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003

ON MOTHER BODIES OF CONVEX POLYHEDRA*

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Abstract. If Ω is a bounded domain in $\mathbb{R}^N$ provided with a mass distribution ρ_Ω (e.g., Lebesgue measure restricted to Ω), another mass distribution μ sitting in Ω and producing the same external Newtonian potential as ρ_Ω is sometimes called a mother body of Ω, provided it is maximally concentrated in some sense. We first discuss the meaning of this and formulate five desirable properties ("axioms") of mother bodies. Then we show that convex polyhedra do have unique mother bodies in that sense made precise in the case that ρ_Ω is either Lebesgue measure on Ω, hypersurface measure on $\partial\Omega$, or any mixture of these two.

Key words. convex polyhedron, ridge, mother body, skeleton, balayage

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1. Introduction. A mother body (or maternal or materic body) in the terminology of the Bulgarian school of geophysics [Zi], [Ko1], [Ko2] is a more concentrated mass distribution sitting in a given body and producing the same external gravitational field as the latter. For example, one good mother body for a ball with constant mass density is a point mass (of appropriate strength) at the center of the ball. The meaning of a mother body being "more concentrated" is quite vague and there is no general agreement of its exact meaning.

Mother bodies are an important computational tool in geophysics (see, e.g., [Zi], [Ko2]). For solid polyhedra with constant mass density there are natural candidates of mother bodies with support on systems of hyperplanes reaching the boundary of the polyhedron at edges and corners. There is a beautiful example of D. Zidarov [Zi, Sect. III.6] (see also section 4 in the present paper) showing that mother bodies of this sort are not unique in general. One purpose of the present paper is to show that for *convex* polyhedra we do have uniqueness. (Zidarov's counterexample is a square in two dimensions with a smaller square at one corner cut away; hence, it is nonconvex.) The same result holds if, instead of constant volume density, the mass of the polyhedron is sitting on its boundary and has constant density there with respect to surface measure and even for any mixture of these two measures.

This paper however starts with a long discussion of what one should reasonably require of a mother body. This results in five "axioms" ((1)–(5) below), which we feel are fairly well motivated. In practice it is usually not possible to satisfy them all, but they could at least be looked upon as guide lines. The formulation of such a system of axioms is a secondary purpose of this paper.

SOME GENERAL NOTATION. If $A \subset \mathbb{R}^N$ we set
$A^c = \mathbb{R}^N \backslash A$,
$A^e = \mathbb{R}^N \backslash \bar{A}$ ($\bar{A}$ = closure of A),
int A = the interior of A,
$B(x,r) = \{y \in \mathbb{R}^N : |y-x| < r\}$,
$\mathcal{L}^N$ = N-dimensional Lebesgue measure,

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$\mathcal{H}^{N-1} = (N-1)$-dimensional Hausdorff measure,
$\mathcal{L}^N\lfloor\Omega$, $\mathcal{H}^{N-1}\lfloor\partial\Omega$: the above measures restricted to Ω and $\partial\Omega$, respectively,
supp $\mu =$ the closed support of a distribution μ.

$$E(x) = \begin{cases} -c_2 \log|x| & (N=2), \\ \ \ c_N|x|^{2-N} & (N \geq 3) \end{cases}$$

is the Newtonian kernel so that $-\Delta E = \delta$, the Dirac measure at the origin.
$U^\mu = E * \mu =$ the Newtonian potential of μ, if μ is a distribution with compact support in $\mathbb{R}^N$. Thus $-\Delta U^\mu = \mu$.

2. Discussion of mother bodies. By a "body" we shall mean a bounded domain $\Omega \subset \mathbb{R}^N$ satisfying $\Omega = \operatorname{int}\left(\overline{\Omega}\right)$, $\mathcal{H}^{N-1}(\partial\Omega) < \infty$, and provided with an associated mass distribution $\rho = \rho_\Omega$. Primarily we think of the mass distribution with density one in the domain and density zero outside; i.e., $\rho = \mathcal{L}^N\lfloor\Omega$. However, the results in this paper work equally well for the case of hypersurface measure on the boundary, i.e., $\rho = \mathcal{H}^{N-1}\lfloor\partial\Omega$, or for any mixture of these two.

Thus given any two constants $a, b \geq 0$ with $a + b > 0$ we associate with any Ω as above the mass distribution

$$\rho_\Omega = a\mathcal{H}^{N-1}\lfloor\partial\Omega + b\mathcal{L}^N\lfloor\Omega.$$

Then ρ_Ω is a positive Radon measure, and we denote by U^Ω its Newtonian potential

$$U^\Omega = U^{\rho_\Omega} = E * \rho_\Omega$$

(a and b will be kept fixed throughout the discussion).

Given a body $\Omega \subset \mathbb{R}^N$, a mother body for it should be a signed measure μ having certain properties. The basic requirement is that

$$U^\mu = U^\Omega \qquad \text{in } \Omega^e. \tag{1}$$

Clearly this implies that supp $\mu \subset \bar{\Omega}$.

One possible additional requirement is that

$$U^\mu \geq U^\Omega \qquad \text{in all } \mathbb{R}^N. \tag{2}$$

Such a condition is natural if one wishes to think of ρ_Ω as being the result of applying some kind of (partial) balayage process to μ (cf. [Zi], [Sa1], [Ko1], [Ko2], [Gu-Sa1], [Gu-Sg], [Gu2]). Indeed, any balayage (or "sweeping") process we know of can be thought of as being composed of elementary steps in which point masses are swept to measures of the kind ρ_B for balls B centered at the support of the point masses, and for each such elementary step the potential of the measure decreases.

Thus in order to have a μ which is as "primitive" as possible with respect to balayage one should ask U^μ to be as large as possible. For this to be sensible one has to have a lower bound on μ because otherwise one can always increase a given U^μ. For example, as is natural, one may ask μ to be positive:

$$\mu \geq 0. \tag{3}$$

Since $\mu = \rho_\Omega$ itself satisfies (1)–(3) and the supremum of any increasing sequence of superharmonic functions is superharmonic (or $\equiv +\infty$), it follows that for any given Ω there exist (plenty of) measures μ satisfying (1)–(3) with U^μ maximal among potentials of such measures. A mother body for Ω should be one of these (cf. Proposition 2.1 at the end of this section).

In order for μ to be a good mother body it should be concentrated or minimal in some sense, such as having small support sitting deeply inside Ω. Although this is to some degree implicit in the desire that U^μ should be as large as possible, we also want to formulate such conditions in direct geometric terms. One way to be concentrated is simply to be singular with respect to Lebesgue measure. We shall find the slightly stronger requirement

$$\mathcal{L}^N(\text{supp } \mu) = 0 \tag{4}$$

convenient to work with.

It is easy to see, however, that (4) does not guarantee a good mother body. For any Ω there are an abundance of measures μ satisfying all of (1)–(4). It is just to fill Ω with (infinitely many) disjoint balls so that the remaining set has measure zero, and then replace the volume part of ρ_Ω by the sum of the appropriate point masses sitting in the centers of these balls. In other words, one writes $\Omega = \bigcup_{j=1}^\infty B(x_j, r_j) \cup (\text{null set})$, where the $B(x_j, r_j)$ are disjoint, and then takes $\mu = a\mathcal{H}^{N-1}\lfloor\partial\Omega + b\sum_{j=1}^\infty \mathcal{L}^N(B(x_j, r_j))\delta_{x_j}$, δ_x denoting the unit point mass at $x \in \mathbb{R}^N$.

One way in which a mother body μ constructed as above, by ball-packing, is not good is that supp μ typically (even if $a = 0$) contains all of $\partial\Omega$, and therefore cuts off the exterior of Ω from the interior. This must necessarily be so in general because when supp μ does not reach $\partial\Omega$ then (1) gives a harmonic continuation of U^Ω across $\partial\Omega$ into Ω, which is not possible unless $\partial\Omega$ is real analytic (roughly speaking). Nevertheless, whenever possible we desire something like the following to hold.

$$\text{Each component of } \mathbb{R}^N \backslash \text{supp } \mu \text{ intersects } \Omega^e. \tag{5}$$

This simply means that for each $x \in \Omega\backslash\text{supp } \mu$ there is a curve in $\mathbb{R}^N\backslash\text{supp } \mu$ joining x with some point in Ω^e.

The requirements (1)–(5) are the "axioms" for a mother body which we propose. As indicated earlier there is neither existence nor uniqueness of mother bodies satisfying (1)–(5) in general (Zidarov's counterexample fulfills all of (1)–(5)). Indeed, the problem of finding a mother body of a given body exhibits all features of an ill-posed problem: existence and uniqueness of solutions only under special conditions and sensitive dependence on given data when solutions do exist. Nevertheless, for certain particular classes of bodies, e.g., various kinds of polyhedra (see sections 3 and 4 below and [Gu-Sa2]) and certain types of algebraic domains [Sav-St-Sv], there are constructive algorithms for computing (candidates of) mother bodies.

For the rest of this section, we discuss in more detail the roles of the axioms (1)–(5) and various ways of relaxing or strengthening them. The axioms naturally fall into three groups: (1); (2) and (3); (4) and (5).

Axiom (1) is the most indispensable one. In the case that Ω^e has more than one component, a possible way to relax it is to require only

$$\nabla U^\mu = \nabla U^\Omega \text{ in } \Omega^e$$

(equality of the corresponding fields), which is actually more physical. An even weaker requirement is to ask (1) to hold only in the unbounded component of Ω^e.

The role of the conditions (2) and (partly) (3) is to guarantee that ρ_Ω is the result of a natural balayage operator applied to μ. When $a = 0$, $b > 0$, such an operator $\mu \mapsto \text{Bal}(\mu; b)$ can be defined by declaring that $\text{Bal}(\mu; b)$ shall be the measure which is closest to μ in the energy norm among all measures ν which satisfy $\nu \leq b\mathcal{L}^N$. This makes plain sense and defines $\mathrm{Bal}(\mu; b)$ uniquely whenever $\mu \geq 0$ has finite energy. The definition can then easily be extended to the case of infinite energy. One can show that if Ω is a body, then $\mathrm{Bal}(\mu; b) = \rho_\Omega$ holds if and only if both (1) and (2) are satisfied. In particular, it is possible to reconstruct Ω from μ when (1) and (2) hold, and both conditions are really necessary for this (there are examples of two different Ω satisfying (1), (3)–(5) for the same μ).

Thus the perhaps abstract-looking condition (2) plays a significant role in the context of balayage. It is probably more important than (3) because it is possible, to a certain extent, to allow nonpositive measures μ in $\mathrm{Bal}(\mu; b)$. We refer to [Sa1], [Gu-Sa1], [Gu2] for details on the above balayage operators.

For a general measure μ, $\mathrm{Bal}(\mu; b)$ will not necessarily be of the form ρ_Ω for some open set Ω, but if μ satisfies (3) and (4), it will. This allows for doing "continuous balayage," as follows. Suppose (1)–(4) hold for the pair (Ω, μ), and define for any $t \in \mathbb{R}$ the open set $\Omega(t)$ by $\mathrm{Bal}(e^t\mu; b) = \rho_{\Omega(t)}$. (This defines $\Omega(t)$ only up to a null set, but one naturally takes the largest possible $\Omega(t)$.) Then $\Omega(s) \subset \Omega(t)$ for $s < t$, $\Omega(0) = \Omega$, and $\Omega(t)$ shrinks down to $\operatorname{supp} \mu$ as $t \to -\infty$. Moreover, the pair $(\Omega(t), e^t\mu)$ satisfies (1)–(4) for each $t \in \mathbb{R}^N$.

One important point with this family $\Omega(t)$ is that its evolution can be described without reference to μ. Indeed, under some smoothness assumptions the evolution can be described by a nonlocal, but μ-independent, differential equation for the motion of $\partial\Omega(t)$: the normal velocity of the boundary $\partial\Omega(t)$ at any particular point is to be equal to the normal derivative at that point of the function $p = p_{\Omega(t)}$ which solves the Dirichlet problem $\Delta p = 1$ in $\Omega(t)$, $p = 0$ on $\partial\Omega(t)$. This is a Hele–Shaw type moving boundary problem, and by solving it (backwards) for $-\infty < t \leq 0$ with $\Omega(0) = \Omega$ as initial domain one should, in principle, get a canonical candidate of a mother body, namely by taking $\mu = \lim_{t\to-\infty} e^{-t}\rho_{\Omega(t)}$. Unfortunately, however, this moving boundary problem is badly ill-posed and existence of global solutions is not to be expected in general. Local in time solutions exist if, and basically only if, the initial domain has a real analytic boundary (see, e.g., [Re-Wo], [Ti]).

It is possible to introduce balayage operators as above and to do continuous balayage also when $a > 0$, but everything is more complicated in that case: the balayage operators are less well behaved and the evolution families are less continuous (cf. [He], [Gu-Sg]). It is not even true that μ determines Ω uniquely via (1)–(5) [He, Prop. 6.2]. Axiom (2) does not quite suffice for this, as it does in the case $a = 0$, and should therefore ideally be replaced by something stronger.

Returning now to the general case $(a, b \geq 0)$, another advantage with conditions (2), (3) is that they guarantee a certain coupling between the geometry of Ω and of $\operatorname{supp} \mu$. One may therefore prove [Gu-Sa1], [Sg], [Gu-Sg] that for any Ω and any point $x \in \partial\Omega$, the inward normal ray of $\partial\Omega$ at x intersects the closed convex hull of the support of any μ satisfying (1)–(3). Without (2), (3) there will be no geometric coupling whatsoever between Ω and $\operatorname{supp} \mu$. For any domain D and any (small) ball $B \subset D$ one can find a domain Ω approximating D arbitrarily well and a signed measure μ with $\operatorname{supp} \mu \subset B$ such that (1), (4), (5) hold for Ω, μ. See [Gu1], [Sa2] for the case $a = 0$.

From what has been said above it should be clear that conditions (2) and (3) have strong potential theoretic significance. There are however other points of view for which they seem less urgent. In certain complex variable and PDE approaches, see, e.g., [Eb], [Kh-Sh], [Sh], [St-Sv], one considers the search for mother bodies mainly as a problem of analytic continuation (of U^Ω), and one is happy if one can find a distribution (or even analytic functional) μ which satisfies (1) and some (usually stronger) form of (4), (5). If supp μ is then sufficiently small there will simply be no other good candidate for a mother body.

Also for questions of uniqueness of mother bodies conditions (2) and (3) appear often to be dispensable.

The last group of axioms, (4) and (5), are requirements only on the set supp μ. They imply that supp μ is minimal as a set (see Proposition 2.1 below), and they are necessary to guarantee any reasonable degree of uniqueness of mother bodies, e.g., to exclude occurrence of continuous families of them. A sharper form of (5), which together with (1) and (4) definitely guarantees uniqueness (see Proposition 2.1), is

$$\text{supp } \mu \text{ does not disconnect any open set} \tag{6}$$

(i.e., $D \setminus \text{supp } \mu$ is connected whenever D is an open connected set). Clearly (6) implies (5). However, with requirement (6) in place of (5), mother bodies will exist more rarely (polyhedra will not admit mother bodies, for example). On the other hand, in cases when one allows distributional mother bodies (5) becomes too weak to even exclude continuous families of mother bodies and therefore has to be replaced by something stronger, like (6).

The strongest reasonable requirement in the direction of (4), (5), (6) is to require supp μ to consist of only finitely many points. This is what one (classically) requires of a "quadrature domain," namely that there exists a measure or distribution μ with finite support and satisfying some form of (1). The word quadrature domain is however also used in wider senses. See [Sh] for an overview.

In two dimensions, quadrature domains in the above (classical) sense can be produced as conformal images of the unit disc under rational mapping functions. (This is for the case $a = 0$, to which we stick for a moment.) Taking for example $\Omega = f(B(0,1))$, where $f(z) = z + c_2z^2 + \cdots + c_nz^n$ is a univalent polynomial of degree $n \geq 2$ ($z = x_1 + ix_2$), (1) will hold with μ a distribution of order $n-1$ supported at the origin. Clearly also (4)–(6) will hold then, but (2) and (3) will fail. By an argument similar to the proof of Proposition 2.1 (iii), one realizes that there cannot simultaneously exist measures satisfying (1), (4), and (5).

Thus such a simple and smooth domain as the conformal image of the unit disc under a quadratic (or higher degree) polynomial does not admit a mother body in our sense. This is of course disappointing, but one has to keep in mind that the problem of finding a mother body is ill-posed and that the requirements (1)–(5) taken all together combine several different aspects of it (balayage, analytic continuation, minimality, etc.).

Indeed, as the following proposition shows, our axioms for a mother body seem to be fairly complete in the sense that they contain or imply many of the criteria for concentration and minimality which have been used previously for similar purposes. Examples of such criteria are minimality of supp μ as a set, largeness of U^μ (e.g., Kounchev [Ko2] maximizes integrallike $\int_\Omega U^\mu dx$ among all μ satisfying (1), (3)), and μ being an extremal point in a suitable convex set [An1],[An2], [Ka-Pi]. Proposition 2.1 shows (in particular) that if (1), (3), (4), (5) hold for a measure μ then, within the class of measures satisfying (1) and (3), supp μ is minimal, U^μ is maximal, and μ is an extremal point.

Proposition 2.1. *Let μ be a measure satisfying* (1), (4), (5) *with respect to a given body Ω, and let ν, μ_1, μ_2 be (possibly) other measures.*

(i) *If ν satisfies* (1) *and supp $\nu \subset$ supp μ, then $\nu = \mu$.*

(ii) *If ν satisfies* (1), (3) *and $U^\nu \geq U^\mu$ in all $\mathbb{R}^N$, then $\nu = \mu$.*

(iii) *If ν satisfies* (1), (4), (6), *then $\nu = \mu$.*

(iv) *If μ_1, μ_2 satisfy* (1), (3) *and $\mu = \frac{1}{2}(\mu_1 + \mu_2)$, then $\mu_1 = \mu_2 = \mu$.*

Proof. We first consider statements (i)–(iii). Since $U^\nu, U^\mu \in L^1_{loc}(\mathbb{R}^N)$ it is enough to prove that $U^\nu = U^\mu$ holds almost everywhere (a.e.), or by (4) a.e. in $\mathbb{R}^N \backslash \text{supp}\ \mu$. So let D be a component of $\mathbb{R}^N \backslash \text{supp}\ \mu$, set $u = U^\nu - U^\mu$, and we shall prove that $u = 0$ a.e. in D. Note that D meets Ω^e by (5) and that $u = 0$ in $D \cap \Omega^e$.

In case (i) u is harmonic in D; hence, it follows by harmonic continuation that $u = 0$ in all D. In case (ii) u is superharmonic and nonnegative in D; hence, it is either strictly positive in all D or vanishes identically in D. But the first alternative has already been excluded, and we again get $u = 0$ in D. In case (iii) u is harmonic in $D \backslash \text{supp}\ \nu$. Using (6) it follows that $u = 0$ in $D \backslash \text{supp}\ \nu$, hence, a.e. in D.

Proof of (iv): Since $\mu_1, \mu_2 \geq 0$ we have supp $\frac{1}{2}(\mu_1 + \mu_2) =$ supp $\mu_1 \cup$ supp μ_2. Thus supp $\mu_j \subset$ supp μ, and the conclusion follows immediately from (i). ☐

Note. If, in (i)–(iii) of the proposition, one allows μ and ν to be general distributions (instead of measures), then one still gets the conclusion that $U^\nu = U^\mu$ outside a compact set K of measure zero ($K =$ supp μ in cases (i) and (ii), $K =$ supp $\mu \cup$supp ν in case (iii)). This means that $\mu - \nu$ annihilates all functions which are harmonic in some neighborhood of K, which is about as close to the conclusion $\nu = \mu$ as one may come in the case of distributions. Note that there are distributions with support at a single point, e.g., the Laplacian of the Dirac measure, whose potential vanishes identically outside that point.

3. Mother bodies for convex polyhedra. Having formulated precise requirements for mother bodies ((1)–(5) above) one naturally wonders which bodies admit mother bodies in that precise sense and when they are unique. This is a question which is largely open, but in this section we at least start answering it by proving that convex polyhedra always have unique mother bodies.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a convex bounded open polyhedron provided with a mass distribution ρ_Ω as in section* 2. *Then there exists a measure μ satisfying* (1)–(5). *Its support is contained in a finite union of hyperplanes and reaches $\partial\Omega$ only at corners and edges (not at faces), it has no mass on $\partial\Omega$, and U^μ is a Lipschitz continuous function. Moreover, μ is unique among all signed measures satisfying* (1), (4), (5).

Note. The support of μ coincides with what is sometimes called the "ridge" of Ω [Ev-Ha], [Ja]. For convex polyhedra this is the set of points in Ω which have at least two closest neighbors on $\partial\Omega$. See Figure 1 for an example in two dimensions.

Proof. Write $\Omega = \bigcap_{j=1}^m H_j$, where H_j are open half spaces and m is minimal. For any j, set

$$\delta_j(x) = \text{dist}\left(x, H_j^c\right),$$

$$u_j(x) = a\delta_j(x) + \frac{b}{2}\,\delta_j(x)^2.$$

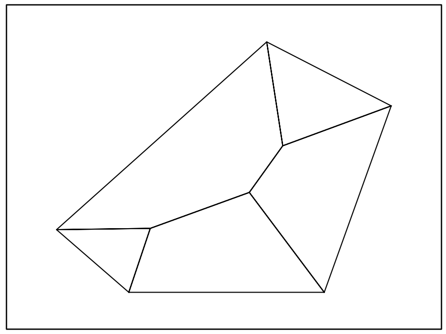

FIG. 1. *The mother body of a convex polyhedron.*

Then $u_j > 0$ in H_j, $u_j = 0$ on H_j^c, and $\Delta u_j = a\mathcal{H}^{N-1}\lfloor\partial H_j + b\mathcal{L}^N\lfloor H_j$. Set also

$$\begin{aligned} \delta(x) &= \operatorname{dist}(x, \Omega^c) \\ &= \inf\{\delta_1(x), \dots, \delta_m(x)\}, \\ u(x) &= a\delta(x) + \frac{b}{2}\,\delta(x)^2 \\ &= \inf\{u_1(x), \dots, u_m(x)\}, \\ R &= \{x \in \Omega : \delta(x) = \delta_j(x) \text{ for at least two different } j\}, \\ D_j &= \{x \in \Omega \backslash R : \delta(x) = \delta_j(x)\} \\ &= \{x \in \Omega : \delta_j(x) < \delta_k(x) \text{ for all } k \neq j\}. \end{aligned}$$

Note that u and u_j are strictly monotone functions of δ and δ_j, respectively (on the range $[0, +\infty)$).

We note that R (the "ridge") is contained in a finite union of hyperplanes, $\Omega = R \cup D_1 \cup \dots \cup D_m$, u is Lipschitz continuous, $u > 0$ in Ω, $u = 0$ on Ω^c. Within Ω we have $\Delta u_j = b$ for all j, hence, $\Delta u \leq b$ in Ω, using the principle that the infimum of a finite family of superharmonic functions (e.g., $u_j(x) - (b/2N)|x|^2$) is superharmonic. The same principle actually gives that $\Delta u \leq b + \sum_{j=1}^m a\mathcal{H}^{N-1}\lfloor\partial H_j$ in all $\mathbb{R}^N$ and hence (since $u = 0$ on Ω^c) that $\Delta u \leq b\mathcal{L}^N\lfloor\Omega + a\mathcal{H}^N\lfloor\partial\Omega = \rho_\Omega$. Outside $\overline{R}$ we have equality in this formula, as is easily seen. Thus $\Delta u = \rho_\Omega - \mu$ where μ is a positive measure with supp $\mu \subset \overline{R}$. Since u vanishes at infinity and $\Delta u = \Delta\left(U^\mu - U^\Omega\right)$ we have $u = U^\mu - U^\Omega$. It now follows that μ satisfies (1)–(4) and that U^μ is Lipschitz continuous.

The latter property implies that μ is absolutely continuous with respect to $\mathcal{H}^{N-1}$. Since $\mathcal{H}^{N-1}\left(\overline{R} \cap \partial\Omega\right) = 0$ it follows that $\mu(\partial\Omega) = 0$.

To see finally that μ satisfies (5), take any $x \in \Omega$ and let $y \in \partial\Omega$ be a closest point on the boundary. Then $y \in \partial H_j$ for a unique j, and it is easy to see that the whole segment (x, y) is in D_j and that $y \notin \bar{R}$. Thus, if $x \notin$ supp μ, the closed segment $[x, y + \varepsilon(y - x)]$ $(\varepsilon > 0)$ connects x with Ω^e without meeting supp μ, proving (5).

It remains to prove the uniqueness part of the theorem. Let μ, u, and u_j be as above, and let ν be any signed measure satisfying (1), (4), (5) (when stated for ν). Set $v = U^\nu - U^\Omega$. Then $v = 0 = u_j$ in H_j^e and $\Delta(v - u_j) = 0$ in $\operatorname{int}(H_j^c \cup \Omega) \setminus$ supp ν. Set

$$\omega_j = \text{ the unbounded component of } \operatorname{int}(H_j^c \cup \Omega) \setminus \text{supp } \nu.$$

It follows that

$$v = u_j \quad \text{in} \quad \omega_j \tag{7}$$

and also, since ω_j is open, that

$$\nabla v = \nabla u_j \quad \text{in} \quad \omega_j. \tag{8}$$

Assumption (5) for ν implies that

$$\omega_1 \cup \cdots \cup \omega_m = \mathbb{R}^N \setminus \operatorname{supp} \nu.$$

Since $\mathcal{L}^N(\operatorname{supp} \nu) = 0$ by (4) it follows that $\bigcup_{j=1}^m \omega_j$ is an open subset of $\mathbb{R}^N$ satisfying

$$\mathcal{L}^N \left(\mathbb{R}^N \setminus \bigcup_{j=1}^m \omega_j \right) = 0, \tag{9}$$

$$\bigcup_{j=1}^m \bar{\omega}_j = \mathbb{R}^N. \tag{10}$$

By (7), (8) v is continuously differentiable in $\bigcup_{j=1}^m \omega_j$ with

$$|\nabla v(x)| \leq C < \infty \qquad \left(x \in \bigcup_{j=1}^m \omega_j \right). \tag{11}$$

Next we claim that the distributional gradient of v is a locally integrable function. To see this, note that $v = U^\nu - U^\Omega = E * (\nu - \rho_\Omega)$; hence,

$$\nabla v = (\nabla E) * (\nu - \rho_\Omega). \tag{12}$$

Here everything is to be interpreted in the sense of distributions. Now, ∇E is a locally integrable (vector) function and $\nu - \rho_\Omega$ is a signed Radon measure with compact support. It then follows (cf. [Do, Sect. 26]) from (12) that ∇v is also a locally integrable (vector) function.

Combining this information with (9), (11) we conclude that the distributional gradient ∇v is in $L^\infty(\mathbb{R}^N)$ and hence that v is a Lipschitz continuous function (i.e., has such a representative).

By continuity, for the Lipschitz continuous version of v, the relation (7) on ω_j extends to hold on all $\bar{\omega}_j$. Thus it follows from (10) that for each $x \in \mathbb{R}^N$ we have $v(x) = u_j(x)$ for some j.

Now let $x \in D_j$, and let y be the closest point on ∂H_j. Then $u_j(\xi) < u_k(\xi)$ for every $\xi \in [x, y]$ and for every $k \neq j$. On H_j^e, $v = u_j = 0$, so by continuity $v(y) = u_j(y)$. Since v is continuous and coincides everywhere with some u_k it follows that $v(\xi) = u_j(\xi)$ for all $\xi \in [x, y]$, in particular $v(x) = u_j(x)$. Thus $v = u_j = u$ in D_j. Since j was arbitrary we conclude that $v = u$ and $\nu = \mu$, completing the proof of the theorem. $\square$

Example. Let Ω be a regular polygon in $\mathbb{R}^2$, say centered at the origin and with $n \geq 3$ corners uniformly distributed on the unit circle. Clearly Ω is convex. Let us compute its mother body μ.

The support of μ, i.e., the ridge R of Ω, consists of the n radii from the origin to the corners of Ω. The density of μ with respect to arclength on R equals the jump of the normal derivative of U^μ across R, or, what is the same, the jump of the normal derivative of $u = U^\mu - U^\Omega$. Since, in the notations of the proof above, $\nabla u = (a+b\delta)\nabla\delta$ and $\nabla\delta$ is a constant unit vector in each component of $\Omega\setminus R$, it follows that the density of μ is proportional to $a + b(1-r)$ on R, where $r = |x|$. Indeed, evaluation of the constant of proportionality gives that

$$d\mu = \frac{2\pi}{n}(a + b(1-r))dr \quad \text{on } R. \tag{13}$$

As n increases, Ω approaches the unit disc $B(0,1)$. One might hope then that μ should approach the unique mother body of the disc, namely $2\pi(a+\frac{b}{2})$ times the Dirac measure at the origin. However, one sees from (13) that, as $n \to \infty$, the μ converge towards that measure on $B(0,1)$ which has density $\frac{a}{r} + b(\frac{1}{r} - 1)$ with respect to area measure. This certainly is more concentrated than the original mass distribution $\rho_{B(0,1)}$, but less concentrated than the Dirac measure. In particular, the mother bodies of the regular polyhedra do not converge towards the mother body of the limiting disc.

This failure of convergence should not be surprising since, as was discussed in section 2, the search for mother bodies is an ill-posed problem with no continuous dependence on initial data, even when unique solutions do exist. The mother bodies for the approximating polyhedra may actually be more useful and more realistic in practical problems than the mother body for the disc itself. Consider, e.g., the case $a = 0$, $b = 1$ and think of the ill-posed Hele–Shaw model briefly discussed in section 2. In experiments with Hele–Shaw flows one never sees an initially circular blob shrinking down to a point. The predominant phenomenon always is that shrinking occurs by development of fingers of the exterior domain penetrating into the fluid (see, e.g., [Ho]). What eventually remains of the fluid domain is not a pointlike blob, but rather a kind of skeleton, which is somewhat reminiscent of the mother body of the approximating polygon Ω for a suitable n.

Thus there is a possibility that mother bodies of polyhedra could be a useful tool for handling ill-posed Hele–Shaw problems: one approximates a given initial fluid domain by a polygon, computes its mother body (uniquely determined and easily computed in the convex case), and then the whole evolution in time is obtained by balayage (section 2). The initial approximation with a polygon of course contains a degree of arbitrariness, but it is also known, for real Hele–Shaw flows, that the onset of the finger development contains a stochastic element.

4. General polyhedra and Zidarov's counterexample. By a (general) polyhedron we mean a domain which is the interior of a finite union of compact convex polyhedra. Mother bodies for general polyhedra will be treated in subsequent papers, e.g., [Gu-Sa2]. The situation in higher dimensions is not completely clear at present, but let us summarize what is known in the two-dimensional case.

When hypersurface measure is present in ρ_Ω, i.e., when $a > 0$, $b \geq 0$, nonconvex polyhedra do not admit mother bodies satisfying all of (1)–(5). Indeed, if Ω is a nonconvex polyhedron in $\mathbb{R}^2$, then Ω must have a nonconvex corner and it is well known that classical balayage of any positive measure μ in Ω onto $\partial\Omega$ will then be a measure on $\partial\Omega$ whose density with respect to $\mathcal{H}^1$ tends to infinity at the corner. When $b = 0$, requirement (1) means that ρ_Ω will have to coincide with this balayage measure; hence, a mother body μ cannot exist in this case. This argument extends to the case $a > 0$, $b > 0$.

On the other hand, extending previous work of G. Choquet and I. Deny [Ch-De] concerning regular polyhedra, D. Siegel [Si] has constructed, in the pure hypersurface case ($a > 0$, $b = 0$), mother bodies (or skeletons, as he calls them) for general polyhedra which satisfy (1)–(2), (4)–(5). The construction actually works for general $a, b \geq 0$. Moreover the construction is canonical (involves no choices) and the shape of the mother body reflects that of the original body. Hence we feel that it is a satisfactory mother body, although the positivity requirement (3) is violated in the nonconvex case.

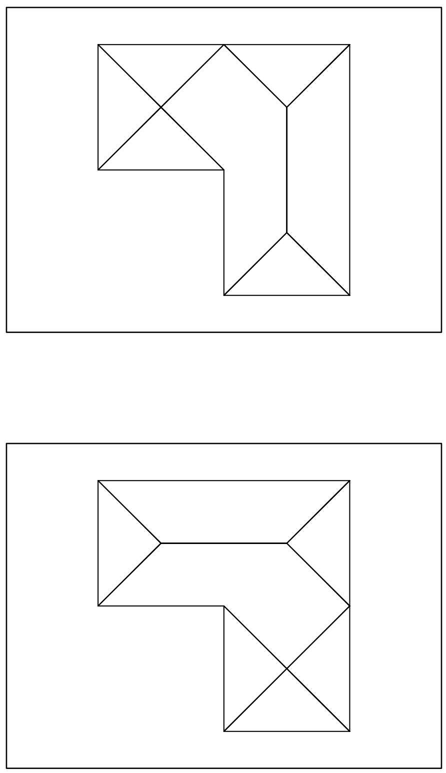

FIG. 2. *Two mother bodies for a nonconvex polyhedron in the case $a = 0$ (Zidarov's example).*

In the case of pure volume measure ($a = 0$, $b > 0$) the function $\Omega \mapsto \rho_\Omega$ is additive under disjoint unions (even after "removal of slits," i.e., after taking the interior of the closure). Therefore a possible way to construct a mother body for a polyhedron Ω is to decompose it into finitely many subpolyhedra, e.g., convex ones, each of which has a mother body satisfying (1)–(5). By adding these up one gets a measure μ which automatically satisfies (1)–(4) for Ω. Requirement (5) is more troublesome, but at least in the two-dimensional case it can be met by choosing the decomposition properly [Gu-Sa2].

In conclusion, mother bodies satisfying all of (1)–(5) do exist for arbitrary polyhedra when $N = 2$ and $a = 0$. However, as Zidarov discovered, there is no uniqueness of mother bodies for nonconvex polyhedra. Zidarov's example is a square in $\mathbb{R}^2$ with a smaller square at one corner removed, say $\Omega = (-1, 1)^2 \backslash (-1, 0]^2$. This can be de-

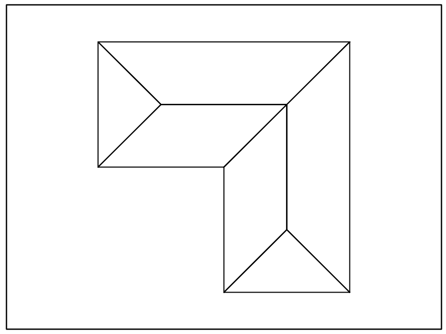

FIG. 3. *Siegel's mother body (with* (3) *violated) for the same nonconvex polyhedron* $(a, b \geq 0)$*.*

composed into three squares (with side length one) in a natural way. Adding up the mother bodies for these, one gets a measure not satisfying (5). But, if Ω is instead decomposed into a rectangle (with side lengths one and two) and a square, the sum of the mother bodies for these will satisfy all of (1)–(5). This decomposition can be made in two different ways, and the result will be two different mother bodies.

These are depicted in Figure 2. Figure 3 shows the mother body obtained by Siegel's procedure for the same Ω. The latter does not satisfy (3), but it has other advantages, namely that supp μ meets $\partial\Omega$ only at corners not at smooth points of $\partial\Omega$, that it shares the symmetry properties of Ω, and that it is indecomposable in a certain sense.

Acknowledgments. I would like to dedicate this paper to the memory of Dimiter Zidarov, who died in 1993. His work in geophysical potential theory has for several years been one of my main sources of inspiration. I also want to thank Ognyan Kounchev and Harold S. Shapiro for many interesting discussions concerning mother bodies and Siv Sandvik for typing my manuscript.

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