ON QUADRATURE DOMAINS AND AN INVERSE PROBLEM IN POTENTIAL THEORY

By

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0. Introduction and background

This paper is a result of the renewed interest in the uniqueness question for so-called quadrature domains (see below) I got by reading a paper [Z] by Lawrence Zalcman. That paper made me aware of the fact that this uniqueness question is basically the same as a long-standing open problem in potential theory (in the Soviet Union called "the (exterior) inverse problem of potential theory" according to Zalcman).

I have not solved any of the problems, but in my efforts on the one hand to construct counterexamples and on the other hand to obtain positive results (the latter primarily by trying to understand Sakai's results on the uniqueness question in [Sa 2]) at least some kind of progress has been made. For example, by introducing a certain partial order on domains in $\mathbf{R}^N$, new insight into the structure of the classes of quadrature domains is obtained; and this gives interesting results for the uniqueness question. The relation to the inverse problem in potential theory also gives a new motivation for the study of quadrature domains.

To set up the problems, let us start from the potential theoretic point of view. If μ is a Radon measure with compact support in $\mathbf{R}^N$ ($\mu \in M_c$), we denote its Newtonian potential by U^μ, i.e. $U^\mu = E * \mu$ where

$$E(x) = \begin{cases} -\dfrac{1}{2\pi}\log|x| & \text{if } N = 2, \\[2ex] \dfrac{C_N}{|x|^{N-2}} & \text{if } N \geqq 3, \end{cases} \tag{0.1}$$

so that $-\Delta U^\mu = \mu$. If $\mu = \chi_\Omega m$, where $\Omega \subset \mathbf{R}^N$ and m denotes Lebesgue measure, we write U^Ω instead of U^μ. Thus U^Ω is the Newtonian potential produced by Ω considered as a body with density one.

The aforesaid inverse problem in potential theory is (see [I], [Z] for history and references):

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(P) Do there exist two different solid domains Ω_1 and Ω_2 in $\mathbf{R}^N$ such that $U^{\Omega_1} = U^{\Omega_2}$ in all $\mathbf{R}^N \setminus (\Omega_1 \cup \Omega_2)$?

By a "solid" domain we mean a domain without cavities. To be precise, let us, for the rest of the paper, call a domain or open set Ω *solid* if Ω is bounded, $\Omega^e = \mathbf{R}^N \setminus \bar{\Omega}$ is connected and $\partial\Omega = \partial\Omega^e$.

Simple examples (e.g., Example 1.2 below) show that, without the assumption that both of Ω_1 and Ω_2 are solid, the answer of (P) is "yes". (There are even examples with Ω_1 a ball and Ω_2 a domain for which $\Omega_2^c = \mathbf{R}^N \setminus \Omega_2$ is connected (but Ω_2^e disconnected) [Sa 6]). Less trivial examples (e.g. [Sa 1]; see also [Z]) show that the answer would be "yes" also if the equality $U^{\Omega_1} = U^{\Omega_2}$ were required to hold only in a neighbourhood of infinity or, equivalently, throughout the unbounded component of $(\Omega_1 \cup \Omega_2)^e$.

Now assume that Ω_1 and Ω_2 are two bounded open sets, not necessarily solid, such that

$$U^{\Omega_1} = U^{\Omega_2} \qquad \text{on } (\Omega_1 \cup \Omega_2)^c. \tag{0.2}$$

Then we can define a new function U in $\mathbf{R}^N$ by

$$U = \begin{cases} U^{\Omega_1} & \text{on } \Omega_1^c, \\ U^{\Omega_2} & \text{on } \Omega_2^c, \\ \text{"arbitrary"} & \text{on } \Omega_1 \cap \Omega_2. \end{cases}$$

By (0.2) this definition is consistent on $\Omega_1^c \cap \Omega_2^c$. Under suitable additional assumptions, the definition of U in $\Omega_1 \cap \Omega_2$ can be chosen such that $-\Delta U \in L^\infty(\mathbf{R}^N)$. Setting $\mu = -\Delta U$ (regarded as a measure), we have $U = U^\mu$ (since U has the behaviour of a potential at infinity); and it follows that

$$U^{\Omega_j} = U^\mu \qquad \text{on } \Omega_j^c$$

($j = 1, 2$) and (if $\partial\Omega_j = \partial(\Omega_j^e)$)

$$\nabla U^{\Omega_j} = \nabla U^\mu \qquad \text{on } \Omega_j^c.$$

In general, if $\mu \in L^\infty(\mathbf{R}^N)$ has compact support (we then write $\mu \in L_c^\infty$), a bounded open set $\Omega \subset \mathbf{R}^N$ is called a quadrature domain for μ (μ is regarded as a measure) with respect to harmonic functions, written $\Omega \in Q(\mu, HL^1)$, if

$$U^\Omega = U^\mu \qquad \text{on } \Omega^c,$$

$$\nabla U^\Omega = \nabla U^\mu \qquad \text{on } \Omega^c.$$

Thus we see that (at least under some additional assumptions) the relation (0.2) implies $\Omega_1, \Omega_2 \in Q(\mu, HL^1)$ for a suitable $\mu \in L_c^\infty$. Conversely, $\Omega_1, \Omega_2 \in Q(\mu, HL^1)$ clearly implies (0.2).

It follows that problem (P) is essentially equivalent to the uniqueness question for quadrature domains:

(Q) Can $Q(\mu, HL^1)$ contain two different solid domains?

One advantage of question (Q) over (P) is that, as it turns out, the relation between Ω_1 and Ω_2 defined by (0.2) is not an equivalence relation, whereas of course being a member in $Q(\mu, HL^1)$ defines an equivalence relation. Therefore, one natural way to try to make progress on (P) is to investigate the general structure of $Q(\mu, HL^1)$ (for every μ).

We also introduce the classes $Q(\mu, AL^1)$ and $Q(\mu, SL^1)$ by saying that $\Omega \in Q(\mu, AL^1)$ iff $\nabla U^\Omega = \nabla U^\mu$ on Ω^c, and $\Omega \in Q(\mu, SL^1)$ iff $U^\Omega = U^\mu$ on Ω^c, $U^\Omega \leqq U^\mu$ everywhere ($\mu \in L_c^\infty$, $\Omega \subset \mathbf{R}^N$ open and bounded). Then $Q(\mu, SL^1) \subset Q(\mu, HL^1) \subset Q(\mu, AL^1)$. Moreover, if Ω is solid then $\Omega \in Q(\mu, AL^1)$ implies $\Omega \in Q(\mu, HL^1)$ so that, in (Q), $Q(\mu, HL^1)$ can be replaced by $Q(\mu, AL^1)$.

The main programme for this paper is to investigate the structure of the classes $Q(\mu, AL^1)$, $Q(\mu, HL^1)$, $Q(\mu, SL^1)$ with the question (Q) in mind. To this end, we introduce a partial order $\prec$ among bounded sets by saying that $\Omega_1 \prec \Omega_2$ iff $U^{\Omega_1} \geqq U^{\Omega_2}$ everywhere. Then it turns out (Corollary 3.5, Corollary 3.8) that any solid $\Omega \in Q(\mu, AL^1)$ is minimal in $Q(\mu, AL^1)$ with respect to $\prec$. (Likewise for $Q(\mu, HL^1)$.) Further, $D \prec \Omega$ whenever $D \in Q(\mu, HL^1)$, $\Omega \in Q(\mu, SL^1)$ (Corollary 3.4) which in particular proves the (already well-known) fact that $Q(\mu, SL^1)$ contains, up to null sets, at most one element.

This $\Omega \in Q(\mu, SL^1)$, when it exists, can be constructed by different kinds of balayage methods. Two such methods have been elaborated by Sakai [Sa 2], [Sa 3] (to some extent also by myself [Gu 1], for one of the methods); and we here generalize Sakai's results by using a third method, related to Perron's method (Theorem 2.1, Theorem 2.2).

In contrast to $Q(\mu, SL^1)$, $Q(\mu, HL^1)$ may have many members ("most" of which are non-solid). There are examples with $Q(\mu, HL^1)$ even uncountably infinite and with cluster points, but still we do not expect that $Q(\mu, HL^1)$ can contain a whole continuum (in a reasonable topology) of open sets. On the other hand, and this is perhaps our main new result, we prove (Corollary 4.1) that $Q(\mu, AL^1)$ is always "connected", namely that any two open sets in $Q(\mu, AL^1)$ can be deformed into each other within $Q(\mu, AL^1)$. In particular, if the answer to question (Q) turns out to be negative, so that there exist two different solid elements in $Q(\mu, AL^1)$, then these at least can be joined through a family of (non-solid) elements in $Q(\mu, AL^1)$. Actually, one of the starting points for the present investigation was my attempt to construct counterexamples to (Q) by deforming a domain in $Q(\mu, AL^1)$ with two cavities into a solid domain in two different ways (cf. Example 3.2) in the hope that two different solid domains would arise. We still do not know if this is possible; but now at least one can say that, if there are counterexamples to (Q), then they can be constructed in that way.

The paper is organized as follows. Section 1 contains the necessary definitions and some fundamental examples and results. Some material here is new: e.g., the extension of Sakai's original definition of a quadrature domain for subharmonic functions to the case of non-positive measures, the generalization to higher dimensions of quadrature domains for analytic functions and part of the approximation result Lemma 1.3 (which is needed, e.g., to relate the standard definition of a quadrature domain to the one used in this introduction).

In Section 2 we use a partly new approach to build up the existence theory for quadrature domains for subharmonic functions, and in Section 3 we start studying the uniqueness question (Q) via the partial order $\prec$. Some of the results in Sections 2 and 3 exist or have counterparts in previous work (mainly that of Sakai), but almost all proofs here are new, as is the idea of considering the partial order $\prec$. It should be pointed out also that we work in arbitrary dimension throughout, whereas much of the previous theory was developed just in two dimensions.

Finally, Section 4 contains some further results related to the partial order $\prec$. This section may be considered the main contribution of the paper, as most results here are entirely new. We prove that the least upper bound with respect to $\prec$ always exists in $Q(\mu, HL^1)$ and $Q(\mu, AL^1)$ (Theorem 4.1) and that whenever two domains in $Q(\mu, AL^1)$ are related by $\prec$ they can be connected by a chain (with respect to $\prec$) in $Q(\mu, AL^1)$ (Theorem 4.2). These results are quite interesting in their own right; combining them gives the connectedness of $Q(\mu, AL^1)$ (which, at least to me, was a surprising result). We also obtain estimates (lower bounds) for the local "dimension" of $Q(\mu, AL^1)$ at an element Ω in terms of the number of cavities and certain "special points" in Ω.

List of notations

$B(a; r) = \{x \in \mathbf{R}^N : |x - a| < r\}$
$\mathscr{D}'(\mathbf{R}^N)$ the distributions in $\mathbf{R}^N$
$\mathscr{E}'(\mathbf{R}^N)$ those with compact support
$M_c = \mathscr{E}'(\mathbf{R}^N)_+ - \mathscr{E}'(\mathbf{R}^N)_+ =$ the Radon measures with compact support in $\mathbf{R}^N$
$L_c^\infty = \{f \in L^\infty(\mathbf{R}^N) : f = 0 \text{ outside a compact set}\}$.

Primarily measures etc. are regarded as distributions, so that e.g. $L_c^\infty \subset M_c \subset \mathscr{E}'(\mathbf{R}^N)$. Accordingly, we usually do not distinguish notationally between an absolutely continuous measure and its density function (with respect to Lebesgue measure). For example, m (Lebesgue measure) and 1 denote the same thing.

E = the spherically symmetric fundamental solution of $-\Delta$
$U^\mu = E * \mu$ (for $\mu \in \mathscr{E}'(\mathbf{R}^N)$)
$U^\Omega = U^{\chi_\Omega} = U^{\chi_\Omega m}$ (for $\Omega \subset \mathbf{R}^N$)
$\Omega_1 < \Omega_2$ means $U^{\Omega_1} \geqq U^{\Omega_2}$ (in all $\mathbf{R}^N$)
$\mu_1 < \mu_2$ means $U^{\mu_1} \geqq U^{\mu_2}$ (in all $\mathbf{R}^N$)
m = Lebesgue measure; we also write
$|\Omega| = m(\Omega)$ (for $\Omega \subset \mathbf{R}^N$)
$[\Omega] = \{x \in \mathbf{R}^N : \exists r > 0$ such that $|B(x; r) \setminus \Omega| = 0\}$ (for $\Omega \subset \mathbf{R}^N$ open)
$SL^p(\Omega)$, $HL^p(\Omega)$, $AL^p(\Omega)$ etc. see beginning of Section 1
$Q(\mu, \Lambda)$ the set of quadrature domains for the test class Λ with respect to $\mu \in M_c$; see Section 1
$\omega(\mu)$ defined by (2.1) ($\mu \in M_c$)
$\Omega(\mu)$ defined by (2.2) ($\mu \in M_c$)
V^μ defined in Theorem 2.1 ($\mu \in M_c$)
F an operator $M_c \to M_c$; see (2.4)
$W^{p,m}$, $H^m = W^{2,m}$, $H_0^m = W_0^{2,m}$ etc. Sobolev spaces
$\Omega^c = \mathbf{R}^N \setminus \Omega$
$\Omega^e = \mathbf{R}^N \setminus \bar{\Omega}$
h_ε mollifiers; see (1.6)
ω_N the volume of the unit ball in $\mathbf{R}^N$

1. Definitions and preliminaries

For Ω and open subset of $\mathbf{R}^N$ ($N \geqq 2$) we set

$$SL^p(\Omega) = \{\text{subharmonic functions in } \Omega\} \cap L^p(\Omega),$$

$$HL^p(\Omega) = \{\text{harmonic functions in } \Omega\} \cap L^p(\Omega),$$

where $1 \leqq p \leqq \infty$ and $L^p(\Omega) = L^p(\Omega; m)$. The elements in $SL^p(\Omega)$, $HL^p(\Omega)$ (and in $AL^p(\Omega)$ defined below) will be regarded as *functions*, not merely equivalence classes of functions. Thus, if e.g. $\varphi \in SL^1(\Omega)$, then φ is an upper semicontinuous function in Ω with values in $\mathbf{R} \cup \{-\infty\}$ (satisfying $\Delta\varphi \geqq 0$ in the distribution sense and being integrable).

When $N = 2$ and $\mathbf{R}^2$ is identified with $\mathbf{C}$ in the usual way, $AL^p(\Omega)$ usually denotes the set of analytic functions in Ω belonging to $L^p(\Omega)$. When generalizing to higher dimensions it is more convenient to consider the antianalytic functions instead, i.e. those (complex-valued) functions f which satisfy $\partial f/\partial z = 0$. Then the natural generalization to $N \geqq 3$ are the *harmonic vector fields*. A harmonic vector field in Ω is a (smooth) vector field $f = (f_1, \ldots, f_N) : \Omega \to \mathbf{R}^N$ satisfying

$$\text{(1.1)} \qquad \operatorname{rot} f = \left(\frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k}\right)_{k,j} = 0,$$

$$\operatorname{div} f = \sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j} = 0. \tag{1.2}$$

In terms of the corresponding one-form $\omega = f_1 dx_1 + \cdots + f_N dx_N$ (1.1) and (1.2) say that $d\omega = 0$ and $d * \omega = 0$ respectively, where $*$ is the Hodge star operator. It is clear that the harmonic vector fields are exactly those vector fields which locally are gradients of harmonic functions. This also shows that each component of a harmonic vector field is a harmonic function. (Conversely, each harmonic function is locally a component of some harmonic vector field.) Some other properties of harmonic vector fields can be found e.g. in [Bg].

We now define, for arbitrary $N \geqq 2$,

$$AL^p(\Omega) = \{\text{harmonic vector fields in } \Omega\} \cap (L^p(\Omega))^N,$$

$$AL^p(\Omega)_j = \{f_j : f = (f_1, \cdots, f_N) \in AL^p(\Omega)\}$$

($j = 1, \ldots, N$). It is clear by the above discussion that $AL^p(\Omega)_j \subset HL^p(\Omega)$ for each j.

When $N = 2$, $AL^p(\Omega)_j$ is independent of j (since the map $(f_1, f_2) \rightarrow (-f_2, f_1)$ acts on $AL^p(\Omega)$), but this is not true in general. Take e.g. $\Omega = \{x \in \mathbf{R}^N : 0 < |x| < 1\}$. Then it is easy to see that the most general kind of singularity at the origin for $f \in AL^1(\Omega)$ is $f(x) = ax/|x|^N + \mathcal{O}(1)$ ($a \in \mathbf{R}$) if $N \geqq 3$, and this readily shows that $AL^1(\Omega)_k \neq AL^1(\Omega)_j$ for $k \neq j$.

Therefore, it is appropriate to also introduce the space

$$AL^p(\Omega)_1 + \cdots + AL^p(\Omega)_N,$$

or better its closure in $HL^p(\Omega)$:

$$AL^p(\Omega)_c = \operatorname{clos}(AL^p(\Omega)_1 + \cdots + AL^p(\Omega)_N)$$

(c for "closure of components of"). Every $\varphi \in AL^p(\Omega)_c$ satisfies

$$\int_\Gamma \frac{\partial \varphi}{\partial n} ds = 0$$

for every closed oriented hypersurface Γ in Ω, as a computation shows (see [Gu 5, p. 238]), and probably $AL^p(\Omega)_c$ actually coincides with the set of all $\varphi \in HL^p(\Omega)$ for which this is true. The space $AL^p(\Omega)_c$ has some technical advantages over $AL^p(\Omega)$ because it is a subspace of $HL^p(\Omega)$ and therefore is equipped with the same norm as $HL^p(\Omega)$ but in order to keep the number spaces down we will in this paper mostly work with just $AL^p(\Omega)$ itself.

Clearly, $HL^p(\Omega)$, $AL^p(\Omega)_c$ and $AL^p(\Omega)$ are Banach spaces with their induced norms (from $L^p(\Omega)$ and $L^p(\Omega)^N$) and $SL^p(\Omega)$ is a closed cone in $L^p(\Omega)$. We note the inclusions

$$AL^p(\Omega)_j \subset AL^p(\Omega)_c \subset HL^p(\Omega) \subset SL^p(\Omega) \tag{1.3}$$

(all j). In the sequel we will stick to the case $p = 1$.

Following and generalizing [Sa 2] we now give the definition of a quadrature domain for a general measure. Let $\mu \in M_c$ (i.e. a Radon measure with compact support). If $\Lambda = AL^1$, $(AL^1)_c$ or HL^1 a *quadrature domain* for the test class Λ is an open bounded set Ω in $\mathbf{R}^N$ satisfying

$$\begin{aligned} &\text{(i) } \mu = 0 \quad \text{on } \Omega^c,\\ &\text{(ii) } \Lambda(\Omega) \subset L^1(|\mu|) \quad \text{and}\\ &\text{(iii) } \textstyle\int \varphi d\mu = \int_\Omega \varphi dm \quad \text{for all } \varphi \in \Lambda(\Omega). \end{aligned} \tag{1.4}$$

Here (ii) means strictly speaking just that $\int |\varphi| d|\mu| < \infty$ for all $\varphi \in \Lambda(\Omega)$; the natural map $\Lambda(\Omega) \to L^1(|\mu|)$ is then not always one-to-one. Also, $L^1(|\mu|)$ should be interpreted as $L^1(|\mu|)^N$ in the case $\Lambda = AL^1$. Observe that we do not require a quadrature domain to be connected, despite the word "domain".

For $\Lambda = SL^1$ we replace the definition (1.4) by

$$\begin{aligned} &\text{(i}'\text{) } \mu = 0 \quad \text{on } \Omega^c,\\ &\text{(ii}'\text{) } \textstyle\int \varphi_+ d\mu_+ + \int \varphi_- d\mu_- < \infty \quad \text{for all } \varphi \in \Lambda(\Omega),\\ &\text{(iii}'\text{) } \textstyle\int \varphi d\mu \leqq \int_\Omega \varphi dm \quad \text{for all } \varphi \in \Lambda(\Omega). \end{aligned} \tag{1.4'}$$

Actually, the latter definition can be used also for $\Lambda = AL^1$, $(AL^1)_c$ and HL^1 because (ii′)–(iii′) reduce to (ii)–(iii) when $\Lambda(\Omega)$ is a linear space. Observe also that (ii) and (ii′) have the roles of making sense to the left members of (iii) and (iii′), with $\int \varphi d\mu = -\infty$ allowed in the case $\Lambda = SL^1$.

The set of quadrature domains for the test class Λ is denoted $Q(\mu, \Lambda)$. Using the definition (1.4′) it is immediate from (1.3) that

$$Q(\mu, SL^1) \subset Q(\mu, HL^1) \subset Q(\mu, (AL^1)_c) \subset Q(\mu, AL^1). \tag{1.5}$$

The difference between $Q(\mu, (AL^1)_c)$ and $Q(\mu, AL^1)$ is minor, if any, and we will usually just work with $Q(\mu, AL^1)$.

It is easy to see that (ii) gives a restriction on μ only near $\partial\Omega$ (hence (ii) is automatically valid if $\operatorname{supp}\mu \subset \Omega$) and that (ii′) gives a restriction on μ_+ only near $\partial\Omega$ (for φ subharmonic in Ω implies that φ is bounded from above on each compact subset of Ω). Thus (ii′) is automatically satisfied if $\mu \geqq 0$ and $\operatorname{supp}\mu \subset \Omega$. On the other hand (ii′) does not allow μ_- to be e.g. a point mass. In fact, we have

Lemma 1.1. *If* (ii′) *holds* (*for* $\Lambda = SL^1$) *then* μ_- *has finite energy.*

Proof. Take $\varphi = -U^{\mu_-}$. Then φ is subharmonic and, in fact, $\varphi \in SL^1(\Omega)$, for the potential of an arbitrary Radon measure with compact support is locally integrable [T, Prop. 30.3]. Thus, if (ii′) holds $\int (U^{\mu_-})_+ d\mu_- = \int \varphi_- d\mu_- < \infty$ which means that μ_- has finite energy. ■

Another useful observation is

Lemma 1.2. *Whenever* (1.4)(ii) *holds* (*for* $\Lambda = HL^1$, $(AL^1)_c$ *or* AL^1) *the "embedding" is continuous, i.e. there exists a constant C such that*

$$\int |\varphi| d|\mu| \leqq C \int_\Omega |\varphi| dm \qquad \text{for all } \varphi \in \Lambda(\Omega).$$

Proof. This is a standard application of the closed graph theorem. (We omit the details.) ■

I have not been able to prove a corresponding result for (1.4′)(ii′).

Since $1 \in AL^1(\Omega)_j$ we have $|\Omega| = \int d\mu$ whenever $\Omega \in Q(\mu, \Lambda)$ for any one of the test classes Λ considered. Also note (using this) the existence of maximal (with respect to inclusion) quadrature domains: for Ω any open set, define

$$[\Omega] = \{x \in \mathbf{R}^N \colon \exists r > 0 \,|\, B(x; r) \setminus \Omega | = 0\};$$

thus $[\Omega]$ is the largest open set D satisfying $\Omega \subset D$, $|D \setminus \Omega| = 0$. Then $\Omega \in Q(\mu, \Lambda)$ implies $[\Omega] \in Q(\mu, \Lambda)$ and an open set $D \supset \Omega$ is in $Q(\mu, \Lambda)$ if and only if $D \subset [\Omega]$.

We will repeatedly in the paper have use for systems of radial mollifiers of the following kind. Let $h \in L^\infty(\mathbf{R}^N)$ be a function depending only on $r = |x|$ such that $h \geqq 0$, supp $h \subset B(0; 1)$, $\int h dm = 1$. For any $\varepsilon > 0$ we set

$$h_\varepsilon(x) = \varepsilon^{-N} h(\varepsilon^{-1} x). \tag{1.6}$$

Then $h_\varepsilon \geqq 0$, supp $h_\varepsilon \subset B(0; \varepsilon)$ and $\int h_\varepsilon dm = 1$.

Note that if φ is a harmonic (or subharmonic) function then, by the (sub-)mean-value property, $\varphi = \varphi * h_\varepsilon$ (or $\varphi \leqq \varphi * h_\varepsilon$ respectively), within the domain of definition of $\varphi * h_\varepsilon$. This easily shows that if $\mu, \nu \in M_c$, supp $\mu \subset \Omega$ then, for $\Lambda = AL^1$, $(AL^1)_c$ and HL^1, $\Omega \in Q(\mu + \nu, \Lambda)$ if and only if $\Omega \in Q(\mu * h_\varepsilon + \nu, \Lambda)$, provided $\varepsilon > 0$ is so small that supp$(\mu * h_\varepsilon) \subset \Omega$. For this reason many results in this paper stated with the assumption $\mu \in L_c^\infty$ also hold with this assumption replaced by $\mu \in M_c$, supp $\mu \subset \Omega$ (or, more generally, $\mu \in M_c$ with $\mu \in L^\infty$ in a neighbourhood of $\partial\Omega$).

For $\Lambda = SL^1$ the above statements are only partially true. If e.g. μ contains negative point masses, then $Q(\mu, SL^1) = \varnothing$ by Lemma 1.1 but $Q(\mu * h_\varepsilon, SL^1)$ may very well be non-empty. For a positive result, see Theorem 2.4(iv).

The reader should be warned that our definition of "quadrature domain" is far from being the only one in use: μ is sometimes allowed to be a more general distribution, often it is required that $\operatorname{supp}\mu \subset \Omega$ or even that $\operatorname{supp}\mu$ shall be a finite point set, Ω may be allowed to be (genuinely) unbounded, the choice of test class Λ may be varied at lot, Lebesgue measure may be replaced by other measures, a quadrature domain is usually required to be connected, etc. For the previous work on (bounded) quadrature domains we refer to [Ah–Sh], [Av], [D], [Gu 2], [Gu 6], [Sa 2], [Sa 3], [Sa 4], [Sa 5], [Sa 6], [Sa 7], [Sa 8], [Sa 9], [Sh 1], [Sh 2].

Lots of examples of quadrature domains are given in the above references. For the moment we shall just give two examples, the first of which is the most classical of all quadrature domains. The second one, which in two dimensions also was given in [Sa 2, Example 1.2], provides a simple but useful illustration to the theory to be developed in the paper.

Example 1.1. Let $\mu = \alpha\delta$ where δ is the Dirac measure at the origin and $\alpha \geqq 0$. Then

$$Q(\mu, AL^1) = Q(\mu, HL^1) = Q(\mu, SL^1) = \{B(0; r)\}$$

where $r \geqq 0$ is determined by $|B(0; r)| = \alpha$. (If $\alpha < 0$ the $Q(\mu, \Lambda)$ are empty.) That $B(0; r) \in Q(\mu, SL^1)$ follows directly from the submeanvalue property of subharmonic functions and that there are no other elements, even in $Q(\mu, AL^1)$, is a by now quite classical result (at least in two dimensions) which will be proved in passing also in this paper (see Example 3.4).

Example 1.2. Let $\mu = \mu_\alpha = \alpha\rho$ where $\alpha > 0$ and ρ is the unit mass uniformly distributed on the unit sphere $S = \partial B(0; 1)$. Let ω_N denote the volume of the unit ball in $\mathbf{R}^N$ and define, for $t \geqq 0$,

$$\Omega_t = \{x \in \mathbf{R}^N : t < \omega_N |x|^N < t + \alpha\},$$

$$\Omega = \Omega_0 \cup \{0\}.$$

Then $|\Omega_t| = |\Omega| = \alpha = \int d\mu_\alpha$, $S \subset \Omega_t$ iff $\omega_N - \alpha < t < \omega_N$, $S \subset \Omega$ iff $\alpha > \omega_N$. It is easy to see that $\Omega_t, \Omega \in Q(\mu_\alpha, AL^1)$ whenever $S \subset \Omega_t$ and $S \subset \Omega$ respectively.

Let $A = e\pi$ if $N = 2$, $A = (N/2)^{N/(N-2)}\omega_N$ if $N \geqq 3$. Then $A > \omega_N$. For each $0 < \alpha \leqq A$ there is unique $t = t_\alpha$ with $\omega_N - \alpha < t_\alpha < \omega_N$ such that $\int_{\Omega_{t_\alpha}} E\, dm = \int E\, d\mu_\alpha$, where E is the fundamental solution (0.1). (If e.g. $N \geqq 3$, the equation for t_α becomes $(N/2)\omega_N^{1-(2/N)}[(t_\alpha + \alpha)^{2/N} - t_\alpha^{2/N}] = \alpha$.) As α increases from zero to A, t_α decreases from one to zero.

Now, for $0 < \alpha \leqq \omega_N$ we have

$$Q(\mu_\alpha, AL^1) = \{\Omega_t : \omega_N - \alpha < t < \omega_N\},$$

$$Q(\mu_\alpha, HL^1) = Q(\mu_\alpha, SL^1) = \{\Omega_{t_\alpha}\}.$$

For all $\alpha > \omega_N$

$$Q(\mu_\alpha, AL^1) = \{\Omega\} \cup \{\Omega_t : 0 \leqq t < \omega_N\}$$

while

$$Q(\mu_\alpha, HL^1) = \{\Omega, \Omega_{t_\alpha}\},$$

$$Q(\mu_\alpha, SL^1) = \{\Omega_{t_\alpha}\}$$

for $\omega_N < \alpha < A$,

$$Q(\mu_A, HL^1) = Q(\mu_A, SL^1) = \{\Omega, \Omega_{t_A}\}$$

and

$$Q(\mu_\alpha, HL^1) = Q(\mu_\alpha, SL^1) = \{\Omega\} \qquad \text{for } \alpha > A.$$

The above statements are proved in [Sa 2, Example 1.2] for $N = 2$. The proofs in higher dimensions are similar and therefore omitted.

The Newtonian field corresponding to the potential $U^\mu = E * \mu$ is $-\nabla U^\mu = -(\nabla E) * \mu$, with the pointwise expression

$$(1.7) \qquad -\nabla U^\mu(x) = C_N \int \frac{x - y}{|x - y|^N} \, d\mu(y)$$

for suitable $C_N > 0$. On open sets ∇U^μ always makes good sense as the distributional gradient of U^μ but we will very often in the sequel need to make pointwise statements about ∇U^μ (as well as about U^μ). Let us then make the convention that statements like "∇U^μ is pointwise well-defined on S", S an arbitrary subset of $\mathbf{R}^N$, will mean that for each $x \in S$ the integral in (1.7) is absolutely convergent (in which case $\nabla U^\mu(x)$ has a definite value, given by (1.7), for each $x \in S$). Similarly for U^μ itself.

It is immediately verified that if Ω is open and bounded then, as functions of $x \in \Omega$,

$$\nabla E(x - y) \in AL^1(\Omega) \qquad \text{for } y \in \Omega^c,$$

$$E(x - y) \in HL^1(\Omega) \qquad \text{for } y \in \Omega^c,$$

$$-E(x - y) \in SL^1(\Omega) \qquad \text{also for } y \in \Omega.$$

Of course, by the inclusions (1.3), also $E_j(x - y) \in AL^1(\Omega)_c \subset HL^1(\Omega)$, $\pm E(x - y) \in SL^1(\Omega)$, $\pm E_j(x - y) \in SL^1(\Omega)$ for $y \in \Omega^c$, $1 \leqq j \leqq N$; here and in the sequel

$$E_j = \partial E / \partial x_j.$$

Using the above functions as test functions in the definition of a quadrature domain gives

Proposition 1.1. *Let $\mu \in M_c$ and let Ω be a bounded open set in $\mathbf{R}^N$.*

(i) *If $\Omega \in Q(\mu, AL^1)$ then ∇U^μ is pointwise well-defined on Ω^c and*

$$\nabla U^\Omega = \nabla U^\mu \qquad \text{on } \Omega^c.$$

(ii) *If $\Omega \in Q(\mu, HL^1)$ then U^μ and ∇U^μ are pointwise well-defined on Ω^c and satisfy*

$$U^\Omega = U^\mu \qquad \text{on } \Omega^c,$$

$$\nabla U^\Omega = \nabla U^\mu \qquad \text{on } \Omega^c.$$

(iii) *If $\Omega \in Q(\mu, SL^1)$ then U^μ and ∇U^μ are pointwise well-defined on Ω^c and*

$$U^\Omega \leqq U^\mu \qquad \text{in } \mathbf{R}^N, \tag{1.8}$$

$$U^\Omega = U^\mu \qquad \text{on } \Omega^c,$$

$$\nabla U^\Omega = \nabla U^\mu \qquad \text{on } \Omega^c.$$

Remark 1.1. The inequality (1.8) can be interpreted either pointwise or in the sense of distributions. These two interpretations actually coincide whenever (ii′) of (1.4′) is fulfilled. In fact, in this case U^{μ_-} is everywhere finite so $U^\mu \geqq U^\Omega$ (in $\mathbf{R}^N$) can be written $U^\Omega + U^{\mu_-} \leqq U^{\mu_+}$, and if this holds in the sense of distributions (i.e. a.e.) then

$$(U^\Omega + U^{\mu_-}) * h_\varepsilon \leqq U^{\mu_+} * h_\varepsilon \qquad (\varepsilon > 0)$$

and letting here $\varepsilon \to 0$ first in the right member and then in the left member we get the pointwise inequality, using that $U^\nu * h_\varepsilon \nearrow U^\nu$ pointwise when ν is a positive measure. The converse statement is obvious.

Remark 1.2. If $\Omega \in Q(\mu, AL^1)$ we also have $U^\mu = U^\Omega$ in the unbounded component of Ω^e. In fact, it is easy to check that, more generally, if $\nu \in M_c$ and ∇U^ν vanishes in a connected neighbourhood of infinity then also U^ν vanishes there ($\nu = \mu - \chi_\Omega m$ above).

To prove a partial converse of Proposition 1.1 we need an approximation theorem which says that the test functions used above are dense.

Lemma 1.3. *Let Ω be a bounded open subset of $\mathbf{R}^N$.*

(i) *The linear combinations with positive coefficients of the functions (of $x \in \Omega$) $\pm E_j(x-y)$ $(1 \leqq j \leqq N)$ and $E(x-y)$ for $y \in \Omega^c$ and $-E(x-y)$ for $y \in \mathbf{R}^N$ are dense in $SL^1(\Omega)$.*

(ii) *The linear combinations with real coefficients of the functions $E_j(x-y)$ $(1 \leqq j \leqq N)$ and $E(x-y)$ for $y \in \Omega^c$ are dense in $HL^1(\Omega)$.*

(iii) *The linear combinations with real coefficients of the functions* $E_j(x-y)$ $(1 \leqq j \leqq N)$ *for* $y \in \Omega^c$ *are dense in* $AL^1(\Omega)_1 + \cdots + AL^1(\Omega)_N$ *and (hence) in* $AL^1(\Omega)_c$.

(iv) *Every* $f \in AL^1(\Omega)$ *can be approximated (in the* $L^1(\Omega)^N$*-norm) by linear combinations with* $N \times N$*-matrix coefficients of the vector fields* $\nabla E(x-y)$ *for* $y \in \Omega^c$. *Moreover, only antisymmetric matrices and scalar multiples of the identity matrix are needed.*

Proof. (i) is proved in [Sa 4] and (ii) is proved in [Sa 2] (for $N = 2$, but the proof works with minor changes also for $N \geqq 3$). (iii) is an immediate consequence of (iv), so it just remains to prove (iv). We then follow the classical paper [Bs] in which the complex analytic case is treated.

The statement then to be proven is intuitively reasonable in view of the representation formula

$$f = -(\operatorname{div} f) * \nabla E - (\operatorname{rot} f) * \nabla E \tag{1.9}$$

(interpreted as below) valid for all $f \in (\mathscr{E}'(\mathbf{R}^N))^N$, in particular for any $f \in AL^1(\Omega)$ extended by zero outside Ω. (1.9) is proved as follows. Using that $\partial E_j/\partial x_k = \partial E_k/\partial x_j$ and that E is a fundamental solution of $-\Delta$ we have, for each k,

$$\begin{aligned} f_k = -(\Delta f_k) * E &= -\sum_{j=1}^{N} \frac{\partial f_k}{\partial x_j} * E_j \\ &= -\sum_{j=1}^{N} \frac{\partial}{\partial x_j}(f_k * E_j) + \sum_{j=1}^{N} \frac{\partial}{\partial x_k}(f_j * E_j) - \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(f_j * E_k) \qquad (1.10) \\ &= -\sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j} * E_k - \sum_{j=1}^{N} \left(\frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k}\right) * E_j \end{aligned}$$

which is what we mean by (1.9). Changing the signs on a cancelling pair of terms in (1.10) we obtain an adjoint formula which will be used later in the proof:

$$f_k = -\sum_{j=1}^{N} \frac{\partial}{\partial x_j}(f_k * E_j) - \sum_{j=1}^{N} \frac{\partial}{\partial x_k}(f_j * E_j) + \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(f_j * E_k). \tag{1.11}$$

Choose a function $\psi \in C^\infty(\mathbf{R})$ satisfying $0 \leqq \psi \leqq 1$, $\psi(t) = 0$ for $t \leqq 1$, $\psi(t) = 1$ for $t \geqq 2$ and set

$$\omega_n(x) = \psi\left(\frac{n}{\log\log(1/\delta(x))}\right) \qquad \text{where } \delta(x) = \frac{\operatorname{dist}(x, \partial\Omega)}{1 + \operatorname{dist}(x, \partial\Omega)}.$$

Then $\omega_n \in C_0^{0,1}(\Omega)$ and $\lim_{n\to\infty} \omega_n(x) = 1$ for each $x \in \Omega$. A straightforward computation shows that

$$(1.12) \qquad |\nabla\omega_n(x)| \leqq \frac{c}{n\delta(x)\log\dfrac{1}{\delta(x)}} \qquad \text{for } x\in\Omega.$$

Now any linear continuous functional on $AL^1(\Omega)$ is of the form

$$L(f)=\sum_{j=1}^{N}\int_\Omega f_k g_k \quad f=(f_1,\ldots,f_N)\in AL^1(\Omega) \text{ for some } g=(g_1,\ldots,g_N)\in L^\infty(\Omega)^N.$$

If we can prove that whenever $L(A\nabla E(\cdot - y))=0$ for every $y\in\Omega^c$ and every $N\times N$ matrix A which either is antisymmetric or is the identity matrix then $L(f)=0$ for all $f\in AL^1(\Omega)$, then (iv) will follow from the Hahn–Banach theorem.

Thus assume that $L(A\nabla E(\cdot - y))=0$ for all y and A as above. This means exactly that

$$(1.13) \qquad g_k * E_j - g_j * E_k = 0 \qquad \text{on } \Omega^c,$$

$$(1.14) \qquad \sum_{j=1}^{N} g_j * E_j = 0 \qquad \text{on } \Omega^c,$$

for all $1\leqq k,j\leqq N$. Here and in the sequel g is extended to all $\mathbf{R}^N$ by setting $g=0$ outside Ω. Thus $g\in(L_c^\infty)^N$. If φ denotes any of the left members in (1.13) or (1.14) we have, by elementary estimates [Gü],

$$|\varphi(x)-\varphi(y)|\leqq C|x-y|\log\frac{1}{|x-y|} \qquad \text{for } |x-y| \text{ small.}$$

Hence (1.13), (1.14) give

$$|\varphi(x)|\leqq C\delta(x)\log\frac{1}{\delta(x)}$$

for $x\in\Omega$ and some C. Thus using also (1.12)

$$(1.15) \qquad \left\|\frac{\partial\omega_n}{\partial x_i}(g_k * E_j - g_j * E_k)\right\|_\infty \leqq \frac{c}{n},$$

$$(1.16) \qquad \left\|\frac{\partial\omega_n}{\partial x_i}\sum_{j=1}^{N} g_j * E_j\right\|_\infty \leqq \frac{c}{n},$$

for all i,j,k.

Let now $f=(f_1,\ldots,f_N)\in AL^1(\Omega)$. Using (1.15), (1.16) and the representation (1.11) (applied to g) we get

$$L(f)=\sum_{k=1}^{N}\int_{\Omega} f_k g_k=\lim_{n\to\infty}\sum\int_{\Omega}\omega_n f_k g_k$$

$$=\lim_{n\to\infty}\sum_{k,j=1}^{N}\int_{\Omega}\omega_n\left[f_k\frac{\partial}{\partial x_j}(g_j*E_k)-f_k\frac{\partial}{\partial x_j}(g_k*E_j)-f_k\frac{\partial}{\partial x_k}(g_j*E_j)\right]$$

$$=\lim_{n\to\infty}\sum\int_{\Omega}\omega_n\left[-\frac{\partial f_k}{\partial x_j}(g_j*E_k)+\frac{\partial f_k}{\partial x_j}(g_k*E_j)+\frac{\partial f_k}{\partial x_k}(g_j*E_j)\right]$$

$$+\lim_{n\to\infty}\sum\int_{\Omega}\left[\frac{\partial\omega_n}{\partial x_j}f_k(-g_j*E_k+g_k*E_j)+\frac{\partial\omega_n}{\partial x_k}f_k(g_j*E_j)\right]$$

$$=\lim_{n\to\infty}\sum\int_{\Omega}\omega_n\left[\left(\frac{\partial f_k}{\partial x_j}-\frac{\partial f_j}{\partial x_k}\right)(g_k*E_j)+\frac{\partial f_k}{\partial x_k}(g_j*E_j)\right]=0.$$

This proves the lemma. ∎

Observe that, when $N=2$, the set of matrices occurring in (iv) are exactly those which correspond to multiplication by complex numbers. The reason that we did not speak of something being dense in $AL^1(\Omega)$ in (iv) is that the vector fields approximating $f\in AL^1(\Omega)$ are not, in general, themselves in $AL^1(\Omega)$ (when $N\geqq 3$).

A particular consequence of (iii) of the lemma is that $A_eL^1(\Omega)_c=AL^1(\Omega)_c$, where $A_eL^1(\Omega)=\{\nabla u\in AL^1(\Omega)$: u is (single-valued) harmonic in $\Omega\}$ (e for "exact") and the subscript c has the same meaning as before. Therefore the smaller test class $A_eL^1(\Omega)$ actually gives the same quadrature domains as $AL^1(\Omega)$.

It is not true, not even in two dimensions (see below), that for fixed k the linear combinations of $E_k(x-y)$, $y\in\Omega^c$, are dense in $AL^1(\Omega)_k$; to approximate an $f_k\in AL^1(\Omega)_k$ one will in general need also the $E_j(x-y)$ for $j\neq k$ (although these $E_j(x-y)$ usually do not belong to $AL^1(\Omega)_k$). To exemplify this statement just take $\Omega=\{x\in\mathbf{R}^2\colon 0<|x|<1\}$. Then

$$f(x)=\left(\frac{-x_2}{|x|^2},\frac{x_1}{|x|^2}\right)$$

is in $AL^1(\Omega)$ but $f_1(x)=-x_2/|x|^2$ obviously cannot be approximated by linear combinations of

$$E_1(x-y)=\frac{x_1-y_1}{|x-y|^2}\qquad\text{for } y\in\Omega^c.$$

In view of Lemma 1.2, (iii) in the definition (1.4) of a quadrature domain for $\Lambda = HL^1$, AL^1 or $(AL^1)_c$ need only be checked for a dense subset of Λ. Thus, by Lemma 1.3, if Ω is a bounded open set such that (i) and (ii) of (1.4) hold (e.g. if $\operatorname{supp}\mu \subset \Omega$) then $\Omega \in Q(\mu, \Lambda)$ if and only if $U^\Omega = U^\mu$ and $\nabla U^\Omega = \nabla U^\mu$ on Ω^c (in the case $\Lambda = HL^1$) or $\nabla U^\Omega = \nabla U^\mu$ (in the cases $\Lambda = AL^1$ and $(AL^1)_c$).

This shows e.g. that if $\Omega \in Q(\mu, AL^1)$ and $\operatorname{supp}\mu \subset \Omega$ then $\Omega \in Q(\mu, (AL^1)_c)$. For similar reasons $Q(\mu, AL^1) = Q(\mu, (AL^1)_c)$ if $\mu \in L_c^\infty$. For $\mu \in L_c^\infty$ the characterizations of the classes $Q(\mu, \Lambda)$ are in fact very simple (cf. the introduction):

Proposition 1.2. *If $\mu \in L_c^\infty$ then, for a bounded open set $\Omega \subset \mathbf{R}^N$,*
(i) *$\Omega \in Q(\mu, AL^1)$ if and only if $\nabla U^\Omega = \nabla U^\mu$ on Ω^c,*
(ii) *$\Omega \in Q(\mu, HL^1)$ if and only if $U^\Omega = U^\mu$ on Ω^c, $\nabla U^\Omega = \nabla U^\mu$ on Ω^c,*
(iii) *$\Omega \in Q(\mu, SL^1)$ if and only if $U^\Omega \leqq U^\mu$ in $\mathbf{R}^N$, $U^\Omega = U^\mu$ on Ω^c.*
Also
(i′) *$[\Omega] \in Q(\mu, AL^1)$ if and only if $\nabla U^\Omega = \nabla U^\mu$ a.e. on Ω^c,*
(ii′) *$[\Omega] \in Q(\mu, HL^1)$ if and only if $U^\Omega = U^\mu$ a.e. on Ω^c,*
(iii′) *$[\Omega] \in Q(\mu, SL^1)$ if and only if $U^\Omega \leqq U^\mu$ in $\mathbf{R}^N$, $U^\Omega = U^\mu$ a.e. on Ω^c.*

Proof. The "only if" parts are special cases of Proposition 1.1. We now prove the "if" part for (iii), those for (i) and (ii) being similar and easier.

By elliptic regularity theory U^Ω, $U^\mu \in W^{2,p}_{\mathrm{loc}}(\mathbf{R}^N)$ for all $p < \infty$, in particular U^Ω and U^μ are continuously differentiable. Therefore $\nabla U^\Omega = \nabla U^\mu$ on Ω^c because $U^\mu - U^\Omega$ attains its minimum there. It also follows (see [Ki–St, Lemma A4]) that $\Delta U^\Omega = \Delta U^\mu$ a.e. on Ω^c, hence that $\mu = \chi_\Omega$ a.e. on Ω^c. This shows that $\mu = 0$ on Ω^c as a measure, i.e. that (i′) of (1.4′) is satisfied.

When $\mu \in L_c^\infty$ (ii′) of (1.4′) is automatically satisfied and (iii′) need only bejechecked for a dense subclass of $SL^1(\Omega)$. Now the hypotheses (including $\nabla U^\Omega = \nabla U^\mu$ on Ω^c) show that (iii′) does hold for the dense subset of $SL^1(\Omega)$ appearing in Lemma 1.3.

As to the "if" parts of (i′)–(iii′) let us prove the one for (ii′), the other ones being similar. From $U^\Omega = U^\mu$ a.e. on Ω^c and the regularity of U^Ω and U^μ we also have $\nabla U^\Omega = \nabla U^\mu$ a.e. on Ω^c. Since U^Ω, U^μ, ∇U^Ω and ∇U^μ are continuous it now follows that $\{x \in \mathbf{R}^N\colon U^\Omega(x) \neq U^\mu(x) \text{ or } \nabla U^\Omega(x) \neq \nabla U^\mu(x)\}$ is an open set whose intersection with Ω^c has measure zero, hence is contained in $[\Omega]$. Applying now (ii) to $[\Omega]$ the result follows. ■

Lemma 1.4. *When Ω is solid $AL^1(\Omega)_c = HL^1(\Omega)$.*

Proof. Suppose $g \in L^\infty(\Omega)$ annihilates $AL^1(\Omega)_c$. Then g annihilates in particular all the $E_j(x-y)$ for $1 \leqq j \leqq N$, $y \in \Omega^c$ so that $\nabla U^g = 0$ on Ω^c. Thus $U^g = 0$ in Ω^e (see Remark 1.2) and by continuity $U^g = 0$ on all Ω^c. This means that g annihilates also all the $E(x-y)$ for $y \in \Omega^c$, hence by Lemma 1.3 annihilates all $HL^1(\Omega)$. This proves the proposition. ■

Thus if Ω is solid $\Omega \in Q(\mu, HL^1)$ iff $\Omega \in Q(\mu, (AL^1)_c)$ and, by earlier remarks, if $\mu \in L_c^\infty$ or supp $\mu \subset \Omega$ this occurs iff $\Omega \in Q(\mu, AL^1)$. Even in two dimensions it may happen that $AL^1(\Omega)_1 + \cdots + AL^1(\Omega)_N$ is strictly smaller than $HL^1(\Omega)$ (with Ω solid) because a harmonic function can be integrable over a domain Ω without its harmonic conjugate being integrable there. (*Example*: $\Omega = \{re^{i\varphi} \in \mathbf{C}: 0 < \varphi < r,\ 0 < r < 1\}$, $u = \cos 3\varphi/r^3$, $v = \sin 3\varphi/r^3$; then $v \in HL^1(\Omega)$ but $v \notin AL^1(\Omega)_1 + AL^1(\Omega)_2$ because $u = -v^*$ and $\| u \| = \infty$.)

We finish this section by giving examples which show that $Q(\mu, SL^1)$ may be empty even if μ is a positive measure and $Q(\mu, HL^1)$ is non-empty, and that $Q(\mu, HL^1)$ may be empty even if $Q(\mu, AL^1)$ is non-empty.

Example 1.3. With notations as in Example 1.2 take $\mu = \alpha\rho + \varepsilon\chi_B m$ where $B = B(0; 1)$, $\alpha = \omega_N = |B|$ and $\varepsilon > 0$ is small enough. Then $B(0; R) \in Q(\mu, HL^1)$ where $R > 1$ is determined by $|B(0; R)| = \int d\mu$. Since $\Omega_{t_0} \in Q(\alpha\rho, SL^1)$ (see Example 1.2) and $\alpha\rho \leqq \mu$ an open set $\Omega \in Q(\mu, SL^1)$ has to satisfy both $\Omega_{t_0} \subset \Omega$ (see Corollary 2.2 below) and $B \subset \Omega$ (at least except for a null set) which is impossible if ε is small enough (because $|\Omega| = \int d\mu$ if $\Omega \in Q(\mu, SL^1)$). Hence $Q(\mu, SL^1) = \varnothing$.

That $B(0; R) \notin Q(\mu, SL^1)$ in this example indicates that $U^\Omega > U^\mu$ holds somewhere in $B(0; R)$. In fact, it is easy to see that $\{x \in \mathbf{R}^N: U^\Omega(x) > U^\mu(x)\} = B(0; r)$ for a suitable $0 < r < 1$. Note that $\mu = \varepsilon m \neq 0$ in $B(0; r)$. In Theorem 4.4 we show that this fact is in some sense responsible for $Q(\mu, SL^1)$ being empty.

Example 1.4. Let $\Omega = \{x \in \mathbf{R}^N: 1 < |x| < 2\}$ and let μ be an absolutely continuous measure on Ω with $\int d\mu = |\Omega|$ and with a nowhere vanishing density function $f \in L^\infty$ depending only on $r = |x|$.

Then $\Omega \in Q(\mu, AL^1)$ for any μ as above. On the other hand $Q(\mu, HL^1) = \varnothing$ if μ is chosen so that $U^\Omega(0) \neq U^\mu(0)$, i.e. so that $\int_1^2 E(r) r^{N-1} dr \neq \int_1^2 f(r) E(r) r^{N-1} dr$, for $D \in Q(\mu, HL^1)$ requires that D coincides with Ω up to a nullset (due to our assumptions on μ) and then $0 \notin D$, $U^D(0) = U^\Omega(0) \neq U^\mu(0)$ which contradicts Proposition 1.1.

2. Construction of quadrature domains

$Q(\mu, SL^1)$ may be empty, even if the measure μ is positive, but when it is non-empty its essentially unique element can be constructed by various kinds of balayage methods. For positive μ one such construction, more or less a kind of sweeping out by hand in an infinite number of steps, was carried out in [Sa 2]. A more elegant construction, using variational inequalities, was later given in [Sa 3] (cf. also [Gu 1]).

Here we shall use a variant of the method of variational inequalities to generalize Sakai's construction. Our method is related to Perron's method for solving Dirichlet's principle and it is quite flexible. It applies to completely arbitrary Radon measures with compact support and it will also be used in later sections of the paper for other (but related) purposes.

Theorem 2.1. *Let $\mu \in M_c$. Then the family*

$$\mathscr{F} = \{u \in \mathscr{D}'(\mathbf{R}^N) \colon u \leqq U^\mu \text{ and } -\Delta u \leqq 1\}$$

contains a (necessarily unique) largest element $u = V^\mu$. This V^μ can be taken to be upper semicontinuous and then satisfies

(i) *$V^\mu = U^\mu$ outside a compact set;*

(ii) *the open sets*

$$\omega(\mu) = \{x \in \mathbf{R}^N \colon \text{there is an } \alpha > V^\mu(x) \text{ such that } U^\mu \geqq \alpha \text{ in some neighbourhood of } x\} \tag{2.1}$$

and

$$\Omega(\mu) = \mathbf{R}^N \setminus \operatorname{supp}(1 + \Delta V^\mu) \tag{2.2}$$

are bounded and $\omega(\mu) \subset \Omega(\mu)$.

Informally speaking $\omega(\mu)$ is the set where $V^\mu < U^\mu$ and $\Omega(\mu)$ the set where $-\Delta V^\mu = 1$. If $\mu \geqq 0$ (or $\mu_- \in L_c^\infty$ e.g.) then U^μ is lower semicontinuous and we simply have $x \in \omega(\mu)$ iff $V^\mu(x) < U^\mu(x)$.

Proof. We first prove that $\mathscr{F} \neq \varnothing$. Consider $u = U^{\mu_+} * h_\varepsilon - U^{\mu_-} = U^{\mu_+ * h_\varepsilon} - U^{\mu_-}$ where h_ε $(\varepsilon > 0)$ are mollifiers as in (1.6). For any $\varepsilon > 0$, $U^{\mu_+} * h_\varepsilon \leqq U^{\mu_+}$ since U^{μ_+} is superharmonic, hence $u \leqq U^\mu$. Moreover $-\Delta u = \mu_+ * h_\varepsilon - \mu_- \leqq \mu_+ * h_\varepsilon$, which is $\leqq 1$ if ε is *large* enough. Thus $u \in \mathscr{F}$ if $\varepsilon > 0$ is large enough.

There are even simpler examples of elements in $\mathscr{F}$ (e.g. $-U^{\mu_-} \in \mathscr{F}$ if $N \geqq 3$) but the above $u \in \mathscr{F}$ has the useful additional property that it coincides with U^μ far away (in particular $-\Delta u$ has compact support). To see this just notice that since μ_+ has compact support $U^{\mu_+} * h_\varepsilon = U^{\mu_+}$ outside a compact set by the mean-value property for harmonic functions.

After addition of the smooth term $\psi(x) = (1/2N)|x|^2$ the members of $\mathscr{F}$ become subharmonic. Therefore standard facts [He] for subharmonic distributions become available and we deduce e.g. that $\mathscr{F} \subset L^1_{\mathrm{loc}}(\mathbf{R}^N)$, that every $\varphi \in \mathscr{F}$ has a unique representative in form of an upper semicontinuous function (with values in $\mathbf{R} \cup \{-\infty\}$) and that $\varphi_1, \varphi_2 \in \mathscr{F}$ implies $\sup(\varphi_1, \varphi_2) \in \mathscr{F}$ ($\mathscr{F}$ is "upward directed"). Also note that the members of $\mathscr{F}$ are locally uniformly bounded from above.

Set $u = \sup\{\varphi \colon \varphi \in \mathscr{F}\}$ (the pointwise supremum of the upper semicontinuous representatives) and $V^\mu = \inf\{\varphi \colon \varphi \text{ is upper semicontinuous and } \varphi \geqq u\}$. Then V^μ is upper semicontinuous and it is easy to check that $V^\mu \leqq U^\mu$ (in the distribution sense) and that $V^\mu + \psi$ is subharmonic (see [He, Theorem 4.16]). Hence $V^\mu \in \mathcal{F}$ and, by construction, $\varphi \leqq V^\mu$ for all $\varphi \in \mathcal{F}$.

To prove that $V^\mu = U^\mu$ outside a compact set it is enough to prove that there exists a function $u \in \mathcal{F}$ with this property, and this was done already in the beginning of the proof.

If $x \in \omega(\mu)$ then $-\Delta V^\mu = 1$ in a neighbourhood of x, for otherwise one could get a larger element in $\mathcal{F}$ by changing V^μ in some small ball B centered at x to $P - \psi$, where P is the Poisson integral of $V^\mu + \psi$ with respect to B (see [He, Lemma 4.17]). This shows that $\omega(\mu) \subset \Omega(\mu)$ and completes the proof of the theorem. ■

Theorem 2.2. *Let $\mu \in M_c$ and define $\omega(\mu)$ and $\Omega(\mu)$ as in Theorem* 1.1. *Then every $D \in Q(\mu, SL^1)$ satisfies*

$$\omega(\mu) \subset D \subset \Omega(\mu), \tag{2.3}$$
$$|\Omega(\mu) \setminus D| = 0.$$

In particular, either $\Omega(\mu) \in Q(\mu, SL^1)$ (and is the maximum open set in $Q(\mu, SL^1)$) or $Q(\mu, SL^1) = \varnothing$.

Proof. Suppose $D \in Q(\mu, SL^1)$. Then by Proposition 1.1, $U^D \leqq U^\mu$ in $\mathbf{R}^N$, $U^D = U^\mu$ on D^c. Thus $U^D \in \mathcal{F}$ with $\mathcal{F}$ as in Theorem 2.1, showing that $U^D \leqq V^\mu \leqq U^\mu$. From this it is clear that $\omega(\mu) \subset D$. To prove that $D \subset \Omega(\mu)$, $|\Omega(\mu) \setminus D| = 0$, i.e. that $[D] = \Omega(\mu)$, it is enough (and necessary) to prove that $U^D \geqq V^\mu$ for then $U^D = V^\mu$ and the definition of $\Omega(\mu)$ will show that $\Omega(\mu)$ is the largest open set in which $\chi_D = -\Delta U^D$ equals one a.e.

Set $w = V^\mu - U^D$. Then $\Delta w = \Delta V^\mu + \chi_D \geqq -1$ shows that w has an upper semicontinuous representative. On D^c, $w = V^\mu - U^\mu \leqq 0$. In D, $\Delta w = \Delta V^\mu + 1 \geqq 0$, i.e. w is subharmonic in D. By the maximum principle and the upper semicontinuity of w it now follows that $w \leqq 0$ also in D. This proves that $V^\mu \leqq U^D$ in $\mathbf{R}^N$ as desired. ■

When $\Omega(\mu) \in Q(\mu, SL^1)$, then typically (but not always) also $\omega(\mu) \in Q(\mu, SL^1)$. Clearly $\omega(\mu)$ then is the minimum open set in $Q(\mu, SL^1)$ and $Q(\mu, SL^1)$ consists of all open sets D in the interval $\omega(\mu) \subset D \subset \Omega(\mu)$. Cf. Remarks 2.4 and 2.6.

Theorem 2.1 shows that one can define a (non-linear) operator $F\colon M_c \to M_c$ by

$$F(\mu) = -\Delta V^\mu \qquad (\mu \in M_c). \tag{2.4}$$

This operator will turn out to be useful in the sequel. Clearly $F(\mu) \leqq 1$ for all $\mu \in M_c$ and $F(\mu) = \mu$ when $\mu \leqq 1$ (since then $U^\mu \in \mathcal{F}$ so that $V^\mu = U^\mu$). Hence $F(F(\mu)) = F(\mu)$ $(\mu \in M_c)$ which shows that F is some kind of projection operator onto the set $\{\mu \in M_c \colon \mu \leqq 1\}$. It is useful to think of F as a balayage operator.

Some further properties of F are

Theorem 2.3. (a) *If* $\mu_2 \geqq 0$ *then* $F(F(\mu_1) + \mu_2) = F(\mu_1 + \mu_2)$.

(b) *If* μ_- *has finite energy then so has* $F(\mu)$. *In this case*

$$\int (U^\mu - V^\mu)(1 + \Delta V^\mu) = 0 \tag{2.5}$$

which shows that $F(\mu)$ *is the orthogonal projection, with respect to the energy inner product, of* μ *onto* $\{\nu \in M_c : \nu \leqq 1\}$.

(c) *If* μ_- *has finite energy then*

$$\min(1, \mu) \leqq F(\mu). \tag{2.6}$$

More generally, if $\mu \leqq \nu$ *and* μ_- *has finite energy then* $F(\mu) \leqq F(\nu)$.

Remark 2.1. (2.5) can be looked upon as a refined version of $\omega(\mu) \subset \Omega(\mu)$ in Theorem 2.1. The operator F has earlier been used in [Gu 4], [Gu 5] to construct weak solutions to certain ill-posed moving boundary problems. Theorem 2.3 (and some subsequent results below) generalize some results in [Gu 4].

In (c) above (as well as in Corollary 2.2 below) the assumption that μ_- shall have finite energy is probably artificial.

Proof. (a) We have to show that

$$V^{F(\mu_1)+\mu_2} = V^{\mu_1+\mu_2} \tag{2.7}$$

if $\mu_2 \geqq 0$. Using that for a general ν, $V^\nu = U^{F(\nu)}$ is the largest function $\leqq U^\nu$ satisfying $-\Delta V^\nu \leqq 1$, we get

$$V^{F(\mu_1)+\mu_2} \leqq U^{F(\mu_1)+\mu_2} = U^{F(\mu_1)} + U^{\mu_2} = V^{\mu_1} + U^{\mu_2} \leqq U^{\mu_1} + U^{\mu_2} = U^{\mu_1+\mu_2}$$

and hence, since $-\Delta V^{F(\mu_1)+\mu_2} \leqq 1$, $V^{F(\mu_1)+\mu_2} \leqq V^{\mu_1+\mu_2}$. On the other hand $V^{\mu_1+\mu_2} - U^{\mu_2} \leqq U^{\mu_1}$ and (when $\mu_2 \geqq 0$) $-\Delta(V^{\mu_1+\mu_2} - U^{\mu_2}) \leqq 1$ which shows that $V^{\mu_1+\mu_2} - U^{\mu_2} \leqq V^{\mu_1}$, hence that

$$V^{\mu_1+\mu_2} \leqq V^{\mu_1} + U^{\mu_2} = U^{F(\mu_1)} + U^{\mu_2} = U^{F(\mu_1)+\mu_2}.$$

Since $-\Delta V^{\mu_1+\mu_2} \leqq 1$ this yields $V^{\mu_1+\mu_2} \leqq V^{F(\mu_1)+\mu_2}$, which completes the proof of (2.7).

(b) Assume μ_- has finite energy and set

$$u_\varepsilon = U^{\mu_+} * h_\varepsilon - U^{\mu_-}$$

for $\varepsilon > 0$. Then $u_\varepsilon \leqq U^\mu$. If moreover ε is large enough, say $\varepsilon \geqq \varepsilon_0$, then $-\Delta u_\varepsilon \leqq 1$ so that $u_\varepsilon \in \mathscr{F}$ and hence $u_\varepsilon \leqq V^\mu$. Choose an open ball B such that $\operatorname{supp} \mu \subset B$ and such that $u_{\varepsilon_0} = U^\mu$ in a neighbourhood of B^c and let φ be the solution of the Dirichlet problem

$$\begin{cases} -\Delta\varphi = 1 & \text{in } B, \\ \varphi = U^{\mu} & \text{on } \partial B. \end{cases}$$

Also set

$$\psi = \varphi - U^{\mu},$$

$$v = \varphi - V^{\mu}$$

$$w_{\varepsilon} = \varphi - u_{\varepsilon},$$

in B. Then $v = \psi = w_{\varepsilon_0} = 0$ on ∂B and v, ψ and w_{ε_0} coincide and are smooth near ∂B. Since $u_{\varepsilon_0} \leqq V^{\mu} \leqq U^{\mu}$, $\psi \leqq v \leqq w_{\varepsilon_0}$. Moreover $-\Delta v \geqq 0$ and $-\Delta w_{\varepsilon_0} \geqq 0$. Thus also $0 \leqq v \leqq w_{\varepsilon_0}$.

To prove that $F(\mu)$ has finite energy we shall take $\varepsilon = \varepsilon_0$ while in proving (2.5) we shall let $\varepsilon \to 0$.

Observe that $\operatorname{supp} F(\mu) \subset B$ by the choice of B. Therefore $F(\mu) = -\Delta V^{\mu}$ having finite energy is equivalent to $-\Delta v$ having finite Green energy in B, which also is the same as saying that v is in the Sobolev space $H_0^1(B)$. Set $\nu = -\Delta v$, $\lambda = -\Delta w_{\varepsilon_0}$. Then λ has finite energy, since μ_- has, and using the above inequalities we obtain

$$\begin{aligned} \iint_{B\times B} G(x,y)d\nu(x)d\nu(y) &= \int_B v d\nu \leqq \int_B w_{\varepsilon_0} d\nu = \int_B v d\lambda \\ &\leqq \int_B w_{\varepsilon_0} d\lambda = \iint_{B\times B} G(x,y)d\lambda(x)d\lambda(y) < \infty \end{aligned} \tag{2.8}$$

where $G(x,y)$ is the Green's function in B (so that $v(x) = \int_B G(x,y)d\nu(y)$ etc.). Thus ν has finite energy and $v \in H_0^1(B)$.

Next let

$$K = \{u \in H_0^1(B) \colon u \geqq \psi\},$$

which is a closed convex cone in the Sobolev space $H_0^1(B)$. Moreover K is non-empty, for w_{ε_0}, $v \in K$. Therefore the minimum norm problem

$$\text{Minimize} \int_B |\nabla u|^2 \qquad \text{for } u \in K$$

or, equivalently, the variational inequality

$$u \in K \quad \text{and}$$

(2.9)

$$\int_B \nabla(w-u)\nabla u \geqq 0 \qquad \text{for all } w \in K$$

has a (unique) solution, which we denote u.

In (2.9) $w - u$ is allowed to be any non-negative function in $H_0^1(B)$. This shows that

$$-\Delta u \geqq 0.$$

We can also choose $w = v$ in (2.9), giving

$$\int_B \nabla(v-u)\nabla u \geqq 0. \tag{2.10}$$

On the other hand $v \leqq u$, for otherwise the function U defined by $U = \varphi - \inf(u, v) = \sup(\varphi - u, \varphi - v)$ in B, $U = U^\mu$ outside B, would be an element in $\mathcal{F}$ larger than V^μ. Therefore

$$\int_B \nabla(v-u)\nabla v = \langle -\Delta v, v-u \rangle \leqq 0, \tag{2.11}$$

where $\langle \ , \ \rangle$ denotes the dual pairing between $H^{-1}(B)$ and $H_0^1(B)$. (2.10) and (2.11) now show that

$$\| v - u \|^2_{H_0^1(B)} = \int_B \nabla(v-u)\nabla(v-u) \leqq 0,$$

hence that

$$u = v. \tag{2.12}$$

Next we choose $w = w_\varepsilon$ in (2.9) and let $\varepsilon \searrow 0$. Note that $w_\varepsilon \in K$ if $\varepsilon > 0$ is so small that $\operatorname{supp}(\mu_+ * h_\varepsilon) \subset B$ (to guarantee that $w_\varepsilon = 0$ on ∂B). This gives

$$0 \leqq \int \nabla(w_\varepsilon - u)\nabla u = \langle -\Delta u, w_\varepsilon - u \rangle = \int_B (w_\varepsilon - u)d\nu \quad \text{where } \nu = -\Delta u \geqq 0,$$

for it is known [Bz–Bw] that when $\nu \in H^{-1}(B)$ is a measure any $\xi \in H_0^1(B)$ is integrable with respect to ν and $\langle \nu, \xi \rangle = \int \xi d\nu$. As $\varepsilon \searrow 0$, $w_\varepsilon \searrow \psi$. It follows that $\psi - u$ is integrable with respect to ν and that $\int (\psi - u)d\nu \geqq 0$. Since $u \geqq \psi$ we conclude that $\int (u - \psi)dv = 0$. By means of (2.12) and the definitions of v and ψ this yields (2.5).

(c) In terms of the notations above for the "obstacle problem" (2.9) and using (2.12) the inequality (2.6) reads

$$-\Delta u \leqq (-\Delta\psi)_+. \tag{2.13}$$

This inequality is known to be true if $-\Delta\psi$ has finite energy. See [Sa 4], [R, Ch. 5]. However the proof in e.g. [Sa 4] works also if merely $(-\Delta\psi)_+$ has finite energy, which is the same as saying that μ_- has finite energy. For convenience let us give the details.

Let $\xi \in H_0^1(B)$ be the unique solution of $-\Delta\xi = (-\Delta\psi)_+$ and let $w \in H_0^1(B)$ be the unique solution of the minimum norm problem ("obstacle problem")

$$\text{Minimize} \int_B |\nabla w|^2 \qquad \text{for } w \in H_0^1(B) \text{ satisfying } w \geqq \xi - u.$$

$w \in H_0^1(B)$ is also characterized as the unique solution of the complementarity problem

$$w \geqq \xi - u, \tag{2.14}$$

$$-\Delta w \geqq 0, \tag{2.15}$$

$$\langle -\Delta w, w - \xi + u \rangle = 0. \tag{2.16}$$

Observe that the present obstacle problem is slightly nicer than the former one (2.9) in that the obstacle function $\xi - u$ belongs to $H_0^1(B)$.

We are now going to prove that $w = \xi - u$. By (2.15) this will imply the desired inequality (2.13). (2.16) can be written $\langle \Delta w, \xi - w \rangle = \langle \Delta w, u \rangle$ and by (2.14) we have $\langle -\Delta\xi, \xi - w \rangle \leqq \langle -\Delta\xi, u \rangle$ (observe that $-\Delta\xi \geqq 0$). Using this we obtain

$$\| \xi - w \|^2 = \int |\nabla(\xi - w)|^2 = \langle -\Delta(\xi - w), \xi - w \rangle \leqq \langle -\Delta(\xi - w), u \rangle =$$
$$\int \nabla(\xi - w)\nabla u \leqq \| \xi - w \| \; \| u \|$$

and thus

$$\| \xi - w \| \leqq \| u \|. \tag{2.17}$$

Next $\xi - \psi \geqq \xi - u$ and $-\Delta(\xi - \psi) = (-\Delta\psi)_- \geqq 0$. Also $\xi - \psi = 0$ on ∂B so $\xi - \psi \geqq 0$ (but $\xi - \psi \notin H_0^1(B)$ in general). This shows that $\xi - \psi \geqq w$ (w is the smallest superharmonic function passing the obstacle). In fact, $\phi = \min(w, \xi - \psi)$ is superharmonic and satisfies $\xi - u \leqq \phi \leqq w$. Moreover $-\Delta\phi$

has finite energy by the same argument (2.8) as for $-\Delta v$, hence $\phi \in H_0^1(B)$. It is now clear that w also solves the obstacle problem with obstacle function ϕ. On the other hand, it is easily seen that this problem is also solved by ϕ itself. Hence, by uniquencess, $w = \phi$, i.e. $\xi - \psi \geqq w$ as claimed.

Thus $\xi - w \geqq \psi$, i.e. $\xi - w$ is a competing function in the first obstacle problem (2.9) and, together with (2.17) and the fact that u is unique, this shows that $\xi - w = u$ as desired. Thus (2.13) is now proved.

The last statement of (c) follows by combining (a) and (2.6). In fact, if $\mu \leqq \nu$ and μ_- has finite energy then, by (b), $F(\mu)$ has finite energy so that, using (a) and (2.6),

$$F(\mu) \leqq \min(F(\mu) + \nu - \mu, 1) \leqq F(F(\mu) + \nu - \mu) = F(\mu + \nu - \mu) = F(\nu).$$

This completes the proof of the theorem. ∎

Corollary 2.1. *If $\mu_- \in L^p(\mathbf{R}^N)$, $1 < p < \infty$, then $V^\mu \in W^{2,p}_{\mathrm{loc}}(\mathbf{R}^N)$. In particular V^μ is continuous if $p > N/2$, continuously differentiable if $p > N$.*

Corollary 2.2. *If $\mu \leqq \nu$ and μ_- has finite energy then $\Omega(\mu) \subset \Omega(\nu)$.*

Remark 2.2. A related, but simpler, result is: if $\mu \leqq \nu$ then $\omega(\mu) \subset \omega(\nu)$. To prove this just notice that $U^\mu - U^\nu + V^\nu \leqq U^\mu$ and $-\Delta(U^\mu - U^\nu + V^\nu) \leqq \mu - \nu + 1 \leqq 1$ so that $U^\mu - U^\nu + V^\nu \leqq V^\mu$ by the definition of V^μ. Hence $U^\mu - V^\mu \leqq U^\nu - V^\nu$ and so $\omega(\mu) \subset \omega(\nu)$.

Proposition 2.1. *If $\mu \in L_c^\infty$ then*

$$F(\mu) = \chi_{\Omega(\mu)} + \mu\chi_{\Omega(\mu)^c}. \tag{2.18}$$

Moreover, the following are equivalent (when $\mu \in L_c^\infty$):

(i) $Q(\mu, SL^1) \neq \varnothing$;

(ii) $\mu = 0$ *on* $\Omega(\mu)^c$;

(iii) $F(\mu) = \chi_{\Omega(\mu)}$;

(iv) $F(\mu) = \chi_D$ *for some open set* D.

Proof. By Theorem 2.3, $F(\mu) \in L^\infty$. That $F(\mu) = 1$ on $\Omega(\mu)$ is obvious from the definition (2.2) of $\Omega(\mu)$. Since $V^\mu \in W^{2,p}_{\mathrm{loc}}$ (all $p < \infty$) by Corollary 2.1 it follows that $F(\mu) = -\Delta V^\mu = -\Delta U^\mu = \mu$ a.e. on $\{x \in \mathbf{R}^N : V^\mu(x) = U^\mu(x)\} = \omega(\mu)^c \supset \Omega(\mu)^c$ (observe also that V^μ and U^μ are both continuous). This proves (2.18).

Now the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) are obvious (using Theorem 2.2 and (2.18)). Suppose (iv) holds. Then $U^D = U^{F(\mu)} = V^\mu \leqq U^\mu$. Moreover $\int (U^\mu - U^D)(1 - \chi_D) = 0$, i.e. $U^D = U^\mu$ a.e. on D^c, by (b) of Theorem 2.3. Proposition 1.2 now shows that $[D] \in Q(\mu, SL^1)$, hence (i) holds. ∎

Remark 2.3. If (iv) holds it does not follow that $D \in Q(\mu, SL^1)$, just that $[D] = \Omega(\mu) \in Q(\mu, SL^1)$. In fact, (iv) is not affected if a single point is removed from D while Theorem 2.1 shows that D can never be in $Q(\mu, SL^1)$ if some point from $\omega(\mu)$ is missing.

Corollary 2.3. *If $\mu \in L_c^\infty$ and, for some bounded open set D, $\mu \geqq 1$ on D, $\mu = 0$ outside D then $D \subset \Omega(\mu)$, $F(\mu) = \chi_{\Omega(\mu)}$ and $Q(\mu, SL^1) \neq \varnothing$. Moreover, $\Omega(\mu)$ is connected if D is. (More generally: every component of $\Omega(\mu)$ meets D.)*

Proof. Since $\Omega(\mu)$ by definition is the largest open set on which $F(\mu) = 1$, the first assertions follow directly from the proposition.

To prove the last statement, suppose on the contrary that $\Omega(\mu)$ has a component G which does not meet D. Then $V^\mu - U^\mu = 0$ on $\partial G \subset \Omega(\mu)^c \subset \omega(\mu)^c$ while in G, $-\Delta(V^\mu - U^\mu) = 1 - \mu = 1$ which, since $V^\mu - U^\mu \leqq 0$, contradicts the minimum principle for superharmonic functions. ■

Remark 2.4. The conclusion of Corollary 2.3 can be slightly refined as follows. Assume for simplicity that D is connected. Then there are two cases: (i) $\mu = 1$ (a.e.) in D. In this case $V^\mu \equiv U^\mu$ hence $\omega(\mu) = \varnothing$, $\Omega(\mu) = [D]$. Thus $Q(\mu, SL^1)$ consists of all open sets $G \subset \Omega(\mu)$ with $|\Omega(\mu) \setminus G| = 0$. In particular $Q(\mu, SL^1)$ does not contain any minimum set. (ii) $\int d\mu > |D|$. Since $-\Delta(U^\mu - V^\mu) = \mu - 1 \geqq 0$ (and $\not\equiv 0$) in D and $U^\mu - V^\mu \geqq 0$, this implies that $U^\mu - V^\mu > 0$ in D by the minimum principle for superharmonic functions. Hence $D \subset \omega(\mu)$. Therefore also $0 = \mu = -\Delta U^\mu = -\Delta V^\mu = 1$ a.e. in $\Omega(\mu) \setminus \omega(\mu)$, i.e. $|\Omega(\mu) \setminus \omega(\mu)| = 0$. Using Proposition 1.2 it now follows that $\omega(\mu) \in Q(\mu, SL^1)$. Obviously $\omega(\mu)$ then is the minimum open set in $Q(\mu, SL^1)$ and $Q(\mu, SL^1)$ consists of all open sets G in the interval $\omega(\mu) \subset G \subset \Omega(\mu)$.

This result, that under the above assumptions $\omega(\mu)$ is the minimum open set in $Q(\mu, SL^1)$, was the original existence theorem of Sakai, proved in [Sa 2] and [Sa 3].

After all the above preparations we can now prove a more general existence result, stating that $Q(\mu, SL^1)$ is non-empty under various assumptions on μ.

Theorem 2.4. *Let $\mu, \mu_1, \mu_2, \ldots, \nu \in M_c$.*

(i) *If $Q(\mu_j, SL^1) \neq \varnothing$ $(j = 1, 2)$ then $Q(\mu_1 + \mu_2, SL^1) \neq \varnothing$.*

(ii) *If $Q(\mu, SL^1) \neq \varnothing$ and $\alpha \geqq 1$ then $Q(\alpha\mu, SL^1) \neq \varnothing$.*

(iii) *If $\mu_n \geqq 0$, $\mu_n \nearrow \mu$ as $n \to \infty$ (in the sense that $\mu_n(E) \nearrow \mu(E)$ for every Borel set E) and $Q(\mu_n, SL^1) \neq \varnothing$ then $Q(\mu, SL^1) \neq \varnothing$ (in fact $\bigcup_{n=1}^\infty \Omega(\mu_n) \in Q(\mu, SL^1)$).*

(iv) *If $\mu, \nu \geqq 0$ and $Q(\mu * h_\varepsilon + \nu, SL^1) \neq \varnothing$ for some $h_\varepsilon \in L^\infty$ as in (1.6) then $Q(\mu + \nu, SL^1) \neq \varnothing$ and $\Omega(\mu + \nu) = \Omega(\mu * h_\varepsilon + \nu)$.*

(v) *If $\mu \geqq 0$ and, for some ball $B(a; r)$, $\mu = 0$ outside $B(a; r)$ and $\mu(B(a; r)) \geqq 6^N m(B(a; r))$ then $Q(\mu, SL^1) \neq \varnothing$. Moreover $\Omega(\mu)$ is connected.*

(vi) *If $\mu \geqq 0$ and μ is singular with respect to Lebesgue measure then $Q(\mu, SL^1) \neq \varnothing$.*

(vii) *If there is an open set* D *such that* $\mu \geqq 1$ *on* D, $\mu = 0$ *outside* D *then* $Q(\mu, SL^1) \neq \varnothing$. *Moreover* $D \subset \Omega(\mu)$ *and* $\Omega(\mu)$ *is connected if* D *is*. (*More generally, every component of* $\Omega(\mu)$ *meets* D.)

Proof. (i) Set $\Omega_j = \Omega(\mu_j)$, $j = 1, 2$. By Corollary 2.3 there exists $\Omega \in Q((\chi_{\Omega_1} + \chi_{\Omega_2})m, SL^1)$. We claim that $\Omega \in Q(\mu_1 + \mu_2, SL^1)$.

First of all $\Omega_1 \cup \Omega_2 \subset \Omega$ by Corollary 2.3, which shows that (i′) and (ii′) in the definition (1.4′) of the class $Q(\mu_1 + \mu_2, SL^1)$ are fulfilled. But also (iii′) is fulfilled, for

$$\int \varphi d(\mu_1 + \mu_2) \leqq \int_{\Omega_1} \varphi + \int_{\Omega_2} \varphi = \int \varphi(\chi_{\Omega_1} + \chi_{\Omega_2}) dm \leqq \int_{\Omega} \varphi$$

for all $\varphi \in SL^1(\Omega)$ by assumption and Corollary 2.3.

(ii) is proved similarly.

As to (iii) set $\Omega = \bigcup_{n=1}^{\infty} \Omega(\mu_n)$. Observe that $\Omega(\mu_1) \subset \Omega(\mu_2) \subset \cdots$ by Corollary 2.2. It is immediate that $\mu = 0$ on Ω^c since this is true for each μ_n. If $\varphi \in SL^1(\Omega)$ then $\varphi_+ \in SL^1(\Omega) \subset SL^1(\Omega(\mu_n))$ for each n and

$$\int \varphi_+ d\mu_n \leqq \int_{\Omega(\mu_n)} \varphi_+ dm \leqq \int_{\Omega} \varphi_+ dm < \infty.$$

Since φ_+ can be approximated (e.g. in the $L^1(\mu)$-norm) by finite linear combinations of characteristic functions of Borel sets it follows, passing to the limit, that $\int \varphi_+ d\mu \leqq \int_\Omega \varphi_+ dm < \infty$.

Also, $\int \varphi d\mu_n \to \int \varphi d\mu$ and $\int_{\Omega(\mu_n)} \varphi \to \int_\Omega \varphi$ as $n \to \infty$. Therefore, passing to the limit in $\int \varphi d\mu_n \leqq \int_{\Omega(\mu_n)} \varphi dm$ we see that the final requirement for $\Omega \in Q(\mu, SL^1)$ is fulfilled.

(iv) Set $\Omega = \Omega(\mu * h_\varepsilon + \nu)$ and $K_\varepsilon = \{x \in \Omega: \operatorname{dist}(x, \Omega^c) \geqq \varepsilon\}$. Then $\mu = 0$ outside K_ε for, in the contrary case, $\mu * h_\varepsilon$ could not vanish outside Ω, which it does since $\Omega \in Q(\mu * h_\varepsilon + \nu, SL^1)$.

Now take $\varphi \in SL^1(\Omega)$. Then also $\varphi_+ \in SL^1(\Omega)$. By the submean-value property we have $\varphi \leqq \varphi * h_\varepsilon$ and $\varphi_+ \leqq \varphi_+ * h_\varepsilon$ in K_ε. Therefore

$$\int_\Omega \varphi d\mu = \int_{K_\varepsilon} \varphi d\mu \leqq \int_{K_\varepsilon} (\varphi * h_\varepsilon) d\mu = \int_\Omega (\varphi * h_\varepsilon) d\mu = \int_\Omega \varphi d(\mu * h_\varepsilon) \quad (< \infty)$$

and similarly for φ_+. From this it follows that $\Omega \in Q(\mu + \nu, SL^1)$ and in fact (since $\Omega = [\Omega]$) that $\Omega = \Omega(\mu + \nu)$.

(v) Let

$$h = \frac{1}{|B(0; 2r)|} \chi_{B(0;2r)}.$$

We claim that $Q(\mu * h, SL^1) \neq \varnothing$, which by (iv) will prove that $Q(\mu, SL^1) \neq \varnothing$.

We have $\mu * h \geqq 0$,

$$\mu * h \geqq \frac{6^N |B(a; r)|}{|B(0; 2r)|} = 3^N \qquad \text{on } B(a; r),$$

$$\mu * h = 0 \qquad \text{outside } B(a; 3r).$$

Set $\nu = 3^N \chi_{B(a;r)}$. Then $B(a; 3r) \in Q(\nu, SL^1)$, hence $F(\nu) = \chi_{B(a;3r)}$. Writing $\mu * h = \nu + (\mu * h - \nu)$ and observing that $\mu * h - \nu \geqq 0$, (a) of Theorem 2.3 therefore gives $F(\mu * h) = F(\chi_{B(a;3r)} + \mu * h - \nu)$. But now Corollary 2.3 applies (to the measure $\chi_{B(a;3r)} + \mu * h - \nu$ and with $D = B(a; 3r)$) and shows that $F(\mu * h) = \chi_\Omega$ for some connected open set Ω. Thus $Q(\mu * h, SL^1) \neq \varnothing$ (by Proposition 2.1) and $\Omega(\mu * h) = [\Omega]$.

(vi) Suppose $\mu \geqq 0$, μ singular with respect to Lebesgue measure and let

$$\rho = \sup\{r \geqq 0: \mu(B(a; r)) \geqq 6^N m(B(a; r)) \text{ for some ball } B(a; r)\}.$$

Choose a ball $B(a; r) = B(a_1; r_1)$ as above with $r_1 \geqq \rho/2$ (actually the supremum is attained so we could take $r_1 = \rho$, but this does not matter) and set

$$\nu_1 = \mu_{|B(a_1;r_1)}.$$

Now repeat the same procedure with $\mu - \nu_1$ in place of μ, let $B(a_2, r_2)$ be a corresponding new ball and set $\nu_2 = (\mu - \nu_1)|_{B(a_2;r_2)}$. Continuing in this way we obtain a sequence $\nu_1, \nu_2, \nu_3, \ldots$: in the nth step the procedure is applied to $\mu - \nu_1 - \cdots - \nu_{n-1}$ and ν_n is defined by

$$\nu_n = (\mu - \nu_1 - \cdots - \nu_{n-1})|_{B(a_n, r_n)}.$$

Set $\mu_n = \nu_1 + \nu_2 + \cdots + \nu_n$.

By the definition of ν_n and (v) of the theorem $Q(\nu_n, SL^1) \neq \varnothing$, hence by repeated use of (i) $Q(\mu_n, SL^1) \neq \varnothing$. We claim that $\mu_n \nearrow \mu$ (setwise) which by (iii) will prove that $Q(\mu, SL^1) \neq \varnothing$.

Let E be a Borel set. Clearly $\mu_n(E)$ increases with n. Set $\nu(E) = \sup_n \mu_n(E)$. The set function ν so defined is easily seen to be a measure and by definition $\mu_n \nearrow \nu$ setwise. Therefore it remains to prove just that $\lambda = \mu - \nu$ is the zero measure.

Clearly $0 \leqq \lambda \leqq \mu$, in particular λ is singular with respect to Lebesgue measure. On the other hand

$$\frac{\lambda(B)}{m(B)} \leqq 6^N \tag{2.19}$$

for every ball $B = B(a; r)$ (with $r > 0$). In fact, since μ is a finite measure the radii r_n in the definition of v_n tend to zero as $n \to \infty$ and if (2.19) did not hold for some ball $B = B(a; r)$ $(r > 0)$ we would have, for all n,

$$(\mu - \mu_n)(B(a; r)) \geqq \lambda(B(a; r)) \geqq 6^N m(B(a; r))$$

which says that the successive suprema $\rho = \rho_n$ above all satisfy $\rho_n \geqq r$, a contradiction.

Now (2.19) implies that λ is absolutely continuous with density function $\leqq 6^N$. Since λ also is singular we conclude that $\lambda = 0$.

(vii) Let $D_n = \{x \in D: \operatorname{dist}(x, D^c) > 1/n\}$ $(n = 1, 2, \ldots)$. By (iii) it is enough to prove that $Q(\mu_n, SL^1) \neq \varnothing$ where $\mu_n = v_n + \chi_D$, $v_n = (\mu - 1)|_{D_n}$. By Corollary 2.3, $Q(v_n * h_\varepsilon + \chi_D) \neq \varnothing$ if $0 < \varepsilon \leqq 1/n$ and by (iv) this implies $Q(v_n + \chi_D) \neq \varnothing$.

Since $\Omega(\mu) = [\bigcup_{n=1}^{\infty} \Omega(\mu_n)]$ by (iii) the last assertions of (vii) follow easily from the corresponding assertion in Corollary 2.3. ■

Remark 2.5. In (v) 6^N is not the best constant. The smallest constant for which our idea of proof works is $(3 + 2\sqrt{2})^N$ (obtained by replacing the radii $2r$ and $3r$ in the proof by $(1 + \sqrt{2})r$ and $(2 + \sqrt{2})r$ respectively).

Remark 2.6. Using Remark 2.2 and Remark 2.4 we also obtain the following, for $\omega(\mu)$. In (iii): if $\omega(\mu_n) \in Q(\mu_n, SL^1)$ for all n then $\omega(\mu) \in Q(\mu, SL^1)$. In (iv): if $\omega(\mu * h_\varepsilon + v) \in Q(\mu + v, SL^1)$ then $\omega(\mu + v) \in Q(\mu + v, SL^1)$. In (v) and (vi): $\omega(\mu) \in Q(\mu, SL^1)$. In (vii): if D is connected and $\int d\mu > |D|$ then $\omega(\mu) \in Q(\mu, SL^1)$.

3. A partial order and some simple results related to it

We define a partial order $\prec$ among bounded open sets in $\mathbf{R}^N$ by saying that

$$D \prec \Omega \qquad \text{iff } U^D \geqq U^\Omega \text{ in all } \mathbf{R}^N.$$

Strictly speaking $\prec$ is a partial order only on equivalence classes of sets since clearly $D \prec \Omega$ and $\Omega \prec D$ hold simultaneously if (and only if) D and Ω differ by a null set.

It is actually natural to regard $\prec$ as a partial order among measures (identifying Ω with $\chi_\Omega m$) and then it extends to all Radon measures with compact support ($\mu \prec v$ iff $U^\mu \geqq U^v$). With respect to the pairing $(\mu, v) = \int U^\mu dv$ $\prec$ then is the order which is dual to the usual partial order $\leqq$. Also note that in terms of the partial order $\prec$ the definition of the operator F (see (2.4)) can be written

$$F(\mu) = \inf\{v \in M_c: \mu \prec v, v \leqq m\}$$

where the infimum is taken with respect to $\prec$. Then notice (obvious) that

$$\mu \prec v \text{ implies } F(\mu) \prec F(v). \tag{3.1}$$

Cf. (c) of Theorem 2.3.

We will use $\prec$ only between sets of equal volume (e.g. on $Q(\mu, AL^1)$ for a fixed μ) and then the usual partial order $\subset$ (inclusion) can be regarded as a refinement of $\prec$: an inclusion $D \subset \Omega$ can hold only if D and Ω are equivalent with respect to $\prec$ (i.e. $D \prec \Omega$ and $\Omega \prec D$).

This section of the paper is largely self-contained and most results are quite simple. Nevertheless I think that they demonstrate the usefulness of considering the partial order $\prec$ on our classes of quadrature domains. In particular they give new proofs of many of Sakai's results in [Sa 2].

We first give a couple of examples which show that within each class $Q(\mu, AL^1)$ an order relationship $D \prec \Omega$ means that D is in some sense "more solid" than Ω.

Example 3.1. Let $\mu = \alpha\rho$ $(\alpha > 0)$, $\Omega_t = \{x \in \mathbf{R}^N : t < \omega_N |x|^N < t + \alpha\}$ $(t \geqq 0)$ as in Example 1.2. Then a simple computation shows that $\Omega_{t_1} \prec \Omega_{t_2}$ iff $t_1 \leqq t_2$. Taking α in the interval $\omega_N < \alpha < A$ (e.g.) we see from Example 1.2 that the solid domain $\Omega = \Omega_0 \cup \{0\}$ is minimal in $Q(\mu, AL^1)$ and $Q(\mu, HL^1)$ with respect to $\prec$, that the unique domain Ω_{t_α} in $Q(\mu, SL^1)$ is the maximum domain in $Q(\mu, HL^1)$ with respect to $\prec$ but that $Q(\mu, AL^1)$ does not contain any maximum (or even maximal) domain.

With Example 3.1 as a starting point one can build up more complicated examples. To do this we first introduce a convenient notation. If Ω_1 and Ω_2 are two bounded open sets we set

$$\Omega_1 + \Omega_2 = \Omega((\chi_{\Omega_1} + \chi_{\Omega_2})m),$$

with $\Omega(\mu)$ defined by (2.2). Thus, by Corollary 2.3, $\Omega_1 + \Omega_2 \in Q((\chi_{\Omega_1} + \chi_{\Omega_2})m, SL^1)$. In terms of F

$$F(\chi_{\Omega_1} + \chi_{\Omega_2}) = \chi_{\Omega_1 + \Omega_2} \tag{3.2}$$

and together with $\Omega_1 + \Omega_2$ being "complete" ($\Omega_1 + \Omega_2 = [\Omega_1 + \Omega_2]$) this characterizes $\Omega_1 + \Omega_2$.

If $\Omega_1 \cap \Omega_2 = \varnothing$ then simply $\Omega_1 + \Omega_2 = [\Omega_1 \cup \Omega_2]$, otherwise $\Omega_1 + \Omega_2$ is an open set strictly including $[\Omega_1 \cup \Omega_2]$. It is immediately verified that if $\Omega_j \in Q(\mu_j, \Lambda)$ $(j = 1, 2, \Lambda = AL^1, HL^1$ or $SL^1)$ then $\Omega_1 + \Omega_2 \in Q(\mu_1 + \mu_2, \Lambda)$. Also it is immediate from (3.1), (3.2) that $\Omega_1 \prec \Omega_1'$, $\Omega_2 \prec \Omega_2'$ implies $\Omega_1 + \Omega_2 \prec \Omega_1' + \Omega_2'$.

Example 3.2. Let $x_1 = (-1, 0, \ldots, 0)$, $x_2 = (1, 0, \ldots, 0)$, let ρ_j be the unit mass uniformly distributed on $S_j = \{x \in \mathbf{R}^N : |x - x_j| = 1\}$ $(j = 1, 2)$ and let $\mu = \alpha(\rho_1 + \rho_2)$ for some α with $\omega_N < \alpha < A$ (see Example 1.2). Then we have a two-parameter family of (connected) domains $Q(\mu, AL^1)$, namely

$$\Omega_{s,t} = \Omega_s^{(1)} + \Omega_t^{(2)}$$

where $\Omega_t^{(j)} = \{x \in \mathbf{R}^N: t < \omega_N |x - x_j|^N < t + \alpha\}$ and $0 \leqq s, t < \omega_N$. This follows from the above discussion and Example 1.2.

Moreover $\Omega_{0,0}$, $\Omega_{t_*,0}$, Ω_{0,t_*} and Ω_{t_*,t_*} are in $Q(\mu, HL^1)$ and $\Omega_{t_*,t_*} \in Q(\mu, SL^1)$. Also, $\Omega_{s,t} \prec \Omega_{s',t'}$ if $s \leqq s'$, $t \leqq t'$. The map $(s, t) \to \Omega_{s,t}$ is not always one-to-one but it is easy to see that, at least if α is close enough to ω_N, the four domains in $Q(\mu, HL^1)$ above are distinct. In fact, this follows from $\Omega_s^{(1)} \cup \Omega_t^{(2)} \subset \Omega_{s,t}$ combined with $|\Omega_s^{(1)}| + |\Omega_t^{(2)}| = |\Omega_{s,t}|$, which shows that, for suitable α, $\Omega_{0,0}$ has no holes at x_1 and x_2, $\Omega_{t_*,0}$ has a hole at x_1 but not at x_2, Ω_{0,t_*} has a hole at x_2 but not at x_1 and Ω_{t_*,t_*} has holes at both x_1 and x_2.

Now we turn to the general theory. In all results below Ω, D etc. denote arbitrary bounded open sets in $\mathbf{R}^N$ and μ (etc.) arbitrary Radon measures with compact support.

Proposition 3.1. *Suppose $D \prec \Omega$. Then $U^\Omega < U^D$ on $\bar{D} \setminus \bar{\Omega}$.*

Proof. Set $u = U^\Omega - U^D$. Then $\Delta u = \chi_D - \chi_\Omega$ and, by assumption, $u \leqq 0$. Suppose, to derive a contradiction, that $u(x) = 0$ for some $x \in \bar{D} \setminus \bar{\Omega} = \bar{D} \cap \Omega^e$. Let B be an open ball in Ω^e with center x. Since u is subharmonic in B we have

$$0 = u(x) \leqq \frac{1}{|B|} \int_B u$$

and it follows that $u \equiv 0$ in B. But this is a contradiction, for $x \in \bar{D} \cap \Omega^e$ implies that $\Delta u \not\equiv 0$ in B. ∎

Remark 3.1. In addition to the conclusion of Proposition 3.1 we have: for each component D_j of D either $U^\Omega < U^D$ in D_j or $|D_j \setminus \Omega| = 0$. In fact, with notations from the proof we have (u being subharmonic in D_j) either $u < 0$ in D_j or $u \equiv 0$ in D_j, and in the latter case $\chi_\Omega = \chi_D - \Delta u = 1$ (a.e.) in D_j.

Corollary 3.1. *Suppose $D \prec \Omega$ and $U^\Omega = U^D$ on $(\Omega \cup D)^c$ (or even just on $(\bar{\Omega} \cup D)^c$). Then $\partial D \subset \bar{\Omega}$.*

Proof. This is immediate from the proposition since $\partial D \setminus \bar{\Omega} \subset (\bar{\Omega} \cup D)^c \cap (\bar{D} \setminus \bar{\Omega})$. ∎

Proposition 3.2. *Suppose $D \not\prec \Omega$ and, in case $N = 2$, $|D| \leqq |\Omega|$. Then there is a point $x \in \bar{\Omega} \setminus D$ satisfying $U^D(x) < U^\Omega(x)$, $\nabla U^D(x) = \nabla U^\Omega(x)$.*

Proof. Set $u = U^\Omega - U^D$, $M = \sup u$, $S = \{x \in \mathbf{R}^N: u(x) = M\}$. By the assumptions $M > 0$ and $\limsup u(x) \leqq 0$ as $|x| \to \infty$. Hence S is non-empty and bounded.

Suppose $S \subset (\bar{\Omega} \setminus D)^c = \Omega^e \cup D$. Since u is subharmonic in $\Omega^e \cup D$ this would imply that S is an open subset of $\mathbf{R}^N$ and since S obviously also is a closed set this leads to a contradiction to the first part of the proof. Hence $S \not\subset (\bar{\Omega} \setminus D)^c$, i.e. $S \cap (\bar{\Omega} \setminus D) \neq \varnothing$. Since $\nabla u = 0$ and $u > 0$ on S this proves the proposition. ■

Corollary 3.2. *Suppose $D \nprec \Omega$ and $U^\Omega = U^D$ on $(\Omega \cup D)^c$. Then there exists a point $x \in \Omega \setminus D$ satisfying $U^D(x) < U^\Omega(x)$, $\nabla U^D(x) = \nabla U^\Omega(x)$.*

Proof. Since $\partial\Omega \setminus D \subset (\Omega \cup D)^c$ the point x obtained in Proposition 3.2 must be in $\Omega \setminus D$. ■

Corollary 3.3. *Suppose Ω, $D \in Q(\mu, HL^1)$. Then*
(i) *if $D \prec \Omega$ then $\partial D \subset \bar{\Omega}$;*
(ii) *if $D \nprec \Omega$ then there exists a point $x \in \Omega \setminus D$ satisfying $U^\Omega(x) > U^\mu(x)$, $\nabla U^\Omega(x) = \nabla U^\mu(x)$.*

Proof. This is immediate from Corollary 3.1 and Corollary 3.2, just observing, for (ii), that $U^D = U^\mu$, $\nabla U^D = \nabla U^\mu$ outside D (Proposition 1.1). ■

Example 3.3. Using the notations of Example 1.2 and Example 3.1 we have, if $\omega_N < \alpha < A$, Ω, $\Omega_{t_\alpha} \in Q(\mu, HL^1)$ and
(i) $\Omega \prec \Omega_{t_\alpha}$, $\partial\Omega \subset \bar{\Omega}_{t_\alpha}$;
(ii) $\Omega_{t_\alpha} \nprec \Omega$ and $0 \in \Omega \setminus \Omega_{t_\alpha}$, $U^\Omega(0) > U^\mu(0)$, $\nabla U^\Omega(0) = \nabla U^\mu(0)$.
This illustrates Corollary 3.3.

Despite it being extremely simple Corollary 3.3 is quite useful. If, e.g., given $\Omega \in Q(\mu, HL^1)$, one finds that there is no point $x \in \Omega$ at all satisfying $U^\Omega(x) > U^\mu(x)$ and $\nabla U^\Omega(x) = \nabla U^\mu(x)$ then Corollary 3.3 shows that Ω is the maximum domain (with respect to $\prec$) in $Q(\mu, HL^1)$ and that $\partial D \subset \bar{\Omega}$ for all $D \in Q(\mu, HL^1)$. This applies e.g. when μ is a (positive) point mass and $\Omega \in Q(\mu, HL^1)$ the appropriate ball and then easily gives that $Q(\mu, HL^1) = \{\Omega\}$. This result was first proved in [E–Sc] and [Ku]. In Example 3.4 below we obtain a stronger result.

The above remarks also apply when $\Omega \in Q(\mu, SL^1)$:

Corollary 3.4. *Suppose $D \in Q(\mu, HL^1)$, $\Omega \in Q(\mu, SL^1)$. Then $D \prec \Omega$.*

Proof. By Proposition 1.1, $U^\Omega \leqq U^\mu$ everywhere. ■

Thus if $Q(\mu, SL^1)$ is non-empty, its elements are the maximum domains, with respect to $\prec$, in $Q(\mu, HL^1)$. Note that this in particular gives (again) the uniqueness up to null-sets of elements in $Q(\mu, SL^1)$.

If inclusion $\subset$ is regarded as a refinement of $\prec$ as indicated in the beginning of this section it seems most natural to consider it going in the opposite direction. Thus the real "top" element in $Q(\mu, HL^1)$ (and $Q(\mu, SL^1)$) would be the *minimum* (with respect to $\subset$) open set in $Q(\mu, SL^1)$ when this exists. Cf. Remarks 2.4 and 2.6.

Clearly $\partial D \subset \bar{\Omega}$ implies $D \subset$ (the unbounded component of $\Omega^c)^c$. Hence

Lemma 3.1. *Suppose Ω is solid and $\partial D \subset \bar{\Omega}$. Then $D \subset \Omega$.*

Corollary 3.5. *Any solid $\Omega \in Q(\mu, HL^1)$ is minimal in $Q(\mu, HL^1)$ with respect to $\prec$. More precisely, $D \subset \Omega$, $|\Omega \setminus D| = 0$ whenever $D \in Q(\mu, HL^1)$, $D \prec \Omega$ (if $\Omega \in Q(\mu, HL^1)$ is solid).*

This is immediate from Corollary 3.3 and Lemma 3.1. Using Corollary 3.4 we also obtain

Corollary 3.6. *Suppose $\Omega \in Q(\mu, SL^1)$ is solid. Then $D \subset \Omega$, $|\Omega \setminus D| = 0$ for every $D \in Q(\mu, HL^1)$.*

Thus the answer of the uniqueness question (Q) in the introduction is negative whenever $Q(\mu, SL^1)$ contains a solid open set. Probably Corollary 3.6 can be sharpened to say that $Q(\mu, HL^1) = Q(\mu, SL^1)$ when $\Omega(\mu, SL^1)$ contains a solid element. (This is at least true if $\mu \in L_c^\infty$.) Corollary 3.6 is similar to [Sa 2, Proposition 9.1].

Next we come to another group of results.

Propositon 3.3. *Suppose $D \prec \Omega$ and $\nabla U^D = \nabla U^\Omega$ far away. Then $D \subset$ (the unbounded component of $\Omega^e)^e$.*

Corollary 3.7. *Suppose Ω is solid, $D \prec \Omega$ and $\nabla U^D = \nabla U^\Omega$ far away. Then $D \subset \Omega$ (and $|\Omega \setminus D| = 0$).*

Corollary 3.8. *Any solid $\Omega \in Q(\mu, AL^1)$ is minimal in $Q(\mu, AL^1)$ with respect to $\prec$.*

Proof. The corollaries are immediate consequences of the proposition so we need only prove the latter. Set $u = U^\Omega - U^D$. Thus $u \leqq 0$ in $\mathbf{R}^N$. Moreover $\nabla u = 0$ far away which implies (Remark 1.2) $u = 0$ far away and, by harmonic continuation, $u = 0$ on $\bar{G}$ where G is the unbounded component of $\Omega^e \cap D^e$. On the other hand $u < 0$ on $\Omega^e \cap \bar{D}$ by Proposition 3.1. Thus $\Omega^e \cap \bar{D} \cap \bar{G} = \varnothing$.

Let H be the unbounded component of Ω^e and we wish to prove that $D \cap \bar{H} = \varnothing$. Supposing that this is not true there must be a point $x \in D \cap H$. Let γ be a curve in H from x to infinity and let y be the last point of γ which belongs to $\bar{D}$. Then $y \in \bar{G}$, for the part of γ coming after y is in $\Omega^e \cap D^e$. Thus $y \in H \cap \bar{D} \cap \bar{G}$, which however contradicts the first part of the proof. Hence $D \cap \bar{H} = \varnothing$. ∎

Proposition 3.3 and its corollaries are a kind of parallel to Proposition 3.1 and its corollaries. We now wish to prove a corresponding parallel to Proposition 3.2.

Proposition 3.4. *Suppose Ω is solid, $\partial\Omega$ is regular for Hopf's maximum principle in Ω^e, $D \not< \Omega$ (or even $D \not\subset \Omega$) and that $\nabla U^\Omega = \nabla U^D$ far away. Then there is a point $x \in \Omega \setminus D$ with $U^\Omega(x) > U^D(x)$, $\nabla U^\Omega(x) = \nabla U^D(x)$.*

Proof. First observe that $D \not\subset \Omega$ implies $D \not< \Omega$ by Corollary 3.7. Now Proposition 3.2 can be applied and we only have to show that the x there can be chosen not on $\partial\Omega$. Referrring to the notations of the proof of Proposition 3.2 it is enough to prove that $S \cap \partial\Omega = \varnothing$. But $x \in S \cap \partial\Omega$ means that $x \in \partial\Omega^e$, $u(x) = M = \sup u$, $\nabla u(x) = 0$ and since u is subharmonic in the connected open set Ω^e this implies, by the Hopf maximum principle [Ho], [Ba–Cp], that $u \equiv M$ in Ω^e. Since $M > 0$ and $u = 0$ far away this is a contradiction. ■

Corollary 3.9. *Suppose $\Omega, D \in Q(\mu, AL^1)$, $D \neq \Omega$, Ω is solid and that $\partial\Omega$ is regular for Hopf's maximum principle in Ω^e. Then there is a point $x \in \Omega \setminus D$ satisfying $\nabla U^\Omega(x) = \nabla U^\mu(x)$ (with ∇U^μ pointwise well-defined at x).*

Proof. If $D \not\subset \Omega$ then Proposition 3.4 combined with the fact that $\nabla U^D = \nabla U^\mu$ outside D (Proposition 1.1) gives the desired conclusion. If $D \subset \Omega$ then $|\Omega \setminus D| = 0$ hence $U^D \equiv U^\Omega$, which shows (by Proposition 1.1) that $\nabla U^\Omega(x) = \nabla U^D(x) = \nabla U^\mu(x)$ for any $x \in \Omega \setminus D$. ■

Corollary 3.9 is similar to [Sa 2, Corollary 9.5]. It shows that the "special points" (in the terminology of [Sh 2]) $x \in \Omega$ for which $\nabla U^\Omega(x) = \nabla U^\mu(x)$ (with ∇U^μ pointwise well-defined at x) play an important role in the uniqueness question. It follows directly from the definition of a quadrature domain that these special points are exactly those points which can be deleted from $\Omega \in Q(\mu, AL^1)$ without destroying it being a quadrature domain. If $N = 2$ and μ has support in a finite number of points the number of special points can be estimated [Sa 7], [Gu 6].

Example 3.4. Let $\mu = \alpha\delta$ $(\alpha > 0)$, $\Omega = B(0; r)$ as in Example 1.1 $(|B(0; r)| = \alpha)$. Then $\Omega \in Q(\mu, AL^1)$ satisfies the hypotheses in Corollary 3.9 and a computation shows that $\nabla U^\Omega(x) \neq \nabla U^\mu(x)$ for all $x \in \Omega$ for which $\nabla U^\mu(x)$ is point-wise well-defined, namely for all $x \in \Omega \setminus \{0\}$. Hence Corollary 3.9 shows that $Q(\mu, AL^1)$ just consists of $\Omega = B(0; r)$.

Remark 3.2. Proposition 3.4 and Corollary 3.9 remain true if the hypothesis "Ω is solid" is replaced by "$\partial\Omega = \partial\Omega^e$, D is connected and D meets every component of Ω^e". For the proof we just have to notice that $D \cup \Omega^e$ is connected and u is subharmonic there; if $S \cap \partial\Omega \neq \varnothing$ this implies the contradiction $u \equiv M$ in $D \cup \Omega^e$. Cf. [Sa 2, Proposition 9.4].

The above results, e.g. Corollary 3.3, can also be formulated directly in terms of inverse problems in potential theory:

Corollary 3.10. *Let Ω be a solid open set with $\partial\Omega$ analytic. By the Cauchy–Kovalevska theorem the Cauchy problem*

$$\begin{cases} \Delta u = 1 & \text{in a neighbourhood of } \partial\Omega \\ u = 0, & \nabla u = 0 \quad \text{on } \partial\Omega \end{cases}$$

has a (unique) solution u in some neighbourhood of $\partial\Omega$. Let K be a compact subset of Ω such that

(i) *$\Omega \setminus K$ is connected,*

(ii) *u is defined in $\Omega \setminus K$,*

(iii) *for each $x \in \Omega \setminus K$ either $u(x) > 0$ or $\nabla u(x) \neq 0$ (or both).*

Then $D = \Omega$ whenever D is a solid open set containing K and satisfying $\nabla U^D = \nabla U^\Omega$ in a neighbourhood of infinity.

Proof. By the hypotheses one can construct a function $v \in C^{1,1}(\mathbf{R}^N)$ satisfying $v = 0$ in Ω^c and $v = u$ in $\Omega \setminus \omega$, where ω is a small neighbourhood of K. Define $\mu = \chi_\Omega - \Delta v$. Then $\mu \in L^\infty$, $\operatorname{supp}\mu \subset \bar{\omega}$ and $v = U^\mu - U^\Omega$. Since $v = 0$ on Ω^c and $\Omega = [\Omega]$ it follows from Proposition 1.2 that $\Omega \in Q(\mu, HL^1)$.

Next we observe that (for D as in the statement) $D \in Q(\mu, HL^1)$, provided ω was chosen so small so that $\bar{\omega} \subset D$. In fact $U^D = U^\Omega = U^\mu$ in a neighbourhood of infinity and by harmonic continuation the relation $U^D = U^\mu$ persists to hold in all D^e. Since $D^c = \operatorname{clos} D^e$ it follows from Proposition 1.2 that $D \in Q(\mu, HL^1)$ and now (ii) of Corollary 3.3 shows that $D \prec \Omega$ which (both D and Ω being solid) implies $D = \Omega$. ■

In analogy with Example 3.4, Corollary 3.10 implies that the only solid open set having the Newtonian field $cx/|x|^N$ far away is the appropriate ball $\Omega = B(0; r)$ ($K = \{0\}$). Cf. [Ah–Sc–Z], [Sa 6].

We finish this section with a result whih generalizes [Sa 2, Theorem 6.4].

Proposition 3.5. *The elements in $Q(\mu, AL^1)$ (and hence those in $Q(\mu, HL^1)$ and $Q(\mu, SL^1)$) are uniformly bounded for fixed μ.*

Proof. Choose an open ball B such that $\operatorname{supp}\mu \subset B$ and numbers $\rho \geqq 1$, $\varepsilon > 0$ such that $\operatorname{supp}(\mu * h_\varepsilon) \subset B$ and $\mu * h_\varepsilon \leqq (\rho - 1)\chi_B$. We claim that every $\Omega \in Q(\mu, AL^1)$ satisfies $\Omega \subset B_1$, where B_1 is the open ball with the same center as B and with volume $\alpha = \rho |B|$.

Suppose $\Omega \in Q(\mu, AL^1)$. The function $\nu = \chi_\Omega + \rho\chi_B - \mu * h_\varepsilon$ satisfies $\nu = 0$ on $(\Omega \cup B)^c$, $\nu \geqq 1$ in $\Omega \cup B$. Thus by Corollary 2.3, $Q(\nu, SL^1) \neq \emptyset$, namely $\Omega(\nu) \in Q(\nu, SL^1)$.

Moreover $\Omega \cup B \subset \Omega(\nu)$. But now $U^\mu = U^{\mu * h_\varepsilon}$, $U^{\rho\chi_B} = U^{\alpha\delta}$ and $\nabla U^\Omega = \nabla U^\mu$ outside $B \cup \Omega$ so that $\nabla U^{\Omega(\nu)} = \nabla U^\nu = \nabla U^\Omega + \nabla U^{\rho\chi_B} - \nabla U^{\mu * h_\varepsilon} = \nabla U^{\alpha\delta}$ outside $\Omega(\nu)$. This shows that $\Omega(\nu) \in Q(\alpha\delta, AL^1)$ and Example 3.4 now gives that $\Omega(\nu)$ is the ball B_1 above. Since $\Omega \subset \Omega(\nu)$ this completes the proof. ∎

4. New quadrature domains from old

In Section 3 we introduced an operation $\Omega_1 + \Omega_2$ between quadrature domains such that if $\Omega_j \in Q(\mu_j, \Lambda)$ ($j = 1, 2$, $\Lambda = AL^1$, HL^1 or SL^1) then $\Omega_1 + \Omega_2 \in Q(\mu_1 + \mu_2, \Lambda)$. The first result in this section concerns another operation on quadrature domains: the least upper bound with respect to $\prec$ exists in $Q(\mu, \Lambda)$. The technique used is a variant of that introduced in the beginning of Section 2. By these methods we then also prove that if $\Omega_1, \Omega_2 \in Q(\mu, AL^1)$, $\Omega_1 \prec \Omega_2$ then Ω_1 and Ω_2 can be joined by a curve in $Q(\mu, AL^1)$ and combining these two results shows that $Q(\mu, AL^1)$ is arc-wise connected.

Theorem 4.1. *Suppose $\mu \in L^\infty_c$, $\Omega_1, \Omega_2 \in Q(\mu, AL^1)$. Then the least upper bound $\Omega_1 \vee \Omega_2$ exists in $Q(\mu, AL^1)$, i.e. there exists a $\Omega \in Q(\mu, AL^1)$ satisfying $\Omega_j \prec \Omega$ ($j = 1, 2$) and $\Omega \prec D$ for every $D \in Q(\mu, AL^1)$ with $\Omega_j \prec D$ ($j = 1, 2$). If $\Omega_1, \Omega_2 \in Q(\mu, HL^1)$ then this $\Omega = \Omega_1 \vee \Omega_2$ also is in $Q(\mu, HL^1)$.*

Proof. Consider the family

$$\mathscr{F} = \{u \in \mathscr{D}'(\mathbf{R}^N) : u \leqq \min(U^{\Omega_1}, U^{\Omega_2}), \ -\Delta u \leqq 1\}.$$

The function $w = \min(U^{\Omega_1}, U^{\Omega_2})$ is superharmonic (since U^{Ω_j} are) and satisfies $w = U^{\Omega_1} = U^{\Omega_2}$ far away. Thus $w = U^\nu$ where $\nu = -\Delta w$ is a positive Radon measure with compact support. This shows that $\mathscr{F}$ is the same type as in Theorem 2.1 (with μ replaced by ν), hence there is a largest element $V = V^\nu$ in $\mathscr{F}$.

Define

$$\omega = \omega(\nu) = \{x \in \mathbf{R}^N : V(x) < w(x)\} = \{x \in \mathbf{R}^N : V(x) < U^\nu(x)\},$$

$$\Omega = \Omega(\nu) = \mathbf{R}^N \setminus \operatorname{supp}(1 + \Delta V),$$

$$D_1 = \{x \in \mathbf{R}^N : U^{\Omega_1}(x) < U^{\Omega_2}(x)\},$$

$$D_2 = \{x \in \mathbf{R}^N : U^{\Omega_1}(x) > U^{\Omega_2}(x)\},$$

$$S = \{x \in \mathbf{R}^N : U^{\Omega_1}(x) = U^{\Omega_2}(x)\}.$$

Observe that all functions above are continuous (recall Corollary 2.1). Thus ω, Ω, D_1, D_2 are open and bounded while S is closed. Moreover $\omega \subset \Omega$ (Theorem 2.1) and $D_1 \cup D_2 \cup S = \mathbf{R}^N$.

Clearly $w = U^{\Omega_j}$ on $D_j \cup S$ ($j = 1, 2$) and $V = w$ on $\Omega^c \subset \omega^c$. Therefore, using that $V, U^{\Omega_j} \in W^{2,p}_{\mathrm{loc}}$ for all $p < \infty$ it follows that

$$V = U^{\Omega_j} \qquad \text{on } \Omega^c \cap (D_j \cup S), \tag{4.1}$$

$$\nabla V = \nabla U^{\Omega_j} \qquad \text{a.e. on } \Omega^c \cap (D_j \cup S), \tag{4.2}$$

$$-\Delta V = -\Delta U^{\Omega_j} = \chi_{\Omega_j} \quad \text{a.e. on } \Omega^c \cap (D_j \cup S). \tag{4.3}$$

In $\Omega_1 \cap \Omega_2$, $-\Delta U^{\Omega_j} = 1$, hence $v = -\Delta w = -\Delta \min(U^{\Omega_1}, U^{\Omega_2}) \geqq 1$ there. By (c) of Theorem 2.3 this shows that $-\Delta V = 1$ in $\Omega_1 \cap \Omega_2$ $(-\Delta V = F(v))$. In other words

$$\Omega_1 \cap \Omega_2 \subset \Omega. \tag{4.4}$$

Similarly, since $v = 1$ in $\Omega_1 \cap D_1$ and in $\Omega_2 \cap D_2$ we have $-\Delta V = 1$ in $(\Omega_1 \cap D_1) \cup (\Omega_2 \cap D_2)$, or

$$(\Omega_1 \cap D_1) \cup (\Omega_2 \cap D_2) \subset \Omega. \tag{4.5}$$

Using that $\Omega^c \cap D_j \cap \Omega_j = \varnothing$ by (4.5) and that $\Omega^c \cap S \cap \Omega_1 \cap \Omega_2 = \varnothing$ by (4.4) we can write

$$\begin{aligned} \Omega^c &= (\Omega^c \cap D_1) \cup (\Omega^c \cap D_2) \cup (\Omega^c \cap S) \\ &= (\Omega^c \cap D_1 \cap \Omega_1^c) \cup (\Omega^c \cap D_2 \cap \Omega_2^c) \cup (\Omega^c \cap S \cap (\Omega_1^c \cup \Omega_2^c)). \end{aligned} \tag{4.6}$$

Now, since $\nabla U^{\Omega_j} = \nabla U^\mu$ on Ω_j^c (Proposition 1.1) and, if $\Omega_j \in Q(\mu, HL^1)$, $U^{\Omega_j} = U^\mu$ on Ω_j^c (4.1), (4.2), (4.3) and (4.6) show that

$$V = U^\mu \quad \text{on } \Omega^c, \tag{4.7}$$

$$\nabla V = \nabla U^\mu \quad \text{a.e. on } \Omega^c, \tag{4.8}$$

$$-\Delta V = 0 \quad \text{a.e. on } \Omega^c, \tag{4.9}$$

(4.7) only in case $\Omega_j \in Q(\mu, HL^1)$. As $-\Delta V = 1$ in Ω by the definition of Ω and $V \in W^{2,p}_{\mathrm{loc}}$ (4.9) shows that $-\Delta V = \chi_\Omega$, i.e. that $V = U^\Omega$. Since moreover $\Omega = [\Omega]$ (also by definition) (4.7), (4.8) combined with Proposition 1.2 now show that $\Omega \in Q(\mu, AL^1)$, or $Q(\mu, HL^1)$ if $\Omega_j \in Q(\mu, HL^1)$.

That $\Omega = \Omega_1 \vee \Omega_2$, $\vee$ taken in $Q(\mu, AL^1)$, is obvious from the construction of Ω. In fact, identifying a (bounded open) set D with the distribution χ_D, χ_Ω is simply constructed as $\chi_{\Omega_1} \vee \chi_{\Omega_2}$, $\vee$ taken in $\{\rho \in \mathscr{E}'(\mathbf{R}^N) : \rho \leqq 1\}$. ■

Note the simple formula for $\Omega = \Omega_1 \vee \Omega_2$ obtained from the proof:

$$\Omega_1 \vee \Omega_2 = \Omega(-\Delta \min(U^{\Omega_1}, U^{\Omega_2})), \quad \text{or}$$

$$\chi_{\Omega_1 \vee \Omega_2} = F(-\Delta \min(U^{\Omega_1}, U^{\Omega_2})).$$

One can construct examples which show that $Q(\mu, AL^1)$ can contain two different (non-solid) minimal domains. (One can e.g. modify Example 3.2 so that (the modifications of) $\Omega_{t_a,0}$ and Ω_{0,t_a} become minimal in $Q(\mu, AL^1)$, namely by giving μ a little extra mass in such a way that $\operatorname{supp} \mu \cap \Gamma_{t_a,0} \cap \Gamma_{0,t_a} \neq \varnothing$ where $\Gamma_{s,t}$ is the outer component of $\partial\Omega_{s,t}$. This prevents $\Omega_{t_a,0}$ and Ω_{0,t_a} becoming "more solid".) This shows that the greatest lower bound with respect to $\prec$ does not always exist in $Q(\mu, AL^1)$. If it did this would by the way immediately solve the uniqueness problem.

Theorem 4.2. *Suppose $\mu \in L_c^\infty$, $\Omega_0, \Omega_1 \in Q(\mu, AL^1)$, $\Omega_0 < \Omega_1$. Then there exists a chain (with respect to $<$) in $Q(\mu, AL^1)$ connecting Ω_0 and Ω_1, i.e. there exists a one-parameter family $\Omega(t) \in Q(\mu, AL^1)$, $0 \leqq t \leqq 1$, satisfying $\Omega(0) = \Omega_0$, $\Omega(1) = \Omega_1$ and $\Omega(t_1) < \Omega(t_2)$ for $t_1 \leqq t_2$. The sets $\Omega(t)$ depend continuously on t with e.g. the energy of $\chi_{\Omega(t_1)} - \chi_{\Omega(t_2)}$ taken as the distance between $\Omega(t_1)$ and $\Omega(t_2)$.*

Notational remark. In Theorem 4.2 and its proof the notations $\Omega(\mu)$, $\omega(\mu)$ introduced in Theorem 2.1 will not be used; $\Omega(t)$ and $\omega(t)$ will denote just one-parameter families of sets.

Proof. For each $0 \leqq t \leqq 1$ consider

$$\mathscr{F}_t = \{u \in \mathscr{D}'(\mathbf{R}^N) \colon u \leqq \min(U^{\Omega_0}, U^{\Omega_1} + c(1-t)), -\Delta u \leqq 1\},$$

where $c = \sup(U^{\Omega_0} - U^{\Omega_1}) \geqq 0$. The function $w_t = \min(U^{\Omega_0}, U^{\Omega_1} + c(1-t))$ equals U^{Ω_0} for $t = 0$, U^{Ω_1} for $t = 1$ and, for any $0 \leqq t \leqq 1$, equals U^{Ω_0} far away (since $U^{\Omega_0} = U^{\Omega_1}$ far away). As usual (Theorem 2.1, Theorem 4.1) it follows that $\mathscr{F}_t$ contains a largest element V_t. Moreover $V_t = U^{\Omega_0} = U^{\Omega_1}$ outside a compact set and $V_0 = U^{\Omega_0}$, $V_1 = U^{\Omega_1}$.

The rest of the proof is similar to that of Theorem 4.1. We set

$$\omega(t) = \{x \in \mathbf{R}^N \colon V_t(x) < w_t(x)\},$$

$$\Omega(t) = \mathbf{R}^N \setminus \operatorname{supp}(1 + \Delta V_t),$$

$$D_0(t) = \{x \in \mathbf{R}^N \colon U^{\Omega_0}(x) < U^{\Omega_1}(x) + c(1-t)\},$$

$$D_1(t) = \{x \in \mathbf{R}^N \colon U^{\Omega_0}(x) > U^{\Omega_1}(x) + c(1-t)\},$$

$$S(t) = \{x \in \mathbf{R}^N \colon U^{\Omega_0}(x) = U^{\Omega_1}(x) + c(1-t)\}.$$

Then $\omega(t)$, $\Omega(t)$, $D_0(t)$, $D_1(t)$ are open, S is closed, $\omega(t) \subset \Omega(t)$ and $D_0(t) \cup D_1(t) \cup S(t) = \mathbf{R}^N$. Also, $\Omega(0) = [\Omega_0]$ and $\Omega(1) = [\Omega_1]$.

Clearly $w_t = U^{\Omega_0}$ on $D_0(t) \cup S(t)$, $w_t = U^{\Omega_1} + c(1-t)$ on $D_1(t) \cup S(t)$ and $V_t = w_t$ on $\Omega(t)^c \subset \omega(t)^c$. Therefore

$$V_t = U^{\Omega_0} \qquad \text{on } \Omega(t)^c \cap (D_0(t) \cup S(t)),$$

$$V_t = U^{\Omega_1} + c(1-t) \qquad \text{on } \Omega(t)^c \cap (D_1(t) \cup S(t)),$$

$$\nabla V_t = \nabla U^{\Omega_j} \qquad \text{a.e. on } \Omega(t)^c \cap (D_j(t) \cup S(t)), \tag{4.10}$$

$$-\Delta V_t = -\Delta U^{\Omega_j} = \chi_{\Omega_j} \qquad \text{a.e. on } \Omega(t)^c \cap (D_j t) \cup S(t)). \tag{4.11}$$

In $\Omega_0 \cap \Omega_1$, $-\Delta U^{\Omega_0} = -\Delta(U^{\Omega_1} + c(1-t)) = 1$, hence $-\Delta w_t \geqq 1$ there. Hence, since $-\Delta V_t = F(-\Delta w_t)$, (c) of Theorem 2.3 shows that $-\Delta V_t = 1$ in $\Omega_0 \cap \Omega_1$, or

$$\Omega_0 \cap \Omega_1 \subset \Omega(t).$$

Similarly, since $-\Delta w_t = 1$ in $\Omega_0 \cap D_0(t)$ and in $\Omega_1 \cap D_1(t)$, $-\Delta V_t = 1$ in $(\Omega_0 \cap D_0(t)) \cup (\Omega_1 \cap D_1(t))$, i.e.

$$(\Omega_0 \cap D_0(t)) \cup (\Omega_1 \cap D_1(t)) \subset \Omega(t).$$

Thus $\Omega(t)^c \cap D_j(t) \cap \Omega_j = \varnothing$ and $\Omega(t)^c \cap S \cap \Omega_0 \cap \Omega_1 = \varnothing$ so that

$$\begin{aligned} \Omega(t)^c &= (\Omega(t)^c \cap D_0(t) \cap \Omega_0^c) \cup (\Omega(t)^c \cap D_1(t) \cap \Omega_1^c) \\ &\cup (\Omega(t)^c \cap S(t) \cap (\Omega_0^c \cup \Omega_1^c)). \end{aligned} \tag{4.12}$$

Since $\nabla U^{\Omega_j} = \nabla U^{\mu}$ on Ω_j^c (4.10)–(4.12) show that

$$\begin{aligned} \nabla V_t &= \nabla U^{\mu} \quad \text{a.e. on } \Omega(t)^c, \\ -\Delta V_t &= 0 \quad \text{a.e. on } \Omega(t)^c. \end{aligned} \tag{4.13}$$

Thus, since $-\Delta V_t = 1$ in $\Omega(t)$, $-\Delta V_t = \chi_{\Omega(t)}$, hence $V_t = U^{\Omega(t)}$. By Proposition 1.2 and $\Omega(t) = [\Omega(t)]$, (4.13) now shows that $\Omega(t) \in Q(\mu, AL^1)$.

That $\Omega(t_1) \prec \Omega(t_2)$ for $t_1 \leqq t_2$ is immediate from the construction ($w_{t_1} \geqq w_{t_2}$, hence $V_{t_1} \geqq V_{t_2}$). We note that in terms of F ((2.4)),

$$\chi_{\Omega(t)} = F(-\Delta \min(U^{\Omega_0}, U^{\Omega_1} + c(1-t))).$$

From this, the last assertion of the theorem easily follows, for since F is an orthogonal projection operator with respect to the energy inner product (Theorem 2.3) F is Lipschitz continuous with respect to the energy norm and it therefore is enough to prove that the map $t \to \min(U^{\Omega_0}, U^{\Omega_1} + c(1-t))$ is continuous in the energy norm on the potential side (namely $\| u \|^2 = \int |\nabla u|^2$). That the latter is true follows readily from well-known facts [An], [Ba–Cp] about normal contractions on Sobolev spaces $H_0^1(G)$. ■

Corollary 4.1. *When $\mu \in L_c^{\infty}$, $Q(\mu, AL^1)$ is arc-wise connected in the metric* $\text{dist}(\Omega_1, \Omega_2) = \| \chi_{\Omega_1} - \chi_{\Omega_2} \|$ *where $\| \cdot \|$ is the energy norm.*

Proof. By Theorem 4.1 and Theorem 4.2 any two $\Omega_1, \Omega_2 \in Q(\mu, AL^1)$ can be joined via $\Omega_1 \vee \Omega_2$. ■

In the context of Corollary 4.1 it seems natural to ask for the dimensionality of $Q(\mu, AL^1)$. Corollary 4.1 shows that $Q(\mu, AL^1)$ is at least one-dimensional (in some sense) whenever $Q(\mu, AL^1)$ contains at least two elements. Below we obtain (usually) better lower bounds for the dimension of $Q(\mu, AL^1)$. Probably these results give essentially the correct picture of $Q(\mu, AL^1)$ as far as dimension is concerned. E.g. the embedding in Corollary 4.2 should in the generic case be a local isomorphism (provided sets which differ by a null set are identified).

Theorem 4.3. *Suppose* $\mu \in M_c$, $\Omega \in Q(\mu, AL^1)$. *Let* K *be either* (a) *an isolated component of* Ω^c *with* $K \cap \operatorname{supp} \mu = \varnothing$ *and with* $\partial K = \partial\Omega \cap K$ *a real analytic hyper-surface* (*without singularities*) *or* (b) $K = \{x\}$ *where* $x \in \Omega \setminus \operatorname{supp} \mu$ *is a "special point"* ($\nabla U^{\Omega}(x) = \nabla U^{\mu}(x)$) *which is a strict local minimum for* $U^{\mu} - U^{\Omega}$. *If in case* (a) K *is the unbounded component of* Ω^c *we assume moreover that* $|\Omega^c \setminus K| > 0$.

Then to K *there corresponds a one-parameter family* $\Omega_t \in Q(\mu, AL^1)$, *defined in some interval* $0 \leqq t < \varepsilon$, *with* $\Omega_0 = \Omega$ *and satisfying*

(i) Ω_t *is a strict chain*: $\Omega_\tau \prec \Omega_t$, *with* $\Omega_t \not\prec \Omega_\tau$, *for* $\tau < t$ ($\prec$ *reversed if* K *is the unbounded component of* Ω^c);

(ii) Ω_t *is monotone also in the following* (*related*) *sense*: *if* L *is a neighbourhood of* K *with* $L \setminus \Omega = K$, $L \cap \operatorname{supp} \mu = \varnothing$ *then, for* $\tau < t$, $\Omega_t \cap L \subset \Omega_\tau \cap L$ *and* $\Omega_t \setminus L \supset \Omega_\tau \setminus L$;

(iii) *in the energy norm* χ_{Ω_t} *is a Lipschitz continuous function of* t.

Also:

(iv) *for any number of different choices of* K *with* K *compact the corresponding chains are "linearly independent".*

Proof. The construction of Ω_t in case (a) has been carried out in [Gu 5] with a different application in mind. Since it was not proved in [Gu 5] that the Ω_t really are in $Q(\mu, AL^1)$ and since anyway case (b) has to be proven, we given here the full proof.

The idea is to replace μ or $\chi_\Omega m$ by another measure ν with suitable extra properties and such that $\Omega \in Q(\nu, SL^1)$. The family Ω_t is then obtained as $\Omega_t = \Omega(\nu_t) \in Q(\nu_t, SL^1)$ for certain modifications ν_t of ν. ν will be constructed by constructing the function $u = U^{\nu} - U^{\Omega}$.

Let G be a connected neighbourhood of K with $G \cap \operatorname{supp} \mu = \varnothing$, $G \setminus \Omega = K$, $\partial G \subset \Omega$ and with ∂G a real analytic hyper-surface. Define $u = 0$ on G^c and extend u to a neighbourhood of G^c by solving the Cauchy problem $\Delta u = 1$ in G (close to ∂G), $u = 0$, $\nabla u = 0$ on ∂G. This is possible by the Cauchy–Kovalevska theorem and the resulting function u satisfies $\Delta u = \chi_G$ in a neighbourhood of G^c. Also, $u > 0$ in G close to ∂G.

On K we have $U^{\mu} - U^{\Omega} + c = 0$ and $\nabla(U^{\mu} - U^{\Omega} + c) = 0$ for some constant c (Proposition 1.1). Define $u = U^{\mu} - U^{\Omega} + c$ in a neighbourhood of K. Then $\Delta u = \chi_\Omega$ in a neighbourhood of K and $u > 0$ in Ω (close to K). The assumed regularity of ∂K in case (a) was needed only for this last conclusion.

It is possible to find connected open sets D_1, D_2, D_3, D_4 with smooth boundaries such that, setting $D = D_4 \setminus \bar{D}_1$, the following hold: $K \subset D_1 \Subset D_2 \Subset D_3 \Subset D_4 \Subset G$; u is defined by the above in D^c and satisfies $u > 0$ in $D^c \cap \Omega \cap G$; u can be redefined in D and extended to all D in such a way that $u \in C^{1,1}(\mathbf{R}^N)$, $u > 0$ in D and, for some $\delta > 0$, $-\Delta u \geqq \delta$ in D. (Only this last point requires some thought; some details on this are given in [Gu 4].)

Now $u \in C^{1,1}(\mathbf{R}^N)$, $u = 0$ on $(\Omega \cap G)^c$, $u > 0$ in $\Omega \cap G$, $-\Delta u = \rho\chi_D - \chi_{\Omega \cap G}$ where $\rho \in L^\infty$, $\rho \geqq 1 + \delta$. Define $\nu = \chi_\Omega - \Delta u$. Then $\nu \in L_c^\infty$, $\nu = 1$ in $\Omega \setminus \bar{G}$, $\nu \geqq 1 + \delta$ in D and $\nu = 0$ (a.e.) elsewhere. Note that $u = U^\nu - U^\Omega$. Since $u \geqq 0$ with $u = 0$ on Ω^c it follows that $\Omega \in Q(\nu, SL^1)$. Also notice that by construction, and since $\Omega \in Q(\mu, AL^1)$,

$$(4.14) \qquad \begin{aligned} U^\nu &= U^\mu + c && \text{on } \bar{D}_1, \\ \nabla U^\nu &= \nabla U^\mu && \text{on } \Omega^c, \end{aligned}$$

with c as earlier.

Next choose a function $\psi \in C^\infty(\mathbf{R}^N)$ with $0 \leqq \psi \leqq 1$ such that $\psi = 1$ in $\bar{D}_2$, $\psi = 0$ on D_3^c. Thus supp $\Delta\psi \subset D$ and it follows that for $|t|$ small enough (namely $|t| < \delta / \|\Delta\psi\|_\infty$) the measure $\nu_t = \nu + t\Delta\psi \in L_c^\infty$ satisfies $\nu_t = 0$ outside $D \cup (\Omega \setminus \bar{G})$ and $\nu_t \geqq 1$ in $D \cup (\Omega \setminus \bar{G})$. Thus by Corollary 2.3, $Q(\nu_t, SL^1) \neq \varnothing$. Define $\Omega_t = \Omega(\nu_t) \in Q(\nu_t, SL^1)$. Then $\Omega_0 = \Omega(\nu) = [\Omega]$ (Theorem 2.2) and $D \cup (\Omega \setminus \bar{G}) \subset \Omega_t$ for all t (Corollary 2.3). We claim that $\{\Omega_t\}$ has the required properties.

First assume that K is not the unbounded component of Ω^c. Then K is compact and $\psi = 0$ far away so that

$$(4.15) \qquad U^{\nu_t} = U^\nu - t\psi.$$

Thus $U^{\nu_\tau} - U^{\nu_t} = (t - \tau)\psi$ so that, for $\tau < t$, $U^{\nu_\tau} - (t - \tau) \leqq U^{\nu_t} \leqq U^{\nu_\tau}$. These inequalities, together with $U^{\Omega_s} \leqq U^{\nu_s}$, show that in the construction of $\Omega_s = \Omega(\nu_s)$ in Theorem 2.1, Theorem 2.2, U^{Ω_t} is a competing function (i.e. is in $\mathscr{F}$ in Theorem 2.1) when $s = \tau$ and $U^{\Omega_\tau} - (t - \tau)$ is a competing function when $s = t$.

Theorefore

$$(4.16) \qquad U^{\Omega_\tau} - (t - \tau) \leqq U^{\Omega_t} \leqq U^{\Omega_\tau}$$

for $\tau < t$. The second inequality says that $\{\Omega_t\}$ is a chain with respect to $\prec$ ($\Omega_\tau \prec \Omega_t$ for $\tau < t$). This could also have been seen directly from (3.1), since $F(\nu_s) = \chi_{\Omega_s}$ (Proposition 2.1). The last remark by the way also shows that, in the energy norm, χ_{Ω_s} is a Lipschitz continuous function of s (since F is an orthogonal projection for the energy inner product). Also (4.16) can be regarded as some kind of continuity statement.

In $D_2 \setminus \Omega_\tau$ we have $U^{\Omega_\tau} = U^{\nu_\tau} = U^{\nu_t} + (t - \tau) \geqq U^{\Omega_t} + (t - \tau)$, hence $U^{\Omega_\tau} = U^{\Omega_t} + (t - \tau)$ by (4.16). Therefore taking the Laplacian, $\chi_{\Omega_\tau} = \chi_{\Omega_t}$ a.e. in $D_2 \setminus \Omega_\tau$, i.e. $|\Omega_t \cap (D_2 \setminus \Omega_\tau)| = 0$. Thus $\Omega_\tau \cup (\Omega_t \cap D_2) \subset [\Omega_\tau]$, and since $\Omega_\tau = [\Omega_\tau]$ this implies

$$\Omega_t \cap D_2 \subset \Omega_\tau \cap D_2 \tag{4.17}$$

(for $\tau < t$).

Similarly, in $\Omega_t^c \setminus \bar{D}_3$, $U^{\Omega_t} = U^{\nu_t} = U^{\nu_\tau} \geqq U^{\Omega_\tau}$ which by (4.15) implies $U^{\Omega_t} = U^{\Omega_\tau}$. Hence as above $|(\Omega_\tau \setminus \bar{D}_3) \setminus \Omega_t| = 0$ and by $\Omega_t = [\Omega_t]$, $\Omega_\tau \cap (\Omega_t^c \cap \bar{D}_3^c) = \varnothing$, or

$$\Omega_t \setminus \bar{D}_3 \supset \Omega_\tau \setminus \bar{D}_3 \tag{4.18}$$

(for $\tau < t$).

(4.17) and (4.18) prove (ii). Choosing $\tau = 0$, $t > 0$ in (4.17) and (4.18) gives $K \subset \Omega_t^c$ and $\Omega_t^c \cap \bar{D}_3^c \subset \Omega^c \cap \bar{D}_3^c = \Omega^c \setminus K$ respectively. Using the latter relation, (4.14), (4.15) and that $U^{\Omega_t} = U^{\nu_t}$, $\nabla U^{\Omega_t} = \nabla U^{\nu_t}$ in Ω_t^c, we find that

$$U^{\Omega_t} = U^{\mu} + c - t \qquad \text{in } \Omega_t^c \cap \bar{D}_1, \tag{4.19}$$

$$U^{\Omega_t} = U^{\nu} \qquad \text{in } \Omega_t^c \setminus \bar{D}_3, \tag{4.20}$$

$$\nabla U^{\Omega_t} = \nabla U^{\mu} \qquad \text{in } \Omega_t^c. \tag{4.21}$$

(4.21) shows that $\Omega_t \in Q(\mu, AL^1)$. In fact using (4.17), (4.18) for $\tau = 0$, $\operatorname{supp} \mu \cap \bar{D}_3 = \varnothing$ and the fact that $\Omega_0 = [\Omega] \in Q(\mu, AL^1)$ it follows that (i) and (ii) in (1.4) hold and (4.21) shows that (iii) in (1.4) holds for a dense subset of $AL^1(\Omega)$. Thus $\Omega_t \in Q(\mu, AL^1)$. Moreover, (4.19) combined with $\varnothing \neq K \subset \Omega_t^c \cap \bar{D}_1$ shows that Ω_t really depends on t. (In fact it shows that $\Omega_t \not< \Omega_\tau$ when $\tau < t$.) (4.20) shows that U^{Ω_t} is independent of t on $\Omega_t^c \setminus \bar{D}_3$ and together with (4.19) it therefore proves (iv).

When K is the unbounded component of Ω^c, the constant c is zero, (4.15) gets replaced by

$$U^{\nu_t} = U^{\nu} - t(\psi - 1)$$

and (4.16) accordingly becomes

$$U^{\Omega_t} - (t - \tau) \leqq U^{\Omega_\tau} \leqq U^{\Omega_t}.$$

Thus we now have $\Omega_t < \Omega_\tau$ for $\tau < t$. (4.17) and (4.18) remain the same and one ends up with

$$U^{\Omega_t} = U^{\mu} \qquad \text{in } \Omega_t^c \cap \bar{D}_1,$$

$$U^{\Omega_t} = U^{\nu} + t \qquad \text{in } \Omega_t^c \setminus \bar{D}_3, \tag{4.22}$$

in place of (4.19), (4.20) ((4.21) remains the same).

Now $\Omega_t^c \setminus \bar{D}_3$ is shrinking for increasing t by (4.18) and (4.22) shows that Ω_t really changes with t as long as $\Omega_t^c \setminus \bar{D}_3 \neq \varnothing$. Since initially $|\Omega_0^c \setminus \bar{D}_3| = |\Omega^c \setminus \bar{D}_3| = |\Omega^c \setminus K| > 0$ it follows from the continuity of $t \to \chi_{\Omega_t}$ that $|\Omega_t^c \setminus \bar{D}_3|$ remains positive for t in some interval $0 \leqq t < \varepsilon$. This finishes the proof. ■

It should be remarked that the assumed regularity of ∂K in case (a) really is supposed to be the generic regularity of ∂K when $K \cap \operatorname{supp} \mu = \varnothing$. E.g. in two dimensions an application of the Riemann mapping theorem combined with a reflexion principle shows that every isolated component of $\partial \Omega$ is an analytic curve, with a few types of singularities allowed. (A complete solution of the regularity problem in two dimensions has in fact recently been given in [Sa 9].) In higher dimensions there are similar results although they are less complete. See [Ki–N], [Ca], [F].

On the other hand the said singularities on ∂K, when they appear, really may prevent the variation $\{\Omega_t\}$ of Ω, namely more or less when they make $U^\mu - U^\Omega$ decrease as $\partial\Omega$ is crossed from K (cf. the proof). See [Gu 6] for an example in two dimensions. Also, it is obvious that $\operatorname{supp} \mu$ reaching K also prevents $\{\Omega_t\}$ since $\partial\Omega_t$ is withdrawn at K. Therefore the assumptions on K in case (a) are really necessary. The same is true for case (b) as there are plenty of examples of special points which are saddle points for $U^\mu - U^\Omega$ and for which a corresponding variation $\{\Omega_t\}$ does not exist [Sh 2], [Sa 7], [Gu 6].

It may be noticed however that the special point $x \in \Omega \setminus D$ appearing in e.g. Corollary 3.9 is constructed as a global minimum point for $U^D - U^\Omega$. Hence if this x is not on ∂D then it is at least a local minimum point for $U^\mu - U^\Omega$ (although perhaps not a *strict* local minimum). In [Sa 7] an "index theorem" for special points is proved (in two dimensions).

Clearly Theorem 4.3 says something about the "dimension" of $Q(\mu, AL^1)$ near $\Omega \in Q(\mu, AL^1)$. E.g. it gives

Corollary 4.2. *Suppose* $\Omega \in Q(\mu, AL^1)$, $\operatorname{supp} \mu \subset \Omega$, *that* $\partial\Omega$ *is the disjoint union of* $m+1$ *real analytic hypersurfaces and that* $U^\mu - U^\Omega$ *has* n *(local) strict minima in* $\Omega \setminus \operatorname{supp} \mu$. *Then there is an order preserving embedding of a neighbourhood of the origin in* $\mathbf{R}^m \times (\mathbf{R}_+)^n$ $(\mathbf{R}_+ = \{t \in \mathbf{R}: t \geqq 0\})$ *into a "neighbourhood" of* Ω *in* $Q(\mu, AL^1)$. *The (partial) order used in* $\mathbf{R}^m \times (\mathbf{R}_+)^n$ *is* $\tau \leqq t$ *iff* $\tau_i \leqq t_i$ *for each* i $(\tau = (\tau_1, \ldots, \tau_{m+n})$ *etc.) and the one used in* $Q(\mu, AL^1)$ *is* $<$.

The case when K is the unbounded component of Ω^c is not the least interesting part of Theorem 4.3. It says that, under the stated assumptions on the outer component of $\partial\Omega$, if $\Omega \in Q(\mu, AL^1)$ is not already solid it can be made "more solid", or:

Corollary 4.3. *Assume* $\Omega \in Q(\mu, AL^1)$. *If the outer component of* $\partial\Omega$ *is isolated, real analytic and does not meet* $\operatorname{supp} \mu$, *then* Ω *is* not *minimal (with respect to* $<$*) in* $Q(\mu, AL^1)$, *unless* $[\Omega]$ *is solid. Similarly,* Ω *is not maximal in* $Q(\mu, AL^1)$ *if there exists an isolated, real analytic interior component of* $\partial\Omega$ *which does not meet* $\operatorname{supp} \mu$ *or if* $U^\mu - U^\Omega$ *has at least one local strict minimum in* $\Omega \setminus \operatorname{supp} \mu$.

Our final result relates to Example 1.3:

Theorem 4.4. *Suppose $\mu \in L_c^\infty$, $D \in Q(\mu, HL^1)$ and that μ vanishes on the closure of the set where $U^D > U^\mu$. Then $Q(\mu, SL^1) \neq \varnothing$.*

Proof. The proof is (partly) similar to that of Theorem 4.1. Set

$$\mathcal{F} = \{u \in \mathcal{D}'(\mathbf{R}^N): u \leqq \min(U^D, U^\mu), \ -\Delta u \leqq 1\}.$$

Then there exists a largest element V in $\mathcal{F}$ and setting $w = \min(U^D, U^\mu)$ we define

$$\omega = \{x \in \mathbf{R}^N: V(x) < w(x)\},$$

$$\Omega = \mathbf{R}^N \setminus \operatorname{supp}(1 + \Delta V),$$

$$D_1 = \{x \in \mathbf{R}^N: U^D(x) < U^\mu(x)\},$$

$$D_2 = \{x \in \mathbf{R}^N: U^D(x) > U^\mu(x)\},$$

$$S = \{x \in \mathbf{R}^N: U^D(x) = U^\mu(x)\}.$$

We are going to prove that $\Omega \in Q(\mu, SL^1)$.

We have $w = U^D$ on $D_1 \cup S$, $w = U^\mu$ on $D_2 \cup S$ and $V = w$ on $\Omega^c \subset \omega^c$. It follows that

$$V = U^D \qquad \text{on } \Omega^c \cap (D_1 \cup S), \tag{4.23}$$

$$V = U^\mu \qquad \text{on } \Omega^c \cap (D_2 \cup S), \tag{4.24}$$

and hence that

$$-\Delta V = -\Delta U^D = \chi_D \qquad \text{a.e. on } \Omega^c \cap (D_1 \cup S), \tag{4.25}$$

$$-\Delta V = -\Delta U^\mu = \mu \qquad \text{a.e. on } \Omega^c \cap (D_2 \cup S). \tag{4.26}$$

In $\mathbf{R}^N \setminus \bar{D}_2 \subset D_1 \cup S$, $w = U^D$, hence $-\Delta w = 1$ in $D \setminus \bar{D}_2$. By (c) of Theorem 2.3 this shows that $-\Delta V = F(-\Delta w) = 1$ in $D \setminus \bar{D}_2$. Hence

$$D \setminus \bar{D}_2 \subset \Omega.$$

It follows that

$$\Omega^c = (\Omega^c \cap \bar{D}_2) \cup (\Omega^c \setminus \bar{D}_2) = (\Omega^c \cap \bar{D}_2) \cup (\Omega^c \setminus (D \cup \bar{D}_2)). \tag{4.27}$$

Observe that $\bar{D}_2 \subset D_2 \cup S$. Thus $V = U^\mu$ on $\Omega^c \cap \bar{D}_2$ by (4.24) and $-\Delta V = 0$ a.e. on $\Omega^c \cap \bar{D}_2$ by (4.26) and the assumption that $\mu = 0$ on $\bar{D}_2$. On $\Omega^c \setminus (D \cup \bar{D}_2) \subset \Omega^c \cap D^c \cap (D_1 \cup S)$, $V = U^D = U^\mu$ by (4.23) and the assumption that $D \in Q(\mu, HL^1)$. By (4.25) we also have $-\Delta V = 0$ a.e. on $\Omega^c \setminus (D \cup \bar{D}_2)$. Thus, by (4.27), $V = U^\mu$ on Ω^c, $-\Delta V = 0$ a.e. on Ω^c. As $-\Delta V = 1$ in Ω (by definition of Ω) we get $-\Delta V = \chi_\Omega$ hence $V = U^\Omega$. Since $V \leqq U^\mu$ by construction, Proposition

1.2 now shows that $\Omega \in Q(\mu, SL^1)$. ■

Some open questions

(1) The inverse problem in potential theory mentioned in the introduction, or in terms of quadrature domains: can $Q(\mu, HL^1)$ (or $Q(\mu, AL^1)$) contain more than one solid open set?

(2) Is it true that

$$AL^1(\Omega)_c = \left\{ \varphi \in HL^1(\Omega)\colon \int_\Gamma \frac{\partial \varphi}{\partial n}\, ds = 0 \right.$$

$$\text{for every closed oriented hypersurface } \Gamma \text{ in } \Omega\}?$$

(See Section 1.)

(3) Another technical question of relevance in Section 1: suppose $\mu \in M_c$, $\Omega \in Q(\mu, HL^1)$ and $U^\Omega \leqq U^\mu$ everywhere. Does it follow that $\Omega \in Q(\mu, SL^1)$? (True if e.g. $\mu \in L_c^\infty$.)

(4) Regularity of $\partial\Omega$: Suppose $\Omega \in Q(\mu, AL^1)$ and $\operatorname{supp} \mu \subset \Omega$. To what extent does it follow that $\partial\Omega$ is real analytic? In two dimensions this problem has been completely solved [Sa 9] but in higher dimensions the theory is still incomplete. It is known [Ki–N], [Ca], [F] that $\partial\Omega$ is a real analytic hypersurface close to any point $x \in \partial\Omega$ at which Ω^c is not too thin in a certain sense, at least provided $\Omega \in Q(\mu, SL^1)$.

(5) In two dimensions the following is known [Ah–Sh], [Gu 2], [Gu 6]: if $\operatorname{supp} \mu$ is a finite set and $\Omega \in Q(\mu, AL^1)$ then $\partial\Omega$ is an algebraic curve. (μ need not be a measure in this result; it is enough that $\mu \in \mathscr{D}'(\mathbf{R}^N)$.) Is there a corresponding result in higher dimensions?

(6) How large can $Q(\mu, HL^1)$ be? There are examples with $Q(\mu, HL^1)$ even uncountably infinite and with cluster points but can $Q(\mu, HL^1)$ contain e.g. a whole continuum (in a reasonable topology) of open sets?

(7) Does $Q(\mu, HL^1)$ always contain a largest element with respect to $\prec$? (True whenever $Q(\mu, SL^1) \neq \varnothing$, by Corollary 3.4, and also true whenever $Q(\mu, HL^1)$ is finite, by Theorem 4.1.)

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