Möbius invariant differential operators on Riemann surfaces

by Björn Gustafsson and Jaak Peetre[1]

0. Introduction and first examples.

In the past seven years or so most of my mathematical activities have in one way or other been connected with Hankel operators (or forms). As there have been at least two talks (Arazy, Janson)[2] at this Summer Schoool devoted to this subject, I have been forced to pick up things left over...

To begin the discussion recall the definition of the classical Schwarz derivative

$$\{F,z\} \equiv -2\sqrt{F'}\frac{d^2}{dz^2}\left(\frac{1}{\sqrt{F'}}\right) = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2. \tag{1}$$

This third order nonlinear DO plays an important rôle in conformal mapping, in uniformization, in the theory of Kleinian groups, in the theory of second order linear DO (all these topics are in fact related!). It has an essentially invariant character, as is manifested in the following identity due to Cayley:

$$\{F,z\} = \{F,\zeta\}\left(\frac{d\zeta}{dz}\right)^2 + \{\zeta,z\}, \tag{2}$$

where $\zeta = \phi(z)$ is a change of coordinate. No workable generalization to higher order of the Schwarz derivative is known. However, if we allow only projective transformations of coordinates the situation changes drastically and we have plenty of other operators which display a similar invariant behavior; in this situation $\{\zeta, z\} = 0$ so (2) simplifies to

$$\{F,z\} = \{F,\zeta\}\left(\frac{d\zeta}{dz}\right)^2. \tag{2'}$$

In the general case the exponent 2 has to be replaced by some other integer. To get a global formulation one is led to consider manifolds (either real or complex) equipped with a *projective structure.* Thus in the complex case - to which case we focus our attention in what follows - we are dealing with a *Riemann surface* with such a structure.

In fact, many examples of such *Möbius invariant* DOs can be found on the basis of a general fact, which I have begun to call Bol's lemma or Bol's theorem and which has fascinated me over a period of several years. In a way all that we say here may be viewed as offshoots of this in itself rather elementary observation:

BOL'S LEMMA. *[Bol] If we let the function F transform according to the rule*

$$F(z) \mapsto F(\phi(z))e(z)^{\mu-1},$$

[1] This compilation, prepared for the Sodankylä Summer School, Aug. 1988, was written by J.P. but as many of the results announced were obtained in cooperation with B.G. it has been judged appropriate, following great examples in the past, to include both names in the heading.

[2] As the latter could not come because of illness, his address was actually delivered by Robert Wallstén.

where $\phi(z) = \dfrac{az+b}{cz+d}$ *(*$ad-bc=1$*) is an arbitrary Möbius transformation and where we have put* $e(z) = cz+d$*,* μ *being a given integer* ≥ 0*, then its* μ*th derivative* $F^{(\mu)}$ *transforms according to the rule*

$$F^{(\mu)}(z) \mapsto F^{(\mu)}(\phi(z))e(z)^{-\mu-1}.$$

Several proofs of this result are listed in [GP1] (see also the discussion in Sec. 2 of the present compilation).[3]

Let us consider in detail some examples, interesting in their own right.

Example 1. In [P1], [P2] I suggested the following generalization of the DO in (1):

$$(3) \qquad (F^{(\mu)})^{\frac{\lambda-1}{\mu+1}} \frac{d^\lambda}{dz^\lambda}\left(\frac{1}{(F^{(\mu)})^{\frac{\lambda-1}{\mu+1}}}\right),$$

where λ is another integer ≥ 0. If $\lambda = 2$, $\mu = 1$ we clearly get back the Schwarz derivative (up to a factor). I proposed there (see also [PK]) that nullsolutions of the corresponding DE might be of some interest in analysis, in particular that linear combinations of nullsolutions might play a similar rôle in, say, approximation theory as rational functions. It is easy to write down the solutions in question: We find that

$$\frac{1}{(F^{(\mu)})^{\frac{\lambda-1}{\mu+1}}} = P \quad \text{(a polynomial of degree } < \lambda)$$

or

$$F^{(\mu)} = \frac{1}{P^{\frac{\mu+1}{\lambda-1}}},$$

whence

$$F(z) = \frac{1}{(\mu-1)!}\int_0^z \frac{(z-\zeta)^{\mu-1}}{(P(\zeta))^{\frac{\mu+1}{\lambda-1}}} d\zeta + Q(z) \quad \text{(a polynomial of degree } < \mu).$$

(Check: P and Q together determine $\lambda + \mu$ independent parameters, which is the order of the equation.) Such functions are, apparently, related to so-called Picard curves (see [Hol]).

Subexample 1. $\mu = 1$, $\lambda = 2$. Then

$$F'(z) = \frac{1}{(cz+d)^2} \Rightarrow F(z) = \frac{az+b}{cz+d} \quad \text{(a fractional linear function).}$$

Example 2. A related construction was suggested by Menahem Schiffer [Sc]. This depends on a "polarized" version of Bol's lemma [P6]. Let $F(z_1, ..., z_{\mu+1})$ denote the μth Newton's divided difference of the function $F(z)$:

$$F(z_1, ..., z_{\mu+1}) = \frac{\mu!}{2\pi i}\int \frac{F(\zeta)}{(\zeta - z_1)...(\zeta - z_{\mu+1})} d\zeta,$$

[3] A proof not included in [GP1] can be based on the idea to prove it first for the function $\phi = 1/z$. Cf. [Di], p. 187, exc. 1; the appearance of it there suggests that the result may be quite old.

where we integrate over a suitable "contour" encircling the points $z_1, \ldots, z_{\mu+1}$. Apparently

$$F(z, ..., z) = F^{(\mu)}(z).$$

More generally (with $D_j = \dfrac{\partial}{\partial z_j}$)

$$D_1^{a_1} ... D_{\mu+1}^{a_{\mu+1}} F(z, ..., z) = \frac{\mu! a_1! ... a_{\mu+1}!}{(\mu + \sum a_j)!} F^{(\mu + \sum a_j)}(z),$$

thus in particular (in slightly abusive notation)

$$D_1 F = \frac{\mu!}{(\mu+1)!} F^{(\mu+1)}, \; D_1 D_2 F = \frac{\mu!}{(\mu+2)!} F^{(\mu+2)}, \ldots. \tag{4}$$

We have the transformation rule (F transforms as in the lemma):

$$F(z_1, ..., z_{\mu+1}) \mapsto F(\phi(z_1), ..., \phi(z_\mu)) e(z_1)^{-1} \ldots e(z_{\mu+1})^{-1}.$$

Thus if all points z_j are equal to z

$$F^{(\mu)}(z) \mapsto F^{(\mu)}(\phi(z)) e(z)^{-(\mu+1)},$$

which is one of the proofs of Bol's lemma (theorem) given in [GP1].

Consider now

$$D_1 \ldots D_\lambda \log F$$

which expression, apparently, if $\lambda > 1$ is changed, under the influence of ϕ, into

$$D_1 \ldots D_\lambda \log F \circ \phi = (D_1 \ldots D_\lambda \log F) \circ \phi \cdot \phi'(z_1) \ldots \phi'(z_\lambda).$$

Denoting by M (M for Menahem) the restriction to the diagonal, the DO obtained is transformed as

$$M \mapsto M \circ \phi \cdot (\phi')^\lambda,$$

thus as a form of degree λ. M is thus a nonlinear homogeneous Möbius invariant DO of order $\lambda + \mu$.

Subexample 2. $\mu = 1, \lambda = 2$. Then

$$D_1 \log F = \frac{D_1 F}{F},$$

$$D_1 D_2 \log F = \frac{D_1 D_2 F}{F} - \frac{D_1 F \cdot D_2 F}{F^2}$$

gives (see (4))

$$M = \frac{1!}{3!} \frac{F'''}{F'} - (\frac{1!}{2!})^2 (\frac{F''}{F'})^2 = \frac{1}{6} \frac{F'''}{F'} - \frac{1}{4} (\frac{F''}{F'})^2,$$

which can be rewritten as

$$-\frac{1}{3}\sqrt{F'}D^2(\frac{1}{\sqrt{F'}}),$$

as

$$D(\frac{1}{\sqrt{F'}}) = -\frac{1}{2}\frac{F''}{(\sqrt{F'})^3},$$

$$D^2(\frac{1}{\sqrt{F'}}) = -\frac{1}{2}\frac{F'''}{(\sqrt{F'})^3} + \frac{1}{2}\cdot\frac{3}{2}\frac{(F'')^3}{(\sqrt{F'})^5}.$$

Therefore we get essentially back the Schwarzian.

Subexample 3. $\mu = 2, \lambda = 3$. Now

$$D_1D_2D_3 \log F = \frac{D_1D_2D_3F}{F} - \frac{D_1D_2F\cdot D_3F}{F^2} -$$

$$-\frac{D_1D_3F\cdot D_2F}{F^2} - \frac{D_2D_3F\cdot D_1F}{F^2} + \frac{2D_1F\cdot D_2F\cdot D_3F}{F^3}$$

gives (see again (4))

$$M = \frac{\frac{2}{120}F^V}{F''} - 3\frac{\frac{2}{6}\cdot\frac{2}{24}F^{IV}F'''}{(F'')^2} + \frac{2(\frac{2}{6}F''')^3}{(F'')^3} =$$

$$= \frac{\frac{1}{60}(F'')^2F^V - \frac{1}{12}F''F'''F^{IV} + \frac{2}{27}(F''')^3}{(F'')^3}.$$

This expression, again, can be transformed into

$$-\frac{1}{40}(F'')^{\frac{2}{3}}D^3[(F'')^{-\frac{2}{3}}],$$

which is readily found by carrying out the last derivation.

The above suggests that the operators in Example 1 and 2 always coincide (up to a factor). However, continuing the calculation in Subexample 2 and 3 shows that this not the case. In Sec. 3 we will write down the general form of a (homogeneous) Möbius invariant DO. Actually, this is already at least implicit in Morikawa [M] in an invariant theoretic context. We believe however that our presentation (we propose in fact two sligthly different avenues) is more transparent. We give also an abundance of concrete examples.

All the DOs encountered previously have been homogeneous (in F). Now we mention an interesting instance of a nonhomogeneous Möbius invariant DO.

Example 3. The following DE was encountered by Jacobi [J2] in the theory of theta functions

$$C^2D(\log C^3C'') = \sqrt{16C^3C''+1} \tag{5}$$

which, apparently, has a Möbius invariant character provided we let C transform as a form of degree $-\frac{1}{2}$ (the case $\mu = 2$ of Bol's lemma). In [J2] Jacobi shows that (5) is satisfied with $C = \frac{2}{\pi}\vartheta_{e\epsilon}^{-2}(0,t)$, where $\vartheta_{e\epsilon}(0,t)$, $e\epsilon = 0$, is any "Thetanullwerth" (theta constant), and that the general solution is obtained by application of a Möbius transformation. This example will be analyzed in detail in App. 2. So far, however, Jacobi's equation (5) stands out as an isolated special case.

The rest of the present compilation is organized as follows. However, generally speaking, the paper has no "plan" so that its various subdivisions (including the appendices), even parts of them, may to some extent be read independently of each other.

In Sec. 1 we briefly recall some salient facts about projective structures on manifolds and, in particular, their connection to uniformization and to second order linear DO.

Sec. 2 reviews the main contents of the paper [GP1], centering around Bol's lemma and related issues, for instance, the notion of transvectant.

In Sec. 3, as we already told above, a description of "all" Möbius invariant operators is obtained.

In Sec. 4 we show how the transvectant can be exploited in connection with Hankel theory. We also point out the parallel between Hankel theory and operator calculi ("quantization"), the latter subject being briefly reviewed in Sec. 5.

In Sec. 6 we discuss reproducing and "coreproducing" kernels in Hilbert spaces of analytic functions.

Finally, the appendices, six in number, contain auxiliary material more or less loosely related to the main body of our paper.

1. Complex manifolds with a projective structure.

We collect here some basic facts about complex manifolds equipped with a projective structure. It is however only out of convenience that we have restricted attention to the complex case only. Projective structures are also of interest in the real case, at least in dimension one, for instance in the oscillatory theory of second order linear DE, for which we refer to the book [Bor]. (A brief mention of projective structure can further be found in the excellent book [A], pp. 42-56, where also (chiefly) nonlinear equations are considered. Compare further [De].)

So let Ω be a complex manifold of (complex) dimension n. We say that we have a *projective structure* on Ω if there is given a covering of Ω with coordinate neighborhoods $\{U\}$ and corresponding local coordinates $\{z = (z^1, ..., z^n)\}$ such that the change of coordinates is mediated by projective (fractional linear) maps: if $U \cap U' \neq \emptyset$ then z and z' are connected by a relation of the form

$$z'^j = \frac{a_{j0} + a_{j1}z^1 + ... + a_{jn}z^n}{a_{00} + a_{01}z^1 + ... + a_{0n}z^n} \quad (j = 1, ..., n); \tag{1}$$

we can always require that $\det(a_{jk}) = 1$.

In the same way one defines for instance *affine structure*. E.g. a complex torus has a canonical affine structure.

More generally, in the book [Gu3] one considers "structures" associated with any Lie pseudogroup of differentiable (smooth) transformations of $\mathbf{C}^n$.

Let us return to the projective situation and fix attention to the case $n = 1$. So we have a Riemann surface with a projective structure. Formula (1) reduces to

$$z' = \frac{az + b}{cz + d} \quad (ad - bc = 1). \tag{1'}$$

In particular, let us make clear the relation to uniformization (for more details see [Gu1,2]). Let us start with some particular projective coordinate z defined in the coordinate neighborhood U. Then if $U \cap U' \neq \emptyset$ the function z can using (1′) be continued analytically to $U \cup U'$ and, in general, along any path issuing from U. In this way one gets a map $\tilde{\Omega} \mapsto \hat{\mathbf{C}}$, where $\tilde{\Omega}$ is the universal cover of Ω and $\hat{\mathbf{C}}$ the extended complex plane (Riemann sphere, conformally equivalent to the projective line $\mathbf{P}^1$), with the property that germs lying over the same point of Ω are related by projective transformations; it is called the geometric realization of Ω by Gunning [Gu1]. Conversely, given any such map we can define a projective structure on Ω by (locally) pulling back to Ω the identity function on $\hat{\mathbf{C}}$. (See also Tyurin's lectures [Ty] which came to our attention at a rather late stage while compiling this report.)

If Ω is a multiply connected planar domain bounded by finitely many smooth or even analytic arcs, a "regular" domain in the sense of [AFP], then there are several natural projective structures on Ω which compete with each other. First, we have the one which comes from the uniformization theorem (we map $\tilde{\Omega}$ onto the unit disk D). Second, we take for the geometric realization simply a "circular" model for Ω; by a classic theorem (see e.g. [He], p. 481-488 for a proof) every such domain is conformally equivalent to one bounded by finitely many (generalized) circles. It is clear that the circular model is unique up to an arbitrary Möbius transformation (an element of the group $PL(2,\mathbf{C})$).

Projective structures arise further classically in connection with linear DE.

First, consider the case of second order. Then as a projective coordinate one can (locally) take the quotient of two independent solutions (the denominator is required not to vanish). Thus, by a local change of independent variable and multiplying the dependent variable with a suitable factor any second order linear DE can be reduced to the normal form

$$\frac{d^2F}{dz^2} = 0.$$

More computationally: assuming that the equation is already in the form

$$\frac{d^2F}{dz^2} + q(z)F = 0$$

(this first reduction is easily achieved by introducing a suitable multiplier), the final reduction is obtained by solving the third order equation

$$\frac{1}{2}\{\zeta, z\} = q(z),$$

where $\{\ ,\ \}$, as before, stands for the Schwarz derivative.

Historical remark. This was already known to Kummer [Ku] and, at least in a special case, it can be found in Jacobi [J1]. (A portion of Kummer's paper is reproduced in German translation in [Bor], p. 102-103.) The Schwarz derivative appears also, before Schwarz, in Riemann's long unpublished lectures as well as in the work of the young Poincaré. More about the history in [Gr] and, more briefly, e.g. in the marvellous book [Hi], Chap. 10.

As for higher (μth) order linear DE one easily proves that they can be brought on the canonical form

(2) $$F^{(\mu)} + a_{\mu-3}(z)F^{(\mu-3)} + ... + a_0(z)F = 0;$$

the coordinate systems in which the equation has this form obviously determine a projective structure on our manifold Ω. More about this in Sec. 2.

2. The Bol operator and Green's formula.

We now review part of the contents of the paper [GP1].

Consider a Riemann surface Ω equipped with a projective structure. Let κ be the canonical sheaf on Ω, i.e. (local) sections of κ are of the form $s = f(z)dz$ where z is a local coordinate and f an analytic function (in the overlap of two coordinate neighborhoods U and V with local coordiantes z and ζ the corresponding coefficients f and g are related by an equation $g(\zeta) = f(\phi(\zeta))\phi'(\zeta)$ if $z = \phi(\zeta)$). More generally, sections of powers κ^n (forms of integer degree n) are of the form $f(z)(dz)^n$, with an analogous transition rule. If we select a square root of κ, i.e. an invertible sheaf λ such that $\lambda^2 = \kappa$, then one can also talk of half-integer forms.

It follows now from Bol's lemma (see Introduction) that for each $\mu \geq 0$ one can define a linear operator L_μ from $\lambda^{1-\mu}$ into $\lambda^{1+\mu}$: if z is a projective coordinate on Ω then the form $F(z)(dz)^{\frac{1-\mu}{2}}$ is mapped onto the form $F^{(\mu)}(z)(dz)^{\frac{1+\mu}{2}}$. (Notice that formally each successive derivation accounts for another factor dz!)

Remark. (on Eichler cohomology). We have a short exact sequence

$$0 \to \Pi_{\mu-1} \to \lambda^{1-\mu} \to \lambda^{1+\mu} \to 0$$

so we can consider the corresponding exact sequence of cohomology groups. It turns out that the only nontrivial cohomology group is $H^1(\Omega, \Pi_{\mu-1})$ (see e.g. [Gu1,2]). Eichler cohomology, introduced by Eichler in [E], plays a great rôle e.g. in Kleinian groups (see e.g. [Kr]). Eichler himself viewed his theory as a sort of amplification of the classical theory of Abelian integrals and Abelian differentials. This is also why we here distinguish between capital letters, such as $F, B, ...$, for "integrals" and small ones, such as $f, b, ..$, for "differentials".

In [GP1] it is investigated how L_μ looks in a general coordiante z (not necessarily a projective one). First of all, it is clear that $L_0 \equiv \mathrm{id}$ (identity) and further that $L_1 = d$ (differential). They are independent of the projective structure. On the other hand, if $F = F(z)(dz)^{-\frac{1}{2}}$ then $L_2F = (F''(z) + q(z)F(z))(dz)^{\frac{3}{2}}$, and the functions $q(z)$, a different function for each coordinate neighborhood, determine the projective structure. In [GP1] it is shown that if $F(z)$ is the coefficient of a form F of degree $\dfrac{1-\mu}{2}$ then the coefficient of $L_\mu F$, a form of degree $\dfrac{1+\mu}{2}$, is of the form

$$F^{(\mu)}(z) + A_2F^{(\mu-2)}(z) + ... + A_\mu F(z),$$

where the coefficients $A_i = A_i^{(\mu)}$ $(i = 2, ..., \mu)$ are certain universal polynomials in the derivatives $q(z)$, $q'(z)$,...,$q^{(i)}(z)$ of $q(z)$ with resepect to z. E.g. we have

$$L_3F = (F''' + 4qF' + 2q'F)\,(dz)^{\frac{5}{2}},$$
$$L_4F = \left(F^{IV} + 10qF'' + 10q'F' + (9q^2 + 3q'')F\right)(dz)^{\frac{7}{2}}$$

and so forth.

Remark. This result can also be formulated as follows. Let F_1, F_2 be a basis for the solution of the DE $F'' + qF = 0$. Then the functions $F = P(F_1, F_2)$, where P runs through all polynomials homogeneous of degree $\mu - 1$ satisfy a linear μth order DE, whose coefficients depend only on q. We were led to this formulation while reading the review [Sa] of the book [PT] (the case $\mu = 3$).

One can further invoke a certain bilinear "covariant" introduced by Gordan [Go] in classical invariant theory, known as the *transvectant* (German: *Überschiebung*). This depends on the following fact:

GORDAN'S LEMMA. *Let f_k $(k = 1, 2)$ transform under Möbius transforms according to the rule*

$$f_k(z) \mapsto f_k(\phi(z))e(z)^{-\nu_k} \quad \text{where} \quad \nu_k \in \mathbf{Z}.$$

Then

$$J_s(z) \stackrel{\text{def}}{=} \sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} \frac{f_1^{(i)}}{(\nu_1)_i} \frac{f_2^{(s-i)}}{(\nu_2)_{s-i}},$$

where $s \geq 0$ is any integer such that $\nu_k \neq 0, -1, \ldots, -(s-1)$ and, generally speaking, $(x)_i = x(x+1)\ldots(x+i-1)$, transforms according the rule

$$J_s(z) \mapsto J_s(\phi(z))e(z)^{-\nu},$$

where $\nu = \nu_1 + \nu_2 + 2s$.

For the proof see [GP1] or give your own. There this is used in the following way in the global situation of a Riemann surface Ω endowed with a projective structure. Taking $\nu_1 = -(\mu - 1)$, $\nu_2 = 2(\mu - s)$ where now $0 \leq s \leq \mu - 1$, one obtains for each form Θ of degree k a linear map

$$M_\Theta^k : \lambda^{1-\mu} \to \lambda^{1+\mu}.$$

Then one can define, given forms $\Theta_1, \ldots, \Theta_\mu$ of degree $1, \ldots, \mu$ respectively, a μth order linear DO $L : \lambda^{1-\mu} \to \lambda^{1+\mu}$ given by

$$L = L_\mu + M_{\Theta_1}^1 + \cdots + M_{\Theta_\mu}^\mu.$$

The point is that this process can be reversed. That is, essentially all μth order linear DOs arise in this way. In fact, thereby one recaptures the classical *Laguerre-Forsyth invariants* (see the remarkable book [Wi], long fallen in oblivion).

In [Bol] a different approach to invariants of higher order linear DO's is indicated. Suppose the DO is already in the normal form (2) of Sec. 1. Subtract from it the $(\mu - 2)$-th order linear DO

$$\sqrt{a_{\mu-2}(z)} \left(\frac{d}{dz}\right)^{\mu-2} \sqrt{a_{\mu-2}(z)}$$

and continue by induction. From Bol's lemma (see Introduction) it is clear that this is an invariant (coordinate independent) procedure. The drawback is of course the ambiguity in the definition of the square roots involved. Even worse, near points where a coefficient vanishes a branch point is introduced. Nevertheless, it might be worth while to make a closer comparison of the invariants arising in this way with the Laguerre-Forsyth invariants.

A further noteworthy thing in [GP1] is an integral formula for the Bol operator. As before, let Ω be a Riemann surface with a projective structure and let $\mathcal{O}$ be an open set on Ω bounded by an analytic curve $\partial\mathcal{O}$. Assume that we have on $\mathcal{O}$ a complete Hermitean metric of constant curvature, say, (in local coordinates) $ds = |dz|/\omega(z)$. It is easy to see that

$$q(z) = -\frac{\partial^2\omega(z)/\partial z^2}{\omega(z)}$$

transforms as the coefficient connected with a projective structure. We assume that this projective structure on $\mathcal{O}$ agrees with the one induced from the given projective structure on Ω. Then one can show that (if $\mu \geq 1$)

$$\int_{\mathcal{O}} L_\mu F\bar{g}\omega^{\mu-1}dz d\bar{z} = \text{const} \cdot \int_{\partial\mathcal{O}} F\bar{g}(dz)^{\frac{1-\mu}{2}}(d\bar{z})^{\frac{1+\mu}{2}}.$$

(The constant depends on μ only.) Obviously, this reduces to the ordinary Green's formula if $\mu = 1$. In [GP2] it is used to prove the theorem that the reproducing kernel in weighted Bergman space $A^{\alpha,2}(\Omega)$ (α integer ≥ 0) over a multiply connected plane domain $\mathcal{O}$ in $\mathbf{C}$ admits a meromorphic continuation to the Schottky double Ω of $\mathcal{O}$, and this result again is used to study Hankel forms over the said space (cf. the discussion in Sec. 4).

3. Determination of all Möbius invariant operators.

We make now an assault to find all Möbius invariant operators of the type appearing in Examples 1 and 2 in the Introduction. First we apply the Bol operator so the "integral" $F(z)(dz)^{-\frac{\mu-1}{2}}$ gets replaced by a "differential" $f(z)(dz)^{\frac{\mu+1}{2}}$, where $f(z) = F^{(\mu)}(z)$. The problem is therefore to find all "covariant" operators of the type

$$f(dz)^{\frac{\mu+1}{2}} \mapsto \sum_{\substack{k_1\geq 0,\ldots,k_\lambda\geq 0\\ k_1+\cdots+k_\lambda=\lambda}} a_{k_1\ldots k_\lambda}\frac{f^{(k_1)}\ldots f^{(k_\lambda)}}{f^\lambda}(dz)^\lambda.$$

These operators form a finite dimensional vector space $\mathcal{M}_\lambda$. One can also multiply two such operators so that one has an operation $\mathcal{M}_\lambda \otimes \mathcal{M}_{\lambda'} \to \mathcal{M}_{\lambda+\lambda'}$. In other words, $\sum_{\lambda\geq 0}^{\oplus} \mathcal{M}_\lambda$ is a commutative graded ring. In what follows we will uncover its structure.

It turns out that the problem is essentially independent of μ. If f transforms according to the rule

$$f \mapsto (f \circ \phi)e^{-\mu-1},$$

then the kth derivative transforms as

$$f^{(k)} \mapsto \sum_{j=0}^{k}\binom{k}{j}\frac{(\mu+k)!}{(\mu+j)!}\epsilon^{k-j}(f^{(j)}\circ\phi)e^{-\mu-1-2k},$$

where $\epsilon = -e \cdot c = \frac{1}{2}\frac{\phi''}{\phi'^2}$ and, as before, $e = cz + d = \frac{1}{\sqrt{\phi'}}$ (generalization of Bol's lemma - the case $k = \mu$; see [GP1] or [M] or [Te]). If we set

$$\hat{D}^k f = \hat{f}^{(k)} = \frac{f^{(k)}}{(\mu+k)!},$$

this can be written as

$$\hat{D}^k f \mapsto (\hat{D} + \epsilon)^k f \circ \phi \cdot e^{-\mu-1-2k},$$

with $\hat{D}$ and ϵ treated as commuting operators. More generally,

$$P(\hat{D}, ..., \hat{D})f \mapsto P(\hat{D} + \epsilon, \ldots, \hat{D} + \epsilon)f \circ \phi \cdot e^{-(\mu+1)\lambda - 2\lambda}$$

if

$$P(x_1, \ldots, x_\lambda) = \sum \hat{a}_{k_1,\ldots,k_\lambda} x_1^{k_1} \ldots x_\lambda^{k_\lambda}$$

with

$$\hat{a}_{k_1,\ldots,k_\lambda} = a_{k_1,\ldots,k_\lambda} \cdot (k_1 + \mu)! \ldots (k_\lambda + \mu)!.$$

Remark. This may be viewed as a generalization of the transformation rule for the bracket $\{\,,\,\}_1$ (cf. [Gu2], [GP1]), viz. $\frac{f'}{f} \mapsto (\frac{f'}{f} \circ \phi + \frac{1}{2}(\mu+1)\frac{\phi''}{(\phi')^2})\phi'$.

Thus the condition for covariance comes in the form

$$\boxed{P(x_1 + 1, \ldots, x_\lambda + 1) = P(x_1, \ldots, x_\lambda)}$$

or, equivalently, as

$$\boxed{\sum_{i=1}^{\lambda} \frac{\partial P}{\partial x_i} = 0}$$

or again, in terms of coefficients, as

$$\sum \hat{a}_{k_1 \ldots k_\lambda} \binom{k_1}{j_1} \cdots \binom{k_\lambda}{j_\lambda} = \begin{cases} \hat{a}_{j_1 \ldots j_\lambda} & \text{if } j_1 + \cdots + j_\lambda = \lambda \\ 0 & \text{if } j_1 + \cdots + j_\lambda < \lambda \end{cases}.$$

In particular, we find

$$\dim \mathcal{M}_\lambda = p(\lambda) - p(\lambda - 1),$$

where $p(\lambda)$ is the number of partitions of λ. This gives the table

λ	$p(\lambda)$	$\dim\mathcal{M}_\lambda$
1	1	0
2	2	1
3	3	1
4	5	2
5	7	2
6	11	4
7	15	4
8	22	7
9	30	8
10	42	12

which, in particular, explains why the operators in the Introduction (Examples 1 and 2) coincide if $\mu = 3, 4$ but not in general.

The calculations are facilitated by the remark that the polynomial P can be taken to be symmetric. Notice also that $P(1, 1, ...1) = 0$. This can be checked at the hand of the examples below.

Example 1. $\lambda = 2$. Write

$$P(x, y) = A(x^2 + y^2) + 2Bxy.$$

Then

$$\frac{\partial P}{\partial x} = 2Ax + 2By, \quad \frac{\partial P}{\partial y} = 2Ay + 2Bx,$$

so that

$$\frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} = 2(A + B)xy.$$

This gives $A = -B$ and $P = A(x - y)^2$. We thus obtain the DO

$$\frac{\hat{f}''}{\hat{f}} - \frac{\hat{f}'^2}{\hat{f}^2}$$

or

$$\frac{\mu!}{(\mu + 2)!} \frac{f''}{f} - \frac{\mu!^2}{(\mu + 1)!^2} \frac{f'^2}{f^2}.$$

$\mu = 1$ corresponds to the Schwarzian:

$$\frac{f''}{f} - \frac{3}{2}(\frac{f'}{f})^2.$$

Example 2. $\lambda = 3$. Now

$$P(x, y, z) = 2A(x^3 + y^3 + z^3) + B(x^2y + x^2z + y^2x + y^2z + z^2x + z^2y) + 6Cxyz$$

and

$$\frac{\partial P}{\partial x} = 6Ax^2 + 2Bx(y + z) + B(y^2 + z^2) + 6Cyz \text{ etc.}$$

Thus

$$\frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} = (6A + 2B)(x^2 + y^2 + z^2) + 2(2B + 3C)(xy + xz + yz),$$

yielding $B = -3A$, $C = -\frac{2}{3}B = 2A$. The corresponding DO is

$$\frac{\hat{f}'''}{\hat{f}} - 3\frac{\hat{f}''\hat{f}'}{\hat{f}^2} + 2\frac{(\hat{f}')^3}{\hat{f}^3}$$

or

$$\frac{\mu!}{(\mu+3)!}\frac{f'''}{f} - 3\frac{(\mu)!^2}{(\mu+2)!(\mu+1)!}\frac{f''f'}{f^2} + 2\frac{(\mu)!^3}{((\mu+1)!)^3}\frac{f'^3}{f^3}.$$

$\mu = 2$ gives the expression we found earlier (see Subexample 3, Introduction):

$$\frac{1}{3\cdot 4\cdot 5}\frac{f'''}{f} - 3\frac{1}{3\cdot 4\cdot 3}\frac{f''f'}{f^2} + 2\frac{1}{3\cdot 3\cdot 3}\frac{f'^3}{f^3}.$$

In these two cases, therefore, these operators much coincide with Schiffer's "logarithmic" operator M (see Introduction).

Example 3. $\lambda = 4$. The direct eliminations become so complicated that we limit ourselves to give the end result. One finds two independent covariant operators:

$$\delta_1 = \frac{\hat{f}^{IV}}{\hat{f}} - 4\frac{\hat{f}'''\hat{f}'}{\hat{f}^2} + 3\frac{\hat{f}''^2}{\hat{f}^2}$$

and

$$\delta_2 = \frac{\hat{f}''^2}{\hat{f}^2} - 2\frac{\hat{f}''\hat{f}'^2}{\hat{f}^3} + \frac{\hat{f}'^4}{\hat{f}^4} =$$

$$= \left(\frac{\hat{f}''}{\hat{f}} - \frac{\hat{f}'^2}{\hat{f}^2}\right)^2 = \text{the square of the operator in the case } \lambda = 2.$$

In this case the operator M is essentially $\delta_1 - 6\delta_2$.

Before continuing this series of examples let us write down the general formula for computing the coefficents. If $P = \sum A_k x^k = \sum A_{k_1,\ldots,k_\lambda} x_1^{k_1}\ldots x_\lambda^{k_\lambda}$ is the polynomial corresponding to a covariant DO holds

$$\sum_i (l_i + 1)A_{l+e_i} = 0,$$

where $e_i = (0,\ldots,\underbrace{1}_{\text{position}\,1},\ldots,0)$.

Example 4. $\lambda = 5$. The possible partitions of the number 4 correspond to the vectors $l = (40000), (31000), (22000), (21100), (11110)$, yielding the system of equations

$$\begin{aligned} 5A_5 + 4A_{41} &= 0,\\ 4A_{41} + 2A_{32} + 3A_{311} &= 0,\\ 2\cdot 3A_{32} + 3A_{211} &= 0,\\ 3A_{311} + 2\cdot 2A_{221} + 2A_{2111} &= 0,\\ 4\cdot 2A_{2111} + A_{11111} &= 0, \end{aligned}$$

"superfluous" zeros being omitted ($A_5 = A_{50000}$ etc.). There are 5 equations and 7 unknowns corresponding to 2 independent solutions, as predicted. Set $A_5 = 24A$, $A_{41} = 6B$,

$A_{32} = 6C$, $A_{311} = 4D$, $A_{211} = 4E$, $A_{2111} = 6F$, $A_{11111} = 120G$. Then we can write our system as

$$\begin{aligned}
5A + B &= 0,\\
2B + C + D &= 0,\\
3C + E &= 0,\\
3D + 4E + 3F &= 0,\\
2F + 5G &= 0,
\end{aligned}$$

which gives at once (equations 1,3,5) $B = -5A$, $E = -3C$, $G = -\frac{2}{5}F$. If $A = B = 0$ we can express the remaining coefficients in terms of C: $D = -C$, $E = -3C$, $F = 5C$, $G = -2C$. Thus one gets the covariant DO

$$f'''f'' - f'''f'^2 - 3f''^2 f' + 5f''(f')^3 - 2(f')^5 \equiv (f'' - f'^2)(f''' - 3f''f' + 2f'^3);$$

here and in the next formula we *omit* the $\hat{}$ in the notation for the derivative. If instead $F = G = 0$ one expresses instead the coefficients in terms of A: $B = -5A$, $C = 2A$, $D = 8A$, $E = -6A$. The operator now reads:

$$f^V f^4 - 5f^{IV} f' f^3 + 2f''' f'' f^3 + 8f''' f'^2 f^2 - 6f''^2 f' f^2.$$

In the same way as above one can also treat multilinear expressions of the type

$$\sum a_{k_1 \ldots k_\lambda} f_1^{(k_\lambda)} \ldots f_\lambda^{(k_\lambda)}$$

yielding an analogous result (P need not any longer be symmetric).

Example 5. $\lambda = 3$. Write

$$P = 2A_1 x^3 + 2A_2 y^3 + 2A_3 z^3 +$$

$$+ B_{12} x^2 y + B_{13} x^2 z + B_{21} y^2 x + B_{23} y^2 z + B_{31} z^2 x + B_{32} z^2 y + 6Cxyz.$$

The conditions for covariance are:

$$\begin{aligned}
6A_1 + B_{12} + B_{13} &= 0,\\
6A_2 + B_{21} + B_{23} &= 0,\\
6A_3 + B_{31} + B_{32} &= 0,\\
2B_{12} + 2B_{21} + 6C &= 0,\\
2B_{31} + 2B_{13} + 6C &= 0,\\
2B_{23} + 2B_{32} + 6C &= 0.
\end{aligned}$$

If $C = 0$ we get $B_{21} = -B_{12}$ etc., whence $A_1 = -\frac{1}{6}(B_{12} - B_{21})$ etc. Thus

$$f = B_{21}(\frac{1}{3}(x^3 - y^3) - xy(x-y)) + \cdots = \frac{1}{3}B_{21}(x-y)^3 + \ldots,$$

which gives the differential expression

$$B_{21}(\frac{1}{3}(f'''gh - fg'''h) - (f''g'h - f'g''h)) + B_{31}(\frac{1}{3}(f'''hg - fgh''') - (f''gh' - f'gh''))+$$

$$+B_{32}(\frac{1}{3}(fg'''h - fgh''') - (fg''h' - fg'h'')).$$

Let us return to the "logarithmic" operator M. Let $F(z)$ be the $(\mu+1)$-st integral of $f(z)$ ($F^{(\mu)} = f$) and let $F \equiv F(z_1, \ldots, z_\mu)$ be the μth divided difference of $F(z)$ (see Introduction). We may write

$$D_1 \ldots D_\lambda \log F = \sum C_{\alpha_1,\ldots,\alpha_r} \frac{D_{\alpha_1}F \ldots D_{\alpha_r}F}{F^r},$$

where the summation is carried over all families $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ of disjoint nonempty subsets α_i of the set $\{1, \ldots, \lambda\}$ with $\alpha_1 \cup \cdots \cup \alpha_r = \{1, \ldots, \lambda\}$ and D_{α_i} stands for the partial derivative with respect to the indices in α_i. Taking $z_1 = \cdots = z_{\mu+1} = z$ we obtain the covariant DO

$$\sum C_{\alpha_1 \ldots \alpha_r} \frac{\hat{f}^{(|\alpha_1|)} \ldots \hat{f}^{(|\alpha_r|)}}{\hat{f}^r}$$

where $|\alpha_i|$ is the number of elements in α_i. The coefficients C can be found recursively as follows:

Case 1. If β is given by $\{1, \ldots, \lambda+1\} = \beta_0 \cup \cdots \cup \beta_r \equiv \{\lambda\} \cup \alpha_1 \cup \ldots \alpha_r$ then

$$C_\beta = -rC_\alpha.$$

Case 2. If β is given by $\{1, \ldots, \lambda+1\} = \beta_1 \cup \cdots \cup \beta_r \equiv \alpha_1 \cup \cdots \cup (\alpha_i \cup \{\lambda+1\}) \cup \cdots \cup \alpha_r$ (for some index i) then

$$C_\beta = C_\alpha.$$

The corresponding polynomial P is obtained recursively according to the scheme

$$P(x_1, ..., x_\lambda) \mapsto \sum_{i=1}^{\lambda+1} (x_1 + \cdots + x_{\lambda+1} - (\lambda+1)x_i)\, P(x_1, \ldots, \hat{x}_i, \ldots, x_{\lambda+1}).$$

Example 6. $P(x,y) = (x-y)^2 \mapsto (x+y-2z)^2(x-y)^2 + (x+z-2y)^2(x-z)^2 + (y+z-2x)(y-z)^2$. Writing $(x+y-2z)(x-y)^2 = (x^2-y^2)(x-y) - 2z(x^2-2xy+y^2) = x^3 - x^2y - xy^2 + y^3 - 2x^2z + 4xyz - 2zy^2$ etc., form the sum. One obtains the polynomial

$$2(x^3+y^3+z^3) - 3(x^2y + \ldots (6 \text{ terms})) + 12xyz,$$

corresponding to the covariant DO

$$\hat{f}'''\hat{f}^2 - 3\hat{f}''\hat{f}'\hat{f} + 2\hat{f}^3$$

found earlier.

It seems more difficult to incorporate the operator of the primitive Example 1 of the Introduction, viz.

$$D^\lambda(f^{-\frac{\lambda-1}{\mu+1}}),$$

in the general picture, as the corresponding polynomials in general are *μ-dependent.*

The derivative of f^α is given by a formula of the type

$$D^\lambda f^\alpha = \sum_{k=0}^{\lambda}[\alpha]_k f^{\alpha-k} Q_k^\lambda f,$$

where $[\alpha]_j = \alpha(\alpha-1)...(\alpha-(j-1))$ and Q_k^λ is the DO given by the recursion

$$Q_k^\lambda f = f' Q_{k-1}^{\lambda-1} f + (Q_k^{\lambda-1} f)'.$$

Example 7. $\lambda = 4$. Taking $\alpha = -\frac{2}{\mu+1}$, one finds, after some calculations (and omitting a constant factor),

$$(\mu+3)(\hat{f}^{IV}\hat{f}^3 - 4\hat{f}'''\hat{f}'\hat{f}^2 + 3\hat{f}''^2\hat{f}^2) - 3(5+2\mu)(\hat{f}'^4 - 2\hat{f}''\hat{f}'^2\hat{f} + \hat{f}''^2\hat{f}^2),$$

where the second term also may be written as a square $(\hat{f}'^2 - \hat{f}''\hat{f})^2$.

Question. Is there a general formula?

Now we proceed to give an entirely different approach to the problem of finding all Möbius invariant operators of the type considered, which is akin to the procedure in [M].

If f is the coefficient of a $\frac{1}{2}$-form ($\mu = 0$ is sufficient!) write (near $z = 0$)

$$f(z) = \sum_{i=0}^{\infty} s_i z^i,$$

so that $s_i = \hat{f}^{(i)}(0)$, and introduce

$$V = s_0\frac{\partial}{\partial s_1} + 2s_1\frac{\partial}{\partial s_2} + 3s_2\frac{\partial}{\partial s_3} + \ldots,$$

one of the *Cayley-Aronhold operators* (cf. [M]), so that $Vs_0 = 0$, $Vs_1 = s_0$, $Vs_2 = 2s_1$, etc. The operators considered by us are automatically invariant for translation and dilation. Thus it suffices to consider transformations of the type

$$z \mapsto \frac{z}{1+\varepsilon z}$$

only. These form, apparently, a group whose infinitesimal generator is up to sign V. Whence the condition

$$(*) \qquad \boxed{VP = 0};$$

the polynomial P is now viewed as a function of $s_0, s_1, \dots$.

Example 8. $\lambda = 2$. $P = as_2s_0 + bs_1^2$. Then $VP = 2a_2s_1s_0 + 2bs_1s_0 = 0$ yielding $a = b$. We obtain the DO $f''f - f'^2$, which of course corresponds to the Schwarzian.

Example 9. $\lambda = 3$. $P = as_3s_0^2 + bs_2s_1s_0 + cs_1^3$. We find $VP = 3as_2s_0^2 + 2bs_1^2s_0 + bs_2s_0^2 + 3cs_1^2s_0 = 0$ yielding $3a + b = 0$, $2b + 3c = 0$ or $b = -3a$, $c = -\frac{2}{3}b = 2a$, corresponding to the DO $f'''f^2 - 3f''f'f + 2f'^3$.

How does the "log" operator enter into the new picture? Set $\sigma_k \stackrel{\text{def}}{=} s_ks_1 - s_{k+1}s_0$, so that $V\sigma_k = k\sigma_{k-1}$. Consider the DOs L_k given by

$$L_k = (\mathrm{ad}V)^{k-1}L_1\,, \quad L_1 = \frac{\partial}{\partial s_1}(1 + \frac{1}{1!}\frac{\partial}{\partial s_1} + \frac{1}{2!}\frac{\partial^2}{\partial s_2^2} + \dots).$$

Then follows that if P satisfies (*) then also $Q = \sum_{k\geq 1}(-1)^k \frac{\sigma_k}{k!}L_kP$ satisfies (*).

Proof. We obtain

$$VQ = \sum(-1)^k(\frac{V\sigma_k}{k!}L_kP + \frac{\sigma_k}{k!}L_kVP - \frac{\sigma_k}{k!}[L_k, V]P) =$$

$$= \sum(-1)^k\frac{\sigma_{k-1}}{(k-1)!}L_kP - \sum(-1)^{k+1}\frac{\sigma_k}{k!}L_{k+1}P = 0.\#$$

Further, we consider briefly the transition between the two models. If $k = (k_1, \dots, k_\lambda)$ is a partition of the number λ, set

$$m_k = \sum x_{i_1}^{k_1}\dots x_{i_\lambda}^{k_\lambda},$$

where the summation extends over all *different* permutations of $x_1^{k_1}\dots x_\lambda^{k_\lambda}$; let the number of such permutations be N_k. If

$$U = \sum\frac{\partial}{\partial x_i},$$

then apparently

$$Um_k = k_1\frac{N_k}{N_{k-e_1}}m_{k-e_1} + \dots,$$

which formula has to be juxtaposed to the relation

$$Vs_k = k_1s_{k-e_1} + \dots,$$

where $s_k = s_{k_1}s_{k_2}\dots s_{k_\lambda}$. If we consider the map

$$T : P = \sum\frac{1}{N_k}a_km_k \mapsto Q = \sum a_ks_k,$$

we therefore have

$$\boxed{TU = VT}.$$

This readily yields the sought relationship.

Let us also say a few words about the (graded) ring $\mathcal{N}^r$ of invariant DO's of given order r. We claim that it is a polynomial ring in s_0, s_0^{-1} and $r-1$ more "unknowns". For instance, $\mathcal{N}^2$ is generated (apart from the identity operator) by the Schwarz operator $G_2 = G_2(f) = f''f - f'^2$, $\mathcal{N}^3$ by G_2 and $G_3 = G_3(f) = f'''f^2 - 3f''f'f + 2f'^3$, and so forth.

Proof. Choose, quite generally, $G_2, G_3, G_4, \ldots$ is such a way that

$$G_\nu(f) = f^{(\nu)} f^{\nu-1} + \text{operators of lower order;}$$

such operators do exist if $\nu \geq 2$; cf. *infra.* Let g_ν be the corresponding polynomial in the variables $s_0, s_1, s_2, \ldots$. A little thinking reveals that one can as well make the substitution $s_0 = 1$; this somewhat facilitates the following computations. If

$$P = \sum_{k=(k_1,\ldots,k_r)} a_k s_1^{k_1} \ldots s_r^{k_r}$$

contains a term with $k_r > 0$ write $s_r = g_r + \text{polynomial in}\, s_1, \ldots, s_{r-1}$. Then P equals an expression which is a polynomial in $g_r, s_1 \ldots, s_{r-1}$. Continue by induction. We see that P can be written in the form

$$P = s_1^m Q_1 + s_1^{m-1} Q_2 + \cdots + Q_m,$$

where $Q_1, Q_2, \ldots, Q_m$ are polynomials in $g_2, ..., g_r$ only. Hence

$$0 = VP = m s_1^{m-1} Q_1 + (m-1) s_1^{m-2} Q_2 + ... + 0.$$

It follows that all Q_i vanish except Q_m. Thus P also is a polynomial in $g_2, ..., g_r$. It is clear that there are no relations between the latters (and s_0).#

Here is an *explicit* system of generators. Make the "Ansatz"

$$g_n = s_n + a_1 s_{n-1} s_1 + a_2 s_{n-2} s_1^2 + \cdots + a_{n-2} s_2 s_1^{n-2} + a_{n-1} s_1^n.$$

Then

$$V g_n = n s_{n-1} + a_1 \left((n-1) s_{n-2} s_1 + 1 \cdot s_{n-1}\right) + a_2 \left((n-2) s_{n-3} s_1^2 + 2 s_{n-2} s_1\right) + \cdots +$$
$$+ a_{n-2}(2 s_1^{n-1} + (n-2) \cdot 2 s_2 s_1^{n-3}) + a_{n-1} n s_1^{n-1},$$

which yields the recursions

$$\begin{aligned}
n + a_1 &= 0 \\
(n-1)a_1 + 2a_2 &= 0 \\
(n-2)a_2 + 3a_3 &= 0 \\
(n-3)a_3 + 4a_4 &= 0 \\
&\ldots \\
3a_{n-3} + (n-2)a_{n-2} &= 0 \\
2a_{n-2} + n a_{n-1} &= 0
\end{aligned}$$

whence

$$a_k = (-1)^k \binom{n}{k} \ (k < n-1) \quad a_{n-1} = (-1)^{n-1}(\binom{n}{n-1} - 1).$$

(Notice that this is in agreement with that the sum of the coefficients has to be 0, as $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.) The corresponding DO is thus

$$G_n = \sum_{k=0}^{n-1}(-1)^k \binom{n}{k} f^{(n-k)}(f')^k f^{n-k-1} + (-1)^n(f')^n.$$

A general invariant DO thus comes as a polynomial

$$\sum_{\substack{k=(k_0,k_2,\ldots,k_r)\\ k_0+k_2+\cdots+k_r=\text{const}}} C_k f^{k_0} G_2^{k_2} G_3^{k_2} \ldots G_p^{k_p}.$$

4. The transvectant and (generalized) Hankel operators.

The classical theory of Hankel operators (or forms), see e.g. [N], App. 4, is usually formulated for operators (or forms) living on the Hardy space $H^2(\mathbf{T})$ (an analytic function f on the unit disk D is in $H^2(\mathbf{T})$ iff its trace on $\mathbf{T} = \partial D$ is in $L^2(\mathbf{T})$). More precisely, if B is an analytic function in D and if P_- denotes the orthogonal projection in $L^2(\mathbf{T})$ onto the complement $H^2_-(\mathbf{T})$ of $H^2(\mathbf{T})$ in that space, the Hankel operator H_B of symbol B is defined by the formula

$$H_B f = P_-(\overline{B} f). \tag{1}$$

Of vital importance for further developments of the theory and its ramifications is the following covariance property

$$U_\phi H_B U_\phi^{-1} = H_{B\circ\phi} \quad (\phi \in SU(1,1)). \tag{2}$$

Here U stands for the natural (unitary) action of the Möbius group $SU(1,1)$ on $H^2(\mathbf{T})$ and $H^2_-(\mathbf{T})$,

$$U_\phi f(z) = f(\phi z) e(z)^{-1}.$$

The general character of formula (1) suggest many generalizations. In the first place, what comes to ones mind is replacing $H^2(\mathbf{T})$ by a weighted Bergman (or Dzhrbashyan) space $A^{\alpha,2}(D)$ ($\alpha > -1$): an analytic function f on D is in $A^{\alpha,2}(D)$ iff it is square integrable with respect to the probability measure

$$d\mu_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z),$$

where $dA(z) = dxdy/\pi$ is the normalized area measure. The action of $SU(1,1)$ is then given by

$$U_\phi f(z) = f(\phi z) e(z)^{-(\alpha+2)},$$

but, due to the usual ambiguity in defining powers of complex numbers, we have only a projective representation (a genuine representation of a suitable covering group). To be on the safe side let us fix our attention on the case $\alpha =$ integer $= 0, 1, \ldots$. Now the symbol too transforms with a weight:

$$B \mapsto (B \circ \phi)e^{\alpha+1},$$

and is unique up to a polynomial of degree $\alpha + 1$. Bol's lemma suggests that we may alternatively, instead of B, take $b = B^{(\alpha+2)}$ as "symbol". This has also several advantages. For instance, b rather than B is used in the general theory of Hankel forms (on arbitrary domains, even in higher dimensions) developed in [JPR].

One can furthermore consider generalizations of higher weight. This was first suggested in [JP]. It is convenient to express things in terms of bilinear forms, rather than linear operators. Consider as in Sec. 2 the transvectant J_s, now with $\nu_1 = \nu_2 = \alpha + 2$. Then the (generalized) Hankel form of weight s and symbol b is defined by the integral

$$\Gamma_B(f_1, f_2) = \int_{\mathbf{T}} \bar{B} J_s z^{2\alpha+2} dz/2\pi i.$$

(Equivalently, one could have studied what in the literature are called "little" Hankel operators.) Notice that the word "weight" here is used in the sense of É. Cartan's theory. From the point of view of group representations the Hankel forms of higher weight are of importance, because they provide the decomposition of the "regular" representation of the group $SU(1,1)$ on the space of Hilbert-Schmidt operators on $A^{\alpha,2}(D)$.

One can also define ([BJP]), mimicking the primitive definition (1), so-called "big" Hankel operators, even in the case of higher weight, operators which map the Hilbert space $A^{\alpha,2}(D)$ into its orthogonal complement $(A^{\alpha,2}(D))^{\perp}$ in $L^2(D, \mu_\alpha)$. More precisely, one considers operators of the form

$$T_B^r f(z) = \int_D \overline{K(z,\zeta)\Delta^r B(z,\zeta)} f(\zeta) d\mu_\alpha(\zeta),$$

where $K(z,\zeta) = (1 - z\bar{\zeta})^{-(\alpha+2)}$ is the reproducing kernel in $A^{\alpha,2}(D)$ and where we have put

$$\text{(3)} \qquad \Delta^{(r)} B(z,\zeta) = \sum_{j+k=r-1} \binom{-r}{j} \frac{1}{k!} (B^{(k)}(z) + (-1)^{k+1} B^{(k)}(\zeta))(z-\zeta)^{-j}.$$

In particular,

$$\begin{aligned} \Delta^{(1)} B(z,\zeta) &= B(z) - B(\zeta), \\ \Delta^{(2)} B(z,\zeta) &= B'(z) + B'(\zeta) - 2\frac{B(z) - B(\zeta)}{z - \zeta} \\ &\ldots \end{aligned}$$

For certain reasons (see App. 1) we call $\Delta^{(r)}$ the *differential-difference operators of Lagrange*. They are related to Newton's divided differences (cf. Sec. 0) in a simple way:

$$\Delta^{(r)}B(z,\zeta) = (z-\zeta)^r B(\underbrace{z,\ldots,z}_{r},\underbrace{\zeta,\ldots,\zeta}_{r}).$$

The properties (boundednes, compactness, membership in Schatten classes etc.) of the operators T_B^r were studied in [AFP] for $r = 1$ and in [BJP] for $r > 1$. If $r = 1$ we have $T_B^1 = B - P_\alpha B = [B, P_\alpha]$, denoting by P_α the orthogonal projection onto $A^{\alpha,2}(D)$ in $L^2(D,\mu_\alpha)$. In this case one can also allow non-analytic symbols [AFP]. The decomposition of $L^2(D,\mu_\alpha)$ into irreducible subspaces is however not yet fully understood. (The operators T_B^r do not do the whole job, as in the case of "little" Hankel operators, because they correspond to discrete summands in the decomposition and there must be some continuous ones too.)

Remark. In [P3] a generalization of the above definition of the operators T_B^r in the case of the unit ball in $\mathbf{C}^d$ is proposed. This involves also a corresponding generalization of the transvectant.

Until now we have confined our attention to the case when the underlying space is the unit disk D in the complex plane (except for the above brief allusion to the ball). Now a few words about "regular" multiply connected planar domains. Hankel forms (or, equivalently, little Hankel operators) [GP2] and big Hankel operators [AFJP] in the case of lowest weight are defined as before. However, if we wish to define the corresponding objects of higher weight, we must first select a projective structure. In the case of big Hankel operators, however, only the projective structure associated with the "circular model" (Sec. 1) seems to work, due to the "global" definition of the operators Δ^r in formula (3).

As for higher dimensions, besides the ball, one can probably do similar considerations with any symmetric domain, not only with one of rank one. As for "curved" situations, what comes to ones mind are in the first place strictly pseudoconvex domains, in some sense "modeled" on the ball. About the only information known to us in that case are some observations due to Ewa Ligocka [L].

5. General operator calculi and quantization.

The general character of the relation (2) in the previous Section not only leads, as we have seen, to various generalization of the classical notion of Hankel operator, but also puts Hankel operator theory with its various offshoots on equal footing with operator calculi (the theory of ΨDO). Let us therefore say a few words about this, in particular, about Unterberger's program of quantization of symmetric spaces (see e.g. [UU], [Un1], [Un2]).

Let us begin by recalling some salient facts about the Weyl calculus, which has origin in Herman Weyl's ideas about quantum mechanics [Wey].

Remark. A different approach to "quantization", also of interest in Hankel (and Toeplitz) theory, was advocated by the late Berezin in a number of publications (see e.g. the survey [Bere] and further the book [Up], Lecture 10).

As everybody knows, quantization is something which has to do with the interplay between complex valued functions on a "phase space" (the symbols) and operators in a Hilbert space. In Weyl's version the phase space is "flat", a symplectic vector space, and the Hilbert space is the one where the "complex wave representation" of the CCR (canonical commutation relations) acts, in other words, Fock space. In the case of two "degrees of freedom" it is the space $F^{a,2}(\mathbf{C})$ of entire functions on $\mathbf{C}$ which are square integrable with respect to the Gaussian measure

$$d\gamma_a(z) = ae^{-a|z|^2}dA(z),$$

where a is a positive real number whose inverse (*sic!*) has the interpretation of "Planck's constant". If g is any given symbol, then its Weyl transform is the operator S_b defined by the formula

$$S_g = \int_{\mathbf{C}} g(\zeta)S_\zeta dA(\zeta),$$

where again the operators S_ζ are defined by the formula

$$(1) \qquad S_\zeta f(z) = f(2\zeta - z)e^{2az\bar{\zeta}}e^{-2a|\zeta|^2} \quad (f \in F^{a,2}(\mathbf{C})).$$

The symplectic group $Sp(2)$ (in this case isomorphic to $SL(2,\mathbf{R})$) or rather a double cover of it, the metaplectic group $Mp(2)$, has a natural action on $F^{a,2}(\mathbf{C})$ via unitary operators which we denote by V_ψ ($\psi \in Mp(2)$), say, and then, in analogy with formula (2) in Sec. 4,

$$(2) \qquad V_\psi S_g V_\psi^{-1} = S_{g\circ\psi} \quad (\psi \in Mp(2)).$$

André Unterberger's basic observation is now that the operators S_ζ are associated with the spacial symmetries $z \mapsto 2\zeta - z$ of the underlying manifold $\mathbf{C}$ (reflexions about the point ζ). Therefore exactly the same game can be played with any symmetric space, in particular, with the classical symmetric domains of É. Cartan.

Example. In the case of the unit disk D the spatial symmetries are given by

$$s_\zeta z = \frac{\eta - z}{1 - z\bar{\eta}},$$

where ζ has to coincide with the hyperbolic midpoint of the line segment with endpoints 0 and η. Therefore the analogue of formula (1) is

$$(1') \qquad S_\zeta f(z) = f(s_\zeta(z))(s'_\zeta(z))^{\frac{\alpha+2}{2}}.$$

The Hilbert space is now of course our friend $A^{\alpha,2}(D)$. (Paranthetically, we remark that in Berezin's interpretation [Bere] it is the quantity $\frac{1}{\alpha+2}$ that plays the rôle of Planck's constant!)

The point we wish to make here is now that the analogy between formula (2) above and (2) of Sec. 4 forces upon us the view that Hankel operators (or forms) and operator calculi ought to be looked upon from a unified point of view. A difference is of course that in the case of calculi one considers linear maps from the Hilbert space into itself, while in the Hankel case one has maps from one Hilbert space into another, in the case of small Hankel operators a space which can be identified with the *dual* (not the anti-dual) of the given

Hilbert space. This is about the same as the distinction between collineations and correlations in classical geometry. Indeed, in quantum theory states may be viewed as points of the associated projective space and observables (usually realized by linear operators) map projective points into projective points, thus correspond to collineations. So one may ask the question what correlations do for quantum theory. Another difference is that in Hankel theory one deals with irreducible families of operators (under the corresponding group action), not so in the case of calculi. This explains why in the case of calculi one in general expects only "onesided" results (S_p-criteria etc.). At any rate, time is still not mature to say if the analogy established really has any deeper implications or not.

6. Reproducing and coreproducing kernels.

Let Ω be any plane domain and μ a suitable positive measure on it. We denote by $A^2(\Omega,\mu)$ the Hilbert space of all analytic functions in Ω which are square integrable with respect to μ. The orthogonal complement $(A^2(\Omega,\mu))^{\perp}$ (in $L^2(\Omega,\mu)$) consists of all functions of the form $\bar{D}^*g$ with g "vanishing" on the boundary. (We write $\bar{D}$ or $\bar{\partial}$ for the Cauchy-Riemann operator $\frac{\partial}{\partial\bar{z}}$, and D or ∂ for $\frac{\partial}{\partial z}$.)

Determination of $\bar{D}^*$: Partial integration yields (g is a test function)

$$\int_\Omega \frac{\partial f}{\partial \bar{z}}\bar{g}\lambda dA = -\int_\Omega f\frac{\partial \bar{g}\lambda}{\partial\bar{z}}dA.$$

Now

$$\frac{\partial\bar{g}\lambda}{\partial\bar{z}} = \overline{\frac{\partial}{\partial z}g\lambda} = \overline{\frac{\partial g}{\partial z}\lambda + g\frac{\partial\lambda}{\partial z}}.$$

Hence

$$\boxed{\bar{D}^* = -D - \frac{\partial}{\partial z}(\log\lambda)}.$$

(Here $\lambda = \frac{d\mu}{dA}$ is the Radon-Nikodym derivative and $dA = dxdy/\pi$.)

Example 1. If $\mu = \mu_\alpha$, $\Omega = D$ (weighted Bergman case; see Sec. 4) then

$$\bar{D}^* = -D + \frac{\alpha\bar{z}}{1-|z|^2}.$$

Remark. This operator appears also in [GP1] as a covaraint derivative taking $\frac{\alpha}{2}$-forms into $\frac{\alpha+2}{2}$-forms.

We write

$$\delta = K + \bar{D}^*J, \tag{1}$$

where δ is *point evaluation* (*not* delta function). This gives

$$f(\zeta) = \int_\Omega \overline{K(z,\bar{\zeta})}f(z)d\mu(z) + \int_\Omega \overline{J(z,\bar{\zeta})}\frac{\partial f(z)}{\partial\bar{z}}d\mu(z),$$

which incidentally solves the $\bar{\partial}$-problem.

Remark. K is usually called the reproducing kernel. Accordingly, J might be termed the "coreproducing" kernel. It is of interest also in Hankel theory (cf. [AFJP]).

Determination of J (weighted Bergman case).

$$(\frac{\partial}{\partial z} - \alpha\frac{\bar{z}}{1-z\bar{z}})J = K = (1-z\bar{\zeta})^{-(\alpha+2)}.$$

"Ansatz": $J = \frac{1}{\bar{z}}f(z\bar{z})$ if $\zeta = 0$.

This gives

$$\frac{1}{\bar{z}}f'(z\bar{z})\bar{z} - \alpha\frac{\bar{z}}{1-z\bar{z}}\cdot\frac{1}{\bar{z}}f(z\bar{z}) = 1.$$

Thus $f = f(t)$ satisfies the DE

$$f' - \frac{\alpha}{1-t}f = 1$$

or

$$((1-t)^\alpha f)' = (1-t)^\alpha.$$

Integration:

$$(1-t)^\alpha f = C - \frac{1}{\alpha+1}(1-t)^{\alpha+1},$$

$$f(1) = 0 \Rightarrow C = 0,$$

$$\boxed{f = -\frac{1}{\alpha+1}(1-t)}.$$

From the transformation properties of J (see *infra*) follows

$$\boxed{J = -\frac{1}{\alpha+1}\cdot\frac{1-z\bar{z}}{(1-z\bar{\zeta})^{\alpha+1}(\bar{z}-\bar{\zeta})}}.$$

Remark. As $\frac{\partial}{\partial z}(\frac{1}{\bar{z}}) = \delta$ (in our normalization of A!) we see that we have the right polar strength. Moreover, if

$$G = 2\log|\frac{1-z\bar{\zeta}}{z-\zeta}| \quad \text{("rationalized" Green's function)}$$

then

$$\frac{\partial G}{\partial\bar{\zeta}} = -\frac{z}{1-z\bar{\zeta}} + \frac{1}{\bar{z}-\bar{\zeta}} = \frac{1-z\bar{z}}{(1-z\bar{\zeta})(\bar{z}-\bar{\zeta})}.$$

Hence the formula can also be written

$$\boxed{K_\alpha = \delta + \frac{1}{\alpha+1}\bar{\partial}^*_{\alpha z}(K_{\alpha-2}\bar{\partial}_\zeta G)}.$$

For $\alpha = 0$ this is well-known [Berg]. One reason for carrying out all these calculations has been precisely to detect possible generalizations of this formula.

The elements of $A^{\alpha,2}(D)$ behave as forms of degree ν, where $\nu = \frac{\alpha+2}{2}$. Thus, if f is in $A^{\alpha}(D)$ then $\bar{D}f$ transforms as a form of bidegree $(\nu, 1)$ and has to be integrated against forms g of degree $\nu - 1$. (In general, the $A^{\alpha,2}$-pairing extends to a pairing $\kappa^{(p,q)} \times \kappa^{(p',q')} \to$ **C** where $p + q' = q + p' = \nu$; here $\kappa^{(,)}$ stands for the (C^{∞}) sheaf of the bidigree indicated. In our case $p = \nu, q = 1, p' = \nu - 1, q' = 0$.) It follows that J transforms likewise in the variable z and as a form of degree $(0, \nu)$ in ζ:

$$J(z,\zeta) \mapsto J(\phi z, \phi\zeta) e(z)^{-\alpha} \bar{\epsilon}(\zeta)^{-(\alpha+2)}$$

where $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $e(z) = cz + d$, $\epsilon(\zeta) = c\zeta + d$.

We check the transformation properties of $\bar{D}^* = -D + \frac{\alpha\bar{z}}{1 - z\bar{z}}$. Let

$$g(z) \mapsto g(\phi(z)) e(z)^{-\alpha}.$$

On the one hand:

$$D(g(\phi(z))e(z)^{-\alpha}) = Dg(\phi(z))e(z)^{-\alpha-2} - \alpha c g(\phi(z)) e(z)^{-\alpha-1},$$

on the other hand

$$\alpha \frac{\bar{z}}{1 - |z|^2} g(\phi(z)) e(z)^{-\alpha}.$$

The difference involves a factor

$$c + \frac{\bar{z}e(z)}{1 - |z|^2} = \frac{c(1 - z\bar{z}) + \bar{z}(cz + d)}{(1 - |\phi z|^2)e(z)\overline{e(z)}} = \frac{c - cz\bar{z} + cz\bar{z} + d\bar{z}}{(1 - |\phi(z)|^2)e(z)\overline{e(z)}} = \frac{\overline{\phi(z)}}{1 - |\phi(z)|^2} \frac{1}{e(z)},$$

as $\phi(z) = \frac{az + b}{e(z)}$, $\overline{\phi(z)} = \frac{c + d\bar{z}}{\overline{e(z)}}$. Thus we find

$$\bar{D}^* g(z) \mapsto -(Dg(\phi(z)) - \alpha \frac{\overline{\phi(z)}}{1 - |\phi z|^2} g(\phi(z))) e(z)^{-\alpha-2}.$$

Limiting case: $\alpha \to \infty$, $R \to \infty$. For a disk D_R of radius R:

$$J = -\frac{R^2}{\alpha + 1} \frac{1 - \frac{z\bar{z}}{R^2}}{(1 - \frac{z\bar{\zeta}}{R^2})^{\alpha+1}(\bar{z} - \bar{\zeta})} \to -\frac{e^{z\bar{\zeta}}}{\bar{z} - \bar{\zeta}},$$

$$\bar{D}^* = -D + \frac{\frac{\alpha\bar{z}}{R^2}}{1 - \frac{|z|^2}{R^2}} \to -D + \bar{z}.$$

We can check directly:

$$(D-\bar{z})\frac{e^{z\bar{\zeta}}}{\bar{z}-\bar{\zeta}} = \frac{\bar{\zeta}e^{z\bar{\zeta}}}{\bar{z}-\bar{\zeta}} - \frac{\bar{z}e^{z\bar{\zeta}}}{\bar{z}-\bar{\zeta}} = -e^{z\bar{\zeta}}.$$

The case of an annulus $\Omega = \Omega_R = \{z : 1 < |z| < R\}$: It is better to rewrite equation (1) as

$$\frac{\partial u}{\partial z} = -\lambda K + \text{usual delta}$$

with $u = \lambda J$; λ is assumed to be radial, $\lambda = \lambda(r^2)$, $r^2 = |z|^2 = z\bar{z}$. Set

$$M_n(r) = \int_1^r r_1^{2n}\lambda(r_1^2)dr_1^2.$$

Then

$$K = \sum_{n=-\infty}^{\infty} \frac{(z\bar{w})^n}{M_n(R)}.$$

We first solve for each n the equation

$$\frac{\partial u_n}{\partial z} = -\frac{(\bar{w}z)^{n-1}}{M_{n-1}(R)}\lambda(r^2).$$

Put

$$u_n = f_n(z\bar{z})z^n$$

so that

$$\frac{\partial u_n}{\partial z} = [f_n'(z\bar{z})z\bar{z} + nf_n(z\bar{z})]z^{n-1}.$$

Hence (writing $t = r^2$)

$$tf_n' + nf_n(t) = -\frac{\bar{w}^{n-1}}{M_{n-1}(R)}\lambda(t)$$

yielding the particular solution ($C = 0$)

$$f_n(t) = -t^{-n}\int_1^t t_1^{n-1}\lambda(t_1)dt_1 \cdot \frac{\bar{w}^{n-1}}{M_{n-1}(R)}$$

or

$$u_n = -\frac{M_{n-1}(r)}{M_{n-1}(R)} \cdot \frac{z^n\bar{w}^{n-1}}{r^{2n}}.$$

On the other hand

$$\frac{\partial v}{\partial z} = \text{delta}$$

is solved by

(2)
$$v = \frac{1}{\bar{z} - \bar{w}} = \begin{cases} -\sum_{n=0}^{\infty} \frac{\bar{z}^n}{\bar{w}^{n+1}} & (1 < |z| < |w|) \\ \sum_{n=-\infty}^{-1} \frac{\bar{z}^n}{\bar{w}^{n+1}} & (|w| < |z| < R) \end{cases}$$

and the homogeneous equation

$$\frac{\partial h}{\partial z} = 0$$

is solved by any anti-analytic function

(3)
$$h = \sum_{-\infty}^{\infty} a_n \bar{z}^n.$$

Summing up ((1)+(2)+(3)) we get

$$u = \sum_{n=-\infty}^{\infty} u_n + v + h$$

where we have to adjust the coefficients a_n so as to meet the boundary condition. Thus, finally, we get

$$u = \begin{cases} -\sum_{-\infty}^{\infty} \frac{M_{n-1}(r)}{M_{n-1}(R)} \cdot \frac{z^n \bar{w}^{n-1}}{r^{2n}} & (1 < |z| < |w|) \\ \sum_{n=-\infty}^{\infty} (1 - \frac{M_{n-1}(r)}{M_{n-1}(R)}) \cdot \frac{z^n \bar{w}^{n-1}}{r^{2n}} & (|w| < |z| < R) \end{cases}.$$

Remark. This technique of finding a fundamental solution via a series development has a general character and can be applied in many other situations.

Example 2. $\lambda \equiv 1$. Then

$$J = \begin{cases} -\sum_{n=-\infty}^{\infty} \frac{1 - r^{-2n}}{R^{2n} - 1} z^n \bar{w}^{n-1} & (|z| < |w|) \\ \sum_{n=-\infty}^{\infty} \frac{(R/r)^{2n} - 1}{R^{2n} - 1} z^n \bar{w}^{n-1} & (|z| > |w|) \end{cases}.$$

It is easy to express these series in terms of theta functions; cf. the formula for Green's function given in [CH], S. 335-357.

Several variables. Take $\Omega = \mathcal{B} =$ unit ball in $\mathbf{C}^n$ (equipped with its usual Hermitean metric $\|\cdot\|$, the corresponding inner product being $(\cdot,\cdot)$) and $\lambda = (\alpha+1)(1 - \|z\|^2)^\alpha$. We have to solve the equation

$$\delta = K + \sum_{k=1}^{n} (-\frac{\partial J^k}{\partial z^k} + \alpha \frac{\bar{z}^k}{1 - \|z\|^2} J^k)$$

corresponding to the integral formula

$$f(\zeta) = \int_{\mathcal{B}} \overline{K(z,\overline{\zeta})} f(z) d\mu(z) + \int_{\mathcal{B}} \sum_{k=1}^{n} \overline{J^k(z,\overline{\zeta})} \frac{\partial f(z)}{\partial z^k} dV(z);$$

dV is Euclidean volume measure conveniently normalized.

"Ansatz." $J^k = z^k f(\|z\|^2)$ for $\zeta = 0$.

Then

$$\frac{\partial J^k}{\partial z^k} = f(\|z\|^2) + z^k \bar{z}^k f'(\|z\|^2),$$

$$\sum \frac{\partial J^k}{\partial z^k} = n f(t) + t f'(t) \quad (t = \|z\|^2),$$

$$\sum \alpha \frac{\bar{z}^k}{1 - \|z\|^2} J^k = \alpha \frac{t f(t)}{1-t},$$

which gives the DE

$$t f'(t) + (n - \frac{\alpha t}{1-t}) f(t) = 1$$

with the integrating factor $t^n (1-t)^\alpha$. Thus we find

(4) $$f(t) = t^{-n}(1-t)^{-\alpha} \int_0^t t_1^{n-1} (1-t_1)^\alpha dt_1.$$

In particular $f(t) = \frac{1}{n}(1 - t^{-n})$ if $\alpha = 0$.

Invariance properties of K and J:

$$K(z,\zeta) = K(\phi(z), \overline{\phi(\zeta)})(\det \phi'(z))^{1+\alpha/(n+1)} (\overline{\det \phi'(\zeta)})^{1+\alpha/(n+1)},$$

$$\sum_{k=1}^{n} J^k(z,\zeta) \frac{\partial \phi^l}{\partial z^k} = J^l(\phi(z), \overline{\phi(\zeta)})(\det \phi'(z))^{1+\alpha/(n+1)} (\overline{\det \phi'(\zeta)})^{1+\alpha/(n+1)}.$$

Here ϕ is any element of the group of biholomorphic automorphisms of $\mathcal{B}$ (known to be isomorphic to the group $PSU(n,1)$).

We apply this to the fundamental symmetry ϕ interchanging 0 and ζ, i.e. $\phi(0) = \zeta$, $\phi(\zeta) = 0$, $\phi^2 = \text{id}$. We omit the calculatiosn, which are very much similar to thew ones in [P3], and write only down the end result:

$$\boxed{J(z,\zeta) = f(\rho(z,\zeta)) \frac{z - \zeta}{(1 - \|\zeta\|)(1 - (z,\zeta))^{\alpha+n}}}.$$

Here the function f is as in (4) and the invariant distance between the points z and ζ is given by

$$\rho(z,\zeta) = 1 - \frac{(1 - \|z\|^2)(1 - \|\zeta\|^2)}{|1 - (z,\zeta)|^2}.$$

Appendix 1. Lagrange's proof of the addition theorem for elliptic functions. In the appendices we take up various issues more or less loosely connected with what we have discussed in the main body of the paper. We begin with Lagrange's beautiful proof of the addition theorem for elliptic functions (due to Euler). It was subsequently superseded by other, more powerful proofs, for instance the one based on Abel's theorem (see e.g. [Web], S. 27-32), so but for a brief mention in Houzel's masterly survey of the classical theory of elliptic and Abelian functions ([Hou], p. 9) it seems to be completely forgotten nowadays (Houzel writes: "ce qui provoqua l'admiration d'Euler").

Consider an elliptic curve given by the equation

$$\xi^2 = P(x) \equiv Ex^4 + Dx^3 + Cx^2 + Bx + A.$$

(Such an equation is invariant under the transformations

$$x' = \frac{ax+b}{cx+d}, \quad \xi' = \frac{\xi}{(cx+d)^2}.$$

Therefore it is natural to consider the curve to lie on the projective completion of the bundle κ^{-1}, where κ is the canonical sheaf over $\mathbf{P}^1$, which is known to be a rational ruled surface (see [GH], p. 514-520).)

Consider the differentials

$$\frac{dx}{X} = dt, \frac{dy}{Y} = ds, \frac{dz}{Z} = dr,$$

where

$$t + s + r = 0 \quad (\Rightarrow dt + ds + dr = 0)$$

and where we have put

$$X = \sqrt{P(x)}, Y = \sqrt{P(y)}, Z = \sqrt{P(z)}.$$

Then

$$dX = \frac{P'(x)}{2\sqrt{P(x)}} \cdot \sqrt{P(x)}dt = \frac{1}{2}P'(x)dt \quad \text{etc.}$$

and we can write

$$(1) \qquad d\left(\frac{X-Y}{x-y}\right)^2 = 2\frac{X-Y}{x-y}\cdot\left(\frac{dX-dY}{x-y} - \frac{(X-Y)(dx-dy)}{(x-y)^2}\right) =$$

$$= \frac{(X-Y)(P'(x)dt - P'(y)ds)}{(x-y)^2} - 2\frac{(X-Y)^2(Xdt - Yds)}{(x-y)^3}.$$

On the other hand, as P is a polynomial of degree at most 4, we have

$$\frac{P'(x) + P'(y) - 2\frac{P(x)-P(y)}{x-y}}{(x-y)^2} = 2E(x+y) + D.$$

We can write this as

$$(2)\qquad d(E(x+y)^2 + D(x+y)) = \frac{P'(x) + P'(y) - 2\dfrac{(X-Y)(X+Y)}{x-y}}{(x-y)^2}(Xdt + Yds).$$

Subtracting (2) from (1) we find

$$(3)\qquad d\left\{(\frac{X-Y}{x-y})^2 - E(x+y)^2 - D(x+y)\right\} =$$

$$= -\left\{\frac{YP'(x) + XP'(y)}{(x-y)^2} - 4\frac{(X-Y)XY}{(x-y)^3}\right\}(dt+ds) =$$

$$= 2\frac{\dfrac{dX}{dx} + \dfrac{dY}{dy} - 2\dfrac{X-Y}{x-y}}{(x-y)^2}XYdr.$$

Thus, the Lagrange differential-difference operator appears in two different ways. Formula (3) is a way of expressing the addition theorem for elliptic functions.

In particular, taking $r = \text{const}$ it follows that

$$\frac{dx}{X} + \frac{dy}{Y} = 0$$

has a solution y which is an algebraic function of x. Indeed, in integrated form (3) gives

$$(3')\qquad (\frac{X-Y}{x-y})^2 = E(x+y)^2 + D(x+y) + G.$$

This is how the addition theorem for elliptic functions was formulated by Euler (cf. [Hou]).

Consider the special case when there is no x^4-term ($E = 0$), that is, one of the four roots of the polynomial P sits at infinity. Then we have virtually the elliptic curve in Weierstrass's normal form. In this case one further takes $D = 4$, $C = 0$, thus the curve has the equation

$$\xi^3 = 4x^3 + Bx + A.$$

Analyzing the behavior of the curve at ∞ one readily sees that $G = -4z$. Thus (3′) takes the symmetric form

$$x + y + z = k^2,$$

where k denotes the slope of the three collinear points (x, ξ) etc. This is how the addition theorem is usually stated, along with its geometric interpretation (see again [Web], loc. cit. or [GH], p. 227).

Appendix 2. Jacobi's DE.

In this Appendix we reproduce the essentials of the proof of Jacobi's theorem [J2] in Example 3 of the Introduction to the effect that the "Thetanullwerthe" (theta constants) satisfy a third order *algebraic* DE. This because it is something which, apparently, is little known nowadays (in this context it is perhaps amusing to have a peek at Rubel's paper [Ru]) and, on the other hand, definitely belongs to our subject. After some thinking it is not that formidable as it looks at the first sight - we should bear in mind that Jacobi was a master of DE's, both O and P, at a time when the theory of DE's still was finding explicit solutions.

The proof goes via the theory of complete elliptic integrals and their representation in terms of the "Thetanullwerthe". Recall that the complete elliptic integral of the first kind is defined as

$$K(k) = \int_0^{2\pi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}};$$

the number k is known as the modulus. In terms of theta values one has (cf. e.g. [BB], th. 2.1, p. 35)

$$K(k) = \frac{2}{\pi}\vartheta_{00}^2(0,t) \quad \text{where} \quad k' = \vartheta_{01}^2(0,t)/\vartheta_{10}^2(0,t).$$

Here and in the sequel $k' = \sqrt{1-k^2}$ is the complementary modulus. The basis of the method is now the fact that K satisfies a second order linear DE (Legendre's or the Fuchs-Picard equation; see e.g. [Cl], p. 58-62) which may be written as

$$\frac{d\left(k^2k'^2\frac{dK}{dk^2}\right)}{dk^2} = \frac{1}{4}K.$$

or with

$$(1) \qquad \frac{dk^2}{k^2k'^2} = d\log\frac{k^2}{k'^2} = dz,$$

again as

$$(2) \qquad \frac{d^2K}{dz^2} = \frac{1}{4}k^2k'^2K.$$

Because of the symmetry it follows that a second solution is $K'(k) = K(k')$.

The first step of the proof is quite general so let us for a while consider the general equation

$$(3) \qquad \frac{d^2F}{dz^2} + q(z)F = 0.$$

We know that a local "uniformizing" parameter can be obtained by putting

$$t = \frac{F_1}{F}$$

where F and F_1 are any two linearly independent solutions of (3). Also, in view of the special form of equation (3) (no "middle" term with the first derivative!), the Wronskian is constant:

$$(4) \qquad [F,F_1] = FF_1' - F'F_1 = \alpha.$$

(In Jacobi's case, viz. equation (2), one takes $F = \frac{2}{\pi}K$, $F_1 = -2K'$, with z and k related by (1), and, considering the expansion of these functions for small values of k, one finds $\alpha = 1$; this was done by Euler [Eu].) Write then (4) as

$$(\frac{F_1}{F})' = \frac{\alpha}{F^2},$$

that is,

(5)
$$dt = \frac{\alpha}{F^2} dz$$

or, with

$$C = \frac{1}{F},$$

again as

(5′)
$$dt = \alpha C^2 dz.$$

It follows that

$$\frac{dF}{dt} = \frac{1}{\alpha} F^2 \frac{dF}{dz}$$

or

$$\frac{dC}{dt} = -\frac{1}{\alpha}\frac{dF}{dz}.$$

Continuing the differentiation gives

$$\frac{d^2C}{dt^2} = -\frac{1}{\alpha^2} F^2 \frac{d^2F}{dz^2} = \frac{1}{\alpha^2} F^3 q = \frac{1}{\alpha^2} C^{-3} q$$

or

(6)
$$\boxed{C^3 C'' = \alpha^{-2} q}.$$

In the case of equation (2) (in which case $q = -\frac{1}{4}k^2k'^2$, $\alpha = 1$) this gives

(7)
$$C^3 C'' = -\frac{1}{4} k^2 k^2.$$

To proceed further we must invoke the inverse function, say, Q to q. Differentiating the relation $z = Q(q)$ and using (5′) and (6) one finally finds

(8)
$$\boxed{\alpha C^2 \frac{dQ(\frac{C^3 C''}{\alpha^2})}{dt} = 1}.$$

It remains to determine Q and Q' in the case of equation (2). As $k^4 - k^2 - 4q = 0$ we find

$$z = \log \frac{k^2}{k'^2} = \log \frac{1+\sqrt{1+16q}}{1-\sqrt{1+16q}}$$

yielding

$$dz = (\frac{1}{1+\sqrt{1+16q}} + \frac{1}{1-\sqrt{1+16q}})\frac{16dq}{2\sqrt{1+16q}} = \frac{d\log q}{\sqrt{1+16q}}.$$

Thus (6) (or (7)) gives

$$C^2 \frac{d}{dt} \log(C^3C'') = \sqrt{1+16C^3C''}$$

as in Example 1 of the Introduction. (This for the principal theta $\vartheta = \vartheta_{00}$; the calculations for the remaining thetas are similar.) If we, following Jacobi, put $C = y^{-2}$ one gets after some "simplifications"

$$(y^2y''' - 15yy'y'' + 30y'^3)^2 + 32(yy'' - 3y'^2)^3 = y^{10}(yy'' - 3y'^2)^2,$$

which does not seem to be very illuminating. This DE is thus satisfied by theta series. The rest of Jacobi's proof is devoted to exhibiting the general solution, but we have not examined this part in detail. Perhaps there is a general principle saying that the solutions of a Möbius third order DE can be generated from a single particular solution? At least a naive count of parameters supports such a belief.

We conjecture that theta series in two variables can be treated in a similar fashion. This is also suggested by the parallel between the arithmetic-geometric means of Gauss and Borchardt (see [P4]).

Appendix 3. A transformation theory for the heat equation.

We know that the second order linear DO has an interesting transformation theory, connected with names such as Jacobi, Kummer, Riemann, Schwarz, etc. (cf. Sec. 1). It is perhaps less known that the heat equation is susceptible to a similar treatment, to which we know approach.

Let us make in the equation

$$\frac{\partial u}{\partial a} = \frac{1}{2}\frac{\partial^2 u}{\partial c^2} + qu \quad (q = q(a,c))$$

the substitution

$$v = u(\varphi, \psi)m \quad (\text{where}\, \varphi = \varphi(a), \psi = \psi(a,c), m = m(a,c)).$$

(The perhaps change looking choice of the letters a and c for the independent variables is in accordance with [P5].) When does it go over into an equation of the same type? Derivation yields

$$\frac{\partial v}{\partial a} - \frac{1}{2}\frac{\partial^2 v}{\partial c^2} - \tilde{q}v =$$

$$= \frac{\partial u}{\partial a}\varphi' m - \frac{1}{2}\frac{\partial^2 u}{\partial c^2}(\frac{\partial \psi}{\partial c})^2 m + \frac{\partial u}{\partial c}(\frac{\partial \psi}{\partial a}m - \frac{1}{2}\frac{\partial^2 \psi}{\partial c^2}m - \frac{\partial \psi}{\partial c}\frac{\partial m}{\partial c}) + u(\frac{\partial m}{\partial a} - \frac{1}{2}\frac{\partial^2 m}{\partial c^2} - \tilde{q}m).$$

Thus one gets the following conditions:

$$1) \quad \varphi' = (\frac{\partial\psi}{\partial c})^2,$$

yielding

$$\boxed{\psi = \sqrt{\varphi'}c + r} \quad (r = r(a)).$$

$$2) \quad \frac{\partial\psi}{\partial a}m - \frac{1}{2}\frac{\partial^2\psi}{\partial c^2}m - \frac{\partial\psi}{\partial c}\frac{\partial m}{\partial c} = 0,$$

which, if we take into account that $\frac{\partial\psi}{\partial a} = \frac{1}{2}\frac{\varphi''}{\sqrt{\varphi'}}c + r'$, $\frac{\partial^2\psi}{\partial c^2} = 0$, $\frac{\partial\psi}{\partial c} = \sqrt{\varphi'}$, gives

$$\boxed{m = Ne^{\frac{1}{4}\frac{\varphi''}{\varphi'}c^2 + \frac{r'}{\sqrt{\varphi'}}c}} \quad (N = N(a).$$

Our transformation is thus determined by the data φ, r, N. Moreover, one obtains the following relation between q and $\tilde{q}$:

$$3) \quad \frac{\partial m}{\partial a} - \frac{1}{2}\frac{\partial^2 m}{\partial c^2} - \tilde{q}m = -q\varphi' m.$$

Thus holds:

$$\frac{\partial v}{\partial a} - \frac{1}{2}\frac{\partial^2 v}{\partial c^2} - \tilde{q}v \equiv (\frac{\partial u}{\partial a} - \frac{1}{2}\frac{\partial^2 u}{\partial c^2} - qu)\varphi' m.$$

Inserting of the two boxed formulae $\square$ into 3) gives

$$\frac{1}{4}(\frac{\varphi''}{\varphi'})'c^2N + \left(\frac{r'}{\sqrt{\varphi'}}\right)' cN + N' =$$

$$= \frac{1}{2}(\frac{1}{2}\frac{\varphi''}{\varphi'}c + \frac{r'}{\sqrt{\varphi'}})^2N + \frac{1}{2}\frac{1}{2}\frac{\varphi''}{\varphi'}N + \tilde{q}N - q\varphi' N.$$

Write:

$$\tilde{q} - q\varphi' = \frac{1}{2}Pc^2 + Qc + R \quad (P = P(a)\text{etc.}).$$

c^2-coefficient:

$$\frac{1}{4}(\frac{\varphi''}{\varphi'})' - \frac{1}{8}(\frac{\varphi''}{\varphi'})^2 = \frac{1}{2}P$$

or

$$\boxed{\frac{1}{2}\{\varphi, a\} = P}.$$

c-coefficient:

$$(\frac{r''}{\sqrt{\varphi'}})' - \frac{1}{2}\frac{\varphi''}{\varphi'}\frac{r'}{\sqrt{\varphi'}} = Q$$

or

$$\boxed{\frac{r''}{\sqrt{\varphi'}}-\frac{r'\varphi''}{(\sqrt{\varphi'})^3}=Q}.$$

It will be convenient to set

$$r=s\sqrt{\varphi'},\ r'=s\frac{\varphi''}{2\sqrt{\varphi'}}+s'\sqrt{\varphi'},\ r''=s(\frac{\varphi'''}{2\sqrt{\varphi'}}-\frac{(\varphi'')^2}{4(\sqrt{\varphi'})^3})+2s'\frac{\varphi''}{2\sqrt{\varphi'}}+s''\sqrt{\varphi'},$$

yielding

$$s(\frac{\varphi'''}{2\varphi'}-\frac{(\varphi'')^2}{4(\varphi')^2})+s'\frac{\varphi''}{\varphi'}+s''-s\frac{(\varphi'')^2}{2(\varphi')^2}-s'\frac{\varphi''}{\varphi'}=Q$$

or

$$s\cdot\frac{1}{2}\{\varphi,a\}+s''=Q$$

or again, taking account of the expression for P,

$$\boxed{s''+Ps=Q}.$$

c^0-coefficient:

$$N'=(\frac{1}{2}\frac{(r')^2}{\varphi'}+\frac{1}{4}\frac{\varphi''}{\varphi'}+R)N$$

or in terms of s

$$\boxed{N'=(\frac{1}{8}s^2(\frac{\varphi''}{\varphi'})^2+\frac{1}{2}\frac{\varphi''}{\varphi'}ss'+\frac{1}{2}(s')^2+\frac{1}{4}\frac{\varphi''}{\varphi'}+R)N}.$$

From the last boxed formulae □ it follows that P,Q,R essentially determine φ,r,N. But the formula for N can be put in a more clearly visible? (convincing?) form. Write

$$\psi=\sqrt{\varphi'}(c+s)$$

with as before $r=\sqrt{\varphi'}s$. We thus may view our transformation as a composition, a "translation" followed by a "dilation". In an analogous way we can reform the expression for m:

$$m=N\exp(\frac{1}{4}\frac{\varphi''}{\varphi'}(c+s)^2+s'c-\frac{1}{4}\frac{\varphi''}{\varphi'}s^2).$$

It is natural to absorb the c-independent term $-\frac{1}{4}\frac{\varphi''}{\varphi'}s^2$ in the factor N. Thus let us write

$$N=N^*\exp(\frac{1}{4}\frac{\varphi''}{\varphi'}s^2)$$

so that

$$\boxed{m = N^* \exp(\frac{1}{4}\frac{\varphi''}{\varphi'}(c+s)^2 + s'c)}.$$

The DE for N^* now becomes

$$\frac{N^{*\prime}}{N^*} + \frac{1}{4}(\frac{\varphi''}{\varphi'})'s^2 + \frac{1}{2}\frac{\varphi''}{\varphi'}ss' = \frac{1}{8}(\frac{\varphi''}{\varphi'})^2 s^2 + \frac{1}{2}\frac{\varphi''}{\varphi'}ss' + \frac{1}{2}(s')^2 + \frac{1}{4}\frac{\varphi''}{\varphi'} + R$$

or

$$\frac{N^{*\prime}}{N^*} = -\frac{1}{4}\{\varphi, a\}s^2 + \frac{1}{2}s'^2 + \frac{1}{4}\frac{\varphi''}{\varphi''} + R = \frac{1}{4}\frac{\varphi''}{\varphi'} + L + R,$$

where $L = \frac{1}{2}[(s')^2 - Ps^2]$ may be viewed as a *Lagrangian*. Integration gives

$$\boxed{N^* = (\varphi')^{\frac{1}{4}} \exp(\int Lda + \int Rda)},$$

where $\int Lda$ again may be interpreted as an *action*.

In the examples below we take $q = 0, Q = R = 0$.

Example 1. $P = 0$. $\dfrac{\partial v}{\partial a} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial c^2}$ (heat equation). Take

$$\varphi(a) = \frac{1}{a} \quad (\text{implying}\, \varphi' = -\frac{1}{a^2}, \varphi'' = \frac{2}{a^3}, \frac{\varphi''}{\varphi'} = -\frac{2}{a})$$

and

$$s(a) = -d = \text{constant} \quad (\text{implying } s' = 0, L = 0).$$

Taking $u = 1$ gives then (the fundamental solution of the heat equation)

$$\boxed{v = \frac{1}{a^{\frac{1}{2}}} \exp(-\frac{(c-d)^2}{2a})}.$$

Example 2 (generalization of example 1). $P = -1$. $\dfrac{\partial v}{\partial a} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial c^2} - v$ (Gibb's equation for the harmonic oscillator). Take

$$\varphi(a) = \coth a \quad (\text{implying}\, \varphi' = -\frac{1}{\sinh^2 a}, \varphi'' = \frac{2\cosh a}{\sinh^3 a}, \frac{\varphi''}{\varphi'} = -2\coth a)$$

and

$$s(a) = -d\cosh a$$

(implying

$$s' = -d\sinh a,\; L = \frac{1}{2}d^2((\cosh a)^2 + (\sinh a)^2) = \frac{d^2}{4}(e^{2a} + e^{-2a}),$$

$$\int L da = \frac{d^2}{8}(e^{2a} - e^{-2a}) = \frac{d^2}{4}\sinh 2a = \frac{1}{2}d^2 \cosh a \sinh a).$$

We find the fundamental solution (Mehler's formula)

$$\boxed{v = \frac{1}{(\sinh a)^{\frac{1}{2}}} \exp(-\frac{\cosh a(c^2 + d^2) - 2cd}{2\sinh a})}.$$

Question. Are there any other interesting examples?

Appendix 4. Wave packets versus Gauss-Weierstrass functions.[4]

In this appendix we will compare wave packets in the sense of Cordoba and Fefferman [CF] with the Gauss-Weierstrass functions (cf. [P5], [JPW]). For the sake of simplicity we shall confine our attention to the one dimensional case. Alternative names for the same objects: Gabor wavelet, (canonical) coherent state, Gaussian density etc.

We consider the Fock space $F^{1,2}(\mathbf{C})$ of entire functions in $\mathbf{C}$ with the metric

$$\|f\|^2 = \frac{1}{\pi}\iint_{\mathbf{C}} |f(z)|^2 e^{-|z|^2} dx dy.$$

Thus, compared to Sec. 5, "Planck's constant" $1/\alpha$ is (out of convenience) taken to be 1. Make the substitution

$$f(z) = f_1(z)e^{\frac{1}{2}z^2}.$$

Using the identity $z^2 = x^2 - y^2 + 2ixy$ the metric then takes the form

$$\|f_1\|^2 = \frac{1}{\pi}\iint_{\mathbf{C}} |f_1(z)|^2 e^{-2y^2} dx dy,$$

which in view of Parseval's theorem and Fubini can be written as

$$\frac{1}{\pi}\int_{\mathbf{R}}(\frac{1}{2\pi}\int_{\mathbf{R}} |\hat{f}_1(\xi)e^{-y\xi}|^2 d\xi)e^{-2y^2} dy =$$

$$= \frac{1}{2\pi^2}\int_{\mathbf{R}} |\hat{f}_1(\xi)|^2 (\int_{\mathbf{R}} e^{-2y\xi - 2y^2} dy) d\xi = \frac{1}{2\pi^2}\sqrt{\frac{\pi}{2}}\int_{\mathbf{R}} |\hat{f}_1(\xi)|^2 e^{\frac{\xi^2}{2}} d\xi.$$

So finally setting

$$\hat{f}_1(\xi) = f_2(\xi)e^{-\frac{\xi^2}{4}}$$

we get the metric

$$\|f_2\|^2 = \int_{\mathbf{R}} |f_2(\xi)|^2 d\xi.$$

Summing up, we have deduced (I hope, in a comprehensive way) the well-known *Bargmann transform.* It connects the Bargmann-Segal and Schrödinger representations of the Heisenberg group.

[4] The following is the outcome of a discussion that the writer had with A. Cordoba, to whom he expresses his sincere thanks for his patience.

Next, take (*Gauss-Weierstrass function* [P5])

$$f = e_{ac} = e^{\frac{1}{2}az^2+cz} \quad (|a| < 1).$$

Then

$$f_1 = e^{\frac{1}{2}az^2+cz}e^{-\frac{1}{2}z^2} = e^{\frac{1}{2}(a-1)z^2+cz};$$

$$\hat{f}_1(\xi) = \int_{\mathbf{R}} \exp(-ix\xi + \frac{1}{2}(a-1)x^2 + cx)dx =$$

$$= \int_{\mathbf{R}} \exp(-\frac{1}{2}(1-a)\left[x + \frac{i\xi - c}{1-a}\right]^2 + \frac{1}{2}\frac{(i\xi-c)^2}{1-a})dx = \sqrt{\frac{2\pi}{1-a}}\exp(\frac{1}{2}\frac{(i\xi-c)^2}{1-a});$$

$$f_2(\xi) = \frac{1}{\sqrt{1-a}}\exp(\frac{1}{2}\frac{(i\xi-c)^2}{1-a} + \frac{\xi^2}{4}) = \frac{1}{\sqrt{1-a}}\exp(-\frac{1}{4}\frac{1+a}{1-a}\xi^2 - \frac{i\xi c}{1-a} + \frac{1}{2}\frac{c^2}{1-a}).$$

This has to be compared with the *wave packet*

$$\phi_{(x_0,\xi_0,g)}(\xi) = \exp\left(ix_0(\xi-\xi_0) + \frac{i}{2}g(\xi-\xi_0)^2\right),$$

where x_0, ξ_0 are *real* quantities and Im$g > 0$ (cf. [CF]; notice that compared to that paper here the Latin and Greek letters happen to have the opposite meaning.) This suggests to put

$$g = \frac{i}{2}\frac{1+a}{1-a}; \quad g\xi_0 - x_0 = \frac{c}{1-a}.$$

The first relation is the Cayley transform, while the second relation expresses, in classical parlance, that $\xi_0, -x_0$ are the *characteristics* of the complex number $\frac{c}{1-a}$. Look now at the constant exponentials. On the one hand one has

$$\frac{1}{2}\frac{c^2}{1-a},$$

on the other hand

$$\frac{1}{2}ig\xi_0^2 - ix_0\xi_0.$$

How to explain this discrepancy?

It will be expedient to pass via the transformation theory of the heat equation (cf. App. 3). Consider quite generally the PDE

$$\frac{\partial F}{\partial a} = \frac{1}{2}\frac{\partial^2 F}{\partial c^2}.$$

The substitution

$$G(g,z) = F(a,c)\cdot\exp - \frac{c^2}{2(1-a)})\sqrt{1-a} \,\text{where}\, g = \frac{i}{2}\frac{1+a}{1-a},\ z = \frac{c}{1-a}$$

gives the equation

$$\frac{\partial G}{\partial g} = -\frac{i}{2}\frac{\partial^2 G}{\partial z^2};$$

this is the "Cayley transformed" heat equation. Next set

$$\phi = G(g, g\xi - x)\exp\left(\frac{i}{2}g\xi^2 - ix\xi\right).$$

Then

$$\frac{\partial \phi}{\partial g} = \left(\frac{\partial G}{\partial g} + \xi\frac{\partial G}{\partial z} + \frac{i}{2}\xi^2 G\right)\cdot \exp\left(-\frac{i}{2}g\xi^2 + ix\xi\right),$$

$$\frac{\partial^2 \phi}{\partial x^2} = \left(\frac{\partial^2 G}{\partial z^2} + 2i\xi\frac{\partial G}{\partial z} - \xi^2 G\right)\cdot \exp\left(-\frac{i}{2}g\xi^2 + ix\xi\right).$$

Thus, taking the difference, we end up with the "Schrödinger equation"

$$\boxed{\frac{\partial \phi}{\partial g} = -\frac{i}{2}\frac{\partial^2 \phi}{\partial x^2}},$$

which occurs in [CF]. The exponential factors are the same as above. We have thus adequately established the essential identity of the two concepts, wave packets and Gauss-Weierstrass functions.

Appendix 5. The cross ratio of four nearby points on a line.

Let $x = x(t)$ be the coordinate of a moving point on a projective line (or on a Riemann surface equipped with a projective structure).

The crossratio of any four of the points is

$$D(x(t_1), \ldots, x(t_4)) = \frac{x(t_1) - x(t_3)}{x(t_1) - x(t_4)} : \frac{x(t_2) - x(t_3)}{x(t_2) - x(t_4)} \prod_{\substack{i=1,2\\k=3,4}} (x(t_i) - x(t_k))^{(-1)^{i+k}}.$$

Near $t = 0$ we have the expansion

$$x(t_i) - x(t_k) = (t_i - t_k)x'(0) + \frac{t_i^2 - t_k^2}{2}x''(0) + \cdots =$$

$$= (t_i - t_k)x'(0)[1 + \frac{1}{2}\frac{t_i^2 - t_k^2}{t_i - t_k}\frac{x''(0)}{x'(0)} + \frac{1}{6}\frac{t_i^3 - t_k^3}{t_i - t_k}\frac{x'''(0)}{x'(0)} + \frac{1}{24}\frac{t_i^4 - t_k^4}{t_i - t_k}\frac{x^{IV}(0)}{x'(0)} + \ldots] =$$

$$= (t_i - t_k)x'(0)[1 + a_1^{ik} + a_2^{ik} + a_3^{ik} + \ldots],$$

where we have put

$$a_r^{ik} \stackrel{\text{def}}{=} \xi_r \frac{t_i^{r+1} - t_k^{r+1}}{t_i - t_k}$$

with

$$\xi_r \stackrel{\text{def}}{=} \frac{1}{(r+1)!}\frac{x^{(r+1)}(0)}{x'(0)}.$$

We therefore find

(1) $$\log D(x(t_1),\dots,x(t_4)) : D(t_1,\dots,t_4) =$$

$$\sum_{\substack{i=1,2\\k=3,4}} (-1)^{i+k}\left\{(a_1^{ik}+a_2^{ik}+\dots) - \frac{1}{2}(a_1^{ik}+a_2^{ik}+\dots)^2 + \frac{1}{3}(a_1^{ik}+a_2^{ik}+\dots)^3 - \dots\right\}.$$

Let us further introduce the notation

$$T^{\alpha\beta} \stackrel{\text{def}}{=} (t_1^\alpha - t_2^\alpha)(t_3^\beta - t_4^\beta) = \sum_{\substack{i=1,2\\k=3,4}} (-1)^{i+k} t_i^\alpha t_k^\beta.$$

Notice that

$$T^{\alpha 0} = 0, T^{0\beta} = 0.$$

We find

$$\sum(-1)^{i+k}a_r^{ik} = \sum(-1)^{i+k}(t_i^r + t_i^{r-1}t_k + \dots t_k^r)\xi_r(T^{r-1,1} + T^{r-2,2} + \dots + T^{1,r-1})\xi_r;$$

$$\sum(-1)^{i+k}a_r^{ik}a_s^{ik} = \sum(-1)^{i+k}(t_i^r + t_i^{r-1}t_k + \dots + t_k^r)(t_i^s + t_i^{s-1}t_k + \dots + t_k^s)\xi_r\xi_s =$$

$$= (0 + T^{r+s-1,1} + T^{r+s-2,2} + \dots + T^{r,s} +$$

$$+ T^{r+s-1,1} + T^{r+s-2,2} + T^{r+s-3,3} + \dots T^{r-1,s+1} + \dots +$$

$$+ T^{s,r} + T^{s-1,r+1} + T^{s-2,r+2} + \dots 0)\xi_r\xi_s =$$

$$= (2T^{r+s-1,1} + 3T^{r+s-2,2} + \dots + 3T^{2,r+s-2} + 2T^{1,r+s-1})\xi_r\xi_s =$$

$$= \left(\sum_{n=1}^{r+s-1}(1+\min(n,r,s,r+s-n))T^{r+s-n,n}\right)\xi_r\xi_s.$$

Keeping terms up to order 2 in (1) this gives

$$(-1)^{i+k}(a_2^{ik} - \frac{1}{2}(a_1^{ik})^2) =$$

$$= T^{1,1}(\xi_2 - \frac{1}{2}2\xi_1^2) = T^{1,1}(\xi_2 - \xi_1^2) = T^{1,1}(\frac{1}{6}\frac{x'''(0)}{x'(0)} - (\frac{1}{2}\frac{x''(0)}{x'(0)})^2) =$$

$$= \frac{1}{6}T^{1,1}(\frac{x'''(0)}{x'(0)} - \frac{3}{2}(\frac{x''(0)}{x'(0)})^2) = \frac{1}{6}T^{1,1}Sx(0),$$

where S stands for the Schwarzian.

Let y be a function of x. Thus, kinematically speaking, we consider a relative motion. Then

$$\log \frac{D(x(t_1),\ldots,x(t_4))}{D(t_1,\ldots,t_4)} = \frac{1}{6}(t_1-t_2)(t_3-t_4)Sx(0)+\ldots \text{ (near } t=0),$$

$$\log \frac{D(y(x_1),\ldots,y(x_4))}{D(x_1\ldots,x_4)} = \frac{1}{6}(x_1-x_2)(x_3-x_4)Sy(x(0))+\ldots \text{ (near } x=x(t)).$$

Thus we find

$$\log \frac{D(y(x(t_1)),\ldots,y(x(t_4)))}{D(t_1,\ldots,t_4)} = \frac{1}{6}(x(t_1)-x(t_2))(x(t_3)-x(t_4))Sy(x(0))+$$

$$+\frac{1}{6}(t_1-t_2)(t_3-t_4)Sx(0)+\ldots.$$

Passing to the limit we thus get as an application Cayley's formula mentioned in the Introduction (rewritten in the present notation)

$$\boxed{(S(y\circ x))(t)(dt)^2 = (Sy)(x(t))(dx(t))^2 + (Sx)(t)(dt)^2}.$$

Remark. This connection between the cross ratio and the Schwarzian is of course classical (see e.g. [Ca]). The point is that we are also interested also in the higher order terms.

The sums of higher order products of factors a_r^{ik} can be treated in analogous way as above in the case of just one or two factors, and it is easy to write down a recursion for the coefficients involved. It suffices to consider the case of three factors. Thus consider the expression

$$\sum(-1)^{i+k}a_r^{ik}a_s^{ik}a_v^{ik} = \sum_{n=1}^{r+s+v-1} D_{n,r,s,v}T^{r+s+v-n,n}\xi_r\xi_s\xi_v$$

where

$$D_{n,r,s,v} = \sum_{\substack{a+b+c=n\\ 0\le a\le r,0\le b\le s,0\le c\le v}} 1.$$

To find a closed expression for $D_{n,r,s,v}$ we consider the identity

$$(1+x+\cdots+x^r)(1+x+\cdots+x^s)(1+x+\cdots+x^v) = \frac{(1-x^{r+1})(1-x^{s+1})(1-x^{v+1})}{(1-x)^3}.$$

Now

$$(1-x)^{-3} = \sum_{n=0}^{\infty} D_n x^n \quad \text{where } D_n = \frac{(n+1)(n+2)}{2},$$

so the r.h.s. can be written

$$\sum_{n=0}^{\infty} D_n x^n\cdot(1-x^{r+1}-x^{s+1}-x^{v+1}+x^{r+s+2}+x^{r+v+2}+x^{s+v+2}-x^{r+s+v+3}).$$

It follows that ($D_n = 0$ if $n < 0$)

$$D_{n,r,s,v} =$$

$$= D_n - D_{n-r-1} - D_{n-s-1} - D_{n-v-1} + D_{n-r-s-2} + D_{n-r-v-2} + D_{n-s-v-2} - D_{n-r-s-v-3}.$$

Consider now the third order terms in (1), viz.

$$\sum(-1)^{i+k}(a_3^{ik} - a_1^{ik}a_2^{ik} + \frac{1}{3}(a_1^{ik})^2) =$$

$$= (T^{2,1} + T^{1,2})\xi_3 - (2T^{1,2} + 2T^{2,1})\xi_1\xi_2 + \frac{1}{3}(3T^{2,1} + 3T^{1,2})\xi_1^3 =$$

$$= (T^{2,1} + T^{1,2})(\xi_3 - 2\xi_1\xi_2 + \xi_1^3) = (T^{2,1} + T^{1,2})(\frac{1}{24}\frac{x^{IV}}{x'} - 2\cdot\frac{1}{2}\frac{x''}{x'}\cdot\frac{1}{6}\frac{x'''}{x'} + (\frac{1}{2}\frac{x''}{x'})^3) =$$

$$= (T^{2,1} + T^{1,2})\cdot\frac{1}{24}(\frac{x^{IV}}{x'} - 4\frac{x''x'''}{(x')^2} + 3(\frac{x''}{x'})^3) = (T^{2,1} + T^{1,2})\cdot\frac{1}{24}(Sx)',$$

as

$$S = \frac{x'''}{x'} - \frac{3}{2}(\frac{x''}{x'})^2,$$

$$S' = \frac{x^{IV}}{x'} - \frac{x'''x''}{x'^2} - \frac{3}{2}\cdot 2\frac{x''}{x'}(\frac{x'''}{x'} - \frac{x''\cdot x''}{x'^2}) = \frac{x^{IV}}{x'} - 4\frac{x'''x''}{x'^3} + 3(\frac{x''}{x'})^3.$$

Thus

$$\log\frac{D}{D} = \frac{1}{6}T^{1,1}S + \frac{1}{24}(T^{2,1} + T^{1,2})S' + \ldots.$$

We have also considered fourth order terms and by similar calculations as those just done we have found that now there appears the additional term

$$\frac{1}{120}(T^{3,1} + T^{2,2} + T^{1,3})(S'' + \frac{1}{3}S^2) - \frac{1}{72}T^{2,2}S^2$$

in the corresponding formula.

We conjecture that in general the coefficient of $T^{\alpha\beta}$ is a polynomial in $S, S', S'', \ldots, S^{(\nu)}$ where $\nu = \alpha + \beta$, but this we have not proved. Another question: how does Cayley's formula generalize? Apparently, results of this kind can be obtained using the method we just used (see ultra) in the case of that formula.

Appendix 6. A Bol's lemma for the "poly-heat" equation.[5]

Bojarski has in [Boj] obtained an interesting result about the transformation of the polyharmonic equation under the Möbius (conformal) group in any number of dimensions. In the case of two (real) variables it essentially reduces to our Bol's lemma (see Introduction). We now state a counterpart of this result for the iterated heat operator, which seems to be new. Introduce the notation

$$\mathcal{H} = \frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}.$$

[5] Outcome of a discussion with Bogdan Bojarski on the occasion of a hike to the Pyhätunturi mountain.

Then one has the formula

$$\mathcal{H}^m \left[(ct+d)^{m-\frac{3}{2}} e^{-\frac{1}{2}\frac{cx^2}{ct+d}} u(\frac{at+b}{ct+d}, \frac{x}{ct+d}) \right] =$$

$$= (ct+d)^{-m-\frac{3}{2}} e^{-\frac{1}{2}\frac{cx^2}{ct+d}} \mathcal{H}^m u(\frac{at+b}{ct+d}, \frac{x}{ct+d}).$$

Here $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any unimodular matrix ($ad - bc = 1$) and $m = 1, 2, \ldots$. The case $m = 1$ is well-known and, of course, implicit in our App. 3, where the variables were denoted a, c. The general case follows easily from it by induction. Notice that this result agrees with the fundamental solution of the "poly-heat" equation which is easy to write down and which generalizes the classical fundamental solution $\frac{1}{\pi} e^{-\frac{x^2}{2t}}$ in the case of the heat equation. (Each iteration produces a new factor t.)

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Abstract

The classical Schwarz derivative, which is a nonlinear differential operator of the third order, plays an important rôle in conformal mapping and in the theory of second order linear differential equations. No workable generalization to higher order is known. But if we restrict to Riemann surfaces equipped with a projective structure we have an ample supply of candidates. In fact, we give a description of the most general (in a sense) Möbius invariant operator. As for linear operators, for any integer $\mu > 0$ and a Riemann surface equipped with a projective structure, one can define the Bol operator, which in a projective coordinate system just amounts to taking the μth derivative. We have (elsewhere) expressed the Bol operator in terms of general coordinates. We have further proved a Green's formula for the Bol operator. With the aid of it we can study certain kernel functions and this again leads to applications to (small) Hankel operators. The report further discusses issues such as the (formal) relation of Hankel theory to operator calculi ("quantization" in the sense of Weyl, Berezin and Unterberger).

Classification: 30F30, 30C40, 47B35, 58H05.

Keywords: Schwarz derivative, Riemann surface, projective structure, Möbius invariance, Bol's lemma, Bol operator, reproducing kernel, Hankel operator, symbolic calculus, quantization, elliptic functions, theta constants.

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