Geometriae Dedicata **76:** 1–30, 1999.

On Potential Theoretic Skeletons of Polyhedra

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(Received: 19 March 1998)

Abstract. We prove that any polyhedron in two dimensions admits a type of potential theoretic skeleton called *mother body*. We also show that the mother bodies of any polyhedron in any number of dimensions are in one-to-one correspondence with certain kinds of decompositions of the polyhedron into convex subpolyhedra. A consequence of this is that there can exist at most finitely many mother bodies of any given polyhedron. The main ingredient in the proof of the first mentioned result consists of showing that any polyhedron in two dimensions contains a convex subpolyhedron which *sticks to* it in the sense that every face of the subpolyhedron has some part in common with a face of the original polyhedron.

Mathematics Subject Classifications (1991): Primary: 52Bxx; Secondary: 31Bxx.

Key words: polyhedron, polygon, medial axis, symmetric axis, decomposition, face, mother body, skeleton, ridge, gravitational field, Newtonian potential.

1. Introduction

This paper is part of a larger programme which concerns existence, uniqueness and structure of a kind of potential theoretic skeletons, called *mother bodies*, for heavy bodies. By a (heavy) body we mean a compact subset of $\mathbb{R}^N$ provided with a mass distribution, typically just Lebesgue measure restricted to the set. A mother body for it is a more concentrated mass distribution sitting inside the body and producing the same external gravitational field as the latter.

Several notions of potential theoretic skeletons have been discussed in the literature. The particular term 'mother body' dates back at least to the work of the Bulgarian geophysicist Dimiter Zidarov in the 1960's. In his book [Zi] it was defined in somewhat vague terms, comparing with mathematical standards. Suggestions for precise requirements of a mother body in the spirit of [Zi] were given in [Gu1], where also a first step in the above mentioned programme was taken by proving that convex polyhedra always have unique mother bodies in the sense made precise.

In this paper, we continue the programme by treating general polyhedra. Our main result is existence of at least one mother body for any polyhedron in two dimensions. For polyhedra in arbitrary dimension, we prove a structure theorem saying that the mother bodies are in one-to-one correspondence with certain types of decompositions of the polyhedron into convex subpolyhedra. A consequence of this is that there can be at most finitely many mother bodies of any polyhedron. However, the question of existence of mother bodies for arbitrary polyhedra in higher dimensions is still open.

Our approach to the existence question leads to geometric questions for polyhedra of possible independent interest. Indeed, we are led to trying to prove that any polyhedron contains a convex subpolyhedron which *sticks to* it in the sense that every face of the subpolyhedron has some part in common with some face of the original polyhedron. We are able to provide such a proof in two dimensions, and that is the main ingredient in the existence proof for mother bodies. In higher dimensions we neither have a proof, nor a counterexample, for existence of convex polyhedra sticking to a general polyhedron.

The organization of the paper is as follows. In Section 2 we give precise definitions of the geometric concepts needed, namely polyhedron, face and sticking of polyhedra. Section 3, which essentially is a summary of [Gu1], gives the necessary potential theoretic background and in particular contains the precise definition of a mother body. The main substance of the paper then is contained in Sections 4 and 5: in Section 4 we prove the above-mentioned structure and finiteness results (in arbitrary dimension) and in Section 5 we prove, in two dimensions, existence of convex subpolyhedra sticking to an arbitrary polyhedron. We even prove that the convex subpolyhedra sticking to the given polyhedron cover the whole boundary of the latter. On the other hand, we give an example showing that they need not cover all of the interior.

As indicated, this work is much inspired by the work in geophysical potential theory by D. Zidarov and his collaborators. It also has strong connections to the theory of quadrature domains (see [Sa], [Sh] for overviews). Indeed, having a mother body of a given body means that the integral of harmonic functions over the body reduces to an integral over the more condensed mother body and thus to a, possibly effective, ‘quadrature formula’ (for harmonic functions). A different sort of quadrature formula for polyhedra in two dimensions is discussed in [Da, Ch. 11]. See also [Gu-Pu].

1.1. SOME GENERAL NOTATION

$$\begin{aligned}
B(x,r) &= \{y \in \mathbb{R}^N : |y-x| < r\},\\
\operatorname{int} P &= \text{the interior in } \mathbb{R}^N \text{ of a set } P \subset \mathbb{R}^N,\\
\overline{P} &= \text{the closure in } \mathbb{R}^N \text{ of a set } P \subset \mathbb{R}^N,\\
\partial P &= \text{the boundary in } \mathbb{R}^N \text{ of a set } P \subset \mathbb{R}^N,\\
P^c &= \mathbb{R}^N \setminus P \text{ for } P \subset \mathbb{R}^N,\\
\operatorname{supp} \mu &= \text{the closed support of a measure } \mu.
\end{aligned}$$

More special notations are explained within the text (mainly in Sections 2 and 3).

2. Preliminaries on Polyhedra

For a general background on polyhedra, see, e.g., [Gr], [Ro], [Zg]. This paper, however, is self-contained as concerns questions of polyhedra. We shall use the following definition.

DEFINITION 2.1. A *convex polyhedron* in R^N is a set of the form

$$K = \bigcap_{i=1}^{n} H_i, \tag{2.1}$$

where the H_i are closed half-spaces in R^N, and which satisfies

$$\operatorname{int} K \neq \varnothing, \tag{2.2}$$

$$K \text{ is compact.} \tag{2.3}$$

A *polyhedron* is a finite union (disjoint or not) of convex polyhedra as above.

We note the following:

(i) $P = \overline{\operatorname{int} P}$ if P is a polyhedron.
(ii) A polyhedron need not be connected.
(iii) A convex polyhedron is a polyhedron which is convex as a set.
(iv) In the representation (2.1) of a convex polyhedron the family $\{H_1, \ldots, H_n\}$ is unique provided n is taken to be minimal. The representation is then called the *minimal representation*.
(v) An equivalent definition of convex polyhedron is that it is a set which is the convex hull of finitely many points and having nonempty interior. Cf. [Hö, Def. 2.1.20, Th. 2.1.21].

We next introduce the notion of a face of a polyhedron. Define, for any set $P \subset \mathbb{R}^N$,

$$\begin{aligned} \partial_{\text{face}} P \;=\; \{x \in \mathbb{R}^N \colon & \text{there exists } r > 0 \\ & \text{and a closed half-space } H \subset \mathbb{R}^N \text{ with } x \in \partial H \\ & \text{such that } P \cap B(x, r) = H \cap B(x, r)\}. \end{aligned} \tag{2.4}$$

Then $\partial_{\text{face}} P$ is a relatively open subset of ∂P.

DEFINITION 2.2. A *face* of a polyhedron P is a connected component of $\partial_{\text{face}} P$.

Note. This definition of ‘face’ does not agree with the one commonly used in, e.g., the theory of convex polytopes (see [Zg]).

The closed half-space H in the definition of $\partial_{\text{face}} P$ is clearly the same throughout a component of $\partial_{\text{face}} P$, i.e. a face of P, and will be called the *associated half-space* of the face.

For convenience we now give a number of equivalent characterizations of polyhedra.

PROPOSITION 2.3. *Let $P \subset \mathbb{R}^N$ be a nonempty compact set. Then the following conditions are equivalent.*

(i) *P is a polyhedron.*
(ii) *$P = \overline{\text{int}\, P}$ and there exist finitely many (affine) hyperplanes $L_1, \dots, L_n$ such that $\partial P \subset \bigcup_{i=1}^n L_i$.*
(iii) *There exist finitely many hyperplanes $L_1, \dots, L_n$ such that P is the closure of the union of a selection of bounded components of $\mathbb{R}^N \setminus \bigcup_{i=1}^n L_i$.*
(iv) *$\partial_{\text{face}} P$ has only finitely many components and is dense in ∂P.*

Proof. (i) $\Rightarrow$ (ii): By Definition 2.1 we have

$$P = \bigcup_{k=1}^{m} \bigcap_{j=1}^{n_k} H_{kj}$$

for suitable closed half-spaces H_{kj}, with int $\bigcap_{j=1}^{n_k} H_{kj}$ nonempty and bounded for each k. Let $L_1, \dots, L_n$ be an enumeration of all the ∂H_{kj}. Then it is clear that $\partial P \subset \bigcup_{i=1}^n L_i$, and it has already been noticed that $P = \overline{\text{int}\, P}$.

(ii) $\Rightarrow$ (iii): With P and L_i as in (ii), let $\omega_1, \dots, \omega_l$ be the components of $\mathbb{R}^N \setminus \bigcup_{i=1}^n L_i$. Since P has no boundary in $\mathbb{R}^N \setminus \bigcup_{i=1}^n L_i$ it follows that for each j, $P \cap \omega_j$ either is empty or equals ω_j. In the latter case ω_j is bounded since P is compact. Thus $P \setminus \bigcup_{i=1}^n L_i$ is a selection of bounded components ω_j, say $P \setminus \bigcup_{i=1}^n L_i = \bigcup_{j=1}^k \omega_j$. Using $P = \overline{\text{int}\, P}$ it follows that

$$P = \overline{P \setminus \bigcup_{i=1}^{n} L_i} = \overline{\bigcup_{j=1}^{k} \omega_j}$$

as desired.

(iii) $\Rightarrow$ (i): If $\omega_1, \dots, \omega_k$ are the selected components of $\mathbb{R}^N \setminus \bigcup_{i=1}^n L_i$ then, since the ω_j clearly are convex, $P = \overline{\bigcup_{j=1}^k \omega_j} = \bigcup_{j=1}^k \overline{\omega}_j$ is a representation of P as a union of compact convex polyhedra, proving (i).

(iii) $\Rightarrow$ (iv): Let L be the subset of $\bigcup_{i=1}^n L_i$ consisting of those points which lie on only one of the L_i. Then L is relatively open and dense in $\bigcup_{i=1}^n L_i$. Moreover, $(\bigcup_{i=1}^n L_i) \setminus L$, which contains $\partial P \setminus L$, is too small to separate int P from P^c, even locally. Therefore $\partial P \cap L$ is dense in ∂P. On the other hand, for any $x \in \partial P \cap L$ exactly one of the two components of $\mathbb{R}^N \setminus (\bigcup_{i=1}^n L_i)$ having x on its boundary is contained in P. Therefore $x \in \partial_{\text{face}} P$. Thus $\partial P \cap L \subset \partial_{\text{face}} P$, proving that $\partial_{\text{face}} P$ is dense in ∂P.

It also follows from the above that every component of $\partial_{\text{face}} P$ contains a component of L. Thus, $\partial_{\text{face}} P$ has only finitely many components, finishing the proof of (iv).

(iv) $\Rightarrow$ (ii): Let $F_1, \ldots, F_n$ be the components of $\partial_{\text{face}} P$, let $H_1, \ldots, H_n$ be the associated half-spaces and set $L_i = \partial H_i$. Then $F_i \subset L_i$, i.e.,

$$\partial_{\text{face}} P \subset \bigcup_{i=1}^{n} L_i.$$

We also have, by definition of $\partial_{\text{face}} P$,

$$\partial_{\text{face}} P \subset \partial(\text{int}\, P) \subset \partial P.$$

Hence, if $\overline{\partial_{\text{face}} P} = \partial P$ it follows that $\partial P \subset \bigcup_{i=1}^{n} L_i$ and also that $\partial(\text{int}\, P) = \partial P$, i.e., $\overline{\text{int}\, P} = P$. This proves (ii). □

Note. Condition (iii) gives a representation of a polyhedron as a finite union of convex polyhedra with pairwise disjoint interiors. Other such decompositions will be considered in Section 4.

DEFINITION 2.4. Let P, Q be two polyhedra in $\mathbb{R}^N$ with $Q \subset P$. We say that Q *sticks to* P if

$$\begin{aligned} &\text{for every face } F \text{ of } Q \text{ there exists } x \in F \text{ and } r > 0 \\ &\quad \text{such that } Q \cap B(x, r) = P \cap B(x, r). \end{aligned} \tag{2.5}$$

An equivalent, and shorter, way of expressing (2.5) is:

every component of $\partial_{\text{face}} Q$ intersects $\partial_{\text{face}} P$.

Clearly every polyhedron sticks to itself. If P is convex then, as is readily verified, P itself is actually the only polyhedron which sticks to P. It is a non-trivial fact that if P is not convex then P contains a proper subpolyhedron which sticks to P. In two dimensions this subpolyhedron can even be taken to be convex (Theorem 5.1).

3. Preliminaries on Mother Bodies

'Mother body' (or maternal or materic body) is a potential theoretic term which seems to have been coined by a Bulgarian school of geophysical potential theory around D. Zidarov [Zi], [Ko1], [Ko2]. It is intended to mean a kind of potential theoretic skeleton for a heavy body. The discussion in [Zi] is largely heuristic, but attempts of formulating precise requirements for a mother body (or for related notions of potential theoretic skeletons) have been given by various authors, e.g. [An1], [An2], [Gu1], [Ko1], [Ko2], [Ka-Pi].

In this paper we shall simply adopt the five axioms for a mother body given in [Gu1] (see also [Gu2]). To describe these we have to introduce some more notation.

If μ is a (signed) Radon measure with compact support in $\mathbb{R}^N$ we define its Newtonian potential as

$$U^\mu(x) = \begin{cases} c_2 \displaystyle\int \log \frac{1}{|x-y|}\,\mathrm{d}\mu(y) \quad (N=2), \\ c_N \displaystyle\int \frac{\mathrm{d}\mu(y)}{|x-y|^{N-2}} \quad (N \geqslant 3), \end{cases}$$

where the constants $c_N > 0$ are chosen so that $-\Delta U^\mu = \mu$ in the distributional sense, Δ denoting the ordinary Laplace operator.

If $K \subset \mathbb{R}^N$ is a (measurable) bounded set we set $U^K = U^{\chi_K}$, where χ_K denotes the measure with density one on K, zero outside K (i.e., Lebesgue measure restricted to K). Thus U^K is the Newtonian potential of K regarded as a body of density one.

DEFINITION 3.1. Let $K \subset \mathbb{R}^N$ be a compact set satisfying $K = \overline{\operatorname{int} K}$. K is regarded as a body with volume density one. A *mother body* for K is a Radon measure μ satisfying

$$U^\mu = U^K \quad \text{in } \mathbb{R}^N \backslash K, \tag{3.1}$$

$$U^\mu \geqslant U^K \quad \text{in } \mathbb{R}^N, \tag{3.2}$$

$$\mu \geqslant 0, \tag{3.3}$$

$$\operatorname{supp}\mu \text{ has Lebesgue measure zero}, \tag{3.4}$$

$$\begin{aligned} &\text{for every } x \in K\backslash\operatorname{supp}\mu \text{ there exists a curve } \gamma \\ &\quad \text{in } \mathbb{R}^N\backslash\operatorname{supp}\mu \text{ joining } x \text{ to some point in } K^c. \end{aligned} \tag{3.5}$$

Comments. It follows from (3.1) that $\operatorname{supp}\mu \subset K$. The first three axioms are of potential theoretic nature, while the last two just are geometric conditions on the closed set $\operatorname{supp}\mu \subset K$. (3.4) says that it is small (a nullset) and (3.5) says that it does not ‘hide’ any components of $K\backslash\operatorname{supp}\mu$. Ample discussions of the above axioms are given in [Gu1]. (In [Gu1], the axioms were stated for a body in the form of an open set rather than a closed set as here, but since we are assuming that $K = \overline{\operatorname{int} K}$ everything can be translated.)

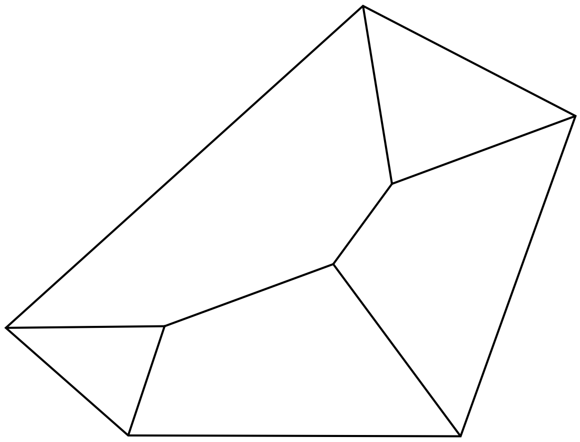

Figure 3.1. The ridge, or support of mother body, of a convex polyhedron.

The question lying behind the investigations carried out in this paper is the question of existence, uniqueness and structure of mother bodies for general polyhedra. For convex polyhedra these questions have simple and complete answers, which were given in [Gu1]. Since it will be needed in the sequel we summarize below the main result in [Gu1].

Let $K \subset \mathbb{R}^N$ be a convex polyhedron, let (2.1) be its minimal representation and define for $x \in \mathbb{R}^N$

$$\delta_i(x) = \mathrm{dist}(x, H_i^c),$$

$$\delta(x) = \min\{\delta_1(x), \dots, \delta_n(x)\} = \mathrm{dist}(x, K^c),$$

$$u_i(x) = \tfrac{1}{2}\delta_i(x)^2,$$

$$u(x) = \min\{u_1(x), \dots, u_n(x)\} = \tfrac{1}{2}\delta(x)^2,$$

$$R = \{x \in K : \delta(x) = \delta_i(x) \text{ for at least two different } i\}, \tag{3.6}$$

$$\begin{aligned} D_i &= \{x \in K \setminus R : \delta(x) = \delta_i(x)\} \\ &= \{x \in K : \delta_i(x) < \delta_j(x) \text{ for all } j \neq i\}. \end{aligned} \tag{3.7}$$

The set R will be called the *ridge* of K because of its geometric interpretation: in the case of two dimensions one may think of K as the top view of a house covered with a roof of height $\delta(x)$. Then R will be the ridge of the roof, cf. Figure 3.1. Other names for R are ‘medial axis’, ‘symmetric axis’ or simply ‘skeleton’. See [Ro, Sect. 5.6].

Note that

$$\partial K \setminus R = \partial_{\text{face}} K = F_1 \cup \cdots \cup F_n,$$

where

$$F_i = \partial K \cap D_i, \tag{3.8}$$

$(1 \leqslant i \leqslant n)$ are the faces of K $(F_i \subset \partial H_i)$.

Since the function $U^K + u$ has the behaviour of a potential at infinity (note that $u = 0$ outside K) we can define the measure μ by the requirement $U^\mu - U^K = u$. This simply means that $\mu = \chi_K - \Delta u$. Then (see [Gu1]) $\operatorname{supp} \mu = R$ and μ is a mother body of K. Moreover, no other signed measure satisfies (3.1), (3.4), (3.5).

4. General Structure of Mother Bodies of Polyhedra

We do not know whether nonconvex polyhedra in higher dimensions always admit mother bodies, but in this section we nevertheless prove a structure theorem for mother bodies when they do exist: every mother body of a polyhedron defines a decomposition of the polyhedron into convex polyhedra such that the mother body is the sum of the mother bodies for the pieces. Moreover, there are only finitely many possibilities of doing this decomposition, hence there are only finitely many mother bodies.

THEOREM 4.1. *Let $P \subset \mathbb{R}^N$ be a polyhedron. Then there are at most finitely many signed measures μ satisfying* (3.1), (3.4), (3.5) *for P. For anyone of these also* (3.2) *holds. If moreover* (3.3) *holds, then there is a decomposition*

$$P = K_1 \cup \cdots \cup K_m, \tag{4.1}$$

where the K_i are convex polyhedra with pairwise disjoint interiors, such that

$$\mu = \mu_1 + \cdots + \mu_m, \tag{4.2}$$

μ_i denoting the unique mother body of K_i.

Remark 4.2. The positivity axiom (3.3) is really necessary in order to ensure the decomposition (4.1): By a procedure due to D. Siegel [Si] (in a slightly different context) one can construct indecomposable 'mother bodies' satisfying (3.1), (3.2), (3.4), (3.5) (but not generally (3.3)) for polyhedra in two dimensions. Figures 4.1 and 4.2 show, for a simple polyhedron P in $\mathbb{R}^2$, two different decomposable mother bodies (this example is due to D. Zidarov [Zi]) and Figure 4.3 shows Siegel's indecomposable one (violating (3.3)).

Proof. (Theorem 4.1). Let μ be a measure satisfying (3.1), (3.4), (3.5) for P. Pick any component ω of $(\operatorname{int} P) \backslash \operatorname{supp} \mu$. By (3.5) there is, for any $x \in \omega$, a curve γ in $\mathbb{R}^N \backslash \operatorname{supp} \mu$ from x to some point in P^c. We may assume that $\gamma \cap \partial P$ consists of only one point, call it y, and that $y \in \partial_{\text{face}} P$. Let H be the associated half-space of the face at y.

Zidarov

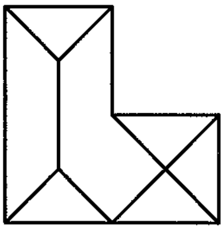

Figure 4.1.

Zidarov

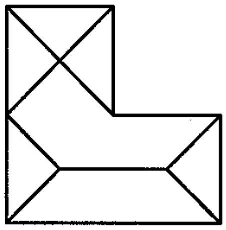

Figure 4.2.

Siegel

Axiom (3.3) violated.
μ is negative on this segment.

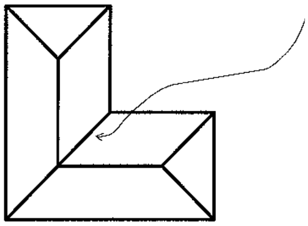

Figure 4.3.

Set

$$u = U^{\mu} - U^{P} \quad \text{in } \mathbb{R}^N, \tag{4.3}$$

so that $\Delta u = \chi_P - \mu$ everywhere and $u = 0$ outside P by (3.1). Since $\Delta(\frac{1}{2}\operatorname{dist}(x, H^c)^2) = \chi_H$ and $\frac{1}{2}\operatorname{dist}(x, H^c)^2 = 0$ on H^c it follows that the function $u(x) - \frac{1}{2}\operatorname{dist}(x, H^c)^2$ is harmonic in a neighbourhood of y and vanishes outside P in this neighbourhood. Thus it vanishes in the full neighbourhood, i.e., $u(x) = \frac{1}{2}\operatorname{dist}(x, H^c)^2$ for x close to y.

Thus, replacing H^c with ∂H, we obtain

$$u(x) = \tfrac{1}{2}\operatorname{dist}(x, \partial H)^2 \tag{4.4}$$

for x in P close to y, hence in an open subset of ω. But $u(x) - \frac{1}{2}\operatorname{dist}(x, \partial H)^2$ is harmonic in all of $(\operatorname{int} P)\setminus\operatorname{supp}\mu$, hence it follows that (4.4) holds in all of ω. In particular, the hyperplane ∂H is uniquely determined by ω.

Let $H_1, \ldots, H_n$ be all the associated half-spaces of the faces of P and let

$$u_i(x) = \tfrac{1}{2}\operatorname{dist}(x, \partial H_i)^2 \tag{4.5}$$

$(1 \leqslant i \leqslant n)$. We assume that $H_i \neq H_j$ for $i \neq j$. Then it follows from the above that on each component ω of $(\operatorname{int} P)\setminus \operatorname{supp}\mu$ u agrees with one of the u_i. In particular,

$$u, \nabla u \in L^{\infty}(\mathbb{R}^N \setminus (\operatorname{supp}\mu \cup \partial P)). \tag{4.6}$$

Next we note that U^{μ} and ∇U^{μ} are locally integrable functions. This is because μ is a measure and the Newtonian kernel and its gradient are locally integrable functions (cf. [Do,§ 26]). Therefore u and its first order distributional derivatives belong to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. By (3.4) and (ii) of Proposition 2.3 $\operatorname{supp}\mu \cup \partial P$ has Lebesgue measure zero. It now follows from (4.6) that

$$u, \nabla u \in L^{\infty}(\mathbb{R}^N) \tag{4.7}$$

in the sense of distributions.

We conclude that u is a Lipschitz continuous function (i.e. has such a representative). Thus, for any component ω of $(\mathrm{int}\, P)\setminus \mathrm{supp}\, \mu$ we even have $u = u_i$ on all of $\overline{\omega}$ for some i. Since $\mathrm{supp}\, \mu$ and ∂P have no interior points, every $x \in P$ is in $\overline{\omega}$ for some ω as above and it follows that u is everywhere in P equal to some u_i.

Outside P, $u = 0$. Since $u_i \geqslant 0$ everywhere, it follows that $u \geqslant 0$ everywhere, proving that μ satisfies (3.2).

Next, set

$$\begin{aligned} R &= \{x \in P \colon \mathrm{dist}(x, \partial H_i) = \mathrm{dist}(x, \partial H_j) \\ &\qquad \text{for some pair } i, j \text{ with } i \neq j\} \\ &= \{x \in P \colon u_i(x) = u_j(x) \text{ for some pair with } i \neq j\}. \end{aligned} \tag{4.8}$$

This is a finite union of (affine) hyperplanes in P. (The present R is not necessarily the same as that in (3.6) when P is convex.) Since u is continuous u can, in $\mathrm{int}\, P$, change representative from one u_i to another only on R. On ∂P u changes representative from one, or several, of the u_i to zero. More precisely, at each point of $\partial P \setminus R$ only one u_i vanishes, showing that u is of the form $u(x) = \frac{1}{2}\,\mathrm{dist}(x, H_i^c)^2$ in a neighbourhood of any point of $\partial P \setminus R$.

Since $P \setminus R$ has only finitely many components there are only finitely many ways of combining the u_i to a continuous function u. This proves that there are only finitely many signed measures satisfying (3.1), (3.4), (3.5). We also conclude that $\Delta u = \chi_P$ in $\mathbb{R}^N \setminus R$ and hence that

$$\mathrm{supp}\, \mu \subset R. \tag{4.9}$$

A further consequence of the fact that u everywhere in P agrees with some u_i is that

$$V \subset \bigcup_{i=1}^{n} \partial H_i, \tag{4.10}$$

where $V = \{x \in P \colon u(x) = 0\}$. Moreover, it follows that

$$\nabla u = 0 \quad \text{on } V. \tag{4.11}$$

From (4.7) and (4.11) we further deduce that $\mu = \chi_P - \Delta u$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$ and that $\mu = 0$ on V.

It is a consequence of (4.10) that $P \setminus V$ has only finitely many components and that their closures, call them $K_1, \ldots, K_m$, are polyhedra. Since u everywhere in P agrees with some u_i, we have $K_i \setminus V = \mathrm{int}\, K_i$. Hence $\mu = 0$ on ∂K_i. Set $\mu_i = \mu\big|_{K_i}$. Then it is immediately verified that μ_i satisfies (3.4) and (3.5) for K_i. Using (4.11) it follows that (3.1) holds for K_i. Applying the above argument to μ_i we see that

(3.2) holds also for K_i. Therefore, in order to prove the theorem it only remains to prove that each K_i is convex under the assumption that $\mu \geqslant 0$.

In order to prove that one K_i is convex we may forget about the other ones. Equivalently, we may prove that P itself is convex under the assumption that $P \setminus V = \operatorname{int} P$ and that it has only one component. This is the content of the following proposition, which contains all that remains of the proof of Theorem 4.1. □

PROPOSITION 4.3. *Let μ be a mother body for a polyhedron P such that* int P *is connected and such that*

$$U^{\mu} > U^{P} \quad \textit{in } \operatorname{int} P.$$

Then P is convex.

The proof of Proposition 4.3 will be based on the following geometrical lemma, which we prove first.

LEMMA 4.4. *Let P be a polyhedron such that* int P *is connected but not convex. Then there exists $x \in \partial P$, $r > 0$ and two distinct half-spaces H_1, H_2 with $x \in \partial H_1 \cap \partial H_2$ such that*

$$P \cap B(x, r) \supset (H_1 \cup H_2) \cap B(x, r),$$

$$\partial P \cap B(x, r) \supset \partial(H_1 \cup H_2) \cap B(x, r).$$

This means that P has a concave edge through x, possibly with other parts of P clustering at the edge.

Proof. There exists a point $y \in \operatorname{int} P$ and a ball $B(z, \varepsilon) \subset \operatorname{int} P$ such that part of, and only part of, $B(z, \varepsilon)$ can be seen from y within P. Precisely, setting $W = \{w \in B(z, \varepsilon) \colon ty + (1 - t)w \in P \text{ for all } 0 \leqslant t \leqslant 1\}$ both W and $B(z, \varepsilon) \setminus W$ have nonempty interiors. The existence of y and $B(z, \varepsilon)$ as above follows from the fact that convexity of connected sets is a local property. More specifically it can be derived from [Hö, Th. 2.1.27]. (One also has to use that $P = \overline{\operatorname{int} P}$.)

Now for $w \in \partial W \cap B(z, \varepsilon)$ there is at least one value of $0 < t < 1$ such that $ty + (1 - t)w \in \partial P$. The set of all such t is compact for a given w. Let t_w denote the smallest of these values of t. Then it is not hard to see that generic points of the set

$$E = \{t_w y + (1 - t_w)w \in \partial P \colon w \in \partial W \cap B(z, \varepsilon)\}$$

are points x of the kind required in the lemma. □

Proof of Proposition 4.3. The proof is basically a continuation of the proof of Theorem 4.1, the only difference being that we are now reduced to the case $m = 1$, i.e. to the case that $P \setminus V$ is connected (and, hence, $= \operatorname{int} P$). Thus we keep all notations from the proof of Theorem 4.1.

We assume that P is not convex and derive a contradiction. By Lemma 4.4, part of ∂P is a concave edge. With appropriate numbering of the H_i and by choosing the point x in Lemma 4.4 generically, namely so that no ∂H_i cuts the edge transversally at x, we get into the following situation at the point $x \in \partial P$: there is an $r > 0$ and an integer k, $2 \leqslant k \leqslant n$, such that

$$x \in \partial H_1 \cap \partial H_2,$$

$$P \cap B(x, r) \supset (H_1 \cup H_2) \cap B(x, r),$$

$$\partial P \cap B(x, r) \supset \partial(H_1 \cup H_2) \cap B(x, r),$$

$$\partial H_1 \cap \partial H_2 \subset \partial H_i \quad \text{for } 1 \leqslant i \leqslant k,$$

$$\partial H_i \cap B(x, r) = \varnothing \quad \text{for } k + 1 \leqslant i \leqslant n.$$

We may further assume that x is the origin and that the edge is 'vertical', i.e. $\partial H_1 \cap \partial H_2 = \{x \in \mathbb{R}^N : x_1 = x_2 = 0\}$. ($x = (x_1, \ldots, x_N)$). The half-spaces $H_1, \ldots, H_k$ are then of the form

$$H_i = \{x \in \mathbb{R}^N : x_1 \cos \theta_i + x_2 \sin \theta_i \geqslant 0\}$$

($1 \leqslant i \leqslant k$) with θ_i the angle to the positive x_1-axis of the inward normal vector of H_i. Letting (r, θ) denote polar coordinates in the (x_1, x_2)-plane, i.e., $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, we have

$$\begin{aligned} u_i(x) &= \tfrac{1}{2} \operatorname{dist}(x, \partial H_i)^2 \\ &= \frac{r^2}{2} \cos^2(\theta - \theta_i) \end{aligned} \tag{4.12}$$

for $1 \leqslant i \leqslant k$.

Since $0 \notin \partial H_i$ for $i > k$, $u_i(0) \neq 0$ for these i. As u is continuous, $u(0) = 0$ (because $0 \in \partial P$) and everywhere in P equals some u_i (all by the proof of Theorem 4.1) it follows that in a neighbourhood of the origin, which we may take to be $B(0, r)$, $u(x)$ only takes values among $u_1(x), \ldots, u_k(x)$ and 0. In particular this shows that u only depends on (x_1, x_2). Thus we have a purely two-dimensional situation in $B(0, r)$, which is easy to analyze. We may work simply in the (x_1, x_2)-plane since everything is constant in $x_3, \ldots, x_N$.

The set R (see (4.8)) on which u can change representative between two of the u_i is in the (x_1, x_2)-plane represented by the set of rays $\theta =$ constant for which

$$\cos^2(\theta - \theta_i) = \cos^2(\theta - \theta_j) \tag{4.13}$$

for some pair $i, j \in \{1, \ldots, k\}$ with $i \neq j$. Moreover, u changes representative from u_1 to $u = 0$ on ∂H_1 and from u_2 to $u = 0$ on ∂H_2.

Now, if u changes representative from u_i to u_j on one of the four rays $\theta = \text{constant} = \frac{1}{2}(\theta_i + \theta_j) + \frac{\pi}{2} \cdot \text{(integer)}$ determined by (4.13), then either

$$u = \max\{u_i, u_j\} \tag{4.14}$$

or

$$u = \min\{u_i, u_j\} \tag{4.15}$$

in a neighbourhood of the ray. However, and this is the crucial point, the first possibility (4.14) cannot occur because it would give Δu a contribution in form of a strictly positive $(N-1)$-dimensional density on the hyperplane in $\mathbb{R}^N$ determined by the ray, and this would contradict the assumption that $\mu \geqslant 0$.

Thus (4.15) is the only possible way of changing representative between u_i and u_j. But keeping r fixed, regarding u as a function of θ and looking at (4.12) we see that if u does not change representative at all, then it will be strictly positive on an θ-interval of length π, while if it does change (according to (4.15)) then it can do it only once and it will be strictly positive only on an interval of length $< \pi$.

Thus in any case, u can never be strictly positive on an angular segment of opening $> \pi$. This contradicts our assumption of having a concave edge and $u > 0$ in int P, and thus finishes the proof of Proposition 4.3. □

By Theorem 4.1, any mother body μ of a polyhedron induces a decomposition of P into convex polyhedra K_j such that $\mu|_{K_j}$ is the mother body of K_j. Next we wish to discuss the opposite question: let

$$P = K_1 \cup \cdots \cup K_m, \tag{4.16}$$

be a decomposition of a polyhedron P into convex polyhedra K_j with pairwise disjoint interiors and let μ_j be the mother body of K_j. Under what circumstances is

$$\mu = \mu_1 + \cdots + \mu_m \tag{4.17}$$

a mother body for P?

It is immediately verified that μ defined by (4.17) always satisfies (3.1)–(3.4). However, (3.5) need not hold. See, e.g., Figure 4.4 where the previously considered polyhedron in Figures 4.1–4.3 is decomposed into too many convex polyhedra. To analyze condition (3.5) closer we shall introduce a notion of *generation* of faces of the K_i for the subdivision (4.16).

Let, as in (3.6)–(3.8), R_i, D_{ij}, F_{ij} ($j = 1, \ldots, m_i$ say) denote, respectively, the ridge of K_i, the components of $K_i \setminus R_i$ and the faces of K_i, so that $R_i = \operatorname{supp} \mu_i$, $F_{ij} = \partial K_i \cap D_{ij}$. Note that the last relation sets up a one-to-one correspondence between the D_{ij} and the F_{ij}.

Axiom (3.5) violated.

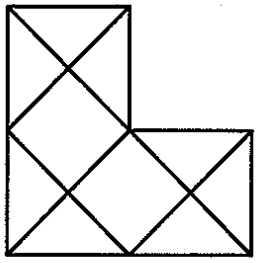

Figure 4.4.

Some of the faces F_{ij} have some part in common with a face of P. These will be the faces of first generation. The corresponding D_{ij} will also be called of first generation. Thus we define

$$\mathrm{gen}(F_{ij}) = \mathrm{gen}(D_{ij}) = 1$$

if and only if $F_{ij} \cap \partial_{\mathrm{face}} P \neq \varnothing$. Next we define

$$\mathrm{gen}(F_{ij}) = \mathrm{gen}(D_{ij}) = 2$$

if and only if $F_{ij} \cap \partial_{\mathrm{face}} P = \varnothing$ but there exists F_{kl} with $\mathrm{gen}(F_{kl}) = 1$ such that $F_{ij} \cap F_{kl} \neq \varnothing$. The significance of the last relation is that it ensures the possibility of passing from D_{kl} to D_{ij} without meeting supp μ.

Proceeding inductively we set $\mathrm{gen}(F_{ij}) = \mathrm{gen}(D_{ij}) = n$ if and only if $\mathrm{gen}(F_{ij}) \neq 1, 2, \ldots, n-1$, but there exists F_{kl} with $\mathrm{gen}(F_{kl}) = n-1$ such that $F_{ij} \cap F_{kl} \neq \varnothing$. After finitely many steps we come to an integer n such that there are no faces of generation n, and then there will of course be no faces of any higher generation $n+1, n+2, \ldots$.

At this stage there are two possibilities:

(i) there are no faces F_{ij} left, i.e. each F_{ij} has been attributed a finite number $\mathrm{gen}(F_{ij})$,

(ii) there are faces F_{ij} which never appeared in the process.

For the latter faces we simply set $\mathrm{gen}(F_{ij}) = \mathrm{gen}(D_{ij}) = +\infty$. See Figure 4.5 for an example of a decomposition (4.16) with the generations of the D_{ij} marked out. Now we just observe

PROPOSITION 4.5. *Referring to the decomposition* (4.16), (4.17), *μ is a mother body of P if and only if every face F_{ij} of the K_i is of finite generation* (*i.e. case* (i) *above occurred*). *Moreover, a necessary condition that a given face F_{ij} is of finite generation is that the hyperplane containing F_{ij} also contains a face of P.*

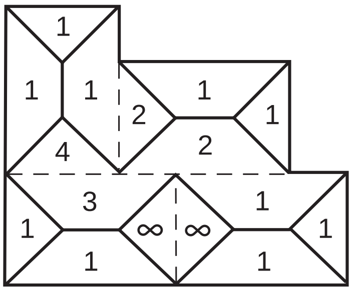

Figure 4.5.

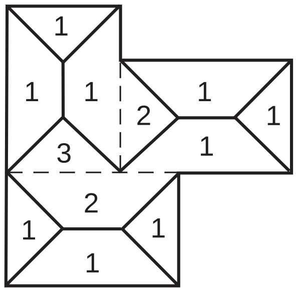

Figure 4.6.

Proof. Since $P\backslash\operatorname{supp}\mu = P\backslash\bigcup_{i=1}^{m} R_i \subset \bigcup_{i,j} D_{ij}$ and since the only requirement for a mother body which needs to be checked is (3.5) it follows that what we have to prove for the first statement of the proposition is the following:

$$\text{a point } x \in D_{ij}\backslash\operatorname{supp}\mu \text{ can be joined to } P^c$$
$$\text{via a curve in } \mathbb{R}^N\backslash\operatorname{supp}\mu \text{ if and only if } \operatorname{gen}(D_{ij}) < +\infty. \tag{4.18}$$

To prove (4.18), suppose first that gen $(D_{ij}) = n < +\infty$. Since $F_{ij} = D_{ij}\backslash(\operatorname{int} D_{ij})$ and $(\operatorname{int} D_{ij}) \cap \operatorname{supp}\mu = \varnothing$, x can be joined to any point y on $F_{ij}\backslash\operatorname{supp}\mu$ by a curve γ in $D_{ij}\backslash\operatorname{supp}\mu$. If $n = 1$ then we can choose $y \in \partial_{\text{face}}P$ and then continue γ into P^c. If $n > 1$ then we can choose $y \in F_{kl}$ for some face F_{kl} with gen $(F_{kl}) = n - 1$. Then $y \in D_{kl}\backslash\operatorname{supp}\mu$, hence we are in the same situation as in the beginning, but with a smaller n. By induction this means that we will finally be able to reach P^c, proving one direction of (4.18).

Conversely, if there is a curve γ from $x \in D_{ij}\backslash\operatorname{supp}\mu$ then γ passes through finitely many D_{kl} on its way out and, by definition, the last D_{kl} is of generation one and two adjacent D_{kl} differ by at most one unit of generation. Hence D_{ij} is of finite generation.

We also note that the hyperplane L containing a face $F_{ij} = \partial K_i \cap D_{ij}$ is the same for two adjacent D_{kl} through which γ passes. Hence it is the same for all F_{ij} crossed by γ, hence it is the hyperplane containing the face of P through which γ leaves P. Thus a necessary condition that gen $(F_{ij}) < +\infty$ is that F_{ij} is contained in a hyperplane which also contains a face of P. □

One special way in which all points of $P\backslash\operatorname{supp}\mu$ could be of finite generation (i.e. μ is a mother body) is that

$$\operatorname{gen}(D_{ij}) \leqslant i \tag{4.19}$$

for all $i = 1, \ldots, m$ and all j. This means that every face of K_1 is of first generation, every face of K_2 of generation at most two etc. Equivalently one can say that, for every $i = 1, \ldots, m$, each face of K_i is of first generation in the polyhedron $P_i = K_i \cup \cdots \cup K_m$. This can be expressed in the previously introduced terminology (Definition 2.4) by simply saying that K_i sticks to P_i, for each $i = 1, \ldots, m$.

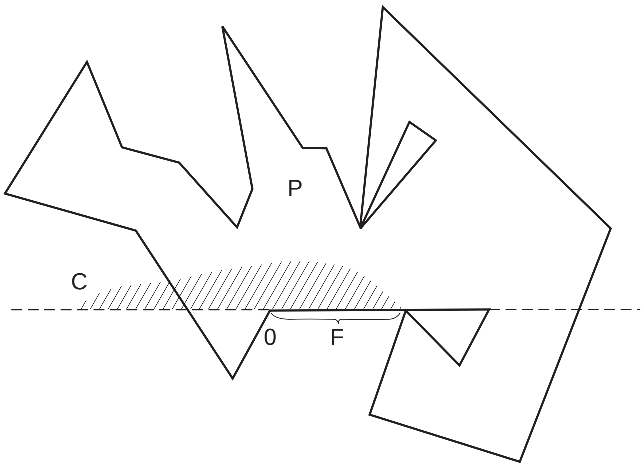

Figure 5.1. Initial data: P is a given polyhedron; $x = 0$ is a given point on ∂P; F is a given face with $x \in \bar{F}$; and C is the largest convex cone in P (near x) satisfying $F \subset \partial C$ (C equals the upper half-plane in the present case).

In Section 5 we will show that in two dimensions any polyhedron admits a convex subpolyhedron which sticks to it. Thus given any $P \subset \mathbb{R}^2$ we can find a decomposition (4.16) satisfying (4.19) by first taking K_1 which sticks to $P_1 = P$, then K_2 which sticks to $P_2 = \overline{P_1 \setminus K_1}$, then K_3 which sticks to $P_3 = \overline{P_2 \setminus K_2}$ etc. Since P_1, P_2,... is a strictly shrinking family of polyhedra with $\partial P_j \subset \bigcup_{i=1}^{n} L_i$, L_i denoting the hyperplanes containing the faces, we will after finitely many steps reach the empty set, which means that we have got the desired decomposition (4.16) satisfying (4.19). Thus it will follow that any polyhedron in two dimensions has a mother body.

Figure 4.6 is an example of a decomposition not of the type (4.19), but still with $\text{gen}(D_{ij}) < +\infty$ for all i, j.

5. Two Dimensions

The main result in this section is that any polyhedron in two dimensions admits a convex polyhedron which sticks to it. As indicated at the end of Section 4 this shows that any polyhedron in two dimensions has a mother body. The proof below is illustrated by Figures 5.1–5.5.

THEOREM 5.1. *For any polyhedron $P \subset \mathbb{R}^2$, the union of the convex polyhedra which stick to P contains a full neighbourhood (in P) of ∂P. In particular, there exists at least one convex polyhedron sticking to P (at least two if P is not convex).*

Proof. It is enough to prove the following. Let x be an arbitrary point on ∂P and let $F \subset \partial P$ be any face of P such that $x \in \overline{F}$. Let C be the largest closed convex cone in $\mathbb{R}^2$ with vertex x satisfying $F \subset \partial C$ and, for $\varepsilon > 0$ sufficiently

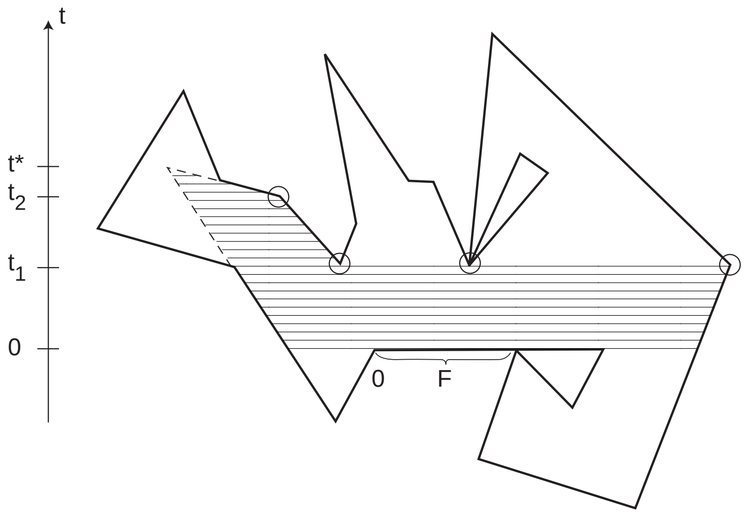

Figure 5.2. Upward phase, selection 1 (of 2): shaded area is Q; horizontal lines are $S(t)$; at t_1, E consists of 3 obstruction points (subcase 2b); the left most segment of $S(t_1)\setminus E$ is selected; at t_2 there is just one obstruction point, and no choice; termination at t^* through case 1.

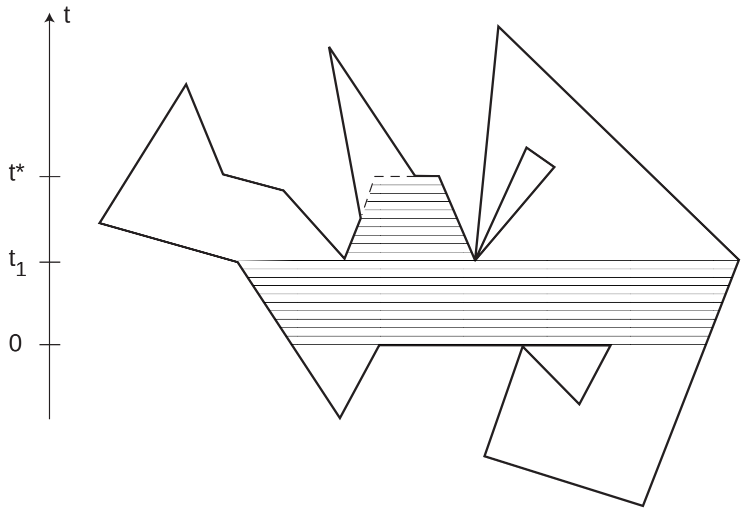

Figure 5.3. Upward phase, selection 2 (of 2): shaded area is Q; horizontal lines are $S(t)$; middle segment of $S(t_1)\setminus E$ is selected; termination at t^* through subcase 2a.

small, $C \cap B(x, \varepsilon) \subset P$. Then (to be proved) there exists a convex polyhedron $K \subset P$ which sticks to P and which contains $C \cap B(x, \varepsilon)$, for $\varepsilon > 0$ small enough (perhaps smaller than above).

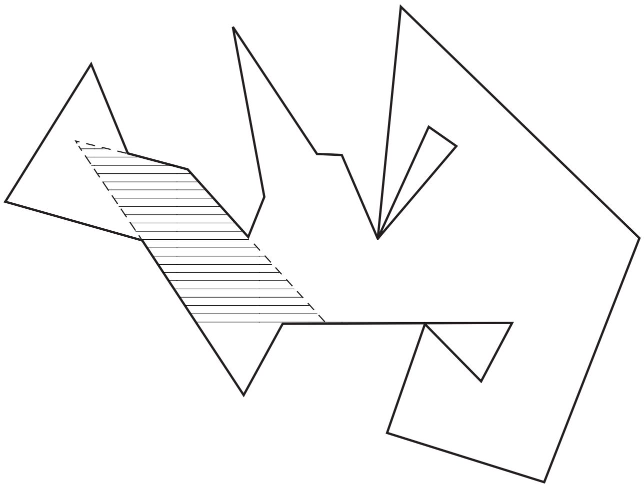

Figure 5.4. Downward phase, with selection 1 in upward phase: shaded area is K; and horizontal lines are $\tilde{S}(t)$.

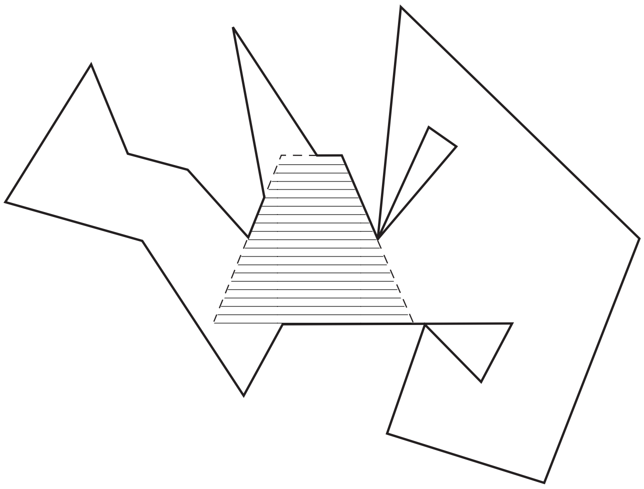

Figure 5.5. Downward phase, with selection 2 in upward phase: shaded area is K; and horizontal lines are $\tilde{S}(t)$.

Note. If the 'corner' of P at x with F as one side has opening $< \pi$ then $C \cap B(x, \varepsilon)$ simply is this corner, if it has opening $\geqslant \pi$ (including the case that x is a face point, $x \in F$) then C is the half-space associated with F.

Let H_0 be the (closed) half-space associated with F and let $H_1, \ldots, H_n$ be the half-spaces associated with the other faces of P. We assume that $H_i \neq H_j$ for $i \neq j$, so there may be several faces belonging to the same H_i. We may assume that

$x = 0$ and that $H_0 = \{x \in \mathbb{R}^2 \colon x_2 \geqslant 0\}$. Then F is the set of those $(x_1, 0) \in \partial H_0$ which satisfy a condition of one of the following three forms:

$$\alpha < x_1 < 0; \tag{5.1}$$

$$\alpha < x_1 < \beta; \tag{5.2}$$

$$0 < x_1 < \beta. \tag{5.3}$$

Here $\alpha < 0 < \beta$.

Set

$$L(t) = \{x \in \mathbb{R}^2 \colon x_2 = t\}.$$

The construction of K consists of two phases: one upward phase in which t increases from $t = 0$ to some final value $t^* > 0$ and for each $0 \leqslant t \leqslant t^*$ a segment $S(t)$ of $P \cap L(t)$ is selected, and then a downward phase at which (unique) subsegments $\tilde{S}(t)$ of $S(t)$ are obtained as t decreases down to zero again. The convex polyhedron K will be the union of the $\tilde{S}(t)$.

The Upward Phase

Choose a point $(\xi_1, 0) \in F$. For $t > 0$ small enough we take

$$S(t) = \text{that component of } P \cap L(t) \text{ which contains the point } (\xi_1, t).$$

This set is of the form

$$S(t) = H_\ell \cap H_r \cap L(t) \tag{5.4}$$

for some $\ell, r \in \{1, \ldots, n\}$ with

$$0 \in \overline{F} \subset H_\ell \cap H_r. \tag{5.5}$$

We use ℓ and r as indices to indicate that H_ℓ delimits $S(t)$ from the left and H_r from the right.

Note that, by (5.5), we have $0 \in \operatorname{int} H_\ell$ unless F is of the form (5.3), and $0 \in \operatorname{int} H_r$ unless F is of the form (5.1). In the continued process we are going to choose half-spaces H_i, $i \in \{1, \ldots, n\}$ which will all satisfy $0 \in H_i$, and which will even satisfy

$$0 \in \operatorname{int} H_i \tag{5.6}$$

with exceptions only for the above mentioned cases in the start of the process, and for the initial face with $i = 0$.

For small $t > 0$ we now have

$$H_\ell \cap H_r \cap L(t) \subset P. \tag{5.7}$$

Moreover, the end points of $H_\ell \cap H_r \cap L(t)$ are on ∂P. As t increases we continue to define $S(t)$ by (5.4) as long as (5.7) holds. Then there are two possibilities.

Case 1: (5.7) holds for all $t > 0$. Since P is compact this must mean that $H_\ell \cap H_r \cap L(t)$ is empty for large t. Then set

$$t^* = \sup\{t : H_\ell \cap H_r \cap L(t) \neq \emptyset\}.$$

Thus $0 < t^* < \infty$ and $H_\ell \cap H_r \cap L(t^*)$ consists of a single point.

In Case 1 we define $S(t)$ by (5.4) for all $0 \leqslant t \leqslant t^*$.

Case 2: (5.7) does not hold for all $t > 0$. Then set

$$t_1 = \inf\{t > 0 \colon H_\ell \cap H_r \cap L(t) \not\subset P\}. \tag{5.8}$$

Thus $0 < t_1 < \infty$, $H_\ell \cap H_r \cap L(t) \subset P$ for $0 \leqslant t < t_1$ and even for $t = t_1$ since P is a closed set, but $H_\ell \cap H_r \cap L(t_1 + \varepsilon) \not\subset P$ for small $\varepsilon > 0$.

In Case 2 we define $S(t)$ by (5.4) for $0 \leqslant t \leqslant t_1$, but for $t > t_1$ we need another definition. Let

$$E = \lim_{\varepsilon \searrow 0} (H_\ell \cap H_r \cap L(t_1 + \varepsilon)) \backslash P$$

be the 'obstruction set' for (5.7). The definition of E means, more precisely, that $x \in E$ if and only if there exist sequences $\varepsilon_j \searrow 0$ and $x_j \in (H_\ell \cap H_r \cap L(t_1 + \varepsilon_j)) \backslash P$ such that $x_j \to x$. Clearly $E \subset S(t_1)$, and since P is a polyhedron E must be a finite union of isolated points and closed segments in $S(t_1)$. By definition of Case 2, E is nonempty.

Subcase 2a: If E contains at least one segment we finish the upward phase here and set $t^* = t_1$. Each segment of E must be part of (the closure of) a face of P, and if H_k is the associated half-space of anyone of these faces, then $E \subset \partial H_k = L(t_1)$, $0 \in \operatorname{int} H_k$.

Subcase 2b: If E contains no segments it consists of just finitely many points and we must continue going upwards. $S(t_1) \backslash E$ consists of one or several components each of which is an open or half-open segment in $L(t_1)$. Choose one point in the interior of each segment, say $(\xi_1, t_1), \ldots, (\xi_m, t_1)$ where $\xi_1 < \xi_2 < \cdots < \xi_m$ and $m \geqslant 1$ is the number of segments.

For $\varepsilon > 0$ small $H_\ell \cap H_r \cap P \cap L(t_1 + \varepsilon)$ consists of at least m closed segments (cf. Figure 5.2). There is exactly one segment for each ξ_k, containing $(\xi_k, t_1 + \varepsilon)$ in the interior and containing no other $(\xi_j, t_1 + \varepsilon)$, but there may also be segments containing no $(\xi_k, t_1 + \varepsilon)$. We shall choose $S(t_1 + \varepsilon)$ to be one of the segments which contains a point $(\xi_k, t_1 + \varepsilon)$.

Each of these m candidates for $S(t_1+\varepsilon)$ will be of the form

$$H_{\ell_k}\cap H_{r_k}\cap L(t_1+\varepsilon)$$

for suitable $\ell_k, r_k \in \{1,\dots,n\}$, where we have chosen the notations so that $(\xi_k, t_1+\varepsilon) \in H_{\ell_k}\cap H_{r_k}$ and H_{ℓ_k} defines the left and H_{r_k} the right end point of the segment. ℓ_1 may be the same as ℓ (in (5.4)) and r_m may be the same as r.

It is easy to see that

$$\begin{aligned}
&0\in H_{\ell_1},\\
&0\in \operatorname{int} H_{r_1}\cup \operatorname{int} H_{\ell_2},\\
&0\in \operatorname{int} H_{r_2}\cup \operatorname{int} H_{\ell_3},\\
&\qquad\dots\\
&0\in \operatorname{int} H_{r_{m-1}}\cup \operatorname{int} H_{\ell_m}\\
&0\in H_{r_m}.
\end{aligned}$$

Here we may also replace H_{ℓ_1} by $\operatorname{int} H_{\ell_1}$ unless $\ell_1=\ell$ and F is of the form (5.3), and H_{r_m} by $\operatorname{int} H_{r_m}$ unless $r_m=r$ and F is of the form (5.1).

From the above it follows that

$$0\in H_{\ell_k}\cap H_{r_k} \tag{5.9}$$

for at least one k, where also $0\in \operatorname{int} H_{\ell_k}$, $0\in \operatorname{int} H_{r_k}$ with the above-mentioned possible exceptions for $\ell_k=\ell$ $(k=1)$ and $r_k=r$ $(k=m)$. Now select a k such that (5.9) holds and define

$$S(t_1+\varepsilon)=H_{\ell_k}\cap H_{r_k}\cap L(t_1+\varepsilon) \tag{5.10}$$

for $\varepsilon>0$ small.

If $m=1$, i.e. E just consists of one or both end points of $S(t_1)$, then $S(t)$ changes in a continuous way as t passes t_1:

$$\lim_{t\searrow t_1} S(t)=S(t_1)=\lim_{t\nearrow t_1} S(t),$$

while if $m\geqslant 2$ $S(t)$ jumps down to a subsegment as t passes t_1. In general we therefore have a kind of semicontinuity at $t=t_1$:

$$\lim_{t\searrow t_1} S(t)\subset S(t_1)=\lim_{t\nearrow t_1} S(t). \tag{5.11}$$

Definition (5.10) will be in force as long as the right member is contained in P. Setting $t=t_1+\varepsilon$ we are then in the same situation as at the beginning of the upward phase: either

$$H_{\ell_k}\cap H_{r_k}\cap L(t)\subset P$$

for all $t > t_1$ (Case 1), in which case there is a last $t = t^*$ for which the left member is nonempty, or (Case 2) an obstruction occurs for some $t = t_2 > t_1$. In the latter case we get subcases as before: in Subcase 2a we are finished and set $t^* = t_2$, in Subcase 2b we must change the definition again.

Proceeding in the above way we end up with a sequence of obstruction levels $0 < t_1 < t_2 < \cdots$. Since P is compact this sequence must terminate, which means that we finally run into Case 1 or Subcase 2a. Let $t^* = t_l$ be the terminating level and set also $t_0 = 0$. On each interval $t_{k-1} < t \leqslant t_k$ $(k = 1, \dots, l)$ $S(t)$ is defined by an expression

$$S(t) = H_{\ell_k} \cap H_{r_k} \cap L(t) \quad (t_{k-1} < t \leqslant t_k). \tag{5.12}$$

(We have here changed the notation for the indices slightly, so that k now refers to the level of t. Thus, e.g., $\ell_1 = \ell$, $r_1 = r$ with ℓ and r the indices in (5.4).)

Let

$$I = \{\ell_1, r_1, \ell_2, r_2, \dots, \ell_l, r_l\}$$

be the set of indices used and recall that

$$0 \in H_i \quad \text{for all } \ i \in I, \tag{5.13}$$

and that $0 \in \operatorname{int} H_i$ with possible exceptions for $i = \ell_1, r_1$. The result of the upward phase is the set

$$Q = \bigcup_{0 \leqslant t \leqslant t^*} S(t).$$

It is easy to see that Q is a polyhedron contained in P. Some of its faces are 'horizontal' (parallel to the x_1-axis), namely the bottom F, possibly the top $S(t^*)$ (if the upward phase is terminated by Subcase 2a) and possibly also intermediate faces arising from cases of strict inclusions in semicontinuities (5.11) (for $t_1, \dots, t_{l-1}$). The associated half-space of F is $H_0 = \{x \in \mathbb{R}^2 \colon x_2 \geqslant 0\}$, the associated half-spaces of the other horizontal faces are all of the form $H = \{x \in \mathbb{R}^2 \colon x_2 \leqslant t_j\}$ for some $j = 1, \dots, l$, hence $0 \in \operatorname{int} H$.

The nonhorizontal faces of Q are those defined by H_{ℓ_k}, H_{r_k} in (5.12), hence their associated half-spaces are exactly the H_i, $i \in I$.

Unfortunately, Q neither is convex nor sticks to P in general. What we have of convexity properties are the following

$$\begin{aligned} &Q \text{ is convex in the } x_1\text{-direction;} \\ &Q \text{ is starshaped with respect to } 0. \end{aligned} \tag{5.14}$$

The first statement just says that each $S(t)$ is connected (which it is by construction) and the second statement follows easily from (5.11), (5.12) and (5.13).

We record here also the semicontinuity property of Q (see (5.11)): For any $0 < t \leqslant t^*$,

$$\lim_{\varepsilon \searrow 0} Q \cap L(t + \varepsilon) \subset Q \cap L(t) = \lim_{\varepsilon \searrow 0} Q \cap L(t - \varepsilon). \tag{5.15}$$

As to sticking properties we note that the bottom face F and, if we finished through Subcase 2a, the top face $S(t^*)$ both stick to P. Moreover, all the nonhorizontal faces of Q stick to P because these are defined by H_i, $i \in I$ as in (5.12) and each time we made a change of some H_i, or when we first chose H_{ℓ_1}, H_{r_1}, we made it by 'necessity', namely that a condition like (5.7) forced us to do it. Therefore, we can even state a little more, as follows.

Given any H_i, $i \in I$, let $t_{k-1} < t \leqslant t_k$ $(1 \leqslant k \leqslant l)$ be the first t-interval at which it was used (e.g., $H_i = H_{\ell_k} = H_{\ell_{k+1}}$ but $H_i \neq H_{\ell_{k-1}}$). Then the corresponding face of Q sticks to P on some interval $t_{k-1} < t < t_{k-1} + \varepsilon$ $(\varepsilon > 0)$. More precisely, with $k = k(i)$ defined as above we have

$$\begin{aligned} &\text{for } \varepsilon > 0 \text{ sufficiently small any } x \in \partial H_i \text{ with} \\ &t_{k-1} < x_2 < t_{k-1} + \varepsilon \text{ has a neighborhood} \\ &B = B(x, r) \text{ such that } Q \cap B = P \cap B = H_i \cap B. \end{aligned} \tag{5.16}$$

The nonconvexity and nonsticking properties of Q will be straightened out in the downward phase.

The downward phase

This can briefly be described as being the upward phase applied to Q turned upside-down, but there are less complications so we prefer to give a direct description.

We shall choose segments $\tilde{S}(t) \subset S(t)$ for t decreasing from t^* to 0. For $t \leqslant t^*$ close to t^* we simply take

$$\tilde{S}(t) = S(t) = Q \cap L(t).$$

Thus

$$\tilde{S}(t) = H_{\ell_l} \cap H_{r_l} \cap L(t) \tag{5.17}$$

for $t^* - \varepsilon < t \leqslant t^*$ say. As t decreases we stick to this definition as long as

$$H_{\ell_l} \cap H_{r_l} \cap L(t) \subset Q. \tag{5.18}$$

If an obstruction occurs, at $t = \tilde{t}_1$ say, we get an obstruction set $E = \lim_{\varepsilon \searrow 0} (H_{\ell_l} \cap H_{r_l} \cap L(\tilde{t}_1 - \varepsilon)) \setminus Q$ as in the upward phase. Because of the properties of Q, namely starshapedness (5.14) or semicontinuity (5.15), the obstruction set E can in the downward phase only consist of one or both end points of $\tilde{S}(\tilde{t}_1)$.

Therefore we simply need to take

$$\tilde{S}(\tilde{t}_1 - \varepsilon) = H_{\ell_l} \cap H_{r_l} \cap Q \cap L(\tilde{t}_1 - \varepsilon) \tag{5.19}$$

for $\varepsilon > 0$ small. This set is of the form

$$\tilde{S}(\tilde{t}_1 - \varepsilon) = H_\ell \cap H_r \cap L(\tilde{t}_1 - \varepsilon) \tag{5.20}$$

for some $\ell, r \in I$, one of which may be the same as ℓ_l or r_l.

By (5.15), (5.19) $\tilde{S}(t)$ changes in a continuous way as t passes $\tilde{t}_1$. Therefore $\partial H_\ell \cap \partial H_{\ell_l} \subset L(\tilde{t}_1)$ or $H_\ell = H_{\ell_l}$, and similarly for r and r_l. Since H_ℓ, H_r were introduced in order to restrict $\tilde{S}(t)$ it follows that

$$H_\ell \cap H_r \cap L(\tilde{t}_1 - \varepsilon) \subset H_{\ell_l} \cap H_{r_l} \cap L(\tilde{t}_1 - \varepsilon)$$

for small $\varepsilon > 0$, and even for all $\varepsilon > 0$. Therefore we could as well have retained the first chosen half-spaces in the definition of $\tilde{S}$, i.e.,

$$\tilde{S}(t) = H_{\ell_l} \cap H_{r_l} \cap H_\ell \cap H_r \cap L(t) \tag{5.21}$$

for $\tilde{t}_1 - \varepsilon < t < \tilde{t}_1$. For similar reasons (5.21) actually holds also for $\tilde{t}_1 \leqslant t \leqslant t^*$.

Now we continue to define $\tilde{S}(t)$ by (5.20) or (5.21) as long as $t \geqslant 0$ and $H_\ell \cap H_r \cap L(t) \subset Q$ $(t < \tilde{t}_1)$, and if obstructions occur we make new modifications according to the same recipe as above. We end up with a finite sequence of obstruction levels $t^* > \tilde{t}_1 > \tilde{t}_2 > \cdots > 0$ and with having $\tilde{S}(t)$ defined for all $0 \leqslant t \leqslant t^*$. Define

$$K = \bigcup_{0 \leqslant t \leqslant t^*} \tilde{S}(t).$$

This will be shown to be a convex polyhedron having the properties stated in the beginning of the proof.

Let us relabel the indices for the downward phase: set $\tilde{\ell}_1 = \ell_l$, $\tilde{r}_1 = r_l$, $\tilde{\ell}_2 = \ell$, $\tilde{r}_2 = r$ (ℓ, r the indices in (5.20)) etc., so that $H_{\tilde{\ell}_k}$, $H_{\tilde{r}_k}$ are the half-spaces used in the interval $\tilde{t}_{k-1} > t \geqslant \tilde{t}_k$ ($k = 1, \ldots, \tilde{l}$, say, and $\tilde{t}_0 = t^*$). Set

$$\tilde{I} = \{\tilde{\ell}_1, \tilde{r}_1, \ldots, \tilde{\ell}_{\tilde{l}}, \tilde{r}_{\tilde{l}}\}.$$

Then $\tilde{I} \subset I$.

For $\tilde{t}_{k-1} > t \geqslant \tilde{t}_k$, $\tilde{S}(t)$ is defined by (cf. (5.17))

$$\tilde{S}(t) = H_{\tilde{\ell}_k} \cap H_{\tilde{r}_k} \cap L(t), \tag{5.22}$$

but as indicated at (5.21) we could also cut by $H_{\tilde{\ell}_{k-1}}$ and $H_{\tilde{r}_{k-1}}$ without changing anything, and also by $H_{\tilde{\ell}_{k+1}}$ and $H_{\tilde{r}_{k+1}}$, etc. Therefore we actually have

$$\tilde{S}(t) = \bigcap_{j=1}^{\tilde{l}} (H_{\tilde{\ell}_j} \cap H_{\tilde{r}_j}) \cap L(t) \tag{5.23}$$

for $\tilde{t}_{k-1} > t \geqslant \tilde{t}_k$, and even for all $t^* \geqslant t \geqslant 0$.

From (5.23) it follows that K is given by

$$K = \bigcap_{j=1}^{l} (H_{\tilde{\ell}_j} \cap H_{\tilde{r}_j}) \cap H_0 \cap H_\infty$$
$$= \bigcap_{i \in \tilde{I}} H_i \cap H_0 \cap H_\infty, \tag{5.24}$$

where $H_\infty = \{x \in \mathbb{R}^2 : x_2 \leqslant t^*\}$, and in particular that K is a convex polyhedron. The half-space H_∞ is needed only if the upward phase terminated through Subcase 2a.

Next we show that K contains $C \cap B(0, \varepsilon)$ with $\varepsilon > 0$ small and C as in the beginning of the proof. Assume first that F is of the form (5.2). Then $C = H_0$ and $0 \in \operatorname{int} H_{\ell_1} \cap \operatorname{int} H_{r_1}$, hence $0 \in \operatorname{int} H_i$ for all $i \in I$. Thus

$$0 \in \operatorname{int} \left(\bigcap_{i \in \tilde{I}} H_i \cap H_\infty \right)$$

which by (5.24) gives that K contains $C \cap B(0, \varepsilon) = H_0 \cap B(0, \varepsilon)$ for $\varepsilon > 0$ small.

If F is of the form (5.1) then $0 \in \operatorname{int} H_{\ell_1}$. It is still possible that $0 \in \operatorname{int} H_{r_1}$, and in this case $C = H_0$ and everything works as in the previous case. The other possibility is that $0 \in \partial H_{r_1}$. Then, by the way H_{r_1} was chosen, $C = H_0 \cap H_{r_1}$. By previous remarks (see before and after (5.9)) the origin is in the interior of all half-spaces in the expression (5.24) for K, except H_0 and H_{r_1}. Thus K contains $C \cap B(0, \varepsilon) = H_0 \cap H_r \cap B(0, \varepsilon)$ for $\varepsilon > 0$ small.

The case that F is of the form (5.3) is analogous to the above case. Hence K has the desired property at the origin. Note that it also follows from the above that K contains $F \cap B(0, \varepsilon)$ for some $\varepsilon > 0$ and hence that $K \cap \partial H_0$ is in the closure of a face of K which sticks to the face F of P.

We finally prove that K sticks to P. The bottom $K \cap \partial H_0$ was treated just above. In a neighbourhood of ∂H_∞, K agrees with Q (the downward phase started by not changing anything) and we have already remarked (after (5.15)) that if $Q \cap \partial H_\infty$ is a segment, i.e. the closure of a face of Q, then it sticks to a face of P. Thus the same is true for $K \cap \partial H_\infty$.

It remains to treat $K \cap \partial H_i$, $i \in \tilde{I}$. Given $i \in \tilde{I}$, let G be the face of K satisfying $K \cap \partial H_i \subset \overline{G}$. By construction K sticks to Q. Therefore $G \cap G' \neq \varnothing$ for some face G' of Q. Thus

$$G \cap G' \cap L(t) \neq \varnothing \tag{5.25}$$

for some $t > 0$. Let t_* be the infimum of the set of $t > 0$ for which (5.25) holds. We distinguish between the following possibilities (exhaustive list).

(i) $t_* = 0$;
(ii) $t_* > 0$, $G \cap L(t_*) \neq \varnothing$;
(iii) $t_* > 0$, $G' \cap L(t_*) \neq \varnothing$;
(iv) $t_* > 0$, $(G \cup G') \cap L(t_*) = \varnothing$.

Note that $G \cap G' \cap L(t_*) = \varnothing$ since G, G' are relatively open in ∂H_i. Item (ii) means that, as t decreases, G survives beyond t_*, (iii) means that G' survives and (iv) means that both G and G' reach their lower end points at t_*.

Now we claim that (iii) cannot occur. This is indeed immediate from the construction of the $\tilde{S}(t)$: we did not change the half-spaces in, e.g., (5.22) unless we were forced to do it in order to keep $\tilde{S}(t)$ inside Q, and if, as in case (iii) above, G' had survived below t_* then there would be no need to change the corresponding half-space and so also G would survive, because $t_* > 0$ and we have already proved that the bottom of K is situated on ∂H_0.

Thus only (i), (ii), (iv) can occur, which means that, as t decreases, (5.25) remains valid until we reach the level t_* of the lower end point of G'. This level is $t_* = t_k$ for some $k = 0, 1, \dots, l-1$ (notations from upward phase). But now, by remarks at (5.16), the lowest part of a face G' of Q always sticks to a face of P. Hence, by (5.25), also G sticks to a face of P just above the level t_*. This finishes the proof that K sticks to P, and also finishes the proof of the theorem. □

COROLLARY 5.2. *Any polyhedron P in two dimensions has at least one mother body. Indeed, there is exactly one mother body if each component of* int P *is convex, at least two if there is a component of* int P *which is not convex.*

Proof. The proof of existence of at least one mother body was indicated at the end of Section 4. If each component of int P is convex, then it follows from [Gu1] (or remarks at the end of Section 3 in the present paper) that the mother body is unique.

If one component of int P is nonconvex then, by Lemma 4.4, P has a 'concave edge', i.e., a concave corner in the present two-dimensional context, in the sense of Lemma 4.4. It follows that if we start the construction in the proof of Theorem 5.1 in that concave corner then there will be at least two possibilities to choose the face F there. Therefore there will exist two different convex polyhedra which stick to P and which partly overlap near the corner. Thus continuing the process as at the end of Section 4 the final result will be two different decompositions (4.16), i.e., two different mother bodies for P. □

Next we give an example showing that the theorem is optimal in one sense: the convex polyhedra sticking to a polyhedron $P \subset \mathbb{R}^2$ do not cover all of P in general, just a neighbourhood of ∂P. By making a 'cylinder' of this example, e.g. by taking $Q = P \times [0, 1] \subset \mathbb{R}^3$, one also sees easily that the theorem does not generalize to $\mathbb{R}^3$ (or higher dimension): the convex polyhedra sticking to a polyhedron $Q \subset \mathbb{R}^3$ do not cover all of ∂Q in general.

As to the latter statement, there are even stronger examples: in our, so far unsuccessful, attempts to construct polyhedra in higher dimensions having no sticking convex subpolyhedra whatsoever, we at least have been able to construct a polyhedron Q in $\mathbb{R}^3$ and a half-space H such that $Q \cap H$ makes up only a small fraction of Q (measured in volume, e.g.) and such that every convex polyhedron which sticks to Q is contained in $Q \cap H$. Moreover, Q has a face which is entirely outside $Q \cap H$, in fact is separated from $Q \cap H$ by a distance close to the diameter of Q. (The details of this example will not be given here.)

EXAMPLE 5.3. We shall construct a polyhedron $P \subset \mathbb{R}^2$ with $0 \in \operatorname{int} P$ such that there is no convex polyhedron K with $0 \in K$ which sticks to P.

The construction depends on two integers, n and k, satisfying

$$k \geqslant 2, \qquad n \geqslant 2k + 3, \quad n \text{ is not divisible by } k.$$

It is easiest to think of the case with k small (say 2 or 3) and n large, and then P may be chosen so that the convex polyhedra which stick to P cover just a thin layer around ∂P.

$n = 16$
$k = 3$

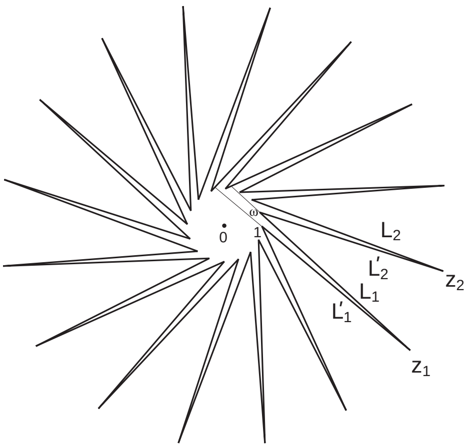

Figure 5.6.

Shaded area is K_1

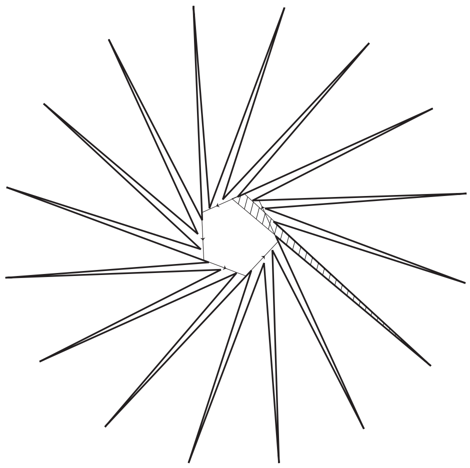

Figure 5.7.

The smallest possible choice of n is $n = 7$ (with $k = 2$), and then for all $n > 7$ except $n = 8$ and $n = 12$ there will be at least one possible value of k. Figures 5.6–5.7 illustrate the construction below for the case $n = 16$, $k = 3$.

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and set $\omega = \mathrm{e}^{2\pi i/n} \in \mathbb{C}$. The nth roots of unity $1, \omega, \ldots, \omega^{n-1}$ will be the nonconvex corners of P. Let L_1' be a straight line through $\omega^0 = 1$ which intersects the unit circle once more between ω^{k+1} and

ω^{k+2}. In the special case that $n = 2k + 3$, which is the only case in which ω^{k+2} lies in the lower half-space, we require more precisely that the latter intersection occurs between ω^{k+1} and -1. Thus, in all cases L_1' will have strictly negative slope. Let H_1' be the closed half-space defined by $\partial H_1' = L_1'$, $0 \notin H_1'$.

Next, let L_1 be a straight line through ω which intersects the unit circle once more between ω^k and ω^{k+1} and chosen to have a more negative slope than L_1'. Then L_1 and L_1' intersect at some point z_1 satisfying $\operatorname{Re} z_1 > 1$, $\operatorname{Im} z_1 < 0$. Let H_1 be the closed half-space defined by $\partial H_1 = L_1$, $0 \in H_1$.

P will be invariant under rotations by $2\pi/n$, and we denote by L_j, L_j', H_j, H_j', z_j $(j = 1, 2, \ldots, n)$ the objects obtained by rotation from the above first chosen ones. Then P is defined to be the polyhedron bounded by the line L_1' from $\omega^0 = 1$ to z_1, the line L_1 from z_1 to $\omega^1 = \omega$, the line L_2' from ω to z_2, the line L_2 from z_2 to ω^2 etc. Set theoretically we have, setting $B = \{z \in \mathbb{C} : |z| < 1\}$ and denoting by C_j the bounded component of $(H_j \cap H_j')\setminus B$,

$$P = B \cup \bigcup_{j=1}^{n} C_j.$$

We extend notations cyclically, so that $z_{n+j} = z_j$ $(j = 1, 2, \ldots, k)$ etc. Clearly, P is a polyhedron.

Now we prove that there is no convex polyhedron containing the origin which sticks to P. Any convex polyhedron K which sticks to P must be of the form

$$K = \bigcap_{i \in I} H_i \cap \bigcap_{j \in J} H_j'$$

for suitable index sets $I, J \subset \{1, \ldots, n\}$. Since $0 \notin H_j'$ we must actually have

$$K = \bigcap_{i \in I} H_i \tag{5.26}$$

in order to have $0 \in K$.

We may assume that $1 \in I$, i.e., that H_1 is present in the description (5.26). Then none of $2, \ldots, k$ can be in I because $K \subset H_1$ and those parts of $\partial H_2, \ldots, \partial H_k$ which stick to ∂P (namely the segments from z_j to ω^j of L_j, $j = 2, \ldots, k$) are all outside H_1. On the other hand we must have $k + 1 \in I$ because otherwise K would contain a full neighbourhood of ω^{k+1}, contradicting $K \subset P$ (note that $\omega^{k+1} \in \operatorname{int} H_j$ for all j except those with $2 \leqslant j \leqslant k + 1$). Continuing in the same way we see that $k+2, \ldots, 2k \notin I$, $2k+1 \in I$ etc. Since n is not divisible by k we reach a contradiction when we come back to $H_{n+1} = H_1$ again. Thus no index set I works.

We remark without proof that the convex polyhedra which stick to P are exactly the following n ones

$$K_j = H_j' \cap H_j \cap H_{j+k}, \quad j = 1, \ldots, n.$$

Of course, $0 \notin K_j$.

6. Open Questions

We list here two main questions which were left unanswered in this paper.

Open Question 1: Does every polyhedron in $\mathbb{R}^N$, $N \geqslant 3$, admit at least one mother body?

Open Question 2: Does every polyhedron in $\mathbb{R}^N$, $N \geqslant 3$, contain a convex polyhedron which sticks to it?

As shown in Section 4, a positive answer to Question 2 implies a positive answer to Question 1.

Acknowledgements

We wish to thank Sadayoshi Kojima, Dmitrij Kozlov, Henrik Shahgholian, Harold S. Shapiro and Tomoyoshi Yoshida for inspiring discussions and important information concerning mother bodies and polyhedra. We also thank Siv Sandvik for typing our manuscript.

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