Linear Analysis of Quadrature Domains. IV

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Paper dedicated to Harold S. Shapiro on the occasion of his seventy-fifth birthday

Abstract. The positive definiteness of the exponential transform of a planar domain is proved by elementary means. This direct approach avoids the heavy machinery of the theory of hyponormal operators and leads to a better understanding of the linear data associated in previous works to a quadrature domain.

1. The exponential transform

Let Ω be a bounded open subset of the complex plane and let dA stand for the Lebesgue planar measure. The *exponential transform* of the set Ω is the function:

$$E_\Omega(z,w) = \exp[-\frac{1}{\pi}\int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}]. \tag{1.1}$$

The integral is convergent for all values of $z, w \in \mathbf{C}$ avoiding the diagonal

$$\Delta = \{(z,w);\ z = w \in \overline{\Omega}\}.$$

In case $(z,w) \in \Delta$ and the integral is divergent (necessarily to infinity) we adopt the convention $\exp(-\infty) = 0$. Thus $E_\Omega(z,w)$ is defined everywhere on $\mathbf{C}^2$ and one proves that the resulting function is uniformly bounded and separately continuous in each variable, see [10]. We shall occasionally use the notation (1.1) also when the set Ω is not open.

The above exponential transform has appeared in operator theory as a determining function for a class of hyponormal operators ([18], [20], [2], [3], [4]). Later it was analyzed in purely function theoretic terms and was used in proving the regularity of certain free boundaries ([10]) or in image reconstruction ([8]). More generally, the exponential transform was regarded as a renormalized Riesz potential, and was instrumental in reconstructing the measure $\chi_\Omega dA$ from its moments,

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see also [13] for a multivariable generalization. In this process, an exact reconstruction algorithm corresponding to the special class of quadrature domains Ω was discovered ([21], [11], [22]). A key positivity property of the exponential transform remained however available only from its operator theoretic origins; and these involved the highly sophisticated theory of the principal function of a semi-normal operator.

The aim of the present note is to make a short cut by proving the basic positivity property of the exponential transform by elementary arguments, accessible to function theorists. We mention that this positivity is a specific phenomenon to two real dimensions, [13].

We recall first some identities satisfied by the transform $E_\Omega(z,w)$. Their simple proofs can be found in [10]. Since we keep the set Ω fixed, we sometimes denote $E = E_\Omega$. Also, to simplify notation we write: $\overline{\partial}_z = \partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}}$ and $\partial_w = \frac{\partial}{\partial w}$.

Remark that $E(z,w) = \overline{E(w,z)}$ for all values of $z, w \in \mathbf{C}$ and that $E(z,w)$ is analytic in $z \in \overline{\Omega}^c$ and antianalytic in $w \in \overline{\Omega}^c$. The Taylor expansion at infinity starts with the terms:

$$E(z,w) = 1 - \frac{1}{\pi}\int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})} + O(z^{-2}, \overline{w}^{-2}). \tag{1.2}$$

The following identities hold in the sense of distributions in $\mathbf{C}^2$:

$$\overline{\partial}_z E(z,w) = E(z,w)\frac{\chi_\Omega(z)}{\overline{z} - \overline{w}},$$

$$\partial_w E(z,w) = -E(z,w)\frac{\chi_\Omega(w)}{z - w}.$$

Note that the right hand members are given by locally integrable functions in $\mathbf{C}^2$. Moreover,

$$\overline{\partial}_z \partial_w E(z,w) = -E(z,w)\frac{\chi_\Omega(z)\chi_\Omega(w)}{|z-w|^2}, \tag{1.3}$$

again as distributions, at least in iterated integrals sense, see formula (2.20) and the related comments in [10].

We define the *interior exponential transform* by:

$$H_\Omega(z,w) = \frac{E_\Omega(z,w)}{|z-w|^2}, \quad z, w \in \Omega, \tag{1.4}$$

so that

$$H(z,w) = -\overline{\partial}_z \partial_w E(z,w), \quad z, w \in \Omega. \tag{1.5}$$

It turns out by elementary computations that $H(z,w)$ is an analytic function in $z \in \Omega$ and antianalytic in $w \in \Omega$. By applying Cauchy's formula twice to the function $1 - E(z,w)$ (which vanishes at infinity in each variable) we obtain the integral representation:

$$1 - E(z,w) = \frac{1}{\pi^2}\int_\Omega \int_\Omega H(u,v)\frac{dA(u)}{u - z}\frac{dA(v)}{\overline{v} - \overline{w}}, \quad z, w \in \mathbf{C}. \tag{1.6}$$

The right member should be interpreted as an iterated convolution in the distribution sense.

A second remarkable feature of the interior transform H is the following complementarity relation, valid for a pair of *disjoint* sets Ω_1 and Ω_2:

$$H_{\Omega_1\cup\Omega_2}(z,w) = H_{\Omega_1}(z,w)E_{\Omega_2}(z,w), \quad z,w\in\Omega_1. \tag{1.7}$$

To prove it one just notices from (1.1) that $E_{\Omega_1\cup\Omega_2}(z,w) = E_{\Omega_1}(z,w)E_{\Omega_2}(z,w)$ holds everywhere. Applying $\overline{\partial}_z\partial_w$ for $z,w\in\Omega_1$ to both members gives (1.7) (in view of (1.5)).

It is not necessary that Ω_2 is open in (1.7), but Ω_1 and $\Omega_1\cup\Omega_2$ should be. If Ω_1 and Ω_2 are both open then $\Omega_1\cup\Omega_2$ is disconnected, and it is interesting to notice that the restriction of $H_{\Omega_1\cup\Omega_2}$ to Ω_1 does not agree with H_{Ω_1}; the other part Ω_2 influences via the factor E_{Ω_2} in (1.7). Thus although H_Ω has some similarity with classical domain functions, like the Szegö kernel, it has drastically different behaviour in some respects. Another example of this is that there seems to be very little of conformal invariance properties for E_Ω and H_Ω (see [10] for behaviour under Möbius transformations).

Example 1. The case of the unit disk $\Omega = \mathbf{D}$ is relevant for the rest of the article. One finds by direct computation:

$$E_{\mathbf{D}}(z,w) = \begin{cases} 1-\frac{1}{z\overline{w}} & |z|\geq 1, |w|\geq 1,\\ 1-\frac{\overline{z}}{\overline{w}} & |z|<1, |w|\geq 1,\\ 1-\frac{w}{z} & |z|\geq 1, |w|<1,\\ \frac{|z-w|^2}{1-z\overline{w}} & |z|,|w|<1. \end{cases} \tag{1.8}$$

Thus the interior transform is:

$$H_{\mathbf{D}}(z,w) = \frac{1}{1-z\overline{w}} = \sum_{n=0}^{\infty} z^n\overline{w}^n, \quad |z|,|w|<1. \tag{1.9}$$

We note that $H_{\mathbf{D}}$ agrees with the Szegö kernel in this case.

A function (thought of as a "kernel") $K:\Omega\times\Omega\longrightarrow\mathbf{C}$ is called *positive semidefinite* if

$$\sum_{i,j=1}^{m} K(z_i,z_j)\lambda_i\overline{\lambda_j}\geq 0.$$

for any finite sequences $z_1,\dots,z_m\in\Omega$ and $\lambda_1,\dots,\lambda_m\in\mathbf{C}$. The kernel is said to be *positive definite* if equality occurs (with the z_i distinct) only when $\lambda_1=\dots=\lambda_m=0$.

It is obvious that any sum (even an infinite one) of positive semidefinite kernels is positive semidefinite. If at least one of the terms is positive definite then the whole sum is definite. We recall also Schur's theorem [15] saying that the pointwise product of two (or more) positive semidefinite kernels is again positive semidefinite.

Any kernel $K(z,w)$ which can be written on the form

$$K(z,w)=\sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)} \tag{1.10}$$

(with absolute convergence for each z and w) for some functions f_n is obviously positive semidefinite. Thus, by (1.9), $H_{\mathbf{D}}(z,w)$ is positive semidefinite. It is even positive definite since it agrees with the Szegö kernel, which is known to be positive definite. The same is true for

$$\frac{1}{E_{\mathbf{D}}(z,w)}=\frac{1}{1-\frac{1}{z\overline{w}}}=\sum_{n=0}^{\infty}\frac{1}{z^n\overline{w}^n},\ \ z,w\notin\overline{\mathbf{D}}.$$

We finally notice that $1-E_{\mathbf{D}}(z,w)=1/z\overline{w}$, for $z,w\notin\overline{\mathbf{D}}$, is positive semidefinite but not definite.

The main result of the note is the following theorem.

Theorem 1.1. *Let Ω be a bounded open planar set. The kernels*

$$\frac{1}{E_\Omega(z,w)},\ \ z,w\in\overline{\Omega}^c,$$

$$H_\Omega(z,w),\quad z,w\in\Omega$$

are positive definite, and

$$1-E_\Omega(z,w),\quad z,w\in\mathbf{C}$$

is positive semidefinite.

Remark 1. Even though $H(z,w)$ is positive definite in the above linear algebra sense there may still be functions $h\neq 0$ such that

$$\int_\Omega\int_\Omega H(z,w)h(z)\overline{h(w)}\,dA(z)dA(w)=0.$$

For example, if $\Omega=\mathbf{D}$ then $h(z)=z$ is such a function. See Proposition 3.3 for a general statement in this respect, and also for a refinement of the statement concerning $1-E_\Omega$.

Proof. By definition (1.1):

$$\frac{1}{E_\Omega(z,w)}=\exp\left[\frac{1}{\pi}\int_\Omega\frac{dA(\zeta)}{(\zeta-z)(\overline{\zeta}-\overline{w})}\right]=\sum_{n=0}^{\infty}\frac{1}{\pi^n n!}\left[\int_\Omega\frac{dA(\zeta)}{(\zeta-z)(\overline{\zeta}-\overline{w})}\right]^n.$$

Clearly the integral kernel here is positive definite:

$$\sum_{i,j=1}^{n}\int_\Omega\frac{dA(\zeta)}{(\zeta-z_i)(\overline{\zeta}-\overline{z_j})}\lambda_i\overline{\lambda_j}=\int_\Omega\left|\sum_{j=1}^{n}\frac{\lambda_j}{\zeta-z_j}\right|^2dA(\zeta)\geq 0$$

with equality only when $\sum_{j=1}^{n}\frac{\lambda_j}{\zeta-z_j}=0$ (identically), i.e., only when all the λ_j are zero.

By Schur's theorem the powers $[\int_\Omega \frac{dA(\zeta)}{(\zeta-z)(\bar\zeta-\bar w)}]^n$ are then also positive semidefinite, and the positivity is preserved under the summation and limit processes. Thus $1/E_\Omega(z,w)$ is positive semidefinite for $z,w \in \overline{\Omega}^c$, and it is even positive definite since the term with $n=1$ is so.

We note from the proof so far that $1/E_\Omega$ will be positive semidefinite even if the set Ω is not open. This will be needed below.

To prove that $H_\Omega(z,w)$ is positive definite, choose a disc $D = D(0,R)$ with $\overline{\Omega} \subset D$. By (1.7) we have

$$H_\Omega(z,w) = H_D(z,w) \cdot \frac{1}{E_{D\setminus\Omega}(z,w)}$$

for $z,w \in \Omega$. Here both factors on the right are positive definite and it follows that the product is positive semidefinite.

Moreover, expanding $1/E_{D\setminus\Omega}$ as in the beginning of the proof and H_D as in (1.9) and multiplying these expansions we get a series of positive semidefinite kernels having at least one term which is positive definite (namely the term coming from the linear term in $1/E_{D\setminus\Omega}$ times the constant term in H_D). Thus H_Ω is positive definite.

Finally, having proved that H_Ω is positive definite the positive semidefiniteness of $1 - E_\Omega(z,w)$ follows from the representation (1.6).

□

In the next section we shall need the following consequence of Theorem 1.1.

Corollary 1.2. *For R sufficiently large, the kernel $(R^2 - z\bar w)H_\Omega(z,w)$ is positive definite.*

Indeed, with R chosen so that $\overline{\Omega} \subset D(0,R)$ we have

$$(R^2 - z\bar w)H_\Omega(z,w) = \frac{H_\Omega(z,w)}{H_{D(0,R)}(z,w)} = \frac{1}{E_{D(0,R)\setminus\Omega}},$$

which is positive definite by the theorem (or rather its proof).

We wish to point out that there is an even more elementary way, not using Schur's theorem, to prove that $1/E_\Omega$ is positive semidefinite when Ω is open. Just exhaust Ω by mutually disjoint discs $D_n = D(a_n, r_n)$ so that

$$\Omega = (\cup_{n=1}^\infty D_n) \cup N$$

where $|N| = 0$. For each finite union $\Delta_n = \cup_{j=1}^n D_j$ we have, outside $\overline{\Omega}$ and using scaled versions of (1.8),

$$\frac{1}{E_{\Delta_n}(z,w)} = \prod_{j=1}^n \frac{1}{E_{D_j}(z,w)} = \prod_{j=1}^n \frac{1}{1 - \frac{r_j^2}{(z-a_j)(\bar w - \bar a_j)}} = \prod_{j=1}^n \sum_{k=0}^\infty \frac{r_j^{2k}}{(z-a_j)^k(\bar w - \bar a_j)^k}$$

$$= \sum_{(k_1,\dots,k_n)} \prod_{j=1}^n \frac{r_j^{2k_j}}{(z-a_j)^{k_j}(\bar w - \bar a_j)^{k_j}} = \sum_{(k_1,\dots,k_n)} \prod_{i=1}^n \frac{r_i^{k_i}}{(z-a_i)^{k_i}} \prod_{j=1}^n \frac{r_j^{k_j}}{(\bar w - \bar a_j)^{k_j}},$$

where $(k_1, \dots, k_n)$ ranges over all n-tuples of nonnegative integers. Choosing an ordering of this set brings $1/E_{\Delta_n}$ onto the form (1.10), hence it is positive semi-definite. Since $\frac{1}{E_{\Delta_n}(z,w)} \to \frac{1}{E_\Omega(z,w)}$ as $n \to \infty$ for $z, w \in \overline{\Omega}^c$ the statement follows.

Theorem 1.1 implies, via a direct computation or standard arguments familiar to complex geometers (or see [13]), that the function $\log(1 - E_\Omega(z, z))$ is subharmonic on the complement of $\overline{\Omega}$. The diagonal versions $E_\Omega(z, z)$ and $H_\Omega(z, z)$ can naturally be extended to any number of variables. However, the subharmonicity of $\log(1 - E_\Omega(z, z))$ does not hold in higher dimensions, although $1 - E_\Omega(z, z)$ remains subharmonic there [13].

2. A Hilbert space factorization

A celebrated and widely used theorem of Kolmogorov asserts that a positive semi-definite kernel $K(i, j)$, $\quad i, j \in I$, can always be factored as

$$K(i, j) = \langle k_i, k_j \rangle,$$

with k_i belonging to an auxiliary Hilbert space. Many spectral decompositions, interpolation and prediction questions, inverse problems depend on such factorizations, see for instance [7], [24].

The positivity results proved in the preceding section invite to study the Hilbert space factorizations of the kernels H_Ω and $1 - E_\Omega$. There are at least three convergent ways of understanding the fine structure of the factorization of these kernels, cf. [4], [19] and respectively [12]. We briefly recall the construction contained in the latter reference.

Throughout this section we assume that Ω is a bounded open set of $\mathbf{C}$ having smooth boundary. Then the boundary behavior of H_Ω is comparable to that of a disk and implies (see more precisely Appendix, Section 5)

$$\int_\Omega \int_\Omega |H_\Omega(u, v)| dA(u) dA(v) < \infty. \tag{2.1}$$

On the space $L^\infty(\Omega)$ we consider the scalar product:

$$\langle f, g \rangle = \frac{1}{\pi^2} \int_\Omega \int_\Omega H_\Omega(u, v) f(u) \overline{g(v)} dA(u) dA(v),$$

and denote by $\mathcal{H}(\Omega)$ the associated separated Hilbert space completion. Thus the map

$$L^\infty(\Omega) \longrightarrow \mathcal{H}(\Omega),$$

has dense range. Notice that even the image of all test functions $\mathcal{D}(\Omega)$ is dense in $\mathcal{H}(\Omega)$.

As a matter of fact $\mathcal{H}(\Omega)$ "contains" many other elements, for instance images of distributions (or even analytic functionals) $\sigma \in \mathcal{E}'(\mathbf{C})$ such that

$$\langle \sigma, \sigma \rangle = \frac{1}{\pi^2} \int_\Omega \int_\Omega H_\Omega(u, v) \sigma(u) \overline{\sigma(v)} dA(u) dA(v) < \infty.$$

A typical example is Dirac's distributions $\delta_z,\ z \in \Omega$, which produce the factorization of H_Ω:

$$H_\Omega(z,w) = \pi^2 \langle \delta_z, \delta_w \rangle, \quad z,w \in \Omega.$$

The constant function $\mathbf{1}$ belongs to $\mathcal{H}(\Omega)$, as well as all simple rational functions

$$k_z(\zeta) = \frac{1}{\zeta - z}, \quad \zeta \in \Omega,\ z \in \mathbf{C} \setminus \partial\Omega.$$

The regularity assumption on the boundary of Ω implies that the map $z \mapsto k_z \in \mathcal{H}(\Omega)$ extends across $\partial\Omega$ and it is weakly continuous on the entire complex plane.

Using (1.6) we have:

$$\langle k_z, k_w \rangle = \frac{1}{\pi^2} \int_\Omega \int_\Omega H(u,v) \frac{dA(u)}{u - z} \frac{dA(v)}{\overline{v} - \overline{w}} = 1 - E_\Omega(z,w), \tag{2.2}$$

for all values $z, w \in \mathbf{C}$. It is worth mentioning at this moment that k_z inherits some regularity from E_Ω. For instance k_z is analytic in $z \in \overline{\Omega}^c$ and bianalytic for $z \in \Omega$ (the latter means that $\overline{\partial}_z^2 k_z = 0$ as an element of $\mathcal{H}(\Omega)$).

By integrating counterclockwise the relation (1.2) on a large circle we obtain:

$$\frac{-1}{\pi} \int_\Omega \frac{dA(u)}{u - z} = \frac{1}{2\pi i} \int_{|w|=R} E_\Omega(z,w) d\overline{w}, \quad z \in \mathbf{C},$$

or equivalently via Stokes' theorem:

$$\frac{-1}{\pi} \int_\Omega \frac{dA(u)}{u - z} = \frac{-1}{\pi} \int_{\mathbf{C}} \partial_w E_\Omega(z,w) dA(w), \quad z \in \mathbf{C}.$$

By taking a partial derivative with respect to $\overline{z}$ this gives:

$$\chi_\Omega(z) = -\frac{1}{\pi} \int_{\mathbf{C}} \overline{\partial}_z \partial_w E_\Omega(z,w) dA(w), \quad z \in \mathbf{C},$$

and hence:

$$1 = \frac{1}{\pi} \int_\Omega H_\Omega(z,w) dA(w), \quad z \in \Omega.$$

Thus, for any $h \in L^\infty(\Omega)$ we find:

$$\langle h, \mathbf{1} \rangle = \frac{1}{\pi} \int_\Omega h dA, \tag{2.3}$$

and more generally:

$$\langle h k_z, \mathbf{1} \rangle = \frac{1}{\pi} \int_\Omega \frac{h(\zeta) dA(\zeta)}{\zeta - z}, \quad z \in \mathbf{C}.$$

As a special case we have the Cauchy transform identity:

$$\langle k_z, \mathbf{1} \rangle = \frac{1}{\pi} \int_\Omega \frac{dA(u)}{u - z}, \quad z \in \mathbf{C}.$$

One step further we can consider the multiplication operator

$$(Tf)(z) = z f(z), \quad f \in \mathcal{H}(\Omega).$$

Corollary 1.2 assures that T is a linear bounded operator on $\mathcal{H}(\Omega)$. The adjoint turns out to be an elementary singular integral operator:

$$(T^*f)(z) = \overline{z}f(z) - \frac{1}{\pi}\int_\Omega \frac{f(\zeta)}{\zeta - z} dA(\zeta).$$

Indeed, notice that $\partial_w[(z-w)H_\Omega(z,w)] = -H(z,w)$ and denote the Cauchy transform by

$$\hat{\psi}(z) = \frac{-1}{\pi}\int_\Omega \frac{\psi(\zeta)dA(\zeta)}{\zeta - z},$$

so that $\partial_{\overline{z}}\hat{\psi} = \psi$. For a pair of test functions $\phi, \psi \in \mathcal{D}(\Omega)$ we find by partial integration (Stokes), and using in the last steps (1.4), the boundedness of $E(z,w)$ and the decay of $\hat{\psi}$ at infinity:

$$\langle z\phi(z), \psi(z)\rangle - \langle \phi(z), \overline{z}\psi(z)\rangle =$$

$$\frac{1}{\pi^2}\int_\Omega\int_\Omega H_\Omega(z,w)(z-w)\phi(z)\overline{\psi(w)}dA(z)dA(w) =$$

$$\frac{1}{\pi^2}\int_\Omega\int_\Omega H_\Omega(z,w)(z-w)\phi(z)\overline{\partial_{\overline{w}}\hat{\psi}(w)}dA(z)dA(w) =$$

$$-\frac{1}{\pi^2}\int_\Omega\int_\Omega \partial_w(H_\Omega(z,w)(z-w))\phi(z)\overline{\hat{\psi}(w)}dA(z)dA(w)$$

$$-\frac{1}{2i\pi^2}\int_\Omega\int_{\partial\Omega} \frac{E_\Omega(z,w)}{\overline{z}-\overline{w}}\phi(z)\overline{\hat{\psi}(w)}d\overline{w}dA(z)$$

$$= \frac{1}{\pi^2}\int_\Omega\int_\Omega H_\Omega(z,w)\phi(z)\overline{\hat{\psi}(w)}dA(z)dA(w) = \langle \phi, \hat{\psi}\rangle.$$

Thus $T^*\psi = \overline{z}\psi + \hat{\psi}$ on test functions ψ and the claimed formula for T^* follows.

A direct computation using (2.3) now leads to the commutator identity

$$[T, T^*] = \mathbf{1}\otimes\mathbf{1} = \mathbf{1}\langle\cdot, \mathbf{1}\rangle,$$

or equivalently, on elements:

$$([T,T^*]f)(z) = \frac{1}{\pi}\int_\Omega f dA, \quad f \in \mathcal{H}(\Omega).$$

In particular this shows that $[T, T^*] \geq 0$, that is, T is a cohyponormal operator.

Remark also the simple identity:

$$(T - z)k_z(\zeta) = (\zeta - z)\frac{1}{\zeta - z} = \mathbf{1}, \quad z \in \mathbf{C}.$$

Thus we can denote by convention $(T - z)^{-1}\mathbf{1} = k_z$ even for points $z \in \overline{\Omega}$.

By putting together the above computations we can state a partial result.

Proposition 2.1. *Let Ω be a bounded open set with smooth boundary. There exists a canonically associated cohyponormal operator $T \in L(\mathcal{H}(\Omega))$ with rank one self-commutator $[T, T^*] = \mathbf{1} \otimes \mathbf{1}$ and whose localized generalized resolvent factors the exponential transform:*

$$1 - E_\Omega(z, w) = \langle (T - z)^{-1}\mathbf{1}, (T - w)^{-1}\mathbf{1} \rangle, \quad z, w \in \mathbf{C}. \tag{2.4}$$

Originally this decomposition was obtained the other way around, from Hilbert space operators to their functional spectral invariants, see [3, 4, 2, 17]. In the case of an arbitrary bounded open set Ω one can use an exhaustion with smooth domains $\Omega_n \uparrow \Omega$ and prove that the weak operator limits $T_n \to T$, $T_n^* \to T^*$ exist, so that the factorization (2.4) holds for Ω and T. Indeed, since for $\Omega_1 \subset \Omega_2$ the difference $H_{\Omega_1}(z, w) - H_{\Omega_2}(z, w) = (1 - E_{\Omega_2 \setminus \Omega_1}(z, w))H_{\Omega_1}(z, w)$ $(z, w \in \Omega_1)$ is positive semidefinite there is a natural embedding ("extension by zero") $\mathcal{H}(\Omega_1) \to \mathcal{H}(\Omega_2)$ which decreases the norm. This gives good enough monotonicity to pass to the limit for $\Omega_n \uparrow \Omega$.

Alternatively, one can argue as in Section VII.3 of [17]. Namely, for an arbitrary domain Ω, by using the positive semidefiniteness of the kernel $1 - E_\Omega(z, w)$, one introduces the Hermitian form:

$$\langle \phi, \psi \rangle = -\int_{\mathbf{C}}\int_{\mathbf{C}} E_\Omega(z, w)\partial_{\overline{z}}\phi(z)\partial_w\overline{\psi(w)}dA(z)dA(w), \quad \phi, \psi \in \mathcal{D}(\mathbf{C}),$$

and consider on the associated Hilbert space the multiplication operator $T = M_z$. Then a formula for the adjoint as before, and the factorization (2.4) will follow. We do not expand here the details of either proof.

3. Quadrature domains

The Hilbert space factorization (2.4) is particularly simple and relevant for the class of quadrature domains. We explore below some constructive aspects of this relationship between quadrature domains and their associated operators T.

A bounded domain $\Omega \subset \mathbf{C}$ is called a *quadrature domain* if there exists a distribution $u \in \mathcal{E}'(\Omega)$ with finite support in Ω satisfying:

$$\int_\Omega h dA = u(h), \quad h \in AL^1(\Omega, dA),$$

where the latter means the space of all integrable analytic functions in Ω.

For instance a disk is a quadrature domain, due to Gauss' mean value property. By abuse of terminology we will accept non-connected open sets Ω carrying such a quadrature identity and still call them quadrature domains. Then a finite disjoint union of disks is also a quadrature domain.

The pioneering work [1] of Aharonov and Shapiro can be considered as the formal birth place of quadrature domains. Since then the study of quadrature domains has achieved maturity; many unexpected ramifications to different fields of pure and applied mathematics were discovered in the last decades. The monograph [26] treats in a unifying format part of these applications of quadrature domains.

The reader can also consult [5, 9, 25, 28]. In the sequel, as a direct continuation of the articles [21, 11, 22], we confine ourselves to investigate quadrature domains and the factorization (2.4) of their exponential transform. As shown elsewhere [8, 23] this study can be motivated by image reconstruction problems.

We recall that quadrature domains have real algebraic boundaries, with a limited variety of possible singular points.

To fix ideas we consider a quadrature domain Ω with distinct quadrature nodes $a_1, \dots a_n \in \Omega$ and corresponding weights $c_1, \dots, c_n \in \mathbf{C}$:

$$\int_\Omega h dA = c_1 h(a_1) + \dots + c_n h(a_n), \quad h \in AL^1(\Omega, dA). \tag{3.1}$$

For fixed $z \in \overline{\Omega}^c$, $v \in \Omega$ the function $u \mapsto \frac{H(u,v)}{u-z}$ is in $AL^1(\Omega, dA)$ (cf. Appendix, Section 5), and similarly with respect to v. Therefore formula (1.6) (or (2.2)) becomes:

$$1 - E_\Omega(z,w) = \frac{1}{\pi^2} \sum_{i,j=1}^n \frac{c_i \overline{c_j} H(a_i, a_j)}{(z - a_i)(\overline{w} - \overline{a_j})}, \quad z, w \in \overline{\Omega}^c. \tag{3.2}$$

According to (2.4), the function $\langle (T-z)^{-1}\mathbf{1}, (T-w)^{-1}\mathbf{1}\rangle$ is then rational for z, w exterior to the closure of Ω. Thus the Hilbert subspace $K = \vee_{j=0}^\infty T^j \mathbf{1}$ is finite dimensional and invariant under the operator T. Let $A \in L(K)$ be the restriction of T to this subspace, so that $A^* = P_K T^*|_K$, where P_K denotes the orthogonal projection onto K. In view of these observations we obtain:

$$\langle (A-z)^{-1}\mathbf{1}, (A-w)^{-1}\mathbf{1}\rangle = \frac{1}{\pi^2} \sum_{i,j=1}^n \frac{c_i \overline{c_j} H(a_i, a_j)}{(z - a_i)(\overline{w} - \overline{a_j})}, \quad z, w \in \mathbf{C}. \tag{3.3}$$

Let

$$P(z) = (z - a_1) \dots (z - a_n)$$

be the monic polynomial of degree n vanishing at the quadrature nodes. It is easy to see from the preceding identities that the matrix A is cyclic, with $\mathbf{1}$ as a cyclic vector, and $P(z)$ is its minimal polynomial. Moreover $\dim K = n$. It turns out that the polynomial

$$Q(z,w) = P(z)\overline{P(w)} E_\Omega(z,w) = \tag{3.4}$$

$$P(z)\overline{P(w)} - P(z)\overline{P(w)}\langle (A-z)^{-1}\mathbf{1}, (A-w)^{-1}\mathbf{1}\rangle, \quad z, w \in \overline{\Omega}^c$$

has minimal degree among all symmetric polynomials describing Ω as:

$$\Omega \equiv \{z \in \mathbf{C};\ Q(z,z) < 0\},$$

where $\equiv$ means equality up to a finite set. For proofs see [11].

In general, a quadrature domain Ω is not determined by its quadrature data $(a_1, \dots, a_n; c_1, \dots, c_n)$. On the other hand the matrix A with the distinguished cyclic vector $\mathbf{1}$ do determine Ω. On this ground, the correspondence

$$(a_1, \dots, a_n; c_1, \dots, c_n) \mapsto (A, \mathbf{1}),$$

when well defined, is fundamental in understanding this break of uniqueness. Section 4 of the paper is devoted to the constructive aspects of the latter correspondence in the very particular case of a disjoint union of disks. Arguably, based on fluid mechanics interpretations, the disjoint unions of disks generate by a natural expansion process all quadrature domains (having positive weights).

As a preparation for this we consider a consequence of Theorem 1.1 which might be of independent interest.

Lemma 3.1. *Let $D_i = D(a_i, r_i),\ 1 \le i \le n$, be disjoint disks and let*

$$Q(z,w) = \prod_{i=1}^{n} [(z-a_i)(\overline{w}-\overline{a_i}) - r_i^2],$$

be the polarized equation defining their union. Then the matrix $(-Q(a_i,a_j))_{i,j=1}^n$ is positive definite.

Proof. Let $\Omega = \cup_{i=1}^n D(a_i, r_i)$. Since the union is disjoint, Ω is a quadrature domain with nodes at $a_1, a_2, \dots, a_n$. Let $P(z)$ be the monic polynomial vanishing at these points.

For large values of $|z|, |w|$, due to the multiplicativity of the exponential transform we find:

$$E_\Omega(z,w) = \prod_{i=1}^{n} E_{D_i}(z,w) = \prod_{i=1}^{n} [1 - \frac{r_i^2}{(z-a_i)(\overline{w}-\overline{a_i})}].$$

Thus we see directly in this case that

$$Q(z,w) = P(z)\overline{P(w)}E_\Omega(z,w).$$

Using (3.2) gives

$$Q(z,w) = P(z)\overline{P(w)} - \frac{1}{\pi^2}\sum_{i,j=1}^{n} \frac{c_i P(z)\overline{c_j}\overline{P(w)} H_\Omega(a_i,a_j)}{(z-a_i)(\overline{w}-\overline{a_j})}.$$

Hence

$$Q(a_i,a_j) = -\frac{1}{\pi^2} P'(a_i) c_i H_\Omega(a_i,a_j) \overline{c_j}\overline{P'(a_j)},$$

which is negative definite by Theorem 1.1. □

It would be interesting to find an elementary proof for Lemma 3.1. For small values of n it is certainly possible to check everything directly (see Example 2 below for the case $n = 2$), but for general n it becomes messy. Anyhow we notice that the above proof works (using (3.4)) for any quadrature domain as in (3.1). We proceed to prove a more general statement.

Let Ω be any quadrature domain as in (3.1). We keep the previous notation and consider for any $w \in \mathbf{C}$ the n solutions $z_1, \dots, z_n$ (some of which could coincide) of

$$Q(z_j, w) = 0. \tag{3.5}$$

For $w=\infty$ we have, with a natural projective interpretation of (3.5) in that case, $z_j = a_j$ (up to a permutation) and for any $w \in \overline{\Omega}^c$ it is known that $z_1, \dots, z_n \in \Omega$ (see [26], Theorem 5.2, for example). The following theorem can be viewed as a strengthened form of that fact, and simultaneously as a generalization of Lemma 3.1.

Theorem 3.2. *The matrix*

$$(-Q(z_i, z_j))_{i,j=1}^n$$

is positive definite for any $w \in \overline{\Omega}^c$ (including $w = \infty$).

Proof. The case $w=\infty$ works exactly as in Lemma 3.1, so we assume from now on that $w \in \mathbf{C} \setminus \overline{\Omega}$. By (3.4),

$$Q(w,w) = |P(w)|^2 (1 - \langle (A-w)^{-1}\mathbf{1}, (A-w)^{-1}\mathbf{1} \rangle)$$

for $w \in \overline{\Omega}^c$. Being an identity between rational functions (see (3.3)) the relation remains valid everywhere. It follows that the assumption $w \notin \overline{\Omega}$ (i.e., $Q(w,w) > 0$) means that

$$\|(A-w)^{-1}\mathbf{1}\| < 1 \tag{3.6}$$

and that the definition (3.5) of $z_1, \dots, z_n$ can be written

$$\langle (A-z_j)^{-1}\mathbf{1}, (A-w)^{-1}\mathbf{1} \rangle = 1.$$

Thus for any complex numbers $t_1, \dots, t_n$:

$$\langle \sum_{j=1}^n t_j (A-z_j)^{-1}\mathbf{1}, (A-w)^{-1}\mathbf{1} \rangle = \sum_{j=1}^n t_j$$

so that

$$|\sum_{j=1}^n t_j| \le \|\sum_{j=1}^n t_j (A-z_j)^{-1}\mathbf{1}\| \cdot \|(A-w)^{-1}\mathbf{1}\| \le \|\sum_{j=1}^n t_j (A-z_j)^{-1}\mathbf{1}\|.$$

By (3.6) the last inequality is strict unless the right member is zero. For any $\lambda_1, \dots, \lambda_n$ we get, setting $t_i = P(z_i)\lambda_i$:

$$\sum_{i,j=1}^n Q(z_i, z_j)\lambda_i \overline{\lambda_j} = \sum_{i,j=1}^n P(z_i)\lambda_i \overline{P(z_j)\lambda_j} (1 - \langle (A-z_i)^{-1}\mathbf{1}, (A-z_j)^{-1}\mathbf{1} \rangle$$

$$= |\sum_{i=1}^n t_i|^2 - \|\sum_{i=1}^n t_i (A-z_i)^{-1}\mathbf{1}\|^2 \le 0,$$

proving the positive semidefinitenes of $(-Q(z_i, z_j))_{i,j=1}^n$.

To show that $(-Q(z_i, z_j))_{i,j=1}^n$ is actually definite assume there is equality in the last inequality. In view of the comment after the previous inequality we then have

$$\|\sum_{i=1}^n t_i (A-z_i)^{-1}\mathbf{1}\| = 0. \tag{3.7}$$

Since we assumed in the beginning that $w \neq \infty$, the points $\{z_i\}$ are not the nodes $\{a_i\}$ of the quadrature identity. Therefore $P(z_i) \neq 0$, so in order to show that $\lambda_1 = \cdots = \lambda_n = 0$ it is enough to show that $t_1 = \cdots = t_n = 0$.

So assume that the t_j are not all zero. Then (3.7) says that the vectors $(A - z_j)^{-1}\mathbf{1}$, $j = 1, \ldots, n$, are linearly dependent. But it follows from the detailed analysis carried out in section 4 of [11] that this is not the case. Indeed, it was shown that the map $z \mapsto (A - z)^{-1}\mathbf{1}$, regarded as rational map from $\mathbf{C}$ to $\mathbf{C}^n$ (or between the corresponding projective spaces), is linearly equivalent to the Veronese embedding $z \mapsto (z, z^2, \ldots, z^n)$, for which the corresponding linear independence is well-known (it amounts to the nonvanishing of a Vandermonde determinant). This finishes the proof.

□

Remark 2. Let $S(z)$ be the algebraic function associated to $Q(z, w)$, i.e., the function defined by

$$Q(z, \overline{S(z)}) = 0, \ z \in \mathbf{C}.$$

Since $Q(z, z) = 0$ on $\partial\Omega$ one of the branches of $S(z)$ satisfies $S(z) = \overline{z}$ on $\partial\Omega$, hence this branch is the Schwarz function [6], [26] of $\partial\Omega$. The definition (3.5) of z_j in terms of w now says that $\overline{S(z_j)} = w$, i.e., that

$$S^{-1}(\overline{w}) = \{z_1, \ldots, z_n\}.$$

Therefore Theorem 3.2 can be conveniently expressed as saying that

$$-Q(S^{-1}(\overline{w}), S^{-1}(\overline{w})) > 0$$

(positive definite) for every $w \notin \overline{\Omega}$.

Example 2. Consider the union Ω of two discs $D_i = D(a_i, r_i)$ $(i = 1, 2)$. When the discs are disjoint we have, keeping the notation from Lemma 3.1 and thereafter,

$$1 - E_\Omega(z, w) = 1 - E_{D_1}(z, w)E_{D_2}(z, w) = \frac{r_1^2}{(z - a_1)(\overline{w} - \overline{a}_1)} +$$
$$+ \frac{r_2^2}{(z - a_2)(\overline{w} - \overline{a}_2)} - \frac{r_1^2 r_2^2}{(z - a_1)(z - a_2)(\overline{w} - \overline{a}_1)(\overline{w} - \overline{a}_2)}$$

for large $|z|$ and $|w|$.

This function is positive semidefinite by Theorem 1.1. It is an interesting fact that it remains positive semidefinite even if the discs overlap a little. Indeed, a straightforward calculation (which we omit) shows that the function $1 - E_{D_1}(z, w)E_{D_2}(z, w)$ is positive semidefinite if and only if

$$r_1^2 + r_2^2 \leq |a_1 - a_2|^2. \tag{3.8}$$

Similarly, with $Q(z, w) = ((z - a_1)(\overline{w} - \overline{a_1}) - r_1^2)((z - a_2)(\overline{w} - \overline{a_2}) - r_2^2)$ the matrix $(-Q(a_i, a_j))$ is positive semidefinite if and only if (3.8) holds. On the other hand, turning to Theorem 3.2, the matrix $(-Q(z_i, z_j))$ will not be positive semidefinite for all choices of $w \in \overline{\Omega}^c$ if the discs overlap. Indeed, in case $D_1 \cap D_2 \neq$

$\emptyset$ we can choose w so that $z_1 \in D_1 \cap D_2$. Then $Q(z_1, z_1) > 0$ and therefore $\sum_{i,j=1}^{2}(-Q(z_i, z_j))\lambda_i\overline{\lambda}_j < 0$ with $\lambda_1 = 1$, $\lambda_2 = 0$.

It also turns out that the induction process to be performed in Section 4 will be destroyed if overlappings are allowed: if $D_1 \cap D_2 \neq \emptyset$ with $1 - E_{D_1}(z,w)E_{D_2}(z,w)$ positive semidefinite then adding a third disc D_3, disjoint from D_1 and D_2, $1 - E_{D_1}(z,w)E_{D_2}(z,w)E_{D_3}(z,w)$ will not always be positive semidefinite.

Finally in this section we wish to make Theorem 1.1 a little more precise. We shall then use the word quadrature domain in its full sense, i.e., we shall allow in the quadrature identity (3.1) also derivatives of h in the right member. For simplicity we keep the notation (3.1) however, thinking of a repeated occurrence of a node a_i as representing a derivative at a_i.

Proposition 3.3. *Let Ω be a bounded planar open set with $\partial\Omega$ smooth. The following statements are equivalent.*

a) Ω is a quadrature domain.

b) $1 - E_\Omega(z, w)$ is not positive definite outside $\overline{\Omega}$ (only semidefinite).

c) There exists a polynomial $p(z)$, not identically zero, such that

$$\int_\Omega \int_\Omega H(z,w)p(z)\overline{p(w)}\, dA(z)dA(w) = 0$$

(i.e., such that $p = 0$ as an element of the Hilbert space $\mathcal{H}(\Omega)$).

Proof. *a)* $\Rightarrow$ *b)*: This follows easily from the representation (3.2) of $1 - E(z,w)$ as a *finite* sum when Ω is a quadrature domain.

b) $\Rightarrow$ *c)*: That $1 - E(z,w)$ is only semidefinite means in view of (2.2) that there exists a rational function

$$R(z) = \sum_{i=1}^{n} \frac{\lambda_i}{z - a_i} = \sum_{i=1}^{n} \lambda_i k_{a_i}(z)$$

($a_i \in \overline{\Omega}^c$), not identically zero, so that

$$\begin{aligned}\langle R, R\rangle &= \frac{1}{\pi^2}\sum_{i,j=1}^{n} \int_\Omega \int_\Omega H(u,v)\frac{\lambda_i}{u - a_i}\frac{\overline{\lambda_j}}{\overline{v} - \overline{a_j}}\, dA(u)dA(v) \\ &= \sum_{i,j=1}^{n} (1 - E(a_i, a_j))\lambda_i\overline{\lambda}_j = 0.\end{aligned}$$

Let $P(z) = \prod_{i=1}^{n}(z - a_i)$. The multiplication operator $h(z) \mapsto P(z)h(z)$ is a bounded linear operator $\mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ because it is a linear combination of repeated uses of the operator T in Section 2 (indeed, it is $P(T)$). Thus from $\langle R, R\rangle = 0$ follows $\langle PR, PR\rangle = 0$, and since PR is a polynomial this is exactly the assertion of *c*).

$c) \Rightarrow a)$: From $\langle p, p \rangle = 0$ (with p a polynomial) it follows that $\langle hp, hp \rangle = 0$ for any function h which is analytic in a neighbourhood of $\overline{\Omega}$. To see this one may e.g. repeat the calculation of the adjoint of T to obtain

$$\langle h\phi, \psi \rangle - \langle \phi, \overline{h}\psi \rangle = \langle \phi, \overline{h'}\hat{\psi} \rangle$$

for say $\phi, \psi \in C^\infty(\overline{\Omega})$. Choosing here $\psi = h\phi$ and using the Cauchy-Schwarz inequality gives

$$\langle h\phi, h\phi \rangle \leq C\langle \phi, \phi \rangle$$

Now take $\phi = p$.

From $\langle hp, hp \rangle = 0$ we deduce, using (2.3), that

$$\int_\Omega hp \, dA = \langle hp, \mathbf{1} \rangle = 0.$$

Having such a relation holding for all h analytic in a neighbourhood of $\overline{\Omega}$ easily implies an identity (3.1), with the a_i being the zeros of p.

□

4. Adding an external disc

We consider the same disjoint union of disks $\Omega_n = \cup_{i=1}^n D(a_i, r_i)$ as in Lemma 3.1, to which we add a new disjoint disk; let $\Omega_{n+1} = \cup_{i=1}^{n+1} D(a_i, r_i)$ be the enlarged set. At each stage we have a finite dimensional Hilbert space K, a cyclic vector $\mathbf{1} \in K$ and an operator $A \in L(K)$ as after (3.2). In terms of matricial representations we write, at stage k:

$$E_{\Omega_k}(z, w) = 1 - \langle (A_k - z)^{-1}\xi_k, (A_k - w)^{-1}\xi_k \rangle, \ |z|, |w| \gg 1,$$

where $A_k \in L(K_k)$ has cyclic vector ξ_k and $\dim K_k = k, \ k = n, n+1$.

Our aim is to understand the structure of the matrix A_{n+1} and its cyclic vector ξ_{n+1} as functions of the previous data (A_n, ξ_n) and the new disk $D(a_{n+1}, r_{n+1})$. Henceforth we assume that the closed disks $\overline{D(a_i, r_i)}$ are still disjoint. In order to simplify notation we suppress for a while the index $n+1$, so that $a = a_{n+1}, \ r = r_{n+1}, \ \xi = \xi_{n+1}, \ A = A_{n+1}$. The following computations are based on standard realization techniques in linear systems theory, see for instance [7].

Due to the multiplicativity of the external exponential transform for disjoint domains we find:

$$[1 - \langle (A_n - z)^{-1}\xi_n, (A_n - w)^{-1}\xi_n \rangle][1 - \frac{r^2}{(z-a)(\overline{w} - \overline{a})}] =$$
$$1 - \langle (A - z)^{-1}\xi, (A - w)^{-1}\xi \rangle.$$

Equivalently,

$$\langle (A_n - z)^{-1}\xi_n, (A_n - w)^{-1}\xi_n \rangle + \frac{r^2}{(z-a)(\overline{w} - \overline{a})} =$$
$$\langle \frac{r}{z-a}(A_n - z)^{-1}\xi_n, \frac{r}{w-a}(A_n - w)^{-1}\xi_n \rangle + \langle (A - z)^{-1}\xi, (A - w)^{-1}\xi \rangle.$$

Thus, for each z avoiding the poles, the norm of the vector

$$f(z) = \begin{pmatrix} (A_n - z)^{-1}\xi_n \\ \frac{r}{z-a} \end{pmatrix} \in K_n \oplus \mathbf{C}$$

equals that of the vector

$$g(z) = \begin{pmatrix} \frac{r}{z-a}(A_n - z)^{-1}\xi_n \\ (A - z)^{-1}\xi \end{pmatrix} \in K_n \oplus K.$$

And moreover, the same is true for any linear combination

$$\|\lambda_1 f(z_1) + \ldots + \lambda_r f(z_r)\| = \|\lambda_1 g(z_1) + \ldots + \lambda_r g(z_r)\|.$$

Because the span of $f(z),\ z \in \mathbf{C}$, is the whole space $K_n \oplus \mathbf{C}$, there exists a unique isometric linear operator $V : K_n \oplus \mathbf{C} \longrightarrow K_n \oplus K$ mapping $f(z)$ to $g(z)$. We write, corresponding to the two direct sum decompositions

$$V = \begin{pmatrix} B & \beta \\ C & \gamma \end{pmatrix},$$

where $B : K_n \longrightarrow K_n,\ \beta \in K_n,\ C : K_n \longrightarrow K,\ \gamma \in K$. Since $Vf(z) = g(z)$ for all z, we find by coefficient identification:

$$B = r(A_n - a)^{-1}, \quad \beta = (A_n - a)^{-1}\xi_n.$$

The isometry condition $V^*V = I$ written at the level of the above 2×2 matrix yields the identities:

$$\begin{cases} r^2(A_n^* - \overline{a})^{-1}(A_n - a)^{-1} + C^*C = I, \\ r(A_n^* - \overline{a})^{-1}(A_n - a)^{-1}\xi_n + C^*\gamma = 0, \\ \|(A_n - a)^{-1}\xi_n\|^2 + \|\gamma\|^2 = 1. \end{cases} \tag{4.1}$$

In particular we deduce that $(A_n^* - \overline{a})^{-1}(A_n - a)^{-1} \leq r^{-2}$ and since this operator inequality is valid for every radius which makes the disks disjoint, we can enlarge slightly r and still have the same inequality. Thus, the defect operator

$$\Delta = [I - r^2(A_n^* - \overline{a})^{-1}(A_n - a)^{-1}]^{1/2} : K_n \longrightarrow K_n \tag{4.2}$$

is strictly positive.

The identity $C^*C = \Delta^2$ shows that the polar decomposition of the matrix $C = U\Delta$ defines without ambiguity an isometric operator $U : K_n \longrightarrow K$. Since $\dim K = \dim K_n + 1$ we will identify $K = K_n \oplus \mathbf{C}$, so that the map U becomes the natural embedding of K_n into the first factor. Thus the second line of the isometry V becomes

$$(C \quad \gamma) = \begin{pmatrix} \Delta & d \\ 0 & \delta \end{pmatrix} : K_n \oplus \mathbf{C} \longrightarrow K_n \oplus \mathbf{C} = K,$$

where $d \in K_n,\ \delta \in \mathbf{C}$. We still have the freedom of a rotation of the last factor, and can assume $\delta \geq 0$. One more time, equations (4.1) imply

$$\begin{cases} d = \frac{1}{r}(\Delta\xi_n - \Delta^{-1}\xi_n), \\ \delta = [1 - \|(A_n - a)^{-1}\xi_n\|^2 - \|d\|^2]^{1/2}. \end{cases} \tag{4.3}$$

From relation $Vf(z) = g(z)$ we deduce:

$$\begin{pmatrix} \Delta & d \\ 0 & \delta \end{pmatrix} \begin{pmatrix} (A_n - z)^{-1}\xi_n \\ \frac{r}{z-a} \end{pmatrix} = (A-z)^{-1}\xi.$$

This shows that $\delta > 0$ because the operator A has the point a in its spectrum.

At this point straightforward matrix computations lead to the following exact description of the couple $(A, \xi) = (A_{n+1}, \xi_{n+1})$:

$$A = \begin{pmatrix} \Delta A_n \Delta^{-1} & -\delta^{-1}\Delta(A_n - a)\Delta^{-1}d \\ 0 & a \end{pmatrix}, \quad \xi = \begin{pmatrix} \Delta^{-1}\xi_n \\ -\delta r \end{pmatrix}. \tag{4.4}$$

It is sufficient to verify these formulas, that is:

$$\begin{pmatrix} \Delta(A_n - z)\Delta^{-1} & -\delta^{-1}\Delta(A_n - a)\Delta^{-1}d \\ 0 & a - z \end{pmatrix} \begin{pmatrix} \Delta & d \\ 0 & \delta \end{pmatrix} \begin{pmatrix} (A_n - z)^{-1}\xi_n \\ \frac{r}{z-a} \end{pmatrix} =$$

$$\begin{pmatrix} \Delta^{-1}\xi_n \\ -\delta r \end{pmatrix}.$$

And this is done by direct multiplication:

$$\Delta\xi_n + \Delta(A_n - z)\Delta^{-1}\frac{rd}{z-a} - \Delta(A_n - a)\Delta^{-1}\frac{rd}{z-a} = \Delta^{-1}\xi_n,$$

which is equivalent to the known relation $dr = \Delta\xi_n - \Delta^{-1}\xi_n$.

Summing up, we can formulate the transition laws of the linear data of a disjoint union of disks.

Proposition 4.1. *Let $\overline{D(a_i, r_i)}, 1 \le i \le n+1$, be a disjoint family of closed disks, and let $\Omega_k = \cup_{i=1}^k D(a_i, r_i),\ 1 \le k \le n+1$.*

The linear data (A_k, ξ_k) of the quadrature domain Ω_k can be inductively obtained by the formula (4.4)*, with the aid of the definitions* (4.2), (4.3)*.*

We remark that letting $n \to \infty$ with $r = r_{n+1} \to 0$ we obtain $\Delta \to I$ and $d \to 0$, which is consistent with the fact that Ω_{n+1} converges in measure to a bounded limit domain Ω. Moreover, in this case the vectors ξ_{n+1} will converge to a vector ξ and A_{n+1} will converge in the weak operator topology to a bounded operator A, namely the ones factoring $1 - E_\Omega$:

$$E_\Omega(z, w) = 1 - \langle (A-z)^{-1}\xi, (A-w)^{-1}\xi \rangle,\ |z|, |w| \gg 1.$$

5. Appendix on integrability of the interior exponential transform

The representation formula (1.6) (or (2.2)) is crucial for the whole theory. It depends on the distributional identity (1.3) together with the definition (1.4) of $H(z,w)$. It is desirable that $H(z,w)$ is integrable over $\Omega\times\Omega$ because then the right member of (1.3) makes immediate sense as a distribution and there is no question about the meaning of (1.6) (the right member will be a convolution between distributions).

Thus setting

$$\|H\|_p = (\int_\Omega\int_\Omega |H(z,w)|^p\, dA(z)dA(w))^{1/p}, \tag{5.1}$$

for $0<p<\infty$, we would like to have at least that $\|H\|_1<\infty$. We do not know whether this is always the case but here are at least some partial results.

Lemma 5.1. *Let Ω be a bounded planar open set. Then*

a) $\|H\|_p<\infty$ for all $p<1$.

b) If $\partial\Omega$ is Lipschitz then $\|H\|_p<\infty$ for all $p<3/2$.

c) If $\partial\Omega$ is smooth real analytic (or if Ω is a quadrature domain) then $\|H\|_p<\infty$ for all $p<3$ (but not for $p=3$).

Proof. Let for $t\geq 0$

$$m(t) = |\{(z,w)\in\Omega\times\Omega : |H(z,w)|>t\}|$$

be the distribution function of H ($|\dots|$ here denotes Lebesgue measure in $\mathbf{C}\times\mathbf{C}$). Then

$$\|H\|_p^p = -\int_0^\infty t^p\, dm(t).$$

The integral from zero to one is certainly finite since Ω is bounded, so $\|H\|_p$ will be finite if and only if $-\int_1^\infty t^p\, dm(t)<\infty$.

For $z\in\Omega$, let $d(z)$ denote the distance from z to Ω^c, and for $\delta>0$ let

$$f(\delta) = |\{(z\in\Omega : d(z)<\delta\}|.$$

Thus $f(\delta)$ is the area of a δ-neighbourhood of $\partial\Omega$. It was shown in [10] (Lemma 2.4 and Lemma 2.5) that

$$|E(z,w)|\leq 2\ \ (z,w\in\mathbf{C})$$

and

$$|H(z,w)|\leq 2\min\{\frac{1}{d(z)^2},\frac{1}{d(w)^2}\}.$$

Combining these estimates gives (with (1.4))

$$|H(z,w)|\leq 2\min\{\frac{1}{d(z)^2},\frac{1}{d(w)^2},\frac{1}{|z-w|^2}\}. \tag{5.2}$$

By (5.2)

$$m(t)\leq M(t),$$

where

$$M(t) = |\{(z,w) \in \Omega \times \Omega : d(z) < \sqrt{\frac{2}{t}},\ |z-w| < \sqrt{\frac{2}{t}}\}|.$$

Clearly

$$M(t) \le f(\sqrt{\frac{2}{t}}) \cdot \pi(\sqrt{\frac{2}{t}})^2 = \frac{2\pi}{t} f(\sqrt{\frac{2}{t}}).$$

Since t^p is an increasing function of t the above inequalities imply

$$-\int_1^\infty t^p\, dm(t) \le -\int_1^\infty t^p\, dM(t) \le -2\pi \int_1^\infty t^p\, d(\frac{f(\sqrt{\frac{2}{t}})}{t}).$$

We now turn to the particular assertions of the lemma. Since $f(\delta) \le C < \infty$ for all δ we have, for $0 < p < 1$,

$$-\int_1^\infty t^p\, d(\frac{f(\sqrt{\frac{2}{t}})}{t}) \le -C\int_1^\infty t^p\, d(\frac{1}{t})) = C\int_1^\infty t^{p-2}\, dt < \infty,$$

proving a).

If $\partial\Omega$ is Lipschitz we have $f(\delta) \le C\delta$, which gives

$$-\int_1^\infty t^p\, d(\frac{f(\sqrt{\frac{2}{t}})}{t}) \le -C\int_1^\infty t^p\, d(\frac{1}{t^{3/2}})) = C\int_1^\infty t^{p-5/2}\, dt < \infty$$

for all $p < 3/2$, proving b).

To prove c) we need a better estimate of $H(z,w)$. What we have when $\partial\Omega$ is analytic is essentially (5.2) without the squares, namely:

$$|H(z,w)| \le C \min\{\frac{1}{d(z)}, \frac{1}{d(w)}, \frac{1}{|z-w|}\}. \tag{5.3}$$

Assuming this for a moment and inserting it in the estimate of $M(t)$ above gives

$$M(t) \le \frac{\pi C^2}{t^2} f(\frac{C}{t}).$$

Thus, still using $f(\delta) \le C\delta$,

$$-\int_1^\infty t^p\, dm(t) \le -C\int_1^\infty t^p\, d(\frac{f(\frac{C}{t})}{t^2}) \le C\int_1 t^{p-4}\, dt < \infty$$

when $p < 3$. For $p = 3$ it is easy to check that $\|H\|_p = +\infty$ even for the unit disc.

It remains to prove (5.3) when $\partial\Omega$ is analytic. Let $S(z)$ be the Schwarz function of $\partial\Omega$, so that $S(z)$ is analytic in a neighbourhood of $\partial\Omega$ and satisfies $S(z) = \overline{z}$ on $\partial\Omega$. When $\partial\Omega$ is smooth real analytic the exponential transform has an analytic/antianalytic continuation from the exterior of Ω across $\partial\Omega$, see [10]. This means that there exists a function $F(z,w)$ analytic/antianalytic in a neighbourhood of $\partial\Omega \times \partial\Omega$ such that $F(z,w) = E(z,w)$ for $z, w \in \Omega^c$.

Inside Ω (but close to $\partial\Omega$) we have

$$F(z,w) = (z - \overline{S(w)})(S(z) - \overline{w})H(z,w)$$

(see [10]). We shall use this to estimate $H(z,w)$. We immediately get

$$H(z,w)=\frac{F(z,w)}{(z-\overline{S(w)})(S(z)-\overline{w})}. \tag{5.4}$$

As $E(z,z)$ vanishes on $\partial\Omega$ so does $F(z,z)$:

$$F(z,z)=0,\ z\in\partial\Omega.$$

Since also $S(z)-\overline{z}=0$ on $\partial\Omega$ it follows that $F(z,w)$ contains $S(z)-\overline{w}$ as a factor. Hence one of the factors in the denominator of (5.4) cancels and we get

$$H(z,w)=\frac{G(z,w)}{z-\overline{S(w)}}, \tag{5.5}$$

where also $G(z,w)$ is analytic/antinalytic in a neighbourhood of $\partial\Omega\times\partial\Omega$. In particular we get the estimate

$$|H(z,w)|\leq\frac{C}{|z-\overline{S(w)}|} \tag{5.6}$$

for $z,w\in\Omega$ close to $\partial\Omega$.

It remains to notice that the estimate (5.6) is equivalent to (5.3). Indeed, $\overline{S(w)}$ is the conformally reflected point of $w\in\Omega$, so $d(w)$ is comparable to $\frac{1}{2}|w-\overline{S(w)}|$, and in addition $|z-w|\leq C|z-\overline{S(w)}|$ (for points $z,w\in\Omega$ close enough to $\partial\Omega$). Now one can pass between (5.6) and (5.3) by using triangle inequalities.

In case Ω is a quadrature domain $\partial\Omega$ is analytic but there may be singular points. However it turns out that these singularities go the right way so that $H(z,w)$ will actually be less singular than at smooth points.

To be a little more precise, when Ω is a quadrature domain $F(z,w)$ is the rational function

$$F(z,w)=\frac{Q(z,w)}{P(z)\overline{P(w)}}$$

(in the notation of Section 3), the Schwarz function $S(z)$ in (5.4) is meromorphic in all Ω and it is one of the branches of the algebraic function defined by the polynomial $Q(z,w)$ (i.e., $Q(z,\overline{S(z)})=0$ identically). Hence $F(z,w)$ still contains $S(z)-\overline{w}$ as a factor and one obtains (5.5).

What happens at a singular point is roughly speaking that $F(z,w)$ contains one more factor $S(z)-\overline{w}$ (for a different branch of $S(z)$). Just think of the simplest example: the touching point of two touching discs. More precisely we have that $G(z_0,z_0)=0$ when $z_0\in\partial\Omega$ is singular, and this really improves the behaviour of $H(z,w)$. Therefore (5.3) remains valid.

□

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