

The analytic continuation of the left hand term across $\partial\Omega$ is then consistent with Proposition 5.1.

Let us return to the case of a quadrature domain Ω with quadrature nodes $a_1, a_2, \ldots, a_d$. Then we know that the kernel L is rational, and it depends, via formula (53), only on the defining equation $Q(z, \bar{w})$ of the domain.

We split the polynomial Q as follows:

$$Q(z, \bar{w}) = (\bar{w} - S_1(z))(\bar{w} - S_2(z)) \dots (\bar{w} - S_d(z)),$$

where $S_1(z) = S(z)$ is the Schwarz function of Ω and it is well defined as a meromorphic function in a neighbourhood of $\bar{\Omega}$. The other functions $S_j(z)$ are in general multivalued, and they can be defined, as an ordered tuple, only locally, except at the ramification points. However, we will work only with symmetric expressions in S_j, and this will eliminate the ordering ambiguity.

For fixed points $z, w \in \mathbf{C} \setminus \overline{\mathbf{D}}$ we obtain

$$\langle R^* R(* - z)^{-1}, (* - w)^{-1} \rangle_{2,\mathbf{D}} = \int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})} = -\pi \log E_\Omega(z, \bar{w}).$$

By differentiating once in $\bar{w}$ and using formula (51) for the exponential kernel we have

$$\langle R^* R(* - z)^{-1}, (* - w)^{-2} \rangle_{2,\mathbf{D}} = -\pi \sum_{j=1}^{d} \left(\frac{1}{\bar{w} - S_j(z)} - \frac{1}{\bar{w} - \overline{a_j}} \right). \tag{56}$$

We recall that for $z \in \mathbf{C} \setminus \bar{\Omega}$ the d values of the multivalued Schwarz symmetry $S_j(z), 1 \leqslant j \leqslant d$, all lie in Ω.

Consequently, by using Cauchy's formula twice we are led to the following result.

Proposition 8.5. *Let Ω be a quadrature domain of order d, relatively compact in the unit disk, let u, v be polynomials and let U be a primitive of u. Then*

$$\langle R^* Ru, v \rangle_{2,\mathbf{D}} = \pi \sum_{j=1}^{d} \langle U(\overline{S_j(z)}) - U(a_j), zv(z) \rangle_{2,\partial\mathbf{D}}, \tag{57}$$

where the torus $\mathbf{T}$ *was endowed with the normalized arc length measure.*

To verify the latter formula on a concentric disk $\Omega = r\mathbf{D}$ we note

$$\begin{aligned}\pi \int_{\mathbf{T}} U\left(\frac{r^2}{\bar{z}}\right) \overline{(zv(z))} \frac{d\theta}{2\pi} &= -\frac{1}{2i} \int_{\mathbf{T}} U(r^2 z)\overline{v(z)}\, d\bar{z} \\ &= r^2 \int_{\mathbf{D}} u(r^2 z)\overline{v(z)}\, dA(z) = \int_{r\mathbf{D}} u(z)\overline{v(z)}\, dA(z).\end{aligned}$$

As a final remark, let us return to the general case of two bounded planar domains $\Omega_0 \subset \overline{\Omega_0} \subset \Omega_1$ and the restriction operator R between the corresponding Bergman spaces. Let λ_n and f_n be as before the eigenvalues of $R^* R$, respectively the eigenfunctions. For a fixed positive constant a we define a new, equivalent, Hilbertian norm on $AL^2(\Omega_1)$ by:

$$\|f\|_a^2 = a\int_{\Omega_0} |f|^2\,dA + \int_{\Omega_1\setminus\Omega_0} |f|^2\,dA, \quad f\in AL^2(\Omega_1).$$

A direct computation shows then that the functions

$$\frac{f_n}{\sqrt{a\lambda_n + 1 - \lambda_n}}, \quad n\geqslant 0,$$

form an orthonormal system in $AL^2(\Omega_1)$, with respect to the norm $\|\cdot\|_a$. Thus the corresponding reproducing kernel is

$$K_a(z,\bar{w}) = \sum_{n=0}^{\infty} \frac{f_n(z)\overline{f_n(w)}}{a\lambda_n + 1 - \lambda_n}. \tag{58}$$

In this way one can compute, for a fixed value $z_0\in\Omega_1$, the extremal value

$$\sup_{\|f\|_a=1} |f(z_0)|^2 = \frac{1}{K_a(z_0,\overline{z_0})}.$$

9. Finite rank restriction operators

The aim of this section is to discuss a few applications of the results proved above to one of the simplest, and in some sense, extreme situation, namely that of finite rank restriction operators. Let Ω be a bounded planar domain and let

$$\mu = \sum_{j=1}^{n} c_j\delta_{a_j},$$

be a positive measure of finite support in Ω. Then the restriction operator

$$R : AL^2(\Omega)\to L^2(\mu) = \mathbf{C}^n$$

is surjective, hence of rank equal to n. Let $K_\Omega(z,\bar{w})$ be the Bergman kernel of the domain Ω.

Lemma 9.1. *The range of $|R|$ is spanned by the functions $K_\Omega(\cdot,\overline{a_j}), 1\leqslant j\leqslant n$, and consequently each eigenfunction of $|R|$ is a linear combination of these vectors.*

Proof. Let $f\in AL^2(\Omega)$ be an eigenfunction of $|R|$, that is there exists $\lambda > 0$ with the property

$$\lambda\int_\Omega f\bar{g}\,dA = \sum_{j=1}^{n} c_j f(a_j)\overline{g(a_j)}, \quad g\in AL^2(\Omega).$$

By choosing $g(z) = K_\Omega(z, \bar{w})$ we find that

$$\lambda f(w) = \sum_{j=1}^{n} c_j f(a_j) K_\Omega(w, \overline{a_j}), \tag{59}$$

which completes the proof.

Actually, the above proof gives more. Namely the eigenvalue problem (59) for $|R|$ is reducible to linear algebra computations, with the only given data being the positive definite Gram matrix $K = [K_\Omega(a_i, \overline{a_j})]_{i,j=1}^n$ and the diagonal positive matrix $C = diag(c_1, c_2, \ldots, c_n)$. Indeed, denoting by f the column vector $(f(a_1), f(a_2), \ldots, f(a_n))$, Eq. (59) implies, by evaluating $w = a_i$:

$$KCf = \lambda f.$$

Once this system is solved, formula (59) explicitly gives the eigenfunctions f.

Corollary 9.2. *With the above notation, the operator $|R|$ is unitarily equivalent to the matrix $\sqrt{C}K\sqrt{C} \in L(\mathbf{C}^n)$, where $\sqrt{C}$ is the positive square root of C.*

Proof. Indeed, an eigenvalue λ of $|R|$ satisfies the system

$$\sqrt{C}K\sqrt{C}\sqrt{C}f = \lambda\sqrt{C}f.$$

Since the matrix $\sqrt{C}K\sqrt{C}$ is positive definite and it has the same spectrum as $|R|$, the unitary equivalence among them follows.

At this point we can combine these observations with the general qualitative analysis of the eigenvalues of $|R|$ and obtain for instance the following result (directly derived from the proof of Theorem 7.5).

Theorem 9.3. *Let Ω be a bounded domain with smooth boundary and let μ be a positive measure of finite support contained in the harmonic kernel positivity set of Ω. Let $f_0, f_1, \ldots, f_{n-1}$ be the eigenfunctions of the restriction operator $|R|$, with corresponding eigenvalues λ_k arranged in decreasing order.*

Then the eigenvalues λ_k are mutually distinct and each f_k, $0 \leqslant k \leqslant n-1$, does not vanish on $\partial\Omega$ and has exactly k zeros.

In the case of the Hardy space, as shown by Fischer–Micchelli, the statement is true without any restriction on the support of the measure μ. Of related interest is Videnskii's recent study [40] of the rather intricate problem of locating the zeros of finite linear combinations of the Bergman kernel of the disk.

To give a simple application of the theorem, which otherwise seems to be difficult to prove, we can consider the case of the unit disk $\mathbf{D}$ and equal weights

$c_1 = c_2 = \cdots = c_n$. Recall that the disk $D(0, \sqrt{2} - 1)$ is a positivity set for the harmonic kernel of $\mathbf{D}$.

Corollary 9.4. *Let $a_1, a_2, \ldots, a_n \in D(0, \sqrt{2} - 1)$ be n distinct points. Then the eigenvalues of the matrix*

$$\left[\frac{1}{(1 - a_i\overline{a_j})^2}\right]_{i,j=1}^{n},$$

are distinct.

The finite rank framework also provides a simple negative answer to the inverse spectral problem discussed at the beginning of Section 8. For instance, take a two point mass measure $\mu = a\delta_\alpha + b\delta_\beta$, where $\alpha, \beta \in \mathbf{D}$ and a, b are positive constants. Then the restriction operator $R : AL^2(\mathbf{D}) \to L^2(\mu)$ has rank two, hence its spectrum depends on 2 real parameters. On the other hand μ depends on 6 free real parameters, and the non-Euclidean rigid motion group (that is the group generated by the Moebius transforms and the complex conjugation) has 3 real parameters. Therefore, the spectrum of R^*R cannot determine in this situation the measure μ, even up to non-Euclidean rigid motions. To be more precise

Lemma 9.5. *There exists a continuum of discrete measures μ_t, $t \in [0, 1]$, each consisting of two atoms, such that the moduli $|R_t|$ of the restriction operators*

$$R_t : AL^2(\mathbf{D}) \to L^2(\mu_t)$$

are all unitarily equivalent $|R_t| \equiv |R_0|$, $t \in [0, 1]$, and such that for every pair $s \neq t$, $0 \leqslant s$, $t \leqslant 1$, there is no rigid non-Euclidean transform T with the property that $T_* \mu_s = \mu_t$.

We do not know whether a similar example exists with characteristic functions of subdomains of the disk instead of atomic measures.

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