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On Domains in Which Harmonic Functions Satisfy Generalized Mean Value Properties

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Abstract. Assume that a bounded domain $\Omega \subset \mathbb{R}^N\,(N \geqslant 2)$ has the property that there exists a signed measure μ with compact support in Ω such that, for every integrable harmonic function h in Ω, $\int_\Omega h\,\mathrm{d}x = \int_\Omega h\,\mathrm{d}\mu$ (Ω is a 'quadrature domain'). The main question studied is whether this implies that Ω has the same property for some *positive* measure (with in general larger, but still compact, support). We show that this is the case provided every positive harmonic function in Ω is the pointwise limit of a sequence of integrable positive harmonic functions in Ω. Moreover, for $N = 2$ we give a complete affirmative answer of the main question. This result is partially based on a previously known explicit description of all quadrature domains in two dimensions.

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1. Introduction

In this paper we investigate the relationship between certain mean value properties for harmonic functions in general domains in $\mathbb{R}^N\,(N \geqslant 2)$. The inspiration comes from a question posed by Lisa Goldberg in connection with Teichmüller theory and a subsequent paper [17] by Burton Randol. In [17] the following (possible) property of a bounded domain $\Omega \subset \mathbb{R}^N$ was studied

(R) There exists a compact subset K of Ω such that every integrable harmonic function h in Ω attains its mean value $(1/|\Omega|)\int_\Omega h\,\mathrm{d}x$ somewhere on K.

Here $\mathrm{d}x$ denotes Lebesgue measure in $\mathbb{R}^N$ and $|\Omega|$ denotes the volume of Ω. Actually, Goldberg and Randol only considered the case $N = 2$.

As an example, any ball Ω has the property (R), with $K = \{\text{the center}\}$, whereas, as we shall see, (R) admits e.g. no domains having 'corners' on the boundary (at least not in two dimensions).

A related property of a domain $\Omega \subset \mathbb{R}^N$ is the following

(QD) There exists a signed measure μ with compact support in Ω such that

$$\int_\Omega h\,\mathrm{d}x = \int h\,\mathrm{d}\mu \tag{1.1}$$

for every integrable harmonic function h in Ω.

Let us call a domain Ω as in (1.1) a *quadrature domain for the measure* μ. If (QD) holds for some μ we simply say that Ω is a *quadrature domain*, or, shorter, Ω is a QD. Other definitions of 'quadrature domain' are also in use, but for the present paper we shall stick to the above one.

An apparently stronger property of Ω is

(PQD) (1.1) holds for some *positive* measure μ (with compact support in Ω).

We write Ω is a PQD if (PQD) holds.

In this paper we prove that the property (PQD) is equivalent to (R). We also show that any domain whose boundary consists of finitely many real analytic hypersurfaces (without singularities) is a PQD. These two results are not hard.

We strongly expect that also (QD) and (PQD) are equivalent (i.e. that $(QD) \Rightarrow (PQD)$), but this we have not been able to prove (in full generality). However, we do have some interesting partial results, e.g. that $(QD) \Rightarrow (PQD)$ holds for domains which satisfy a weak additional condition, like

(PAI) every positive harmonic function in Ω is the pointwise limit of a sequence of integrable positive harmonic functions

(or the still weaker condition (2.6) below).

In two dimensions, a complete geometric description (or classification) of all QD:s has been given by one of the authors [19, 20]. From this one can check off (PAI) and infer that $(QD) \Rightarrow (PQD)$ indeed holds when $N = 2$. The above mentioned classification is a quite hard result, but if one is willing to assume in advance that Ω is finitely connected (when $N = 2$) then one can manage with simpler methods, and in the present paper we give basically complete proofs for this case. Let us state the complete answer in the simply connected case:

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain, and let $f\colon B(0,1) \to \Omega$ be any Riemann mapping function, $B(0,1)$ denoting the unit disc. Then the properties $(R), (QD), (PQD)$ for Ω are all equivalent to the property that f extends analytically to some neighbourhood of $\overline{B(0,1)}$ (the closed disc).

Note that the above property allows certain singularities of $\partial\Omega$, namely inward cusps (corresponding to simple zeros of f' on $\partial B(0,1)$) and double points (f taking the same value at two different points on $\partial B(0,1)$).

2. Basic Results

Throughout the paper Ω denotes a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$. Let us introduce some notation

$$\begin{aligned}
H(\Omega) &= \{h : \Omega \to \mathbb{R}, h \text{ is harmonic}\}, \\
HL^1(\Omega) &= \{h \in H(\Omega) : h \text{ is integrable with respect} \\
&\quad\ \text{to Lebesgue measure}\}, \\
HP(\Omega) &= \{h \in H(\Omega) : h \geqslant 0\} \\
HPL^1(\Omega) &= HP(\Omega) \cap HL^1(\Omega).
\end{aligned}$$

In integrals (with respect to Lebesgue measure) Lebesgue measure is denoted $\mathrm{d}x$, or (in two dimensions) $\mathrm{d}A$, or is omitted. It is convenient also to have a notation for the mean value of a function $h \in HL^1(\Omega)$

$$M[h] = \frac{1}{|\Omega|}\int_\Omega h\,\mathrm{d}x.$$

$B(a,r)$ denotes the open ball with center $a \in \mathbb{R}^N$ and radius $r > 0$, δ_a denotes the Dirac measure at the point $a \in \mathbb{R}^N$, $\delta = \delta_0$.

The space $H(\Omega)$ is naturally equipped with the topology of uniform convergence on compact sets. The notion of a QD has a simple interpretation in terms of this topology.

PROPOSITION 2.1. *Ω is a* QD *if and only if the functional $M : HL^1(\Omega) \to \mathbb{R}$ is continuous with respect to uniform convergence on compact sets. More precisely, given a compact $K \subset \Omega$ the following two assertions are equivalent*

(i) *there exists a signed measure μ with* $\operatorname{supp}\mu \subset K$ *such that $M[h] = \int h\,\mathrm{d}\mu$ for all $h \in HL^1(\Omega)$;*
(ii) *there exists a constant C such that $M[h] \leqslant C \sup_K |h|$ for all $h \in HL^1(\Omega)$.*

Proof. (i)⇒(ii) is obvious and (ii)⇒(i) follows by standard applications of the Hahn–Banach and the Riesz representation theorems. □

There are also a number of similar characterizations of the property of being a PQD.

PROPOSITION 2.2. *Given a compact $K \subset \Omega$ the following assertions are equivalent*

(i) *there exists a positive measure* μ *with* $\operatorname{supp}\mu \subset K$ *such that* $M[h] = \int h\,\mathrm{d}\mu$ *for all* $h \in HL^1(\Omega)$;
(ii) $M[h] \leqslant \sup_K |h|$ *for all* $h \in HL^1(\Omega)$;
(iii) $M[h] \leqslant \sup_K h$ *for all* $h \in HL^1(\Omega)$;
(iv) *there exists a constant* C *such that* $M[h] \leqslant C \sup_K h$ *for all* $h \in HL^1(\Omega)$;
(v) $h \geqslant 0$ *on* $K \Rightarrow M[h] \geqslant 0$ *for* $h \in HL^1(\Omega)$.

Moreover, the assertion

(vi) *every* $h \in HL^1(\Omega)$ *attains its mean value* $M[h]$ *on* K

implies (i)–(v)*, and if* K *is connected* (i)–(v) *imply* (vi).

Proof. The proof only uses that M is a linear functional on $HL^1(\Omega)$ and that $1 \in HL^1(\Omega), M1 = 1$. We first prove (ii)⇔(iii)⇔(iv), then (v)⇒(iii)⇒(i)⇒(v) and finally that (vi) implies (iii) and that (iii) implies (vi) if K is connected.

(ii)⇒(iii): Set $\alpha = \inf_K h, \beta = \sup_K h$ and apply (ii) to $h - (\alpha + \beta)/2$. This gives $M[h - (\alpha + \beta)/2] \leqslant (\beta - \alpha)/2$, i.e. $M[h] \leqslant \beta$.

(iii)⇒(ii): Obvious.

(iii)⇒(iv): Obvious.

(iv)⇒(iii): Choosing $h = \pm 1$ gives $C = 1$.

(v)⇒(iii): Apply (v) to $(\sup_K h) - h$.

(iii)⇒(i): (iii) says that the functional M on $HL^1(\Omega)$ is majorized by the sublinear functional $p(h) = \sup_K h$. It then follows from one version [5], Theorem II.3.10, p. 62 of the Hahn–Banach theorem that M extends to a linear functional $L : C(K) \to \mathbb{R}$ also majorized by p. (Strictly speaking, the ‘embedding’ $HL^1(\Omega) \to C(K)$ is not always injective, but the Hahn–Banach ‘extension’ works anyway.) Then L is a positive functional (since $L(h) \leqslant p(h) \leqslant 0$ if $h \leqslant 0$ on K) and an application of the Riesz representation theorem gives the desired measure $\mu \geqslant 0$.

(i)⇒(v): Obvious.

(vi)⇒(iii): Obvious.

(iii)⇒(vi) if K is connected: Applying (iii) to $\pm h$ gives $\inf_K h \leqslant M[h] \leqslant \sup_K h$ and since K is connected h attains all values in the interval $[\inf_K h, \sup_K h]$ on K. □

NOTE. For any domain Ω and any subset $K \subset \Omega$ let C_K(possibly $= +\infty$) denote the best constant in the estimate

$$M[h] \leqslant C_K \sup_K |h| \quad \text{for all} \quad h \in HL^1(\Omega).$$

Then $1 \leqslant C_K \leqslant +\infty$ and, by the above two propositions, Ω is a QD iff $C_K < +\infty$ for some compact $K \subset \Omega$ and Ω is a PQD iff $C_K = 1$ for some compact $K \subset \Omega$. Clearly C_K decreases (in the nonstrict sense) as K increases, and the main question for this paper is whether $C_K < +\infty$ for some compact K implies that $C_K = 1$ for some larger, but still compact, K. (Note that $C_\Omega = 1$.)

We now give an example (formulated as a proposition) which shows the drastic difference beween having a signed, respectively having a positive, measure μ in a quadrature identity (1.1). Basically, the example says that quadrature domains for signed measures with support in any given small ball $B(0,r)$ are 'dense' in the set of all domains, whereas quadrature domains for positive measures with support in $B(0,r)$ are subject to severe geometrical restrictions. (They are essentially ball shaped.) It follows that even if a QD is also a PQD, the support of the measure in general has to be tremendously enlarged in order to get a representation in terms of a positive measure.

PROPOSITION 2.3. (i) *Let $N=2$, let $r>0$ and $\varepsilon>0$ be arbitrarily small and let D be any bounded domain containing $\overline{B(0,r)}$ and with ∂D consisting of finitely many disjoint analytic Jordan curves. Then there exists a univalent function φ in D with $|\varphi(z)-z|<\varepsilon$ for $z\in D$ such that $\Omega=\varphi(D)$ is a quadrature domain for a signed measure with support in $B(0,r)$.*

(ii) *Let $N\geqslant 2$ and let Ω be a quadrature domain for a positive measure with support in $B(0,r)$. Define $R>0$ by $|\Omega|=|B(0,R)|$. Then, if $R\geqslant 2r$ the following holds*

$B(0,R-r)\subset\Omega\subset B(0,R+r)$;
$\mathbb{R}^N\setminus\bar{\Omega}$ is connected;
$\partial\Omega=\partial(\mathbb{R}^N\setminus\bar{\Omega})$ is a real analytic hypersurface;
for any $x\in\partial\Omega$, the inward normal of $\partial\Omega$ at x intersects $B(0,r)$.

Proof of (i). We prove (i) only in the case D is simply connected. The proof in the multiply connected case is given in [8]. Let $\psi: B(0,1)\to D$ be a conformal map with $\psi(0)=0$ and let $p(z)$ be a polynomial approximating ψ uniformly on $B(0,1)$, sufficiently closely so that p is injective on $\overline{B(0,1)}$, and satisfying $p(0)=0$ (p may e.g. be a truncation of ψ:s Taylor series at 0). Then $\Omega=p(B(0,1))$ has the desired property, with $\varphi=p\circ\psi^{-1}: D\to\Omega$ the mapping function in the statement of (i).

Indeed, if m is the degree of p it is well-known [1, 3, 24] (and easy to check) that an identity (for suitable $a_j\in\mathbb{C}$)

$$\int_\Omega f\,\mathrm{d}A=\sum_{j=0}^{m-1}a_j f^{(j)}(0) \tag{2.1}$$

holds for all f analytic in Ω and (say) smooth up to $\partial\Omega$. If u is harmonic in Ω and smooth up to $\partial\Omega$ then we may apply (2.1) with $f=u+iv$, where v is a harmonic conjugate of u. Since $f^{(j)}(0)=2(\partial^j u/\partial z^j)(0)$ for $j\geqslant 1$, taking real parts of (2.1) with this f gives

$$\int_\Omega u\,\mathrm{d}A=a_0u(0)+2\sum_{j=1}^{m-1}\mathrm{Re}\left(a_j\frac{\partial^j u}{\partial z_j}(0)\right), \tag{2.2}$$

(a_0 is necessarily real). The right member of (2.2) is a distribution, with support at the origin, acting on u. By mollifying this distribution with a radially symmetric test function (with support in $B(0,r)$) one gets a signed measure μ supported in $B(0,r)$ such that

$$\int_\Omega u\,\mathrm{d}A = \int u\,\mathrm{d}\mu$$

holds for all u as above, hence, by approximation, for all $u \in HL^1(\Omega)$.

Proof of (ii). It follows from the results in [7, 9, 18] that when $R \geqslant 6r$ there exists a unique quadrature domain Ω for $\mu \geqslant 0$ (when $\operatorname{supp}\mu \subset B(0,r)$ and $\int \mathrm{d}\mu = |B(0,R)|$) and that Ω has the three last of the stated properties. Later [14, 15, 22], the constant 6 above was improved to 2 (which is best possible), and also the first property was proved [22]. □

The following theorem, although quite simple, is our basic result on the implication $(QD) \Rightarrow (PQD)$.

THEOREM 2.4. *Suppose that Ω is a* QD *and that moreover $HP(\Omega) \subset L^1(\Omega)$, or even that*

$$HP(\Omega) \cap \overline{HL^1(\Omega)} \subset L^1(\Omega), \tag{2.3}$$

(closure with respect to the topology of uniform convergence on compacts of $H(\Omega)$). Then Ω is a PQD.

Proof. We shall argue by contradiction. Assume that Ω is not a PQD. Choose a regular exhaustion $\{\Omega_n\}_{n=1}^\infty$ of Ω, i.e. $\bar{\Omega}_n \subset \Omega_{n+1}, \partial\Omega_n$ smooth for all $n, \bigcup_{n=1}^\infty \Omega_n = \Omega$. Pick a point $a \in \Omega_1$. By (v) of Proposition 2.2 there exists, for every $n, h_n \in HL^1(\Omega)$ with $h_n \geqslant 0$ in $\Omega_n, \int_\Omega h_n < 0$. Then $h_n(a) > 0$ and we may assume that $h_n(a) = 1$.

By Harnack's principle there exists a subsequence, which we denote again $\{h_n\}$, which converges in $H(\Omega)$, say $h_n \to h$. (Strictly speaking, Harnack gives for each fixed m a subsequence which converges in $H(\Omega_m)$, but then one applies the Cantor diagonalization procedure.) Clearly $h \in HP(\Omega) \cap \overline{HL^1(\Omega)}$ and $h(a) = 1$. Thus

$$0 < \int_\Omega h \leqslant +\infty. \tag{2.4}$$

On the other hand (since $\int_\Omega h_n < 0$),

$$\overline{\lim} \int_\Omega h_n \leqslant 0. \tag{2.5}$$

Now, if (2.3) holds, then $h \in HL^1(\Omega)$ and if moreover Ω is a QD, then by Proposition 2.1 we have

$$\int_\Omega h = \lim \int_\Omega h_n.$$

This contradicts (2.4), (2.5). □

Let us augment Theorem 2.4 a little by using the following observation.

LEMMA 2.5. *If* Ω *is a* QD *then* $HPL^1(\Omega)$ *is closed in* $H(\Omega)$.

Proof. Assume $h_n \to h, h_n \in HPL^1(\Omega), h \in H(\Omega)$. Then clearly $h \geqslant 0$, and by Fatou's lemma and Proposition 2.1

$$\int_\Omega h \leqslant \varliminf_{n\to\infty} \int_\Omega h_n \leqslant C \varliminf_{n\to\infty} \sup_K |h_n|$$

$$\leqslant C \sup_K |h| < +\infty,$$

for some $K \subset \Omega$. Hence $h \in HPL^1(\Omega)$. □

Now, since (by definition) $HPL^1(\Omega) \subset HP(\Omega) \cap \overline{HL^1(\Omega)} \subset HP(\Omega)$, the requirement (2.3) is equivalent to that actually $HPL^1(\Omega) = HP(\Omega) \cap \overline{HL^1(\Omega)}$. Therefore, in order to establish (2.3) when Ω is a QD it is, in view of Lemma 2.5, enough (and necessary) to show that

$$HPL^1(\Omega) \text{ is dense in } HP(\Omega) \cap \overline{HL^1(\Omega)}, \tag{2.6}$$

i.e. that any positive harmonic function which can be approximated, uniformly on compacts, by integrable harmonic functions can be approximated also by integrable positive harmonic functions (uniformly on compacts or, which is equivalent on $HP(\Omega)$, pointwise). We do not know of any domain whatsoever not having this property (2.6).

3. Balayage onto Analytic Hypersurface

Let Ω denote a bounded domain in $\mathbb{R}^N$ whose boundary consists of finitely many real analytic hypersurfaces. No singular points are allowed in $\partial\Omega$, so in the neighbourhood of each point $y \in \partial\Omega, \partial\Omega$ is the level surface of a real analytic function whose gradient does not vanish at y.

THEOREM 3.1. *With Ω as above, let f be real analytic on a neighbourhood of $\partial\Omega$ and real-valued, with $f(y) > 0$ for $y \in \partial\Omega$. Let σ denote surface measure on $\partial\Omega$. Then the measure $f\sigma$ is the balayage of a bounded positive measure μ, satisfying* $\operatorname{supp}\mu \subset \Omega$*, onto $\partial\Omega$. Equivalently,*

$$\int_{\partial\Omega} hf \, \mathrm{d}\sigma = \int h \, \mathrm{d}\mu \tag{3.1}$$

holds for every $h \in C(\bar{\Omega})$ that is harmonic in Ω.

REMARKS. (1) In view of known approximation theorems (and the regularity of $\partial\Omega$), (3.1) is equivalent to the formally weaker requirement that the corresponding identity holds for h harmonic on a neighbourhood of $\bar{\Omega}$, or even for h of the form $h(x) = E(x - y), y \notin \bar{\Omega}$, where E is the fundamental solution for Δ. The latter condition can be restated as: the Newtonian potentials of μ and $f\,\mathrm{d}\sigma$ coincide outside $\bar{\Omega}$.

(2) We emphasize that we use $\operatorname{supp}\mu$ to denote support in the sense of Schwartz distributions, so this set is closed.

(3) As in Proposition 2.1, Equation (3.1) is equivalent to the estimate

$$\left| \int_{\partial\Omega} hf \, \mathrm{d}\sigma \right| \leqslant C \sup_K |h|,$$

where $K = \operatorname{supp}\mu$. Some related results, with more explicit estimates and with emphasis on questions of harmonic duality (i.e., when both f and h are harmonic), have recently (and independently) been proved by E. L. Stout. See in particular [26] Theorems 1 and 2.

Proof. By the Cauchy–Kovalevskaya theorem, there is, in some neighbourhood $\mathcal{N}$ of $\partial\Omega$, a (unique) solution to the initial value problem

$$\Delta u = 0 \quad \text{on } \mathcal{N}, \tag{3.2}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3.3}$$

$$\frac{\partial u}{\partial n} = f \quad \text{on } \partial\Omega, \tag{3.4}$$

where $\partial/\partial n$ denotes the inward-directed normal derivative. The following is a simple deduction from (3.2, 3.3, 3.4) whose proof we leave to the reader (cf. [6], Lemma 2).

PROPOSITION 3.2. *There is an open set $W \subset \Omega$ (a neighbourhood of $\partial\Omega$) and $\varepsilon > 0$ such that*

(i) *u is harmonic and positive on W,*

(ii) $\partial\Omega \subset \partial W$,
(iii) $u(x) = \varepsilon$ *for* $x \in \Gamma' := (\partial W)\backslash(\partial\Omega)$,
(iv) *grad* u *vanishes nowhere on* Γ' *(so,* Γ' *is real analytic).*

REMARK. We can achieve (iv) by changing ε to a neighbouring value if necessary.

We can now easily complete the proof of Theorem 3.1. As already remarked we may, in proving (3.1), assume h is harmonic on a neighbourhood of $\bar{\Omega}$. Now, denoting $\partial\Omega$ by Γ we have, applying Green's identity to W

$$\int_W (h\Delta u - u\Delta h)\,\mathrm{d}x = -\int_{\partial W}\left(h\frac{\partial u}{\partial n} - u\frac{\partial h}{\partial n}\right)\mathrm{d}\sigma$$

and the left side is 0. Hence

$$\int_{\partial W} h\frac{\partial u}{\partial n}\mathrm{d}\sigma = \int_{\partial W} u\frac{\partial h}{\partial n}\mathrm{d}\sigma.$$

Now, $\partial W = \Gamma \cup \Gamma'$ so, with appropriate orientation of Γ and Γ'

$$\int_\Gamma h\frac{\partial u}{\partial n}\mathrm{d}\sigma - \int_{\Gamma'} h\frac{\partial u}{\partial n}\mathrm{d}\sigma = \int_\Gamma u\frac{\partial h}{\partial n}\mathrm{d}\sigma - \int_{\Gamma'} u\frac{\partial h}{\partial n}\mathrm{d}\sigma. \tag{3.5}$$

The first integral on the right vanishes (because of (3.3)) and the second is $\varepsilon\int_{\Gamma'}(\partial h/\partial n)\,\mathrm{d}\sigma$ which also vanishes, since h is harmonic in the domain bounded by Γ'. Thus, from (3.4),

$$\int_\Gamma hf\,\mathrm{d}\sigma = \int_{\Gamma'} h\frac{\partial u}{\partial n}\mathrm{d}\sigma = \int h\,\mathrm{d}\mu, \tag{3.6}$$

where μ is the measure $(\partial u/\partial n)\,\mathrm{d}\sigma$ on Γ'. This is non-negative, and supported on the compact subset Γ' of Ω. The theorem is proved. □

THEOREM 3.3. *With* Ω *as above, let* g *be a non-negative integrable function on a neighbourhood of* $\overline{\Omega}$*, which is real-analytic on some neighbourhood of* $\partial\Omega$*. Then, there is a positive bounded measure* μ *on* Ω *with* $\operatorname{supp}\mu \subset \Omega$ *such that*

$$\int_\Omega hg\,\mathrm{d}x = \int h\,\mathrm{d}\mu \tag{3.7}$$

holds for every $h \in HL^1(\Omega)$ *or, in other terms,* μ *is equigravitational with the measure* $g\,\mathrm{d}x$ *on* Ω*.*

Proof. Again, there is no loss of generality if we prove (3.7) under the assumption that h is harmonic on a neighbourhood of $\bar{\Omega}$. Let v denote the (unique) solution to the Dirichlet problem

$$\Delta v = g \quad \text{on } \Omega, \tag{3.8}$$

$$v = 0 \quad \text{on } \partial\Omega. \tag{3.9}$$

Since the theorem is trivially true if g vanishes identically on a neighbourhood of $\partial\Omega$, we may assume this is not the case. Then (3.8), (3.9) imply $v(x) < 0$ for $x \in \Omega$.

By the analyticity theorem for elliptic equations, v extends real analytically to some neighbourhood of $\partial\Omega$. The Hopf maximum principle shows $(\partial v/\partial n) < 0$ on $\partial\Omega$. Now, by Green's identity, again denoting $\partial\Omega$ by Γ,

$$\int_\Omega h\Delta v \,\mathrm{d}x - \int_\Omega v\Delta h \,\mathrm{d}x = -\int_\Gamma h\frac{\partial v}{\partial n}\mathrm{d}\sigma + \int_\Gamma v\frac{\partial h}{\partial n}\mathrm{d}\sigma. \tag{3.10}$$

The second integral on the right vanishes because of (3.9), so (3.10) reduces to

$$\int_\Omega hg \,\mathrm{d}x = -\int_\Gamma h\frac{\partial v}{\partial n}\mathrm{d}\sigma. \tag{3.11}$$

Now, locally Γ has a representation as $\{y : \varphi(y) = 0\}$ for some real-analytic φ with non-vanishing gradient. We may choose φ to be positive outside Ω, so

$$-\frac{\partial v}{\partial n} = (\operatorname{grad} v) \cdot \frac{\operatorname{grad}\varphi}{|\operatorname{grad}\varphi|},$$

which shows that $-(\partial v/\partial n)$, coincides on Γ with a function f that is real analytic on a neighbourhood of Γ, also it is strictly positive, so Theorem 3.1 is applicable, and shows the right hand term in (3.11) can be written $\int h\,\mathrm{d}\mu$ for some positive bounded measure with compact support in Ω.

Taking $g = 1$ in Theorem 3.3 we obtain in particular

COROLLARY 3.4. *Any bounded domain in $\mathbb{R}^N$ whose boundary consists of finitely many real analytic hypersurfaces is a* PQD. □

4. The Two-Dimensional Problem

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$ be a bounded domain whose boundary consists of finitely many continua. Then, the following are equivalent*

(i) Ω *is a* QD.

(ii) Ω *admits a Schwarz function $S(z)$, i.e. there exists S holomorphic and single-valued outside a compact subset of Ω, continuously extendable to $\partial\Omega$, with $S(z) = \bar{z}$ on $\partial\Omega$.*

(iii) *Let D denote a domain conformally equivalent to Ω and bounded by analytic Jordan curves (it is a well-known elementary consequence of the Riemann mapping theorem that such D exist). Then, the conformal map of D on Ω extends analytically to a neighbourhood of $\bar{D}$.*

(iv) Ω *is a* PQD.

NOTE. By definition, a continuum is a closed connected set consisting of more than one point.

REMARK. With $\partial\Omega$ assumed a priori to be smooth of class C^2 and working with analytic test functions instead of harmonic ones, the equivalence between (i) and (iii) has also been proved by P. Zorn [28], Theorem IV.3. Actually, Zorn is mainly concerned with questions in several complex variables and he obtains, in particular, versions of the implications (iii)⇒(i) and (i)⇒(iii) for pseudoconvex domains in $\mathbb{C}^N$ [28], Theorems III.14 and IV.7.

Proof. (i)⇒(ii) is well-known [24]. See also Lemma 5.1 (below) and the remark following it.

(ii)⇒(iii) (also well-known) – here are details: Denote the conformal map by $z = \varphi(\zeta)$ for $\zeta \in D$. Then

$$S(\varphi(\zeta)) = \overline{\varphi(\zeta)}, \quad \zeta \in \partial D.$$

Hence $S \circ \varphi + \varphi$ is real on ∂D, so by the Schwarz principle of reflection it is analytically continuable to a neighbourhood of $\overline{D}$. Likewise, $S \circ \varphi - \varphi$ is pure imaginary and continuable to a neighbourhood of $\overline{D}$. Thus, the same is true of

$$2\varphi = (S \circ \varphi + \varphi) - (S \circ \varphi - \varphi).$$

(iii)⇒(iv) We shall show there exists a positive measure μ with compact support in Ω such that

$$\int_\Omega h \,\mathrm{d}A = \int h \,\mathrm{d}\mu, \tag{4.1}$$

for all $h \in HL^1(\Omega)$, where $\mathrm{d}A$ is area measure. In view of a known approximation theroem of L. I. Hedberg (cf. [25], p. 112) $HL^\infty(\Omega)$ is dense in $HL^1(\Omega)$ (with respect to the $L^1(\Omega)$ topology) under our assumption, so w.l.o.g. we may assume h bounded in proving (4.1).

Now,

$$\int_\Omega h \,\mathrm{d}A = \int_D h(\varphi(\zeta))|\varphi'(\zeta)|^2 \,\mathrm{d}A_\zeta \tag{4.2}$$

and, in view of assumption (iii), $|\varphi'|^2$ is real analytic on a neighbourhood of $\bar{D}$. Therefore, since $h \circ \varphi \in HL^\infty(D)$, by Theorem 3.3 the right side of (4.2) equals $\int (h \circ \varphi) \,\mathrm{d}\nu$ for some positive measure ν with compact support in D, which also can be written as $\int h \,\mathrm{d}(\nu \circ \psi)$ where ψ is the conformal map inverse to φ. Since the support of $\nu \circ \psi$ is compact in Ω, we see that (iv) holds.

(iv)⇒(i) is trivial, and the theorem is proven. □

In general, a QD may be infinitely connected. However, the main result of [19, 20] shows that this can occur only in a rather trivial way, namely as in the following example.

EXAMPLE. Let $\Omega_1 = \{z \in \mathbb{C} : |z| < 1\}, \Omega_2 = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Then Ω_1 is a quadrature domain for $\mu_1 = \pi\delta$ and Ω_2 is a quadrature domain for a measure μ_2 with constant density (with respect to arc length) on the circle $|z| = \rho$, for a suitable $1 < \rho < 2$. Set $\mu = \mu_1 + \mu_2$.

Then $\Omega_1 \cup \Omega_2$ fails to be a quadrature domain just because it is disconnected, but for any nonempty relatively open subset U of $\{|z| = 1\}$,

$$\Omega = \Omega_1 \cup \Omega_2 \cup U$$

is a quadrature domain for μ. Clearly Ω may be infinitely connected.

Note, however, that Ω always contains a finitely connected quadrature domain for μ, namely $D = \Omega_1 \cup \Omega_2 \cup I$, where I is any open interval of $\{|z| = 1\}$ contained in U.

Now, in the general case, [20], Theorem 1.7 shows that the following holds.

LEMMA 4.2. *Let $\Omega \subset \mathbb{R}^2$ be a quadrature domain for a signed measure μ. Then Ω contains a finitely connected quadrature domain for μ.*

An alternative way of establishing the main implication (i)⇒(iv) in Theorem 4.1 has been suggested to one of the authors by D. Khavinson. By combining this approach with Lemma 4.2 a proof of (i)⇒(iv) without any extra assumptions on Ω is obtained.

THEOREM 4.3. *In two dimensions any* QD *is a* PQD.

Proof. Let $\Omega \subset \mathbb{R}^2$ be a quadrature domain for μ, a signed measure. In order to show that Ω is a PQD we need, by Theorem 2.4, only to show that $HP(\Omega) \subset L^1(\Omega)$.

By Lemma 4.2 there exists a finitely connected quadrature domain $D \subset \Omega$ for μ. Then $HP(\Omega)|_D \subset HP(D)$ and $|\Omega \backslash D| = 0$. Thus it is enough to show that $HP(D) \subset L^1(D)$, and by remarks after Theorem 2.4 it is even enough to show that an approximation statement like (2.6), or the slightly stronger one (PAI) (in Sect. 1), holds for D. Thus, to complete the proof it is enough to prove the following lemma.

LEMMA 4.4. *(PAI) holds for any finitely connected, bounded domain in $\mathbb{R}^2$.*

Proof of Lemma 4.4. The idea of D. Khavinson is to prove a statement which is stronger than (PAI), but which is conformally invariant. Let D be the domain in question. We will have to distinguish between those components of ∂D which are continua and those which are singleton sets. We are going to prove that

(PAB) every $h \in HP(D)$ can be approximated (uniformly on compacts) by functions $h_n \in HP(D)$ which are bounded in a neighbourhood of each continuum component of ∂D.

Note that since a function which is harmonic and positive in a punctured neighbourhood of a point has at most a logarithmic singularity there, all the approximating functions h_n above are integrable. Therefore (PAB) implies (PAI). Also note that the property of being a singleton component of ∂D is preserved under conformal mappings, so the statement (PAB) is indeed conformally invariant.

By the above discussion we need only to prove (PAB) for domains D such that each component of ∂D is either a regular analytic curve or consists of a single point. As is well known (cf. [4, 10, 11]) every positive function on any domain is the pointwise limit of a sequence, each element of which is a finite linear combination of Martin (or minimal) functions, with positive coefficients. Now, under the above assumptions on D it is well-known that each Martin function is analytically continuable across each boundary point, with one exception. Let us denote a Martin function by v, and the exceptional boundary point by ζ.

If the boundary component to which ζ belongs consists of just ζ, then v is itself allowed in the approximation (PAB).

In the opposite case ∂D is an analytic curve, with D on just one side, near $\zeta \in \partial D$ and v simply is a Poisson kernel with singularity at ζ. If we consider the translated domain $D_\varepsilon = D + \varepsilon\omega$, where ω is the unit vector directed along the inner normal to D at ζ, and $\varepsilon > 0$ is sufficiently small, then v is harmonic and bounded on D_ε and bounded below there by a constant $-C(\varepsilon)$, where $C(\varepsilon) > 0$ and $C(\varepsilon) \to 0$ as $\varepsilon \searrow 0$. This is the same as saying that $v_\varepsilon(z) := v(z + \varepsilon\omega) + C(\varepsilon)$ is harmonic, bounded and positive on D. Clearly $v_\varepsilon(z) \to v(z)$ for $z \in D$. Thus, each Martin function in D has the property of being approximable, uniformly on compact sets (in view of Harnack's theorem), by a sequence of individually bounded positive harmonic functions, and this implies, in view of the preceeding discussion, that Ω has the property (PAB). $\square$

5. The Higher Dimensional Case

In this final section we shall find some geometrical conditions on $\partial\Omega$ which ensure $HP(\Omega) \subset L^1(\Omega)$ (or (2.6)) when Ω is a QD.

Let $E(x)$ denote the usual Newtonian kernel so that $-\Delta E = \delta$ (Dirac measure). If μ is a signed measure with compact support its Newtonian potential is

$$U^\mu(x) = \int E(x-y)\,\mathrm{d}\mu(y).$$

If $\Omega \subset \mathbb{R}^N$ we write

$$U^\Omega(x) = \int_\Omega E(x-y)\,\mathrm{d}y.$$

Thus $-\Delta U^{\mu} = \mu, -\Delta U^{\Omega} = \chi_{\Omega}$.

Let $\Omega \subset \mathbb{R}^N$ be any bounded domain. Then the Green's function $G(x, y)(x, y \in \Omega)$ for Ω can be defined as

$$G(x, y) = E(x - y) - H(x, y) \tag{5.1}$$

where, for fixed $y \in \Omega, x \mapsto H(x, y)$ is the largest harmonic function in Ω which is $\leqslant E(x - y)$ (cf. [4, 11]). Then $G > 0$ in $\Omega \times \Omega$. We write

$$G^{\mu}(x) = \int G(x, y)\, \mathrm{d}\mu(y),$$

$$G^{\Omega}(x) = \int_{\Omega} G(x, y)\, \mathrm{d}y,$$

$$G^{a}(x) = G(x, a) = G^{\delta_a}(x),$$

if $\operatorname{supp} \mu \subset \Omega, a \in \Omega$.

LEMMA 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, μ a signed measure with* $\operatorname{supp} \mu \subset \Omega$ *and set $u = U^{\mu} - U^{\Omega}$. Then Ω is a quadrature domain for μ if and only if*

$$u = |\operatorname{grad} u| = 0 \quad \textit{on} \;\; \mathbb{R}^N \backslash \Omega. \tag{5.2}$$

(Note that u is continuously differentiable outside $\operatorname{supp} \mu$*). When this is the case we also have*

$$u = G^{\mu} - G^{\Omega} \quad \textit{in} \;\; \Omega.$$

Proof. The Equation (5.2) is exactly the statement that the quadrature identity (1.1) holds for all functions $h(x) = E(x - y)$ and $h(x) = D_i E(x - y)$ (D_i any first order derivative) with $y \in \mathbb{R}^N \backslash \Omega$. Note that $h \in HL^1(\Omega)$. Since the linear span of all these functions is known [18], Lemma 7.3, to be dense in $HL^1(\Omega)$ the first assertion of the lemma follows.

When Ω is a quadrature domain for μ we have

$$\int H(x, y)\, \mathrm{d}\mu(x) = \int_{\Omega} H(x, y)\, \mathrm{d}x,$$

(since $H(\cdot, y) \in HL^1(\Omega)$ for any $y \in \Omega$). Hence $G^{\mu} - G^{\Omega} = U^{\mu} - U^{\Omega}$ by (5.1), proving the final assertion. □

REMARK. The function u is (when Ω is a QD) sometimes called the '(modified) Schwarz potential' for Ω. It is the unique compactly supported solution to $\Delta u = \chi_{\Omega} - \mu$. In two dimensions it is related to the Schwarz function $S(z)$ for Ω (or $\partial\Omega$) by

$$S(z) = \bar{z} - 4(\partial u / \partial z),$$

for $z \in \bar{\Omega} \backslash \operatorname{supp} \mu$.

LEMMA 5.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $a \in \Omega$. If*

$$G^\Omega / G^a \leqslant C < +\infty \tag{5.3}$$

in a neighbourhood of $\partial\Omega$ in Ω, then $HP(\Omega) \subset L^1(\Omega)$.

Proof. The lemma is naturally connected with the Martin theory for positive harmonic functions, but we prefer to give a short direct proof.

Choose a regular exhaustion $\{\Omega_n\}_{n=1}^\infty$ of Ω with $a \in \Omega_1$. For any $h \in HP(\Omega)$, let h_n be the smallest positive superharmonic function in Ω which is $\geqslant h$ on $\overline{\Omega}_n$ (i.e. h_n is the 'reduction of h over $\overline{\Omega}_n$' in potential theoretic language [4, 11]). Then h_n has the representation

$$h_n(x) = \int G(x,y)\,\mathrm{d}\eta_n(y) \quad (x \in \Omega), \tag{5.4}$$

where $\eta_n = -\Delta h_n$ is a positive measure with support on $\partial\Omega_n$.

If (5.3) holds and if n is sufficiently large, integration of (5.4) gives (using also (5.4) for $x = a$)

$$\begin{aligned}\int_\Omega h_n(x)\,\mathrm{d}x &= \int_\Omega \int \frac{G(x,y)}{G(a,y)} G(a,y)\,\mathrm{d}\eta_n(y)\,\mathrm{d}x \\ &\leqslant C \int G(a,y)\,\mathrm{d}\eta_n(y) \\ &= Ch_n(a) = Ch(a) < +\infty.\end{aligned}$$

Thus $h \in L^1(\Omega)$ by Fatou's lemma. □

COROLLARY 5.3. *Let Ω be a* QD *for a signed measure μ and let $u = U^\mu - U^\Omega$ as in Lemma 5.1. If*

$$\frac{u(x)}{G(a,x)} \geqslant C > -\infty, \tag{5.5}$$

for $x \in \Omega$ in a neighbourhood of $\partial\Omega$, then $HP(\Omega) \subset L^1(\Omega)$ (and hence Ω is a PQD *by Theorem 2.4).*

Proof. By Lemma 5.2 it is enough to prove that G^Ω/G^a is bounded in a neighbourhood of $\partial\Omega$. But (Lemma 5.1)

$$\frac{G^\Omega}{G^a} = \frac{G^\mu}{G^a} - \frac{u}{G^a},$$

and it is easy to see (cf. [4], 1.VII.3c) that G^μ/G^a is bounded in a neighbourhood of $\partial\Omega$. □

The normal case, when Ω is a QD, is that the function $u = U^\mu - U^\Omega$ becomes positive inside Ω, at least near $\partial\Omega$. (Note that $u = |\operatorname{grad} u| = 0$ on $\mathbb{R}^N \setminus \Omega$ and that $\Delta u = \chi_\Omega$ in a neighbourhood of $\partial\Omega$.) In this case (5.5) is trivially satisfied. However, there are examples of QD:s having singular points on the boundary behind which u becomes negative. The only example of this sort known to us is when $N = 2$ and $\partial\Omega$ has a 'generic' inward pointing cusp. See [21, 23].

In any case, the fact that $u = |\operatorname{grad} u| = 0$ on $\mathbb{R}^N \setminus \Omega, \Delta u$ is bounded in a neighbourhood of $\partial\Omega$, gives rise to the estimate

$$|u(x)| \leqslant C\delta(x)^2 \log \frac{1}{\delta(x)}$$

($x \in \Omega$ close to $\partial\Omega$), when Ω is a QD. Here $\delta(x) = \operatorname{dist}(x, \mathbb{R}^N \setminus \Omega)$. Inserting this into (5.5) gives

COROLLARY 5.4. *Let Ω be a* QD *and assume that*

$$G(a, x) \geqslant C\delta(x)^2 \log \frac{1}{\delta(x)},$$

for $x \in \Omega$ close to $\partial\Omega$. Then $HP(\Omega) \subset L^1(\Omega)$ and Ω is a PQD.

REMARK. In view of (2.6), in order to show that $HP(\Omega) \subset L^1(\Omega)$ when Ω is a QD it is enough to show that every Martin function is integrable (or even that every such function can be approximated by functions in $HPL^1(\Omega)$). It is known [11] (cf. also the proof of Lemma 5.2 above) that every Martin function is the limit of $G(x, x_n)/G(a, x_n)$ for some sequence $\{x_n\} \subset \Omega$ tending to $\partial\Omega$. (Then $\{x_n\}$ tends to a 'Martin boundary point'). Therefore, when Ω is a QD, (5.3) in Lemma 5.2 can be replaced by the weaker statement that for each sequence $\{x_n\} \subset \Omega$ tending to a Martin boundary point, $G^\Omega(x_n)/G^a(x_n)$ is bounded from above (with the bound allowed to depend on $\{x_n\}$).

It follows that in Corollaries 5.3 and 5.4 the bounds can be replaced by individual bounds for each sequence $\{x_n\}$ tending to a Martin boundary point, with the constant C depending on the sequence.

Even with this remark taken into account it seems that the assumption, in Corollary 5.4, that Ω is a QD is not efficiently exploited. Let us for example consider domains such that Ω and $\partial\Omega$ are locally given by a Lipschitz function with Lipschitz constant at most k. Let $\alpha = \alpha(\psi)$ denote the so-called maximal order of barriers [12, 16]. This means that α is the order of homogeneity of any function harmonic in a circular cone with (half) aperture ψ and vanishing on the boundary of the cone. It is known that $\alpha(\psi)$ is a strictly decreasing function of ψ for $0 < \psi < \pi$ with $\lim_{\psi\to 0} \alpha(\psi) = +\infty$, $\lim_{\psi\to\pi} \alpha(\psi) = 0$ (if $N \geqslant 3$) and $\alpha(\pi/2) = 1$. For $N = 2, \alpha(\psi) = \pi/(2\psi)$.

According to [2, 13] (cf. also [27]), we have for any Lipschitz domain as above

$$G(x, a) \geqslant C\delta(x)^{\alpha},$$

with $\alpha = \alpha$ (arctan$(1/k)$). Thus Corollary 5.4 applies if α(arctan$(1/k)) < 2$.

On the other hand it is shown in [2] that without assuming that Ω is a QD, $HP(\Omega) \subset L^1(\Omega)$ holds when α(arctan$(1/k)) < 2$. Thus the assumption that Ω is a QD does not give us anything extra in this case. Note by the way that in two dimensions, the condition α(arctan$(1/k)) < 2$ becomes $k < 1$, which basically means that corners with interior angle $> \pi/2$ are allowed.

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