What is a quadrature domain?

Björn Gustafsson and Harold S. Shapiro

Abstract. We give an overview of the theory of quadrature domains with indications of some if its ramifications.

1. Introduction

With this introductory paper the authors wish to give a general overview of the theory of quadrature domains, along with indications of some of its ramifications. There will be no proofs, but an extensive bibliography. More details for most of the material known up to 1992 can be found in the book [146]. We shall start by saying a few words of the birth of the theory.

The word 'quadrature' goes back to the latin noun 'quadratura', which means 'making square-shaped', 'constructing squares' or, more specifically, 'the division of land into squares' [43]. Accordingly, 'quadrature' in mathematics traditionally refers to constructive or numerical methods for determining areas, and (more recently) for computing integrals in general.

In the theory discussed in this article 'quadrature' has a related meaning. For example, a 'quadrature identity' will typically be an exact formula for the integral of harmonic or analytic functions in terms of simpler functionals, like point evaluations. The domain of integration is then a quadrature domain.

Specifically we shall call a bounded domain Ω in the complex plane a **(classical) quadrature domain** if there exists finitely many points $a_1, \dots, a_m \in \Omega$ and coefficients $c_{kj} \in \mathbf{C}$ so that

$$\int_\Omega f\,dA = \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} c_{kj} f^{(j)}(a_k) \tag{1.1}$$

Received by the editors November 11, 2004.

1991 *Mathematics Subject Classification.* Primary 30-02; Secondary 31-02, 32Dxx, 32Sxx, 35A21, 35Jxx, 35R35, 47B20, 76D27.

Key words and phrases. Quadrature domain, quadrature identity, quadrature surface, Schwarz function, Schwarz potential, mean value property, Schottky double, balayage, mother body, free boundary problem, special point, Hele-Shaw flow, exponential transform, hyponormal operator, subnormal operator, Friedrichs operator, Bergman kernel.

Paper supported by the Swedish Research Council VR.

for all integrable analytic functions f in Ω (dA denotes area measure). The identity (1.1) is then called a **quadrature identity** and the integer $n = \sum_{k=1}^{m} n_k$ is the **order** of the quadrature identity (assuming $c_{k,n_k-1} \neq 0$).

Embryonic theories of quadrature domains can be found already in papers of C. Neumann [100], [101] from the beginning of the last century. See [146] for discussions. However, the first person to more systematically study general quadrature identities of the kind (1.1) seems to be P. Davis, in the 1960's, see [26] and references therein. Independently, D. Aharonov and H. S. Shapiro [1], [3] discovered a few years later that solutions of certain extremal problems for univalent functions map the unit disc onto domains which allow quadrature identities for (somewhat restricted) classes of analytic functions. (Not having all analytic functions available caused some technical problems, which were finally resolved in [4], [5].) In [2] the authors started studying such quadrature domains for their own sake.

Around the same time and partially influenced by the Aharonov-Shapiro ideas, Y. Avci, a doctoral student of M. Schiffer, was writing his doctoral thesis [8] about quadrature identities, and in another corner of the world M. Sakai was working with related matters from a more potential theoretic point of view [116], [118], [119], [121]. In both cases, like in the Aharonov-Shapiro case, the motivating problems were extremal problems for analytic functions (see the introduction of [8] and the appendix of [121]).

After this initiation in the 1970's (essentially), the subject was pursued mainly by H. S. Shapiro and his students and by M. Sakai. Part of the subject was developed in parallel with certain topics in fluid mechanics, mainly Hele-Shaw flow, which was linked to quadrature domain theory via a discovery by S. Richardson [112].

After a conference, organized by D. Khavinson, at the University of Arkansas in 1988 and the subsequent book [146] a kind of rebirth took place in the mid 1990's with new influences from operator theory [102], [103], [104], [98]. Also, more links to fluid mechanics were discovered, see [23] for an overview.

2. Classical quadrature domains

We shall find it convenient in the sequel to write the quadrature identity (1.1) simply as

$$\int_{\Omega} f\, dA = \sum_{k=1}^{n} c_k f(a_k), \tag{2.1}$$

where in the sequence $a_1, \dots, a_n \in \Omega$ repetitions are allowed, a repeated a_k being interpreted as the occurence of derivatives of f at a_k.

In the simplest case, $n = 1$, it is known that discs $\mathbf{D}(a, r)$ are the only quadrature domains [36], [37], [2], [121], [146] and the quadrature identity then reduces to the ordinary mean value property for analytic functions:

$$f(a) = \frac{1}{|\mathbf{D}(a,r)|} \int_{\mathbf{D}(a,r)} f dA. \tag{2.2}$$

Thus a quadrature identity can be thought of as a generalized mean value property and a quadrature domain as a generalized disc (at least if all the c_k are positive).

By choosing $f(\zeta) = \frac{1}{z-\zeta}$ for $z \in \mathbf{C} \setminus \Omega$ in (2.1) one realizes that the Cauchy transform

$$\hat{\chi}_\Omega(z) = -\frac{1}{\pi} \int_\Omega \frac{dA(\zeta)}{z-\zeta} \tag{2.3}$$

of the characteristic function of a quadrature domain Ω is a rational function outside Ω. To be exact, there exists a rational function $R(z)$ of the form

$$R(z) = \frac{Q(z)}{P(z)}, \tag{2.4}$$

where $P(z) = \prod_{k=1}^{n} (z - a_k)$ and where $Q(z)$ is a polynomial of degree $n-1$, such that

$$\hat{\chi}_\Omega = R \quad \text{on } \mathbf{C} \setminus \Omega. \tag{2.5}$$

Conversely, by an approximation theorem of L. Bers [14] the linear combinations of the above Cauchy kernels are dense in the integrable analytic functions, so (2.5) holding for some rational function R is actually equivalent to Ω being a quadrature domain.

The main tool for the approximation used in [14] is a system of ingenuously constructed mollifiers, nowadays often named the "Ahlfors-Bers mollifiers". These mollifiers play an important role in the theory of quadrature domains, also in higher dimensions. See e.g. [121], [48], [82] for their use. Alternative techniques for proving Bers-type approximations are a lemma by V. Havin [64], [143], [146] and "quasi-balayage" [147].

3. The Schwarz function

Assume that Ω is a quadrature domain (2.1). Since $\frac{\partial}{\partial \bar{z}} \hat{\chi}_\Omega = \chi_\Omega$ in the sense of distributions the function $S(z)$ defined on $\overline{\Omega}$ by

$$S(z) = \overline{z} + R(z) - \hat{\chi}_\Omega(z) \tag{3.1}$$

is meromorphic in Ω, and by (2.5),

$$S(z) = \overline{z} \quad \text{for } z \in \partial\Omega. \tag{3.2}$$

Thus $S(z)$ is a one-sided **Schwarz function** of $\partial\Omega$. To be a two-sided Schwarz function one requires that $S(z)$ is defined and analytic in a full neighbourhood of $\partial\Omega$. The use of such a function associated to an analytic curve can be traced back to G. Herglotz [72]. It was named, after H. A. Schwarz' reflection principle, in [27]. Full accounts are given in [26], [146].

A two-sided Schwarz function, defined in some neighbourhood of $\partial\Omega$, exists if and only if $\partial\Omega$ is analytic, while a one-sided Schwarz function allows for certain singular points of $\partial\Omega$. A complete characterization of those boundaries allowing a one-sided Schwarz function was given by M. Sakai in [127].

The previous argument also goes the other way around, so a bounded domain Ω is a quadrature domain if and only if there exists a meromorphic function $S(z)$ in Ω, continuous up to $\partial\Omega$, so that (3.2) holds. This result was obtained in [26] under some smoothness assumptions on $\partial\Omega$ and in full generality in [2].

Indeed, one of the major achievements in [2] is that the authors prove, without any regularity assumptions whatsoever, that $\partial\Omega$ is a subset of an algebraic curve: there exists a nontrivial polynomial $Q(z,w)$ such that

$$\partial\Omega \subset \{z \in \mathbf{C} : Q(z,\bar{z}) = 0\}. \tag{3.3}$$

From this, additional regularity of $\partial\Omega$ follows easily.

Differentiating (3.2) gives

$$s(z)dz = d\bar{z} \quad \text{along } \partial\Omega, \tag{3.4}$$

where $s(z) = S'(z)$. The relation (3.4) holds for some meromorphic function $s(z)$ (not necessarily having a single-valued primitive) if and only if a quadrature identity of the kind

$$\int_\Omega f\,dA = \sum_{k=1}^{m}\sum_{j=0}^{n_k-1} c_{kj} f^{(j)}(a_k) + \sum_{i=1}^{r} b_i \int_{\gamma_i} f dz \tag{3.5}$$

holds, where the γ_i are (smooth) closed or nonclosed curves, compactly contained in Ω, and $b_i \in \mathbf{C}$. See [2], [8], [44], [81].

4. Riemann surfaces (the Schottky double)

One way to understand the algebraicity (3.3) of $\partial\Omega$ is as follows. The relation (3.2), with $S(z)$ meromorphic in Ω, may be interpreted as saying that the pair of functions $(S(z),\bar{z})$ constitutes a meromorphic function on the **Schottky double** [133], [135] $\hat{\Omega}$ of Ω, namely the compact Riemann surface obtained by completing Ω with a backside $\tilde{\Omega}$ having the opposite conformal structure, and glueing the two together along $\partial\Omega$. (We are here assuming *a priori* some mild regularity of $\partial\Omega$.) Then $S(z)$ represents the values of the function on Ω and $\bar{z}$ the values on $\tilde{\Omega}$.

The opposite pair $(z,\overline{S(z)})$ will also represent a meromorphic function on $\hat{\Omega}$, and since any two meromorphic functions on a compact Riemann surface are related by a polynomial equation it follows that there exists a polynomial $Q(z,w)$ such that

$$Q(z,S(z)) = 0 \quad (z \in \Omega).$$

In particular $Q(z,\bar{z}) = 0$ on $\partial\Omega$, i.e., we obtain (3.3) again.

More detailed investigations [44] show that $Q(z,w)$ (if chosen without extraneous factors) has degree exactly n in each of z and w (total degree $2n$) and that the difference set in (3.3) consists of at most finitely many points (called special points, see Section 12). Additional information on the structure of $Q(z,w)$ can be derived from properties of the exponential transform, see Section 13 and also [103], [107], [53], [55].

If Ω is any simply connected bounded domain, then Ω is a quadrature domain if and only if any conformal map $g : \mathbf{D} \to \Omega$ is a rational function. This was proved in [26] under some regularity assumptions on $\partial\Omega$ and in [2] without such assumptions. The idea for the nontrivial direction is to use the fact that $z \mapsto \overline{S(z)}$ $(z \in \Omega)$ is the anticonformal reflection in Ω to extend the mapping function g from $\mathbf{D}$ to the entire Riemann sphere.

For multiply connected domains there are analogous results [44]: Let D be a finitely connected domain representing a certain conformal type and let $(g, \overline{h})$ represent a meromorphic function on the Schottky double $\hat{D}$, so that g and h are meromorphic in D and $g = \overline{h}$ on ∂D. If then g is holomorphic and univalent on D it will map D onto a quadrature domain, because the relation $g = \overline{h}$ on ∂D will become a relation of the form (3.2) in the image domain.

Other techniques to construct quadrature domains, based on Riemann surface ideas (automorphic functions, Schottky groups, Poincaré series etc) have been developed by Avci [8], Richardson, [113], Crowdy and Marshall [20], [21], [22].

5. Subharmonic quadrature domains

When all the c_k in (2.1) are positive (with the a_k then distinct) it is natural to think of a quadrature domain Ω satisfying (2.1) as something obtained by glueing the discs $\mathbf{D}(a_k, \sqrt{\frac{c_k}{\pi}})$ together in a potential theoretic way, or as the result of some kind of balayage (sweeping) process applied to the measure

$$\mu = \sum_{k=1}^{n} c_k \delta_{a_k}. \tag{5.1}$$

Here δ_a denotes the unit point mass at a. However, it turns out that this picture is fully correct only if one requires (2.1) to hold in the stronger sense that the inequality

$$\int_\Omega f\, dA \geq \sum_{k=1}^{n} c_k f(a_k) \tag{5.2}$$

holds for all integrable subharmonic functions f in Ω. Then (2.1) will automatically hold for all harmonic f (because both f and $-f$ will be subharmonic).

The importance of considering such subharmonic quadrature domains was realized by M. Sakai [119], [121]. The following facts illustrate why subharmonic quadrature domains are natural.

(i) A subharmonic quadrature domain Ω is uniquely determined, up to null-sets, by its measure μ. Such a statement is not true if only (2.1) is required to hold for, e.g., harmonic test functions.

(ii) There are natural ways to construct this unique Ω from μ by a sort of balayage we call **partial balayage** (because the measure is not swept completely, only down to a certain density). The first construction of partial balayage, by M. Sakai, was rather involved [119], [121]. Later more streamlined methods were found, based on techniques such as minimization of energy functionals, variational inequalities or Perron family arguments [122], [123], [45], [48], [57], [51]. See also Section 7 below. Numerical schemes for similar processes of 'equigravitational mass scattering' go back at least to the work of D. Zidarov in the 1960's (see [163] and references therein).

(iii) The geometry of Ω reflects that of μ very well (which need not be the case for analytic or harmonic quadrature domains) [57], [130], [59], [60]. For example, each inward normal ray from $\partial\Omega$ in (5.2) intersects the the convex hull K of the support of μ, and Ω can be written as a union of discs with centers in K. Thus, if μ is concentrated to a small set and the total mass of μ is large, then Ω has to be nearly circular.

6. Harmonic quadrature domains

An intermediate test class for (2.1) is the class of integrable harmonic functions in Ω. In this case we assume that the c_k are real and that the a_k are distinct (no derivatives). Thus μ in (5.1) is a signed measure and (2.1) may be written

$$\int_\Omega h\,dA = \int h\,d\mu, \tag{6.1}$$

to hold for all h harmonic and integrable in Ω.

It becomes natural at this point to allow more general measures μ than in (5.1), for example arbitrary signed measures with compact support in Ω, and also to spell out the various quadrature identities in terms of Newtonian potentials and statements of graviequivalence.

The Newtonian (or logarithmic) potential of any (signed) measure μ is

$$U^\mu(z) = \frac{1}{2\pi}\int \log\frac{1}{|z-\zeta|}\,d\mu(\zeta),$$

where the constant is chosen so that $-\Delta U^\mu = \mu$. Its gradient ∇U^μ is, apart from a constant factor and a complex conjugation, the same thing as the Cauchy transform $\hat{\mu}$ (defined analogously to (2.3)). Let U^Ω denote the potential of the measure $\chi_\Omega\,dA$.

The linear combinations of the functions $h(\zeta) = \log\frac{1}{|z-\zeta|}$ for $z \in \mathbf{C}\setminus\Omega$ and their first order derivatives are dense in $HL^1(\Omega)$ [121], so (6.1) is equivalent to

$$U^\Omega = U^\mu \quad \text{on } \mathbf{C}\setminus\Omega, \tag{6.2}$$

$$\nabla U^\Omega = \nabla U^\mu \quad \text{on } \mathbf{C}\setminus\Omega. \tag{6.3}$$

Here (6.3) is a consequence of (6.2) except possibly at certain singular points on $\partial\Omega$.

In terms of the above notations Ω is a quadrature domain for analytic functions if and only if (6.3) alone holds (because this equation is the same as $\hat{\chi}_\Omega = \hat{\mu}$, cf. (2.5)). Also, Ω is a quadrature domain for subharmonic functions if and only if (6.2) holds together with

$$U^\Omega \leq U^\mu \quad \text{in all } \mathbf{C}. \tag{6.4}$$

In this case (6.3) follows automatically because $U^\mu - U^\Omega$ attains its minimum on $\mathbf{C} \setminus \Omega$.

There is an interesting difference between allowing signed measures μ in (6.1) or just positive ones: quadrature domains for signed measures with support in a small disc are very flexible, whereas those for positive measures are subject to strong geometric restrictions. For example, the following is true: given any disc $\mathbf{D}(a, r)$ (think of $r > 0$ as being small) and any smoothly bounded domain $D \supset \overline{\mathbf{D}(a, r)}$ one can find domains Ω arbitrarily close to D (with respect to Hausdorff distance for example), which are quadrature domains for harmonic functions with respect to signed measures of the form (5.1) and with support in $\mathbf{D}(a, r)$ [49], [131]. See also [12].

On the other hand, if μ is a positive measure with support in $D(a, r)$ and $R \geq 2r$, where R is defined by $\pi R^2 = \int d\mu$, then any quadrature domain for harmonic functions for Ω is actually a subharmonic quadrature domain and is almost circular, for example $\mathbf{D}(a, R - r) \subset \Omega \subset \mathbf{D}(a, R + r)$. See [130].

Assume that (6.1) holds for harmonic h and a signed measure μ. Let $K \subset \Omega$ be a compact set containing supp μ. Then (6.1) continues to hold if μ is swept, by classical balayage, to ∂K. The question naturally arises whether the swept measure on ∂K will be positive if K is chosen large enough. This question was solved in the affirmative in [62] in the two-dimensional case, but the corresponding question in higher dimensions remains open. See [150] for some recent progress.

7. Free boundary problems and PDE

Keeping the notations of the previous section, the difference

$$u = U^\mu - U^\Omega$$

is sometimes called the **modified Schwarz potential** of $\partial\Omega$ when (6.2), (6.3) hold. It satisfies

$$\begin{cases} \Delta u = 1 - \mu & \text{in } \Omega, \\ u = |\nabla u| = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

The Schwarz potential itself (i.e., nonmodified) is

$$w(z) = \frac{1}{2}|z|^2 - 2u(z).$$

It is harmonic in $\Omega \setminus \operatorname{supp} \mu$, agrees together with its first derivatives with $\frac{1}{2}|z|^2$ on $\partial\Omega$ and it is a real-valued potential of the Schwarz function $S(z)$ in the sense that

$$2\frac{\partial w}{\partial z} = S(z).$$

The system (7.1) can be looked upon in several ways. If Ω is given, but not μ, then considering (7.1) in a small neighbourhood (in Ω) of $\partial\Omega$ we have the elliptic equation $\Delta u = 1$ with the Cauchy data $u = |\nabla u| = 0$ on $\partial\Omega$. This is an ill-posed problem which admits a local solution if and (essentially) only if $\partial\Omega$ is analytic ("if" by the Cauchy-Kovalevskaya theorem, "(essentially) only if" by regularity theory of free boundaries [88], [15], [16], [39], [127], [129]). When this local solution u exists it is natural to try to extend it as far as possible, satisfying $\Delta u = 1$. This is never possible throughout Ω (since, by Green's formula, $\int_\Omega \Delta u dA = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds = 0$), so eventually one has to allow for some singularities, in (7.1) represented by μ. The ideal case is of course that the singularities can be confined to finitely many points, as in (5.1), and in general it is natural to look for μ in (7.1) having as small support as possible. Having achieved this one may think of μ as a potential theoretic skeleton of Ω. See for example [163], [90], [91], [132], [50], [149] for constructions of such a μ using essentially real variable methods (potential theory).

If $\partial\Omega$ is globally analytic in a good enough sense (e.g., is algebraic) another way to try to construct μ, or $\operatorname{supp} \mu$, is to complexify the whole picture, from $\mathbf{R}^2$ (or, more generally, $\mathbf{R}^N$) to $\mathbf{C}^2$ (resp. $\mathbf{C}^N$) and to consider (7.1) near $\partial\Omega$ as a holomorphic Cauchy problem for the complexified Laplacian. The complexification of $\partial\Omega$ always contains points (in the complex) which are characteristic for the Laplacian. These characteristic points create singularities for the Cauchy problem in (7.1) (cf. [92]) and the idea is that these singularities propagate along bicharacteristics of the complex Laplacian, to hit the real at points of $\operatorname{supp} \mu$. In the case of an ellipse the latter points turn out to be the ordinary foci, so in general $\operatorname{supp} \mu$ (or some part of it) could be thought of as a generalized "focal set" of $\partial\Omega$.

Indeed, in the two dimensional case, if the boundary is an algebraic curve, already Herglotz remarked in [72] that the singularities of its analytically continued Schwarz function are the focal points of that curve, as defined earlier by Plücker – this focal concept is purely algebraic, however, and is not yet related to propagation of singularities for the holomorphic Cauchy problem.

A programme along the above lines has been advocated by D. Khavinson and H. S. Shapiro [86] and has been performed in certain cases. For example, in two dimensions it works out well [87], and G. Johnsson [76], [77] has obtain almost complete results for quadric surfaces in any number of dimensions. See further [144], [87], [84], [34], [151] and the books [146], [85]. However, for algebraic boundaries of degree higher than two this programme is fraught with difficulties (especially so in more than two dimensions) since in that setting even the purely algebraic-geometric theory of focal sets is largely undeveloped.

Returning to (7.1) in general, another way to view it is to consider μ as given. Then u is uniquely determined by only the Dirichlet data in (7.1), i.e., without the condition $|\nabla u| = 0$ on $\partial\Omega$. Thus in order to have also this condition satisfied, $\partial\Omega$ must be able to adjust itself. In other words, we have a free boundary problem for $\partial\Omega$.

If we add the requirement $u \geq 0$, which means that Ω is to be a quadrature domain for subharmonic functions, then this free boundary problem is an "obstacle problem". In order to explain the terminology, choose a function ψ (representing the obstacle) satisfying $\Delta\psi = \mu - 1$, e.g.,

$$\psi(z) = -\frac{1}{4}|z|^2 + \frac{1}{2\pi}\int \log|z-\zeta| d\mu(\zeta).$$

The obstacle problem in question is the problem of finding the smallest superharmonic function v satisfying $v \geq \psi$. Equivalently, the function v is to minimize the energy $\int_D |\nabla v|^2 dA$ among all $v \geq \psi$ which agree with ψ outside a sufficiently large disc D.

The above obstacle problem has a unique solution v [89], [39], [114], [122], and in terms of it, $u = v - \psi$ solves (7.1) with Ω given by

$$\Omega = \{v > \psi\} = \{u > 0\},$$

provided this open set covers the support of μ ($\operatorname{supp}\mu \subset \Omega$). The latter is not always the case, but it holds if μ is big enough on its support, e.g., if $\mu \geq 0$ is of the form (5.1).

The free boundary point of view on quadrature domains has been particularly emphasized and developed by H. Shahgholian, and also L. Karp. See for example [137], [83], [17].

8. Quadrature domains for arc length and quadrature surfaces

A variant in the definition of a quadrature domain is to replace Lebesgue measure in Ω by arc length measure ds on $\partial\Omega$:

$$\int_{\partial\Omega} f\, ds = \sum_{k=1}^{n} c_k f(a_k). \tag{8.1}$$

Such identities were studied already in [8].

Let $T(z) = \frac{dz}{ds}$ denote the unit tangent vector on $\partial\Omega$, oriented so that Ω is to the left. Then inserting $ds = \frac{dz}{T(z)}$ in (8.1) one finds that Ω is a quadrature domain for arc length (8.1) if and only if $1/T(z)$ has a meromorphic extension to all of Ω. The definition of $T(z)$ can be written formally as

$$\frac{1}{T(z)}\sqrt{dz} = \sqrt{d\bar{z}} \quad \text{along } \partial\Omega.$$

Here, the notion of a half-order differential (like $\sqrt{dz}$) can be made precise [134] and the above statement can be reformulated: Ω is a quadrature domain for arc length if and only if the half-order differential $\sqrt{dz}$ has a meromorphic extension (as a half-order differential) to the Schottky double $\hat{\Omega}$ [46].

In terms of conformal maps $g : D \to \Omega$ from a standard domain D this gives that Ω is a quadrature domain for arc length if and only if $\sqrt{dg}$ extends meromorphically as a half-order differential to $\hat{D}$. In the simply connected case with $D = \mathbf{D}$, the unit disc, the statement becomes very explicit: $\Omega = g(\mathbf{D})$ is such a quadrature domain if and only if g' is the square of a rational function. This result was first obtained in [148].

The free boundary problem corresponding to quadrature identities for arc length and their counterpart in higher dimensions, **quadrature surfaces** [141], is called the Bernoulli problem (because of an interpretation of it within hydrodynamics). The equations replacing (7.1) are

$$\begin{cases} \Delta u = -\mu \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ |\nabla u| = 1 \quad \text{on } \partial\Omega. \end{cases}$$

Similar remarks apply to the above system as to (7.1), e.g., given μ, solutions (u, Ω) can be constructed by certain variational methods. Indeed, for $\mu > 0$ of the form (5.1) an open set Ω satisfying (8.1) is obtained as $\Omega = \{u > 0\}$ where $u \geq 0$ is any minimizer of the nonconvex functional

$$F(u) = \int_{\mathbf{C}} (|\nabla u|^2 + \chi_{\{u>0\}}) dA - 2 \int u d\mu.$$

See [142], [61]. Techniques using methods of sub- and supersolutions can also be used [71].

9. Inverse problems

As a model case, we consider quadrature identities for harmonic functions as in (6.1). Corresponding to the two points of view discussed after (7.1), namely as attempts of maps $\mu \mapsto \Omega$ ("balayage") respectively $\Omega \mapsto \mu$ ("inverse balayage"), there are two natural uniqueness questions:

1) To what extent is Ω uniquely determined by μ?

2) To what extent is μ uniquely determined by Ω?

Let us consider the second question first. If we require μ to be of the form (5.1) then certainly μ is uniquely determined by Ω (and the relation (6.1)). On the other hand, quadrature domains for measures as in (5.1) are exceptionally rare, and it is natural to try to relax (5.1) while retaining some degree of uniqueness. Attempts in this direction were made in [50], leading to concepts of potential theoretic skeletons, or "**mother bodies**", which are reasonably unique when they exist. The basic requirements are that Ω should be a quadrature domain for subharmonic functions for μ, that $\operatorname{supp} \mu$ should have Lebesgue measure zero and that it should not disconnect any part of Ω from the complement of $\overline{\Omega}$. As an example, convex polyhedra (in any number of dimensions) turn out to have unique mother bodies, while nonconvex polyhedra have at most finitely many of them [58]. (It should be remarked that the definition of a mother body allows the support of μ to reach $\partial\Omega$).

Question 1) above is closely related to general inverse problems in potential theory, e.g. the following (see [162], [75] for more information):

Do there exist two different domains Ω_1 and Ω_2 such that

$$U^{\Omega_1} = U^{\Omega_2} \quad \text{outside } \Omega_1 \cup \Omega_2\,? \tag{9.1}$$

By (6.2), a negative answer to 1) implies a negative answer to (9.1), and conversely, at least under some mild extra assumptions, every counterexample for (9.1) can be made into an example of nonuniqueness for 1) (compare [48], [150]).

It is easy to give counterexamples for (9.1) if one allows multiply connected domains (a disc and a concentric annulus with the same area will do), but as far as we know there are no examples with $\overline{\Omega}_j$ both having connected complements (if Ω_j are just simply connected, which is a little weaker, there are examples [124]). However, for several similar questions [71] (arc length instead of area measure), [99] (bodies of nonconstant density) there do exist counterexamples, so we do not really expect any uniqueness for question 1). Still it would be of interest to have a definite answer to the question whether there can be two different simply connected quadrature domains both satisfying the same identity (2.1). Further results on the uniqueness question can be found in [121], [125], [48], [139], for example.

A weaker version of (9.1) is obtained by asking the potentials to agree only in a neighbourhood of infinity. This is equivalent to asking the complex moments to agree

$$\int_{\Omega_1} z^m dA = \int_{\Omega_2} z^m dA \quad \text{for all } m = 0, 1, 2, \dots.$$

For this uniqueness question there are explicit counterexamples (with $\mathbf{C} \setminus \overline{\Omega}_j$ connected) in form of circular polygons [117], [162].

If one asks all the real moments $\int_{\Omega_j} x^m y^n dA$ to agree, or equivalently

$$\int_{\Omega_1} z^m \bar{z}^n dA = \int_{\Omega_2} z^m \bar{z}^n dA \quad \text{for all } m, n = 0, 1, 2, \dots,$$

then certainly $\Omega_1 = \Omega_2$ (up to nullsets). In case Ω is a (classical) quadrature domain (2.1) then Ω can be effectively recovered from the knowledge of these moments by an algorithm based on the exponential transform. See [56].

10. Fluid dynamics (Hele-Shaw flow)

In [23] D. Crowdy discusses connections between quadrature domains and several different problems in fluid dynamics. Here we shall just look at one of these, the Hele-Shaw flow problem, which is the one which so far has been most important.

A Hele-Shaw evolution is, in the simplest case, a family of domains $\Omega(t)$ containing the origin (say) and which develops in time t according to the rule that the normal velocity of the boundary $\partial\Omega(t)$ is proportional to the normal derivative of the Green function (with logarithmic pole at the origin), in other words, proportional to the density of harmonic measure. (Thus Hele-Shaw evolution may be named "motion by harmonic measure".) Physically it models the growth of a blob of a viscous fluid (like oil) confined within a thin gap between two parallel planes and when more fluid is injected (or sucked) at one point (the origin).

One easily finds that the Hele-Shaw evolution is characterized by

$$\frac{d}{dt}\int_{\Omega(t)} h\,dA = qh(0) \tag{10.1}$$

holding for any harmonic function h defined in a neighbourhood of $\overline{\Omega(t)}$. Here $q > 0$ is the strength of the source. Allowing $q < 0$ gives the corresponding problem with a sink instead of a source. In any case it follows that the complex moments of $\Omega(t)$ are preserved quantities:

$$\frac{d}{dt}\int_{\Omega(t)} z^m\,dA = 0$$

for any $m \geq 1$. This property was discovered in the seminal paper [112] by S. Richardson.

The Hele-Shaw problem is that of finding $\{\Omega(t) : 0 \leq t < T\}$ (with T as large as possible) when $\Omega(0)$ is given. This problem turns out to be well-posed when $q > 0$, for example it has a unique global ($T = +\infty$) solution in the weak formulation that

$$\int_{\Omega(t)} h\,dA - \int_{\Omega(0)} h\,dA = qth(0) \tag{10.2}$$

holds for all h harmonic in a neighbourhood of $\overline{\Omega(t)}$. Even better, the inequality $\geq$ holds for all subharmonic h ((10.1) can be amplified in the same way), and this is really the standard requirement on the weak solution.

The above discussion about weak solution only applies for $q > 0$. When $q < 0$ the Hele-Shaw problem is ill-posed and not yet fully understood.

From (10.2) it is clear how Hele-Shaw flow is related to quadrature domains: if $\Omega(0)$ is a quadrature domain (2.1), then the $\Omega(t)$ remain quadrature domains for all t, with the origin as a new quadrature point if it was not already such a point from start.

The literature on Hele-Shaw flow is vast, see [42] for a "complete" bibliography up to 1998. A short selection is [69], [70], [41], [115], [112], [35], [28], [45], [73], [156], [94], [153], [68], [63].

11. Unbounded quadrature domains

Nothing prevents us from allowing also unbounded domains Ω in (2.1). Then there are even simpler quadrature domains than discs. Indeed, there is a whole class of

null quadrature domains, i.e., domains Ω for which

$$\int_\Omega f dA = 0 \tag{11.1}$$

holds for all integrable analytic functions f in Ω. This class consists of half-spaces, exteriors of ellipses, exteriors of parabolas and some degenerate cases. See [120] for a complete classification. In higher dimensions there are similar results [40], but they are less complete.

Null quadrature domains are relevant for Hele-Shaw flow problems in which the fluid occupies a full neighbourhood of infinity, e.g., they exactly comprise those initial fluid domains which can be completely emptied by suction at infinity [74], [29], [156].

Null quadrature domains also come up when investigating regularity of free boundaries, namely when finite boundary points to be investigated are "blown up" under scale changes. See [83], [17] for example.

In general, if a quadrature domain (satisfying (2.1) or (6.1)) is not bounded it must occupy a good portion of a neighbourhood of infinity in the sense that

$$\int_\Omega \frac{dA}{1+|z|^2} = +\infty.$$

See [121], and for the corresponding condition in higher dimensions [136]. More specifically, the unbounded quadrature domains which are not dense in the complex plane turn out to be exactly those domains which can be obtained from bounded quadrature domains by inversion $z \mapsto 1/(z-a)$ at a point a on the boundary or in the interior [145], [146], [129].

A substantial theory for general unbounded quadrature domains (in any number of dimensions) has been elaborated by L. Karp and A. Margulis [82]. It is partly based on modifications of the Newtonian kernel at infinity, to make it integrable there [120], [79].

12. Special points

If Ω is a quadrature domain (2.1) and $a \in \Omega$ is a point distinct from the quadrature nodes a_k then $\Omega \setminus \{a\}$ is usually not a quadrature domain because the deletion of the point a allows for the new test function $f(z) = \frac{1}{z-a}$. In case $\Omega \setminus \{a\}$ does remain a quadrature domain then a is called a **special point**.

It is fairly immediate that $a \in \Omega$ is a special point if and only if (2.5) or (3.2) remains to hold at $z = a$. Also, as already mentioned, the special points exactly constitute the difference set in (3.3).

The last statements above referred to quadrature domains for analytic functions. Special points in the case of quadrature domains for harmonic functions are defined similarly, and they constitute the set of points in $\Omega \setminus \operatorname{supp} \mu$ at which (6.2) and (6.3) remain to hold.

Special points turn out to be important for questions of uniqueness. For example, if for a quadrature domain Ω for harmonic functions (6.1) there are no special points at all, then this Ω is the unique quadrature domain for the measure μ in question. See [121], section 9 (and also [48]).

The special points also provide additional information for determining the polynomial $Q(z,w)$ in (3.3) from the knowledge of the (generally insufficient) data $\{(a_k, c_k)\}$ in (2.1). See [47], [21], [22].

The number of special points were estimated in [47] and [126] after some initial conjectures in [145], where also the terminology was coined.

13. Operator theory and the exponential transform

An operator T on a Hilbert space is called hyponormal if its self-commutator $[T^*, T] = T^*T - TT^*$ is a positive operator [96]. Thirty years ago R. W. Carey and J. D. Pincus [18] found that if $[T, T^*]$ moreover has rank one then T can be characterized up to unitary equivalence by a function $0 \le \rho \le 1$, called the principal function and related to T by

$$\det[(T^* - \overline{w})^{-1}(T - z)(T^* - \overline{w})(T - z)^{-1}] = E_\rho(z, w)$$

for large $z, w \in \mathbf{C}$, where

$$E_\rho(z,w) = \exp[-\frac{1}{\pi}\int \frac{\rho(\zeta)\, dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}] \tag{13.1}$$

is the **exponential transform** of ρ.

In [102], [104] M. Putinar discovered a beautiful connection between operator theory and the theory of quadrature domains (this was actually an independent rediscovery of quadrature domains): $E_\rho(z, w)$ is a rational function of the form

$$E_\rho(z,w) = \frac{Q(z,w)}{P(z)\overline{P(w)}}$$

for large z and w if and only if $\rho = \chi_\Omega$ where Ω is a quadrature domain. Then Q is the same as in (3.3) and P is the same as in (2.4)

Using the exponential transform and results from operator theory more insight into the nature of quadrature domains has been gained. For example, the equation $Q(z, \overline{z}) = 0$ for the boundary can be written in the lemniscate form

$$|P(z)|^2 = \sum_{k=0}^{n-1} |Q_k(z)|^2,$$

where $P(z)$ is the same as above (and in (2.4)) and, for each $0 \le k \le n-1$, $Q_k(z)$ is a polynomial of degree k. Here $Q_{n-1}(z)$ is up to a constant factor the same as $Q(z)$ in (2.4) (see [53]).

It follows from the above that P and Q_{n-1} are directly (and bijectively) related to the quadrature data $\{(a_k, c_k)\}$ in (2.1). However, even if Ω is a quadrature domain for subharmonic functions, so that it is uniquely determined by $\{(a_k, c_k)\}$, no general method is known for effectively determining the remaining polynomials $Q_0, \dots, Q_{n-2}$ from $\{(a_k, c_k)\}$.

The exponential transform is also useful for quadrature domains in a wider sense, like (6.1) for analytic h. For example, it can be used to prove analyticity of $\partial\Omega$, see [52], thereby providing an alternative approach to [127], [128], [129]. For further results on hyponormal operators and quadrature domains, see [103], [105], [157], [161], [159], and for connections to moment problems also the survey [108].

Other classes of operators which are linked to quadrature domains in different ways are subnormal operators (those operators which can be extended to be normal operators on a larger Hilbert space), see [97], [157], [98], [160], [19], [38], [152] and the Friedrichs operator (essentially the orthogonal projection in $L^2(\Omega)$ of the analytic functions onto the antianalytic ones) [143], [146], [106], [110], [111].

14. The Bergman kernel

In terms of the Bergman kernel $K(z, w)$ for Ω the quadrature identity (2.1) becomes

$$\sum_{k=1}^{n} c_k K(z, a_k) = 1 \quad (z \in \Omega).$$

Quadrature identities for arc length (8.1) are related in the same way to the Szegö kernel. The methods of Avci [8] are largely based on these relations.

Recently, S. Bell [12], [13] has developed further the connections to the Bergman and other kernel functions. To give an example, the Bergman kernel for any domain extends to the Schottky double of the domain as a meromorphic differential:

$$K(z, a)dz = \overline{L(z, a)dz} \quad \text{along } \partial\Omega$$

($a \in \Omega$ fixed), where $L(z, a)$ is the adjoint kernel. Since, by (3.4), dz itself extends to the double if and only if Ω is a quadrature domain in the sense (3.5) it follows that that $K(z, a)$ extends meromorphically to the double as a *function* if and only if Ω is a quadrature domain in that sense. From this conclusions can be drawn concerning algebraicity properties of $K(z, w)$, for example.

The reproducing property of the Bergman kernel says that

$$f(a) = \int_\Omega f(z)\overline{K(z, a)}dA(z)$$

for all $f \in L^2_a(\Omega)$ (the Bergman space) and for any $a \in \Omega$. Choosing here $f(z) = g(z)K(z, a)$ where g is a bounded analytic function gives

$$g(a) = \frac{1}{K(a, a)} \int_\Omega g(z)|K(z, a)|^2 \, dA(z). \tag{14.1}$$

Thus we have a one-point quadrature identity for bounded analytic functions and with weight $|K(z, a)|^2$ (in place of pure Lebesgue measure). The same arguments work if $K(z, w)$ is the reproducing kernel for any subspace of $L^2_a(\Omega)$ which is invariant under multiplication by bounded analytic functions, e.g., the subspace consisting of those functions which vanish on a given finite subset Ω. In case Ω is the unit disc and $a = 0$ then (14.1) is exactly H. Hedenmalm's definition of an inner divisor in Bergman space, see e.g. [67], [66]. In this way quadrature identities (in a wide sense) are related to Hedenmalm's theory of contractive zero divisors [65], [30] [31]. (For a general domain Ω, the divisor $K(z, a)$ will possibly not be contractive, however.)

We finally remark that some of the extremal problems which initiated the theory of quadrature domains were problems of giving estimates (e.g. of the Gaussian curvature) for metrics of the kind

$$ds^2 = K(z, z)|dz|^2,$$

on Riemann surfaces, actually with $K(z, w)$ the reproducing kernel for those analytic functions which have a single-valued integral (the reduced Bergman kernel). See [118], [119] and the appendix of [121].

15. Other aspects

a) Quadrature identities of the form

$$\int_\Omega f'' dA = \sum_{k=1}^{n} c_j f(a_j)$$

for f analytic in a neighbourhood of Ω hold for polygons $\Omega \subset \mathbf{C}$, where $a_k \in \partial\Omega$ are the vertices of the polygon [24], [26]. To a limited extent such formulas can be extended to higher dimensions [54].

b) P. Ebenfelt has studied behaviour of solutions of the Dirichlet problems on quadrature domains, e.g., to what extent solutions are analytically continuable outside the domain [32], [33].

c) The ellipse is not a classical quadrature domain, but it is a quadrature domain for a measure on the segment joining the foci, and the complement of it is a null quadrature domain. Similarly in higher dimensions. Many authors have studied quadrature related properties for ellipses (and ellipsoids): [74], [29], [140], [80]. See further [146], [85].

d) For other specific types and more examples of quadrature identities and quadrature domains, see [25], [93], [154], [155], [9], [47], [95], [78], [138], [6], [7], [109], [21], [22], plus the books [26], [121], [146].

e) Studies of quadrature domains in a nonlinear setting (for the p-Laplacian) can be found in [10], [11].

16. Notations

The following notations are in common use in the theory of quadrature domains.

- $AL^p(\Omega) = L^p_a(\Omega) =$ the set of p-integrable analytic functions in Ω ($1 \le p \le \infty$).
- $HL^p(\Omega) = L^p_h(\Omega) =$ the set of p-integrable harmonic functions in Ω
- $SL^p(\Omega) =$ the set of p-integrable subharmonic functions in Ω
- $Q(\mu, AL^p) =$ the class of quadrature domains for the measure (or distribution) μ and the test class of p-integrable analytic functions.
- Similarly for $Q(\mu, HL^p)$, $Q(\mu, SL^p)$ etc. The exact definitions of these classes differ a little between different authors.

References

[1] D. Aharonov, H. S. Shapiro, *A minimal-area problem in conformal mapping - preliminary report*, Research bulletin TRITA-MAT-1973-7, Royal Institute of Technology, 34 pp.

[2] D. Aharonov, H. S. Shapiro, *Domains in which analytic functions satisfy quadrature identities*, J. Analyse Math. **30** (1976), 39–73.

[3] D. Aharonov, H. S. Shapiro, *A minimal-area problem in conformal mapping - preliminary report: Part II*, Research bulletin TRITA-MAT-1978-5, Royal Institute of Technology, 70 pp.

[4] D. Aharonov, H. S. Shapiro, A. Solynin, *A minimal area problem in conformal mapping*, J.Analyse Math. **78** (1999), 157–176.

[5] D. Aharonov, H. S. Shapiro, A. Solynin, *A minimal area problem in conformal mapping. II*, J.Analyse Math. **83** (2001), 239–259.

[6] D. Armitage, M. Goldstein, *Quadrature and harmonic approximation of subharmonic functions in strips*, J. London Math. Soc. **46** (1992), 171–179.

[7] D. Armitage, S. Gardiner, *Best one-sided L^1-approximation by harmonic and subharmonic functions*, Advances in multivariate approximation (Witten-Bommenholz, 1998), 43–56, Math. Res., 107, Wiley-VCH, Berlin, 1999.

[8] Y. Avci, *Quadrature identities and the Schwarz Function*, Doctoral disseertation, Stanford, 1977.

[9] Y. Avci, *Characterization of shell domains by quadrature identities*, J. London Math. Soc. **23** (1980), 123–128.

[10] F. Bahrami, A. Chademan, *Existence of unbounded quadrature domains for the p-Laplace operator*, Bull. Iranian Math. Soc. **14**, (1998), 623–681.

[11] F. Bahrami, A. Chademan, *Existence of a uniform bound for quadrature domains associated to p-Laplacian*, Analysis (Munich) **19**, (1999), 319–326.

[12] S. Bell, *Quadrature domains and kernel function zipping* preprint 2003.

[13] S. Bell, *The Bergman kernel and quadrature domains in the plane*, this volume.

[14] L. Bers: *An approximation theorem*, J. Analyse Math. **14** (1965), 1–4.

[15] L. A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta. Math. **139** (1977) 155–184.

[16] L. A. Caffarelli, *Compactness methods in free boundary problems*, Comm. Partial Diff. Eq. **5** (1980), 427–448.

[17] L. A. Caffarelli, L. Karp, H. Shahgholian, *Regularity of a free boundary problem with application to the Pompeiu problem*, Ann. Math. **151**, (2000), 269–292.

[18] R. W. Carey, J. D. Pincus, *An exponential formula for determining functions,* Indiana Univ. Math.J. **23** (1974), 1031-1042.

[19] J. Conway, L. Yang, *Some open problems in the theory of subnormal operators*, Holomorphic spaces (Berkeley, CA, 1995), 201–209, Math. Sci. Res. Inst. Publ., 33, Cambridge Univ. Press, Cambridge, 1998.

[20] D. Crowdy, *Multipolar vortices and algebraic curves*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **457** (2001), 2337–2359.

[21] D. Crowdy, *Constructing multiply-connected quadrature domains I: algebraic curves*, preprint 2003.

[22] D. Crowdy, J. Marshall, *Constructing multiply-connected quadrature domains II: the Schottky-Klein prime function*, preprint 2003.

[23] D. Crowdy, *Quadrature domains and fluid dynamics*, this volume.

[24] P. J. Davis, *Triangle formulas in the complex plane*, Math. of Comp. **18** (1964), 569–577.

[25] P. J. Davis, *Double integrals expressed as single integrals or interpolatory functionals*, J. Approx. Theory **5** (1972), 276–307.

[26] P. J. Davis, *The Schwarz Function and its Applications*, Carus Math. Mongraphs No.17, Math. Assoc. Amer., 1974.

[27] P. J. Davis, H. Pollak *On the analytic continuation of mapping functions*, Trans. Amer. Math. Soc. **87** (1958), 198–225.

[28] E. DiBenedetto, A. Friedman, *The illposed Hele-Shaw model and the Stefan problem for supercooled water*, Trans. Amer. Math. Soc., **282** (1984) 183–204.

[29] E. DiBenedetto, A. Friedman, *Bubble growth in porous media*, Indiana Univ. Math. J. **35** (1986), 573-606.

[30] P. Duren, D. Khavinson, H. S. Shapiro, C. Sundberg, *Contractive zero-divisors in Bergman spaces*, Pacific J. Math. **157** (1993), 37–56.

[31] P. Duren, A. Schuster, *Bergman spaces*, Math. Surveys and Monographs vol. 100, Amer. Math. Soc., 2004..

[32] P. Ebenfelt, *Singularities encountered by the analytic continuation of solutions to Dirichlet's problem*, Complex Variables **20** (1992), 75–91.

[33] P. Ebenfelt, *Analytic continuation of certain domain functions in quadrature domains and domains bounded by ellipses*, Complex Variables **22** (1993), 69–83.

[34] P. Ebenfelt, *Singularities of the solution to a certain Cauchy problem and applications to the Pompeiu problem*, Duke Math. J. **71** (1993), 119–142.

[35] C. M. Elliott, J. R. Ockendon, *Weak and Variational Methods for Moving Boundary Problems*, Pitman, London (1982).

[36] B. Epstein, *On the mean-value property of harmonic functions*, Proc. Amer. Math. Soc. **13** (1962), 830.

[37] B. Epstein, M. Schiffer, *On the mean-value property of harmonic functions*, J. Analyse Math. **14** (1965), 109–111.

[38] N. Feldman, *Subnormal operators, self-commutators and pseudocontinuations*, Integral Equations Operator Theory **37**, (2000), 402–422.

[39] A. Friedman, *Variational Principles and Free Boundaries*, Wiley and Sons, 1982.

[40] A. Friedman, M. Sakai, *A characterization of null quadrature domains in* $\mathbf{R}^n$, Indiana Univ. Math. J. **35** (1986), 607-610.

[41] L. A. Galin, *Unsteady seepage with a free surface*, Dokl. Akad. Nauk SSSR **47** (1945), 250-253. (Russian)

[42] K. A. Gillow and S. D. Howison, *A bibliography of free and moving boundary problems for Hele-Shaw and Stokes flow*, published electronically at URL http://www.maths.ox.ac.uk/˜howison/Hele-Shaw.

[43] P. G. W. Glare (ed.) *Oxford Latin Dictionary*, Clarendon Press, Oxford, 1982.

[44] B. Gustafsson, *Quadrature identities and the Schottky double*, Acta Appl. Math. **1** (1983), 209–240.

[45] B. Gustafsson, *Applications of variational inequalities to a moving boundary problem for Hele-Shaw flows*, SIAM J. Math. Anal. **16** (1985), 279–300.

[46] B. Gustafsson: *Applications of half-order differentials on Riemann surfaces to quadrature identities for arc-length*, J. Analyse Math. **49** (1987), 54–89.

[47] B. Gustafsson, *Singular and special points on quadrature domains from an algebraic geometric point of view*, J. Analyse Math. **51** (1988), 91-117.

[48] B. Gustafsson, *On quadrature domains and an inverse problem in potential theory*, J. Analyse Math. **55** (1990), 172–216.

[49] B. Gustafsson: *A distortion theorem for quadrature domains for harmonic functions*, J. Math. Anal. Appl. **202** (1996), 169-182.

[50] B. Gustafsson, *On mother bodies of convex polyhedra*, SIAM J. Math.Anal. **29:5** (1998), 1106-1117.

[51] B. Gustafsson, *Lectures on Balayage*, preprint 2003.

[52] B. Gustafsson, M. Putinar, *An exponential transform and regularity of free boundaries in two dimensions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci (4) **26** (1998), 507-543.

[53] B. Gustafsson, M. Putinar, *Linear analysis of quadrature domains II*, Israel J.Math. **119** (2000), 187–216.

[54] B. Gustafsson, M. Putinar, *On exact quadrature formulas for harmonic functions on polyhedra*, Proc. Amer. Math. Soc. **128** (1999), 1427–1432.

[55] B. Gustafsson, M. Putinar, *Linear analysis of quadrature domains IV*, this volume.

[56] B. Gustafsson, C. He, P. Milanfar, M. Putinar, *Reconstructing planar domains from their moments*, Inverse Problems **16** (2000), 1053–1070.

[57] B. Gustafsson, M. Sakai, *Properties of some balayage operators with applications to quadrature domains and moving boundary problems*, Nonlinear Anal. **22** (1994), 1221–1245.

[58] B. Gustafsson, M. Sakai, *On potential theoretic skeletons of polyhedra*, Geom. Didicata **76** (1999), 1–30.

[59] B. Gustafsson, M. Sakai, *Sharp estimates of the curvature of some free boundaries in two dimensions*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 123–142.

[60] B. Gustafsson, M. Sakai, *On the curvature of the free boundary for the obstacle problem in two dimensions*, Monatshefte für Mathematik, to appear.

[61] B. Gustafsson, H. Shahgholian, *Existence and geometric properties of solutions of a free boundary problem in potential theory*, J. Reine Angew. Math. **473** (1996), 137–179.

[62] B. Gustafsson, M. Sakai, H.S. Shapiro: *On domains in which harmonic functions satisfy generalized mean-value properties*, Potential Anal. **7** (1997), 467–484.

[63] B. Gustafsson, A. Vasilév, *Complex and Potential Analysis in Hele-Shaw cells*, Lecture notes, in preparation, 2004.

[64] V. P. Havin, *Approximation in the mean by analytic functions*, Dokl. Akad. Nauk. SSSR 178, pp. 1025–1028 (Russion; Engl. transl. in Siviet Math. Dokl. **9** (1968), 245–258.

[65] H. Hedenmalm, *A factorization theorem for square area-integrable analytic functions*, J. Reine Angew. Math. **422** (1991), 45–68.

[66] H. Hedenmalm, *Recent progress and open problems in the Bergman space*, this volume.

[67] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, **199**, Springer-Verlag, New York 2000.

[68] H. Hedenmalm, S. Shimorin, *Hele-Shaw flow on hyperbolic surfaces*, J. Math Pures Appl. **81** (2002), 187–222.

[69] H. S. Hele-Shaw, *The flow of water*, Nature **58** (1898), 34-36.

[70] H. S. Hele-Shaw, *On the motion of a viscous fluid between two parallel plates*, Trans. Royal Inst. Nav. Archit., London **40** (1898), 218.

[71] A. Henrot, *Subsolutions and supersolutions in free boundary problems* Ark. Mat. **32** (1994), 79–98.

[72] G. Herglotz, *Über die analytische Fortsetzung des Potentials ins Innere der anziehenden Massen, Gekrönte Preisschr. der Jablonowskischen Gesellsch. zu Leipzig*, 56 pp., Reproduced in 'Gustav Herglotz-Gesammelte Schriften', Vandenhoeck& Ruprecht, Göttingen, 1979, pp. 299–355.

[73] Y. Hohlov, S.D. Howison, C. Huntingford, J. R. Ockendon, A. A. Lacey, *A model for non-smooth free boundaries in Hele-Shaw flows*, Quart. J. Appl. Math. **47** (1994), 107-128.

[74] S. D. Howison, *Bubble growth in porous media and Hele Shaw flow*, Proc. Roy. Soc. Edinburgh **A102** (1985), 141-148.

[75] V. Isakov, *Inverse Source Problems*, AMS Math. Surveys and Monographs 34, Providence Rhode Island, 1990.

[76] G. Johnsson, *The Cauchy problem in* $\mathbf{C}^N$ *for linear second order partial differential equations with data on a quadric surface*, Trans. Amer. Math. Soc. **344** (1994), 1–48.

[77] G. Johnsson, *Global existence results for linear partial differential equations*, J. Differential Equations **115** (1995), 416–440.

[78] L. Karp, *Construction of quadrature domains in* $\mathbf{R}^n$ *from quadrature domains in* $\mathbf{R}^2$, Complex variables **17** (1992), 179-188.

[79] L. Karp, *Generalized Newtonian potential and its applications*, J. Math. Anal. Appl. **174** (1993), 480-497.

[80] L. Karp, *On the Newtonian potential of ellipsoids*, Complex Variables **25** (1995), 367–372.

[81] L. Karp, *Multivalued analytic continuation of the Cauchy transform*, preprint 2003.

[82] L. Karp, A. Margulis, *Newtonian potential theory for unbounded sources and applications to free boundary problems*, J. Analyse Math. **70** (1996), 1–63.

[83] L. Karp, H. Shahgholian, *Regularity of a free boundary problem*, J. Geom. Anal. **9**, (1999), 653–669.

[84] D. Khavinson, *Singularities of harmonic functions in* $\mathbf{C}^n$, Several complex variables and complex geometry, Part 3 (Santa Cruz, CA, 1989), 207–217, Proc. Sympos. Pure Math., 52, Part 3, Amer. Math. Soc., Providence, RI, 1991.

[85] D. Khavinson, *Holomorphic Partial Differential Equations and Classical Potential Theory*, Departamento de Análisis Matemático, Universidad de La Laguna, Tenerife, 1996.

[86] D. Khavinson, H. S. Shapiro, *The Schwarz potential in* $\mathbf{R}^n$ *and Cauchy's problem for the Laplace equation*, Research bulletin TRITA-MAT-1986-36, Royal Institute of Thechnology, 112 pp.

[87] D. Khavinson, H. S. Shapiro, *The Vekua hull of a plane domain*, Complex Variables **14** (1990), 117–128.

[88] D. Kinderlehrer, L. Nirenberg, *Regularity in free boundary problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4** (1977), 373–391.

[89] D. Kinderlehrer, G. Stampacchia, *Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[90] O. I. Kounchev, *The partial balayage materic bodies and optimization problems in gravimetry*, in Inverse Modeling in Exploration Geophysics, Proceedings of the 6th International Mathematical Geophysics Seminar in Berlin, Feb. 3–6, 1988 (eds. A. Vogel, R. Gorenflo, B. Kummer, C.O. Ofoegbu), Vieweg & Sohn, Braunschweig/Wiesbaden.

[91] O. I. Kounchev, *Obtaining materic bodies through concentration and optimization of a linear functional*, in Geophysical Data Inversion Methods and Applications, Proceedings of the 7th International Mathematical Geophysics Seminar in Berlin, Feb. 8–11, 1989 (eds. A. Vogel, R. Gorenflo, C.O. Ofoegbu, B. Ursin), Vieweg & Sohn, Braunschweig/Wiesbaden.

[92] J. Leray, *Uniformisation de la solution du problème linéaire analyticque de Cauchy, près de la varieté qui porte les données de Cauchy*, Bull. Soc. Math. France **85**, 389–430.

[93] A. A. Levin, *An example of a doubly connected domain which admits a quadrature identity*, Proc. Amer. Math. Soc. **60** (1976), 163–168.

[94] A. S. Margulis, *The moving boundary problem of potential theory*, Adv. Math. Sci. Appl. **5(2)** (1995), 603-629.

[95] R. Marrero, I. Maria, *Analytic characterization of annuli*, Rev. Acad. Canaria Cienc. **1** (1990), 147–153.

[96] M. Martin, M. Putinar, *Lectures on Hyponormal Operators*, Birkhäuser, Basel, 1989.

[97] J. McCarthy, L. Yang, *Cyclic subnormal operators with finite-rank self-commutators*, Proc. Roy. Irish Acad. Sect. A **95** (1995), 173–177.

[98] J. McCarthy, L. Yang, *Subnormal operators and quadrature domains*, Adv. Math. **127** (1997), 52–72.

[99] N. S. Nadirashvili, *Universal classes of uniqueness of domains in the inverse problem of Newtonian potential theory*, Soviet Math. Dokl. **44** (1992), 287–290.

[100] C. Neumann, *Über das logarithmische Potential einer gewissen Ovalfläche*, Abh. der math.-phys. Klasse der Königl. Sächs. Gesellsch. der Wiss. zu Leibzig 59 (1907), pp. 278–312.

[101] C. Neumann, *Über das logarithmische Potential einer gewissen Ovalfläche*, Zweite Mitteilung, ibib. vol. 60 (1908), pp. 53–56. Dritte Mitteilung, ibid. pp. 240–247.

[102] M. Putinar, *On a class of finitely determined planar domains*, Math. Res. Lett. **1** (1994), 389–398.

[103] M. Putinar, *Linear analysis of quadrature domains*, Ark. Mat. **33** (1995), 357–376.

[104] M. Putinar, *Extremal solutions of the two-dimensional L-problem of moments*, J.Funct.An. **136** (1996), 331–364.

[105] M. Putinar, *Extremal solutions of the two-dimensional L-problem of moments II*, J.Approx.Th. **92** (1998), 38–58.

[106] M. Putinar, *Matrix analysis and the Friedrichs operator of a quadrature domain*, Linear Algebra Appl. **270** (1998), 215–229.

[107] M. Putinar, *Linear analysis of quadrature domains III*, Math. Anal. Appl. **239** (1999), 101–117.

[108] M. Putinar, *A renormalized Riesz transform and applications*, Advances in Constructive Approximation, to appear.

[109] G. Putinar, M. Putinar, *Root separation on generalized lemniscates*, Hokkaido Math. J. **30** (2001), 705–716.

[110] M. Putinar, H. S. Shapiro, *The Friedrichs operator of a planar domain*, Complex analysis, operators, and related topics, 303–330, Oper. Theory Adv. Appl. **113**, Birkhäuser, Basel, 2000.

[111] M. Putinar, H. S. Shapiro, *The Friedrichs operator of a planar domain II*, Recent advances in operator theory and related topics (Szeged, 1999), 519–551, Oper. Theory Adv. Appl. **127**, Birkhäuser, Basel, 2001.

[112] S. Richardson, *Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel*, J. Fluid Mech. **56** (1972), 609-618.

[113] S. Richardson, *Hele-shaw flows with time-dependent free boundaries involving a multiply connected fluid region*, European J. Appl. Math.. **12** (2001), 571–599.

[114] J. F. Rodrigues, *Obstacle problems in Mathematical Physics*, North-Holland, Amsterdam 1987.

[115] P. G. Saffman, F. P. Taylor, *The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous fluid*, Proc. Royal Soc. London Ser. A **245** (1958), 312-329.

[116] M. Sakai, *On basic domains of extremal functions*, Kodai Math. Sem. Rep. **24** (1972), 251-258.

[117] M. Sakai, *A moment problem on Jordan domains*, Proc. Amer. Math. Soc. **70** (1978), 35–38.

[118] M. Sakai, *Analytic functions with finite Dirichlet integrals on Riemann surfaces*, Acta Math. **142** (1979), 199-220.

[119] M. Sakai, *The submeanvalue property of subharmonic functions and its application to the estimation of the Gaussian curvature of the span metric*, Hiroshima Math. J. **9** (1979), 555-593

[120] M. Sakai, *Null quadrature domains*, J.Analyse Math. **40** (1981), 144-154.

[121] M. Sakai, *Quadrature Domains*, Lect. Notes Math. **934**, Springer-Verlag, Berlin-Heidelberg 1982.

[122] M. Sakai, *Application of variational inequalities to the existence theorem on quadrature domains*, Trans. Amer. Math. Soc. **276** (1983), 276 267–279.

[123] M. Sakai, *Solutions to the obstacle problem as Green potentials*, J. Analyse Math. **44** (1984/85) 97–116.

[124] M. Sakai, *Domains having null complex moments*, Complex Variables **7** (1987), 313–319.

[125] M. Sakai, *Finiteness of the family of simply connected quadrature domains*, pp. 295–305 in "Potential Theory" (editors J. Kral, I. Netuka, J. Vesely), Plenum Publishing Corporation, 1988.

[126] M. Sakai, *An index theorem on singular points and cusps of quadrature doamins*, Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986), 119–131, Math. Res. Inst. Publ., 10, Springer, New York, 1988.

[127] M. Sakai, *Regularity of boundary having a Schwarz function*, Acta Math. **166** (1991), 263–297.

[128] M. Sakai, *Regularity of free boundaries in two dimensions*, Ann. Scuola Norm Sup. Pisa Cl. Sci. (4) **20** (1993), 323–339.

[129] M. Sakai: *Regularity of boundaries of quadrature domains in two dimensions*, SIAM J. Math. Anal. **24** (1994), 341–364.

[130] M. Sakai, *Sharp estimates of the distance from a fixed point to the frontier of a Hele-Shaw flow*, Potential Anal. **8** (1998), 277-302.

[131] M. Sakai, *Linear combinations of harmonic measures and quadrature domains of signed measures with small supports*, Proc. Edinburgh Math. Soc. (2) **42** (1999), 433–444.

[132] T. Savina, B. Sternin, V. Shatalov, *Notes on "mother body" problem in geographics*, preprint 1995.

[133] F. Schottky *Über die conforme Abbildung mehrfach zusammenhängender ebener Flächen*, Crelles Journal **83** (1877), 300–351.

[134] M. Schiffer, N. S. Hawley, *Half-order differentials on Riemann surfaces*, Acta Math. **115** (1966), 199–236.

[135] M. Schiffer, D. C. Spencer, *Functionals of Finite Riemann Surfaces*, Princeton Univ. Press, Princeton, 1954.

[136] H. Shahgholian, *Unbounded quadrature domains in* $\mathbf{R}^n$ $(n \geq 3)$, J.Analyse Math. **57** (1991), 281–291, 297–298.

[137] H. Shahgholian, *On quadrature domains and the Schwarz potential*, J. Math. Anal. Appl. **171** (1992), 61–78.

[138] H. Shahgholian, *A characterization of the sphere in terms of single-layer potentials*, Proc. Amer. Math. Soc. **115** (1992), 1167–1168.

[139] H. Shahgholian, *Convexity and uniqueness in an inverse problem of potential theory*, Proc. Amer. Math. Soc. **116** (1992), 1097–1100.

[140] H. Shahgholian, *On the Newtonian potential of heterogeneous ellipsoids*, SIAM J. Math. Anal. **24** (1993), 341–364.

[141] H. Shahgholian, *Quadrature surfaces as free boundaries* Ark. Mat. **32** (1994), 475–492.

[142] H. Shahgholian, *Existence of quadrature surfaces for positive measures with finite support* Potential Analysis **3** (1994), 245–255.

[143] H. S. Shapiro, *Domains allowing exact quadrature identities for harmonic functions – an approach based on p.d.e.*, Anniversary volume on approximation theory and functional analysis (Oberwolfach, 1983), 335–354, Internat. Schriftenreihe Numer. Math., 65, Birkhäuser, Basel-Boston, Mass., 1984.

[144] H. S. Shapiro, *Global aspects of the Cauchy's problem for the Laplace operator*, Geometrical and algebraical aspects in several complex variables (Cetraro, 1989), 309–324, Sem. Conf., 8, EditEl, Rende, 1991.

[145] H. S. Shapiro, *Unbounded quadrature domains*, Complex analysis, I (College Park, Md., 1985-86), 287-331, Lecture Notes in Math., 1275, Springer-Verlag, Berlin 1987.

[146] H. S. Shapiro, *The Schwarz function and its generalization to higher dimensions*, Uni. of Arkansas Lect. Notes Math. Vol. 9, Wiley, New York, 1992.

[147] H. S. Shapiro, *Quasi-balayage and a priori estimates for the Laplace operator*, Multivariate approximation (Witten-Bommerholz 1996), 203-230, 231-254, Math. Res. **101**, Akademie Verlag, Berlin, 1997.

[148] H. S. Shapiro, C. Ullemar, *Conformal mappings satisfying certain extremal properties, and associated quadrature identities*, Research bulletin TRITA-MAT-1981-6, Royal Institute of Technology, 40 pp.

[149] T. Sjödin, *Mother body for the ellipse*, manuscript.

[150] T. Sjödin, *Quadrature identities and deformation of quadrature domains*, this volume.

[151] B. Sternin, V. Shatalov, *On the problem of balayage in the space* $\mathbf{R}^n$, (Russian) Mat. Zametki **54** (1993), 90–112, 160; translation in Math. Notes **54** (1993), 1246–1260.

[152] S. A. Stewart, D. Xia, *A class of subnormal operators with finite rank self-commutators*, Integral Equations Operator Theory **44** (2002), 370–382.

[153] F. R. Tian, *A Cauchy integral approach to Hele-Shaw flow problems with a free boundary: the case of zero surface tension*, Arch. Rational Mech. Anal. **135** (1996), 175–195.

[154] C. Ullemar, *Symmetric plane domains satisfying two-point quadrature identities for analytic functions*, Research bulletin TRITA-MAT-1977-24, Royal Institute of Technology, 23 pp.

[155] C. Ullemar, *A uniqueness theorem for domains satisfying a quadrature identity for analytic functions*, Research bulletin TRITA-MAT-1980-37, Royal Institute of Technology, 59 pp.

[156] A. N. Varchenko, P. I. Etingof, *Why the Boundary of a Round Drop Becomes a Curve of Order Four*, American Mathematical Society AMS University Lecture Series, Volume 3, Providence, Rhode Island 1992.

[157] D. Xia, *Hyponormal operators with finite rank self-commutators and quadrature domains*, J. Math. Anal. Appl. **203** (1996), 540–559.

[158] D. Xia, *On pure subnormal operators with finite rank self-commutators and related operator-tuples*, Integral Equations Operator Theory **24** (1996), 106–125.

[159] D. Xia, *Trace formulas for some operators related to quadrature domains in Riemann surfaces*, Integral Equations Operator Theory **47** (2003), 123–130.

[160] D. V. Yakubovich, *Subnormal operators of finite type, II Structure theorems*, Rev. Mat. Iberoamericana **14** (1998), 623–681.

[161] D. V. Yakubovich, *A note on hyponormal operators and associated with quadrature domains*, Operator theory, system theory and related topics Beer-Sheva/Rehovot, 1997), 513–525, Oper. Theory Adv. Appl., **123**, Birkhäuser, Basel, 2001.

[162] L. Zalcman, *Some inverse problems of potential theory*, Contemp. Math. **63** (1987), 337-350.

[163] D. Zidarov, *Inverse Gravimetric Problem in Geoprospecting and Geodesy*, Elsevier, Amsterdam, 1990. (First edition 1968, in Russian.).

Department of Mathematics, The Royal Institute of Technology, S-10044 Stockholm, Sweden

E-mail address: `gbjorn@math.kth.se`

E-mail address: `shapiro@math.kth.se`