

## INFINITE LIFETIME FOR THE STARLIKE DYNAMICS IN HELE-SHAW CELLS

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ABSTRACT. One of the “folklore” questions in the theory of free boundary problems is the lifetime of the starlike dynamics in a Hele-Shaw cell. We prove precisely that, starting with a starlike analytic phase domain  $\Omega_0$ , the Hele-Shaw chain of subordinating domains  $\Omega(t)$ ,  $\Omega_0 = \Omega(0)$ , exists for an infinite time under injection at the point of starlikeness.

### 1. INTRODUCTION

We consider the flow of a viscous fluid in a plane Hele-Shaw cell under injection through a unique well which can be placed at the origin. Suppose that at the initial time the phase domain  $\Omega_0$  occupied by the fluid is simply connected and bounded by a smooth analytic curve  $\Gamma_0$ . The evolution of the phase domains  $\Omega(t)$ ,  $\Omega(0) = \Omega_0$ , is described by an auxiliary conformal mapping  $f(\zeta, t)$ ,  $f(\zeta, 0) = f_0(\zeta)$ , of the unit disk  $U = \{\zeta, |\zeta| < 1\}$  onto  $\Omega(t)$ ,  $\Gamma(t) = \partial\Omega(t)$ , normalized by  $f(0, t) = 0$ ,  $f'(0, t) > 0$ . Here we denote the derivatives by  $f' = \partial f / \partial \zeta$ ,  $\dot{f} = \partial f / \partial t$ , and  $t$  is a time parameter. This mapping satisfies the equation

$$(1.1) \quad \operatorname{Re} \left[ \dot{f}(\zeta, t) \overline{\zeta f'(\zeta, t)} \right] = 1, \quad \zeta = e^{i\theta},$$

under a suitable rescaling. L. A. Galin [4] and P. Ya. Polubarinova-Kochina [11, 12] first derived the equation (1.1) and stimulated deep investigations in the complex variable approach to free boundary problems (see, e.g., [9, 20] and the references therein).

By a *classical solution* in the interval  $t \in [0, T)$  to the equation (1.1) we mean a map  $f(\zeta, t)$  that is conformal and univalent as a function of  $\zeta$  in a neighbourhood of the closure  $\bar{U}$  of the unit disk  $U$  and  $C^1$  with respect to  $t$  in  $[0, T)$  (one-sided at 0). These assumptions about the classical solution can be found, e.g., in [14].

One of the main features of the solution to the equation (1.1) is that, starting with an analytic boundary  $\Gamma_0$ , we obtain a one-parameter ( $t$ ) chain of classical

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solutions  $f(\zeta, t)$  that exists during a period  $t \in [0, T)$ , developing possible cusps or double points on the boundary  $\Gamma(t)$ ,  $\Gamma(0) = \Gamma_0$ , at a blow-up time  $T$ . It is known [21] that the classical solution exists and is unique locally in time. Recently, in [14], it became clear that this model could be interpreted as a particular case of the abstract Cauchy problem; thus, the classical solvability (locally in time) may be proved using the nonlinear abstract Cauchy-Kovalevskaya Theorem. We also mention here that the problem is Hadamard well-posed in our case [3].

The problem of estimating  $T$  is of primary importance. One of the “folklore” questions is the lifetime of the starlike dynamics in a Hele-Shaw cell. The majority of investigators believe that this lifetime is infinite in the case of a starlike initial domain whereas some of them do not. Our aim is to settle this question. A domain  $\Omega \subset \mathbb{C}$ ,  $0 \in \Omega$ , is said to be *starlike* (with respect to the origin) if each ray starting at the origin intersects  $\Omega$  in a set that is either a line segment or a full ray. If a function  $f(\zeta)$  maps  $U$  onto a domain that is starlike,  $f(0) = 0$ , then we say that  $f(\zeta)$  is a *starlike function*. We denote the class of starlike functions by  $S^*$ . A criterion for a function  $f(\zeta)$ ,  $\zeta \in U$ ,  $f(0) = 0$ ,  $f'(0) > 0$ , to be starlike is the following inequality:

$$(1.2) \quad \operatorname{Re} \frac{\zeta f'(\zeta)}{f(\zeta)} > 0, \quad \zeta \in U.$$

This standard criterion can be found, e.g., in [5, 2, 7, 13].

We prove rigorously (Theorem 3.4) that, starting with a smooth analytic starlike phase domain  $\Omega_0$ , the Hele-Shaw chain of subordinating domains  $\Omega(t)$  exists as a classical solution for an infinite time under injection.

## 2. MONOTONE CHANGE OF STRONG $\alpha$ -STARLIKENESS

The class  $S^*$  is a union of classes  $S_\alpha^*$  of so-called *strongly starlike functions of order  $\alpha$* ,  $0 < \alpha \leq 1$ , defined by D. A. Brannan and W. E. Kirwan [1] and J. Stankiewicz [19]. A function  $f : U \rightarrow \mathbb{C}$ ,  $f(0) = 0$ ,  $f'(0) > 0$ , is said to be from  $S_\alpha^*$  if for all  $\zeta \in U$ ,

$$(2.1) \quad \left| \arg \frac{\zeta f'(\zeta)}{f(\zeta)} \right| < \alpha \frac{\pi}{2}.$$

This class of functions is characterized as follows. Every level line  $f(re^{i\theta})$ ,  $\theta \in [0, 2\pi)$ ,  $f \in S_\alpha^*$ , is reachable from outside by the radial angle  $\pi(1 - \alpha)$ . We will call the images of the unit disk under functions from  $S_\alpha^*$  strongly starlike of order  $\alpha$  as well. Clearly,  $S^* = S_1^*$ .

We prove that, starting with a phase domain  $\Omega_0$  that is strongly starlike of order  $\alpha$  and bounded by an analytic curve, we obtain a subordination chain of domains  $\Omega(t)$  (and functions  $f(\zeta, t)$ ) strongly starlike of order  $\alpha(t)$  with a decreasing order  $\alpha(t)$ .

**Theorem 2.1.** *Let  $f_0 \in S_\alpha^*$ ,  $\alpha \in (0, 1]$ , be analytic and univalent in a neighbourhood of  $\bar{U}$ . Then the classical solution  $f(\zeta, t)$  to the Polubarinova-Galin equation (1.1) forms a subordination chain of strongly starlike functions of order  $\alpha(t)$  with a strictly decreasing  $\alpha(t)$  during the time of existence.*

*Proof.* Let  $T$  be such that the classical solution  $f(\zeta, t)$  exists during the time  $t \in [0, T)$ ,  $T > 0$ . Since all functions  $f(\zeta, t)$  have analytic univalent extension into a neighbourhood of  $\bar{U}$  during the time of the existence of the classical solution to

(1.1), their derivatives  $f'(\zeta, t)$  are continuous and do not vanish in  $\bar{U}$ . Moreover,  $f(\zeta, t)$  are starlike in  $U$  (see [8, 20]). Therefore, there exists  $\alpha(t)$ ,  $0 < \alpha(t) \leq 1$ , such that  $f(\zeta, t) \in S_{\alpha(t)}^*$  and  $f(\zeta, t) \notin S_{\alpha(t)-\varepsilon}^*$  for any  $\varepsilon > 0$ .

Let us fix  $t_0 \in [0, T)$  and consider the set  $A$  of all points  $\zeta$ ,  $|\zeta| = 1$ , for which  $|\arg \frac{\zeta f'(\zeta, t_0)}{f(\zeta, t_0)}| = \alpha\pi/2$ . First, we deal with the subset  $A^+$  of  $A$  where

$$(2.2) \quad \arg \frac{\zeta f'(\zeta, t_0)}{f(\zeta, t_0)} = \frac{\alpha\pi}{2}.$$

The sets  $A^+$  and  $A^- = A \setminus A^+$  are closed and do not intersect. One of the sets  $A^+$  and  $A^-$  is allowed to be empty. Without loss of generality, we suppose that  $A^+ \neq \emptyset$ . For any point  $\zeta \in A^+$ , we have

$$(2.3) \quad \operatorname{Im} \frac{\zeta f'(\zeta, t_0)}{f(\zeta, t_0)} > 0.$$

The argument  $\arg \frac{\zeta f'(\zeta, t_0)}{f(\zeta, t_0)}$  attains its maximum on  $\zeta \in \partial U$  at the points of  $A^+$ . Therefore,

$$\frac{\partial}{\partial \theta} \arg \frac{e^{i\theta} f'(e^{i\theta}, t_0)}{f(e^{i\theta}, t_0)} = 0, \quad \zeta = e^{i\theta} \in A^+.$$

The argument  $\arg \frac{re^{i\theta} f'(re^{i\theta}, t_0)}{f(re^{i\theta}, t_0)}$ ,  $e^{i\theta} \in A^+$ , attains its maximum on  $r \in [0, 1]$  at  $r = 1$ . Hence,

$$\frac{\partial}{\partial r} \arg \frac{re^{i\theta} f'(re^{i\theta}, t_0)}{f(re^{i\theta}, t_0)} \Big|_{r=1} \geq 0.$$

We calculate

$$(2.4) \quad \operatorname{Re} \left[ 1 + \frac{\zeta f''(\zeta, t_0)}{f'(\zeta, t_0)} - \frac{\zeta f'(\zeta, t_0)}{f(\zeta, t_0)} \right] = 0,$$

$$(2.5) \quad \operatorname{Im} \left[ 1 + \frac{\zeta f''(\zeta, t_0)}{f'(\zeta, t_0)} - \frac{\zeta f'(\zeta, t_0)}{f(\zeta, t_0)} \right] \geq 0,$$

where  $\zeta \in A^+$ .

Let us represent the derivative

$$(2.6) \quad \frac{\partial}{\partial t} \arg \frac{\zeta f'(\zeta, t)}{f(\zeta, t)} = \operatorname{Im} \frac{\partial}{\partial t} \log \frac{f'(\zeta, t)}{f(\zeta, t)} = \operatorname{Im} \left( \frac{\frac{\partial}{\partial t} f'(\zeta, t)}{f'(\zeta, t)} - \frac{\frac{\partial}{\partial t} f(\zeta, t)}{f(\zeta, t)} \right).$$

Now we differentiate the Polubarinova-Galin equation (1.1) with respect to  $\theta$  as

$$(2.7) \quad \operatorname{Im} \left( \overline{f'(\zeta, t)} \frac{\partial}{\partial t} f'(\zeta, t) - \zeta \overline{f'(\zeta, t)} \dot{f}(\zeta, t) - \overline{\zeta^2 f''(\zeta, t)} \dot{f}(\zeta, t) \right) = 0, \quad \zeta = e^{i\theta}.$$

This equality is equivalent to

$$\begin{aligned} & |f'(\zeta, t)|^2 \operatorname{Im} \left( \frac{\frac{\partial}{\partial t} f'(\zeta, t)}{f'(\zeta, t)} - \frac{\frac{\partial}{\partial t} f(\zeta, t)}{f(\zeta, t)} \right) \\ &= \operatorname{Im} \left[ \overline{\zeta f'(\zeta, t)} \dot{f}(\zeta, t) \left( \overline{\left( \frac{\zeta f''(\zeta, t)}{f'(\zeta, t)} \right)} - \frac{\zeta f'(\zeta, t)}{f(\zeta, t)} + 1 \right) \right]. \end{aligned}$$

Substituting (1.1) and (2.4) in the latter expression, we have

$$\frac{\partial}{\partial t} \arg \frac{\zeta f'(\zeta, t)}{f(\zeta, t)} \Big|_{\zeta \in A^+, t=t_0} = \frac{-1}{|f'(\zeta, t_0)|^2} \operatorname{Im} \left( \frac{\zeta f'(\zeta, t_0)}{f(\zeta, t_0)} + \frac{\zeta f''(\zeta, t_0)}{f'(\zeta, t_0)} \right).$$

The right-hand side of this equality is continuous on  $A^+$  and strictly negative because of (2.3) and (2.5). Therefore,

$$\max_{\zeta \in A^+} \frac{\partial}{\partial t} \arg \frac{\zeta f'(\zeta, t)}{f(\zeta, t)} \Big|_{t=t_0} = -\delta < 0.$$

There exists a neighborhood  $A^+(\delta)$  on the unit circle of  $A^+$  such that  $A^+(\delta)$  and  $A^-$  do not intersect and

$$\frac{\partial}{\partial t} \arg \frac{\zeta f'(\zeta, t)}{f(\zeta, t)} \Big|_{\zeta \in A^+(\delta), t=t_0} < -\frac{\delta}{2}.$$

There is a positive number  $\sigma$  such that

$$\max_{\zeta \in \partial U \setminus A^+(\delta)} \arg \frac{\zeta f'(\zeta, t)}{f(\zeta, t)} \Big|_{t=t_0} = \frac{\alpha\pi}{2} - \sigma.$$

We choose such  $s > 0$  that

- (i)  $t_0 + s < T$ ;
- (ii)  $\frac{\partial}{\partial t} \arg \frac{\zeta f'(\zeta, t)}{f(\zeta, t)} \Big|_{\zeta \in A^+(\delta)} < 0, \quad t \in [t_0, t_0 + s]$ ;
- (iii)  $\max_{\zeta \in \partial U \setminus A^+(\delta)} \arg \frac{\zeta f'(\zeta, t)}{f(\zeta, t)} \leq \frac{\alpha\pi}{2} - \frac{\sigma}{2}, \quad t \in [t_0, t_0 + s]$ .

The condition (ii) implies that

$$\arg \frac{\zeta f'(\zeta, t)}{f(\zeta, t)} < \frac{\alpha\pi}{2}, \quad t \in (t_0, t_0 + s], \quad \zeta \in A^+(\delta).$$

Thus, the condition (iii) yields

$$\alpha^+(t) := \max_{\zeta \in \partial U} \arg \frac{\zeta f'(\zeta, t)}{f(\zeta, t)} < \frac{\alpha\pi}{2} = \alpha(t_0), \quad \text{for all } t \in (t_0, t_0 + s].$$

This means that  $\alpha^+(t)$  is strictly decreasing in  $[0, T)$ .

If the set  $A^- \neq \emptyset$ , then we can define the function

$$\alpha^-(t) := - \min_{\zeta \in \partial U} \arg \frac{\zeta f'(\zeta, t)}{f(\zeta, t)}.$$

Similar argumentation shows that  $\alpha^-(t)$  is strictly decreasing.

If  $A^- = \emptyset$  (or  $A^+ = \emptyset$ ), then  $\alpha(t) = \alpha^+(t)$  (or  $= \alpha^-(t)$ ) for  $t \in [t_0, t_0 + s]$ ,  $s$  sufficiently small.

We set the function  $\alpha(t) = \max\{\alpha^+(t), \alpha^-(t)\}$  in the case  $A^+ \neq \emptyset$  and  $A^- \neq \emptyset$ . This function  $\alpha(t)$  is strictly decreasing, and the proof is complete.  $\square$

We remark that a similar result, with estimates of  $\alpha(t)$ , has been independently obtained by O. Kuznetsova [10].

3. INFINITE LIFETIME

In this section first we will prove that if the classical solution to (1.1) exists during the time interval  $[0, T)$ , then the limiting function  $\lim_{t \rightarrow T-0} f(\zeta, t) \equiv f(\zeta, T)$  is analytic in some neighbourhood of the unit disk  $U$ . Here the limit is taken with respect to the uniform convergence on compacts of the unit disk  $U$ . It exists because  $f(\zeta, t)$  is a subordination chain, and due to the Carathéodory Kernel Theorem. Then we will obtain the main result about the infinite lifetime.

**Lemma 3.1.** *Let the classical solution to (1.1) exist during the time interval  $[0, T)$ ,  $0 < T < \infty$ ,  $\Omega(t) = f(U, t)$ , and let the initial function  $f(\zeta, 0)$  be analytic and univalent in a neighbourhood of the closure of the unit disk  $U$ . Then, there is  $\eta > 0$  such that the function  $f(\zeta, t)$  is analytic in  $U_{1+\eta} = \{\zeta : |\zeta| < 1+\eta\}$  for all  $t \in [0, T]$ , univalent in  $U$ , and possibly  $f(\zeta, T)$  has a vanishing derivative at some points of the unit circle  $\partial U$  or is not univalent on  $\partial U$ . It follows that  $\Omega(T) \equiv f(U, T)$  is a simply connected domain with an analytic boundary  $\Gamma(T) = \partial\Omega(T)$  with possible analytic singularities in the form of finitely many cusps and double points.*

*Proof.* By the Carathéodory Kernel Theorem the domain

$$\Omega(T) = \bigcup_{t \in [0, T)} \Omega(t)$$

is just the same as in the formulation of the lemma, and  $\Omega(T)$  is a simply connected domain.

It is well known (see, e.g., [6, 15, 16]) and easily verified directly that for any analytic and integrable function  $\Phi$  in  $\Omega(T)$  the identity

$$(3.1) \quad \iint_{\Omega(t)} \Phi(z) d\sigma_z = \iint_{\Omega(0)} \Phi(z) d\sigma_z + 2\pi t \Phi(0)$$

holds for  $t \in [0, T)$ . We note also that since the normal velocity on the boundary never vanishes, we have the strict monotonicity of the subordination chain of domains:

$$(3.2) \quad \overline{\Omega(s)} \subset \Omega(t) \quad \text{for } s < t \text{ and } s, t \in (0, T).$$

Letting  $t \rightarrow T$ , we see that (3.1) and (3.2) hold for  $t = T$ , i.e., for  $\Omega(T)$  as well.

In order to give a rigorous proof of the statement about the properties of  $f(\zeta, T)$  we analytically extend the mapping  $f(\zeta, T)$  following the scheme proposed in [6, Theorem 9]; see also [17, Theorem 1.7]. Let us choose the function  $\Phi(z) = (z-w)^{-1}$  for  $w \notin \Omega(t)$ ,  $t \in (0, T]$ , in (3.1). This gives

$$(3.3) \quad \hat{\chi}_{\Omega(t)} = \hat{\chi}_{\Omega(0)} + 2\pi t \hat{\delta},$$

where

$$\hat{\mu}(w) = -\frac{1}{\pi} \iint_{\Omega(t)} \frac{d\mu(z)}{z-w}$$

denotes the Cauchy transformation of a measure  $\mu$ , normalized so that  $\partial\hat{\mu}/\partial\bar{w} = \mu(w)$ ,  $\chi$  stands for the characteristic function of the corresponding domain, and  $\delta$  is the Dirac distribution.

We define a function  $S(z, t)$  by

$$S(z, t) = \bar{z} - \hat{\chi}_{\Omega(t)}(z) + \hat{\chi}_{\Omega(0)}(z) + \frac{2t}{z}$$

for  $z \in \overline{\Omega(t)} \setminus \overline{\Omega(0)}$ , for each  $t \in (0, T]$ . Then  $S(z, t)$  as a function of  $z$  is continuous in  $\overline{\Omega(t)} \setminus \overline{\Omega(0)}$ , analytic in  $\Omega(t) \setminus \overline{\Omega(0)}$  (one easily calculates that  $\partial S/\partial \bar{z} = 0$ ), and  $S(z, t) = \bar{z}$  on  $\Gamma(t) = \partial\Omega(t)$  by (3.3). Thus,  $\overline{S(z, t)}$  is a one-sided Schwarz function of  $\Gamma(t)$  [16, 18]. The transformation  $z \mapsto \overline{S(z, t)}$  is a one-sided reflection over  $\Gamma(t)$ , which we will use to extend  $f(\zeta, t)$ ,  $t \in (0, T]$ . Let us fix  $t \in (0, T]$ . We set  $r(t) = \max_{z \in \partial\Omega(0)} |f^{-1}(z, t)|$  and consider a point  $\zeta$ ,  $r(t) < |\zeta| < 1$ . The point  $1/\bar{\zeta}$  is its reflection through  $\partial U$ .

We define the function  $f(\zeta, t)$  in a neighbourhood of the unit circle  $1 < |\zeta| < 1/r(t)$  outside  $U$  by  $f(1/\bar{\zeta}, t) \equiv \overline{S(f(\zeta, t), t)}$ . This defines  $f$  analytically in the annulus  $1 < |\zeta| < 1/r(t)$ . Across  $\partial U$  we have a certain form of continuity because of the continuity of  $S(z, t)$ . Indeed, as  $|\zeta| \rightarrow 1$  with  $\zeta \in U$  we have

$$(3.4) \quad |f(\zeta, t) - f(1/\bar{\zeta}, t)| = |f(\zeta, t) - \overline{S(f(\zeta, t), t)}| \rightarrow 0,$$

where  $z = \overline{S(z, t)}$  on  $\Gamma(t)$ , and therefore, given  $\varepsilon > 0$ , we have  $|z - \overline{S(z, t)}| < \varepsilon$  for  $z \in \Omega(t)$  in some neighbourhood of  $\Gamma(t)$ . By now the function  $f(\zeta, t)$  is defined in  $U$  as well as in the annulus  $1 < |\zeta| < 1/r(t)$ , and hence, almost everywhere in the disk  $|\zeta| < 1/r(t)$ . Let us prove that the distributional derivative  $\partial f(\zeta, t)/\partial \bar{\zeta}$  vanishes in  $|\zeta| < 1/r(t)$ , using (3.4). Obviously, we must verify this across the circle  $|\zeta| = 1$ . Given a test function  $\varphi$  with compact support in  $|\zeta| < 1/r(t)$ , we have

$$\begin{aligned} \left\langle \frac{\partial f}{\partial \bar{\zeta}}, \varphi \right\rangle &= - \iint_{\mathbb{C}} f(\zeta, t) \frac{\partial \varphi}{\partial \bar{\zeta}} d\sigma_{\zeta} \\ &= -\frac{1}{2i} \iint_U f(\zeta, t) \frac{\partial \varphi}{\partial \bar{\zeta}} d\bar{\zeta} d\zeta - \frac{1}{2i} \iint_{|\zeta|>1} f(\zeta, t) \frac{\partial \varphi}{\partial \bar{\zeta}} d\bar{\zeta} d\zeta \\ &= -\frac{1}{2i} \lim_{\varepsilon \downarrow 0} \left( \int_{|\zeta|=1-\varepsilon} f(\zeta, t) \varphi(\zeta) d\zeta - \int_{|\zeta|=1+\varepsilon} f(\zeta, t) \varphi(\zeta) d\zeta \right) \\ &= -\frac{1}{2i} \lim_{\varepsilon \downarrow 0} \int_{|\zeta|=1-\varepsilon} (f(\zeta, t) - f(1/\bar{\zeta}, t)) \varphi(\zeta) d\zeta = 0. \end{aligned}$$

In the above curve integrals we take the counterclockwise direction on the circles. Thus, the function  $f(\zeta, t)$  is analytic in the disk  $|\zeta| < 1/r(t)$ .

For any pair of numbers  $s, t$  such that  $0 < s < t \leq T$ , we have that the function  $h(\zeta, s, t) \equiv f^{-1}(f(\zeta, s), t)$  maps the unit disk into itself and  $h(0, s, t) \equiv 0$ . A simple application of the Schwarz Lemma to the function  $h$  shows that

$$f^{-1}(\Omega(0), t) \subset U_{r(s)}.$$

Therefore,  $r(t) \leq r(s)$ . For a sufficiently small  $\varepsilon > 0$  we choose  $\delta$  such that the solution  $f(\zeta, t)$  is analytic in the disk  $U_{1+\varepsilon}$  for any  $t \in [0, \delta)$ . Then we define

$$\eta = \min \left( \varepsilon, \frac{1}{r(\delta)} - 1 \right).$$

Thus, we have defined the function  $f(\zeta, t)$  in the disk  $U_{1+\eta}$  for all  $t \in [0, T]$ . To get a pointwise definition of  $f$  on the circle  $|\zeta| = 1$  one may use the Cauchy integral formula for the disk  $|\zeta| < 1 + \eta/2$ . This finishes the proof of the lemma.  $\square$

**Lemma 3.2.** *Let  $f_0 \in S^*$  be analytic and univalent in a neighbourhood of the closure  $\bar{U}$  of the unit disk  $U$ . If the solution  $f(\zeta, t)$  to the Polubarinova-Galin equation (1.1) exists during the time interval  $[0, T)$ , then it forms a subordination chain of starlike functions such that the limiting domain  $\Omega(T)$  has a smooth analytic boundary.*

*Proof.* By the assumptions on  $f_0$  the initial domain  $\Omega(0)$  has a smooth analytic boundary and the solution  $f(\zeta, t)$  to (1.1) exists locally in time  $t \in [0, s)$ . The function  $f_0 \in S^*_\alpha$  for some  $\alpha \in (0, 1]$ . The function  $f(\zeta, t)$  belongs to the class  $S^*_{\alpha(t)}$  with  $\alpha(t) < 1$  for any  $t \in (0, s)$ , due to Theorem 2.1. Therefore, we can prove Lemma 3.2 choosing  $f(\zeta, t)$  with some  $t \in (0, s)$  as an initial mapping,  $f(\zeta, t) \in S^*_{\alpha(t)}$  with  $\alpha(t) < 1$ . We prove that  $\partial\Omega(T)$  does not contain a cusp or a double point on its boundary.

Define the limiting function  $f(\zeta, T) = \lim_{t \rightarrow T-0} f(\zeta, t)$ , where the limit is taken locally uniformly in  $U$ . The function  $f(\zeta, T)$  is univalent, strongly starlike of order  $\alpha(T) = \lim_{t \rightarrow T-0} \alpha(t) < 1$ , and has a continuous extension on  $\bar{U}$ . According to the geometric characterization of the class  $S^*_{\alpha(T)}$ , the boundary of the domain  $\Omega(T) = f(U, T)$  is reachable by the radial external angles  $\pi(1 - \alpha)$ , which implies that there is no cusp or double point on the boundary of  $\Omega(T)$ . This completes the proof.  $\square$

**Lemma 3.3.** *Let  $f(\zeta, t)$  be the classical solution to the equation (1.1) that exists during the time interval  $[0, T)$  with a starlike initial mapping  $f_0$  as in Lemma 3.2. Then, there exists  $\varepsilon > 0$  such that the classical solution exists during the time interval  $[0, T + \varepsilon)$ .*

*Proof.* The limiting domain  $\Omega(T)$  is simply connected and has an analytic boundary. The limiting mapping  $f(\zeta, T)$  is analytic in a neighbourhood of  $\bar{U}$  by Lemma 3.1 and strongly starlike of order  $\alpha(T) < 1$  by Lemma 3.2. Therefore, there exists an  $\eta > 0$  such that  $f(\zeta, T)$  is starlike and univalent in the disk  $|\zeta| < 1 + \eta$ . Let us construct the subordination chain of mappings  $f_2(\zeta, t)$  satisfying the Polubarinova-Galin equation (1.1) with the initial data  $f_2(\zeta, 0) \equiv f(\zeta, T)$ . The classical solution exists and is unique locally in time, say  $t \in [0, \varepsilon)$ . Moreover, we have

$$\lim_{t \rightarrow T-0} f(\zeta, t) = \lim_{t \rightarrow 0+0} f_2(\zeta, t) = f(\zeta, T)$$

and

$$\lim_{t \rightarrow T-0} f'(\zeta, t) = \lim_{t \rightarrow 0+0} f'_2(\zeta, t) = f'(\zeta, T)$$

locally uniformly in  $U_{1+\eta}$ . We rewrite the equation (1.1) in  $U$  using the Schwarz kernel as

$$\dot{f}(\zeta, t) = \zeta f'(\zeta, t) \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} d\theta, \quad t \in [0, T), \quad |\zeta| < 1.$$

A similar equation is valid for the chain  $f_2(\zeta, t)$  in the time interval  $[0, \varepsilon)$ . Taking the limit in the above equation as  $t \rightarrow T - 0$  we observe that there exists the one-sided limit  $\dot{f}(\zeta, T - 0)$ . Similarly, the one-sided limit  $\dot{f}_2(\zeta, 0 + 0)$  exists, and

they are equal. Let us define  $f(\zeta, t) \equiv f_2(\zeta, t - T)$  in the interval  $t \in [T, T + \varepsilon)$ . The above observations yield that the function thus extended is continuous in the interval  $t \in [0, T + \varepsilon)$ , analytic, univalent and starlike in some neighbourhood of  $\bar{U}$ . Moreover, it is differentiable at the point  $t = T$ , and being extended onto the unit circle, satisfies the equation (1.1). Thus, it is a unique classical solution in the interval  $t \in [0, T + \varepsilon)$ , and the lemma is proved.  $\square$

**Theorem 3.4.** *Starting with a starlike phase domain  $\Omega_0$  with an analytic boundary, the lifetime of the classical Hele-Shaw starlike dynamics  $\Omega(t)$  is infinite.*

*Proof.* Indeed, if the classical solution exists during the finite interval  $t \in [0, T)$  and does not exist in  $t \in [T, T + \varepsilon)$  for any  $\varepsilon > 0$ , then this contradicts Lemma 3.3.  $\square$

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