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ON APPROXIMATION BY HARMONIC VECTOR FIELDS

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1. Introduction.

Let Ω be a finitely connected region in $\mathbb{C}$, bounded by simple analytic curves. Let $\mathcal{A}(\Omega)$ denote the Banach algebra of functions analytic in Ω and continuous in $\bar{\Omega}$ with the norm

$$\|f\|_\infty = \|f\|_{\mathcal{A}} = \sup_{z\in\Omega} |f(z)| .$$

The *analytic content* $\lambda(\Omega)$ introduced in [4], [6, 7] is defined by

$$\lambda(\Omega) = \inf_{\varphi\in\mathcal{A}} \|\bar{z} - \varphi\|_\infty . \tag{1.1}$$

H. Alexander [1] and the second author [6] have shown that $\lambda(\Omega)$ can be estimated in terms of simple geometric quantities $V(\Omega) =$ the area of Ω and $P(\partial\Omega) =$ the perimeter of Ω. Namely,

$$\lambda(\Omega) \leq \left(\frac{V(\Omega)}{\pi}\right)^{1/2} \text{ and } \lambda(\Omega) \geq 2\frac{V(\Omega)}{P(\partial\Omega)}. \tag{1.2}$$

One of the nice upshots of (1.2) is that it implies the isoperimetric inequality

$$P^2 \geq 4\pi V$$

(see [4] for further discussion and references). In this paper, we are making an attempt to extend some of the two-dimensional results to higher dimensions. To do this, note that (1.1) can be rewritten in the form

$$\lambda(\Omega) = \inf_{\varphi\in\mathcal{A}} \|z - \bar{\varphi}\|_\infty .$$

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Now a smooth up to the boundary anti-analytic function $\bar{\varphi} = f_1 + if_2$ can be identified with the *harmonic* vector field $f = (f_1, f_2)$ defined by the conditions

$$(1.3)\qquad \begin{aligned} \operatorname{div} f &:= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0; \\ \operatorname{curl} f &:= \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = 0. \end{aligned}$$

Conversely, every vector field u satisfying (1.3) gives rise to an anti-analytic function $\bar{\varphi} = f_1 + if_2$ (cf. [11, Ch. III, §1]). Thus, if we denote by $A(\Omega)$ the closure of smooth harmonic vector fields (1.3) with respect to the norm

$$(1.4)\qquad \|f\|_\infty := \sup_{x\in\bar{\Omega}} \sqrt{f_1^2(x) + f_2^2(x)},\ x = (x_1, x_2)$$

we arrive at an equivalent definition for $\lambda(\Omega)$:

$$(1.5)\qquad \lambda(\Omega) = \inf_{f\in A(\Omega)} \|x - f\|_\infty,$$

where x denotes the identity vector field $x = (x_1, x_2)$, $(z = x_1 + ix_2)$,

$$(1.6)\qquad \|x - f\|_\infty := \sup_{x\in\Omega} \sqrt{(x_1 - f_1)^2 + (x_2 - f_2)^2}$$

and since $(x_1 - f_1)^2 + (x_2 - f_2)^2$ is subharmonic in Ω, the supremum in (1.6) can be restricted to the boundary $\partial\Omega$.

Note that if Ω is *simply connected,* $A(\Omega)$ coincides with the space of *harmonic gradients* $B(\Omega)$ obtained by completion of $\{f = (f_1, f_2) = \operatorname{grad} u : u \in H(\Omega) \cap C^1(\bar{\Omega})\}$ with respect to the norm (1.4). Here, $H(\Omega) = \{u : u$ is a harmonic in Ω and continuous in $\bar{\Omega}\}$. For multiply-connected Ω, $B(\Omega) \subsetneq A(\Omega)$.)

Now the definition (1.5) easily extends to higher dimensions and we set for a smoothly bounded domain $\Omega \subset \mathbb{R}^N$,

$$(1.7)\qquad \lambda(\Omega) := \inf_{f\in A(\Omega)} \|x - f\|_\infty,$$

where the space of *harmonic vector fields* $A(\Omega)$ is defined similarly to (1.3), (1.4), as consisting of all vector fields in Ω, $f = (f_1, \dots, f_N) \in C^1(\Omega) \cap C(\bar{\Omega})$ satisfying

$$\text{div} f = 0 \text{ and } \text{curl } f = 0. \tag{1.8}$$

This last equation means

$$\frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j} = 0, \text{ for all } j, k = 1, \dots, N.$$

$A(\Omega)$ is equipped with the norm $\|f\|_\infty = \sup_{x \in \Omega} \left(\sum_1^N f_i(x)^2 \right)^{1/2}$. Note that all the components f_j of $f \in A(\Omega)$ are harmonic in Ω (cf. [11]). The space $B(\Omega)$ of harmonic gradients is defined accordingly. Again, $B(\Omega) \subsetneqq A(\Omega)$ unless Ω is simply connected.

Let us outline briefly the contents of the paper. In section 2, we extend the upper estimate (1.2) of $\lambda(\Omega)$ to $\mathbb{R}^N$, $N \geq 3$. Unfortunately, although we obtain along the way a sharp N-dimensional analogue of the Ahlfors-Beurling estimate (cf. [3], [4]) of the $\max_{\Omega} \|\text{grad } u^\Omega\|_\infty$ of the gravitational potential u^Ω of Ω, in higher dimensions this route does not lead to a sharp estimate for $\lambda(\Omega)$ as it does for $N = 2$ (cf. [4]). The desired estimate is stated as Conjecture 2.2.

In section 3, we extend the lower estimate (1.2) to $\mathbb{R}^N$, $N \geq 3$ and consider the problem of finding all extremal domains Ω for which $\lambda(\Omega)$ assumes its lower bound ($NV(\Omega)/P(\partial\Omega)$–Thm. 3.1). Unfortunately, this problem remains unsolved even for $N = 2$. In Theorem 3.2 we formulate a number of equivalent conditions satisfied by extremal domains extending the two-dimensional results from [7] and [8]. We conjecture that the only extremal domains are either balls or spherical shells, and show that the only extremal domains topologically equivalent to a ball are indeed balls.

A few words concerning the notation: $\Omega \subset \mathbb{R}^N$ always stands for a finitely connected domain with a smooth, even real analytic boundary, consisting of $n+1$ pieces Γ_j, $\partial\Omega = \cup_{j=0}^n \Gamma_j$. Also, we agree on having Γ_0 to designate the outer boundary component. V is the Lebesgue measure in $\mathbb{R}^N$, σ denotes the Lebesgue measure on $\partial\Omega$, and $n(\vec{n})$ is the outer unit normal vector to $\partial\Omega$. Hopefully avoiding ambiguity, we omit the arrow "$\rightarrow$" on the top of vectors in order for the formulae to be more readable, and try instead to specify each time precisely which quantities we are dealing with: scalar or vector. $|f|$ denotes the magnitude of a vector f. For a continuous vector function f we use the norm $\|f\| = \|f\|_\infty := \sup_{x\in\partial\Omega}|f(x)|$ unless specified otherwise. $M(\partial\Omega)$ is the Banach space of all vector-valued measures $\mu = (\mu_1, \dots, \mu_N)$, regarded as a dual space of the Banach space of all continuous vector fields $f = (f_1, \dots, f_n)$ on $\partial\Omega$ with the norm $\|f\|_\infty$ via the obvious pairing

$$\mu(f) = \int_{\partial\Omega} f \cdot d\mu = \sum_1^N \int_{\partial\Omega} f_i d\mu_i.$$

2. Upper Estimates for $\lambda(\Omega)$.

Theorem 2.1. *Let $V(\Omega)$ denote the volume of $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Then*

$$\lambda(\Omega) \leq \frac{N^{1+1/N}\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{2N-1}{2N-2}\right)^{1-1/N}}{2\pi^{\frac{2N-1}{2N}}\Gamma\left(\frac{N^2}{2N-2}\right)^{1-1/N}} V(\Omega)^{1/N}. \tag{2.1}$$

Proof. For the sake of brevity, we restrict ourselves to the case $N \geq 3$. The vector field $x + N$ grad $u^\Omega(x)$ is in $A(\Omega)$, where, as usual,

$$u^\Omega(x) = C_N \int_\Omega \frac{dV(y)}{|x-y|^{N-2}},$$

is the gravitational potential of Ω, $C_N = 1/(N-2)\omega_{N-1}$, and $\omega_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$ is the surface area of the unit sphere in $\mathbb{R}^N$. (In fact, this vector field belongs even to $B(\Omega)$-cf. §1.) So, $\Delta u^\Omega = \text{div grad } u^\Omega = \begin{cases} -1, & \text{in } \Omega \\ 0, & \text{in } \mathbb{R}^N \setminus \bar{\Omega} \end{cases}$. Recall that (cf. (1.7))

$$\lambda(\Omega) := \inf_{f\in A(\Omega)} \|x - f(x)\|_\infty = \inf_{f\in A(\Omega)} \|x - f(x)\|_{\partial\Omega},$$

since $|x - f(x)|^2 = \sum_1^N (x_i - f_i(x))^2$ is a subharmonic function in Ω, and therefore attains its maximum on the boundary $\partial\Omega$ of Ω. Hence, by the

Hahn-Banach duality we have

$$\lambda(\Omega) = \sup_{\substack{\mu\in M(\partial\Omega),\mu\perp A(\Omega)\\ \|\mu\|\le 1}} \int x\cdot d\mu(x)$$

$$= \sup_{\substack{\mu\in M(\partial\Omega)\\ \mu\perp A(\Omega),\ \|\mu\|\le 1}} \left\{-N\int \operatorname{grad} u^\Omega\cdot d\mu\right\} \le N\sup_{\partial\bar{\Omega}}\left|\operatorname{grad} u^\Omega\right|$$

$$(2.2)\qquad = \frac{N\Gamma(N/2)}{2\pi^{N/2}}\max_{x\in\mathbb{R}^N}\left|\int_\Omega \frac{y-x}{|y-x|^N}\,dy\right|.$$

To justify the last equality note that u^Ω is harmonic in $\mathbb{R}^N\setminus\bar{\Omega}$, continuously differentiable in $\mathbb{R}^N$, vanishes at infinity, and so $\left|\operatorname{grad} u^\Omega\right|^2$ is subharmonic and continuous in $\mathbb{R}^N\setminus\Omega$. Hence, it assumes its maximum in $\mathbb{R}^N\setminus\Omega$ on $\partial\Omega$. On the other hand, components of grad u^Ω are harmonic in Ω, and therefore $\left|\operatorname{grad} u^\Omega\right|^2$ is subharmonic in Ω as well. So, in fact

$$\max_{\mathbb{R}^N}\left|\operatorname{grad} u^\Omega\right| = \max_{\partial\Omega}\left|\operatorname{grad} u^\Omega\right|.$$

The Ahlfors-Beurling Estimate: The proof now reduces to obtaining the sharp estimate for

$$(2.3)\qquad \frac{\Gamma(N/2)}{2\pi^{N/2}}\max_{x\in\mathbb{R}^N}\left|\int_\Omega \frac{y-x}{|y-x|^N}\,dy\right| = \max_{x\in\mathbb{R}^N}\left|\operatorname{grad} u^\Omega(x)\right|$$

in terms of $V(\Omega)$ which we shall call the *Ahlfors-Beurling* estimate (cf. [3], and the discussion in [4]).

Since (2.3) is invariant under translations and rotations, we can assume without loss of generality that the maximum on $\partial\Omega$ is assumed at the origin and that grad $u^\Omega(0) = \frac{\partial}{\partial x_1}u^\Omega(0)$. Thus,

$$(2.4)\qquad \lambda(\Omega) \le \frac{N\Gamma(N/2)}{2\pi^{N/2}}\sup_{\Omega'}\int_{\Omega'}\frac{y_1}{|y|^N}\,dy$$

where the supremum is taken over all domains $\Omega' :\ V(\Omega') = V(\Omega) := V$. Setting $\varphi(x) = x_1|x|^{-N}$, we conclude from (2.4) that the supremum in (2.4) is attained for

$$(2.5)\qquad \Omega' = \Omega_t := \{x\in\mathbb{R}^N : \varphi(x) > t\}$$

where t is chosen so that $V(\Omega_t) = V$. Passing to cylindrical coordinates (x_1, r, ω), where

$$(x_2, \dots, x_N) = r\omega,\ \omega \in S^{N-2},\ r \geq 0,$$
$$dx = r^{N-2}\ dx_1 dr\ d\omega,$$

we can rewrite (2.5) as

$$\Omega_t = \{x \in \mathbb{R}^N :\ x_1 > t(x_1^2 + r^2)^{N/2},\ \omega \in S^{N-2}\}$$

$$\text{or,} \qquad \Omega_t = D_t \times S^{N-2}, \qquad \text{where}$$

$$D_t = \{(x_1, r) : 0 < r < (t^{-2/N} x_1^{2/N} - x_1^2)^{1/2},\ 0 < x_1 < t^{-1/(N-1)}\}.$$

Set

$$\alpha_N = \int_{S^{N-2}} d\omega = \frac{2\pi^{(N-1)/2}}{\Gamma\left(\frac{N-1}{2}\right)},$$

$$\Phi(t) = \int_{\Omega_t} \varphi(x)\ dx = \alpha_N \int_{D_t} \frac{x_1 r^{N-2}}{(x_1^2 + r^2)^{N/2}}\ dx_1\ dr. \tag{2.6}$$

Then

$$V(\Omega_t) = \alpha_N \int_{D_t} r^{N-2}\ dx_1\ dr. \tag{2.7}$$

Introducing a change of variables in the "meridian plane"

$$(x_1, r) \leftrightarrow (r, s) : s = \frac{r^2}{(x_1^2 + r^2)},$$

so that

$$ds\ dr = \frac{\partial s}{\partial x_1}\ dx_1\ dr = -\frac{2x_1 r^2}{(x_1^2 + r^2)^2}\ dx_1\ dr,$$

we have

$$D_t = \{(r, s) : 0 < s < 1,\ 0 < r < t^{-1/(N-1)}(s^{N-1} - s^N)^{1/(2N-2)}\}.$$

We obtain from (2.6)

$$\begin{aligned}\Phi(t) &= \tfrac{1}{2}\alpha_N \int_{D_t} s^{(N-4)/2}\, ds\, dr \\ &= \tfrac{1}{2}\alpha_N \int_0^1 s^{(N-4)/2} t^{-1/(N-1)} \left(s^{N-1} - s^N\right)^{1/(2N-2)}\, ds \\ &= \tfrac{1}{2}\alpha_N t^{-1/(N-1)} \int_0^1 s^{(N-3)/2}(1-s)^{1/(2N-2)}\, ds \\ &= \tfrac{1}{2}\alpha_N t^{-1/(N-1)} B\left(\tfrac{N-1}{2}, \tfrac{2N-1}{2N-2}\right), \end{aligned} \tag{2.8}$$

where $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 s^{p-1}(1-s)^{q-l}\, ds$ is Euler's beta function.

Similarly, from (2.7) it follows that

$$\begin{aligned}V(\Omega_t) &= \tfrac{1}{2}\alpha_N \iint_{D_t} x_1^{-1} r^{N-4} \left(x_1^2 + r^2\right)^2\, dr\, ds \\ &= \tfrac{1}{2}\alpha_N \iint_{D_t} s^{-3/2}(1-s)^{-1/2} r^{N-1}\, dr\, ds \\ &= \tfrac{1}{2}\alpha_N \int_0^1 s^{-3/2}(1-s)^{-1/2} t^{-N/(N-1)} \left(s^{N-1} - s^N\right)^{N/(2N-2)}\, ds \\ &= (2N)^{-1}\alpha_N t^{-N/N-1} \int_0^1 s^{(N-3)/2}(1-s)^{1/(2N-2)}\, ds \\ &= (2N)^{-1}\alpha_N t^{-N/(N-1)} B\left(\frac{N-1}{2}, \frac{2N-1}{2N-2}\right). \end{aligned}$$

Therefore,

$$t^{-1/N-1} = \left(\frac{2N}{\alpha_N}\right)^{1/N} B\left(\frac{N-1}{2}, \frac{2N-1}{2N-2}\right)^{-1/N} V^{1/N}. \tag{2.9}$$

Substituting (2.9) into (2.8) we obtain from (2.4) that

$$\lambda(\Omega) \le \frac{N\Gamma(N/2)}{2\pi^{N/2}}\Phi(t)$$

$$= \frac{N\Gamma(N/2)}{2\pi^{N/2}}\left(\frac{\alpha_N}{2}\right)^{1-1/N} N^{1/N} B\left(\frac{N-1}{2}, \frac{2N-1}{2N-2}\right)^{1-1/N} \cdot V^{1/N}$$

$$= \frac{N^{1+1/N}\Gamma\left(\frac{N}{2}\right)\pi^{\frac{(N-1)^2}{2N}}\Gamma\left(\frac{N-1}{2}\right)^{1-1/N}\Gamma\left(\frac{2N-1}{2N-2}\right)^{1-1/N} V^{1/N}}{2\pi^{N/2}\Gamma\left(\frac{N-1}{2}\right)^{1-1/N}\Gamma\left(\frac{N^2}{2N-2}\right)^{1-1/N}}$$

$$= \frac{N^{1+1/N}\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{2N-1}{2N-2}\right)^{1-1/N}}{2\pi^{(2N-1)/2N}\Gamma\left(\frac{N^2}{2N-2}\right)^{1-1/N}} V(\Omega)^{1/N}. \tag{2.10}$$

Remark. . We do not know whether estimate (2.1) is sharp for $N \geq 3$. For $N = 2$ though, it is sharp (cf. [1], [4], [6], [7]). (2.1) becomes

$$\lambda(\Omega) \leq \sqrt{\text{Area }(\Omega)/\pi}. \tag{2.11}$$

Now (cf. (1.1), (1.2)) since $\lambda(\Omega) = \inf\{\|z - \varphi\|_\infty ; \varphi$ is antiholomorphic in $\Omega\}$ $= \inf\{\|\bar{z} - g\| : g \in \mathcal{A}(\Omega)\}$, equality holds here when Ω is a disk (and only in that case–cf. [4]). The heart of the matter is that for $N = 2$ (and only for $N = 2$) the extremal domain Ω_t for the Alfors-Beurling estimate (2.4) is in fact a ball. For $N \geq 3$, Ω_t is an axially symmetric solid which comes "tighter" as $N \to \infty$ (cf. (2.9)) in contact with the $\{x : x_1 = 0\}$-plane tangent to it. However, we suspect that the inequality in (2.2) is in fact a strict inequality and that the extremal domain for $\lambda(\Omega)$ (not $\max_\Omega |\text{grad } u^\Omega|$) among domains with a fixed volume is (similarly to the two-dimensional case) still a ball. Thus, we propose the following:

Conjecture 2.2.

$$\lambda(\Omega) \leq \pi^{-1/2}(\tfrac{1}{2}N\Gamma(\tfrac{1}{2}N)V(\Omega))^{1/N} = R_{\text{vol}}, \tag{2.12}$$

where R_{vol} (the "volume radius") denotes the radius of the ball with the same volume $V(\Omega)$.

Remark. In fact we have proven a stronger statement, namely that $\lambda_1(\Omega) := \inf_{f\in B(\Omega)} \|x - f\|_\infty$, which is obviously not less than $\lambda(\Omega)$, still satisfies (2.1). Therefore, one is tempted to conjecture that the extremal domain for $\lambda_1(\Omega)$ is a ball as well. In this regard, it is natural to ask whether for "general" Ω,

$$\lambda(\Omega) = C_\Omega \lambda_1(\Omega), \qquad \text{where } 0 < C_\Omega < 1.$$

The question *for which domains* Ω does $\lambda(\Omega) = \lambda_1(\Omega)$ is also puzzling. In addition to all simply connected domains, this equality holds for spherical shells (see §**3**), which for $N = 2$ are not simply connected.

3. Lower estimates for $\lambda(\Omega)$.

Theorem 3.1.

$$\lambda(\Omega) \geq \frac{N\,V(\Omega)}{P(\partial\Omega)}, \tag{3.1}$$

where $P(\partial\Omega) = \int_{\partial\Omega} d\sigma$ denotes the perimeter of Ω. (3.1) is sharp, since it becomes equality for balls.

Proof. Let $f \in A(\Omega)$. Applying the divergence theorem, we have from (1.8)

$$\|x - f\|_\infty \geq \frac{1}{P(\partial\Omega)} \int_{\partial\Omega} |x - f|\, d\sigma \geq \frac{1}{P(\partial\Omega)} \left| \int_{\partial\Omega} (x - f) \cdot n d\sigma \right|$$

$$= \frac{1}{P(\partial\Omega)} \left| \int_{\Omega} \operatorname{div}(x - f)\; dV \right| = \frac{NV(\Omega)}{P(\partial\Omega)}. \tag{3.2}$$

Taking the infimum over all $f \in A(\Omega)$, we obtain (3.1). If Ω is a ball of radius R, say centered at the origin, then taking $f \equiv 0$ it is seen that both sides in (3.1) equal R so the estimate (3.1) is indeed the best possible. □

The question arises whether balls are the only solids for which the equality (3.1) is attained. As we see shortly, this is not the case; equality in (3.1) also holds for spherical shells. The following theorem characterizes the extremal domains for (3.1). For $N = 2$, a similar characterization has been obtained in [2].

Theorem 3.2. *The following are equivalent:*

(i) $\lambda(\Omega) = NV(\Omega)/P(\partial\Omega)$

(ii) *There exists $g^* \in B(\Omega)$ such that*

$$x - \lambda n(x) = g^*(x) \text{ on } \partial\Omega,$$

where $\lambda = \|x - g^*\|_\infty$.

(iii) *For any harmonic function v in $\bar{\Omega}$ satisfying $\int_S \frac{\partial v}{\partial n} d\sigma = 0$ for all compact smooth oriented hypersurfaces S in Ω the following quadrature identity holds*

$$\frac{1}{V(\Omega)} \int_\Omega v dV = \frac{1}{P(\partial\Omega)} \int_{\partial\Omega} v d\sigma.$$

(iv) *There exists* u *in* $C^1(\bar{\Omega})$ *such that*

$$\begin{cases} \Delta u &= 1 \quad \text{in } \Omega; \\ \frac{\partial u}{\partial n} &= const \quad \text{on } \partial\Omega; \\ u &= const \quad \text{on each component of } \partial\Omega. \end{cases}$$

For $N = 2$, the equivalence of (i) - (iii) and (iv) has been independently observed by I. Marrero [8, 9].

Proof. (i) $\Rightarrow$ (ii). The major step is to establish the existence of g^*. For that purpose, pose a similar extremal problem in the context of the Hilbert space $L^2(\sigma)$. Namely, define

$$(3.3) \qquad \Lambda_2(\Omega) = \inf_{g \in A(\Omega)} \|x - g\|_{L^2(\sigma)} = \inf_{g \in A(\Omega)} \left(\int_{\partial\Omega} |x - g|^2 \, d\sigma \right)^{1/2}.$$

Then, a standard convergence argument shows that there exists a vector field $g^* \in L^2(\sigma)$ on $\partial\Omega$ such that

$$\Lambda_2(\Omega) = \|x - g^*\|_{L^2(\sigma)}$$

and whose harmonic extension to Ω (i.e., a vector field whose components are harmonic extensions by means, say, of the Poisson integral of components of g^*) is a *harmonic* vector field in Ω. Also, it is obvious that

$$(3.4) \qquad \Lambda_2(\Omega) \leq \lambda(\Omega)\sqrt{P(\partial\Omega)}.$$

Applying Jensen's inequality and the divergence theorem to a given g in $A(\Omega)$ we have, similarly to (3.2), ($P = P(\partial\Omega)$) that

$$\|x - g\|_{L^2(\sigma)} = \left(\int_{\partial\Omega} |x - g|^2 \frac{d\sigma}{P} \right)^{1/2} P^{1/2} \geq$$

$$(3.5) \qquad \int_{\partial\Omega} |x - g| \frac{d\sigma}{P^{1/2}} \geq P^{-1/2} \left| \int_{\partial\Omega} (x - g) \cdot n d\sigma \right| = N P^{-1/2} V(\Omega).$$

and hence,

$$\Lambda_2(\Omega) \geq \frac{NV(\Omega)}{[P(\partial\Omega)]^{1/2}}. \tag{3.6}$$

Now if $\lambda(\Omega) = \frac{NV(\Omega)}{P(\partial\Omega)}$, it follows then from (3.4) that we must have equality in (3.6). This implies that for $g^* :=$ the extremal vector field in (3.3) we must have equality everywhere in (3.5), i.e.,

$$x - g^* = \lambda n \text{ a.e. on } \partial\Omega. \tag{3.7}$$

Although a priori g^* is only assumed to belong to the L^2-closure of $A(\Omega)$ on $\partial\Omega$, (3.7) together with the smoothness hypothesis imposed on $\partial\Omega$ imply that g^* is in fact continuous in $\bar{\Omega}$, and (3.7) holds everywhere on $\partial\Omega$. Moreover, since for any closed smooth curve γ on $\partial\Omega$, denoting by τ the unit tangent vector to γ and by ds the arclength on γ, we have by (3.7)

$$\int_\gamma (g^* \cdot \tau) ds = \int_\gamma (x - \lambda n) \cdot \tau ds = \int_\gamma x \cdot \tau ds = 0$$

$\left(x = \frac{1}{2}\text{grad } |x|^2 !\right)$, it follows that g^* is in fact a gradient field. This proves (ii). ($|x - g^*| = \lambda$ on $\partial\Omega$, so $\|x - g^*\|_\infty = \lambda$.)

(ii) $\Rightarrow$ (i). Assuming (ii) and writing $g^* = \text{grad } \psi^*$, we have $x \cdot n - \lambda = \frac{\partial\psi}{\partial n}$ on $\partial\Omega$. Therefore, using the divergence theorem, we obtain

$$0 = \int_{\partial\Omega} \frac{\partial\psi^*}{\partial n} d\sigma = \int_{\partial\Omega} (x \cdot n) d\sigma - \lambda P(\partial\Omega) = NV(\Omega) - \lambda P(\partial\Omega).$$

So $\lambda = NV(\Omega)/P(\partial\Omega)$, and since $\lambda = \|x - g^*\|_\infty \geq \lambda(\Omega) \geq NV(\Omega)/P(\partial\Omega)$ (cf. Thm. 3.1), it follows that $\lambda = \lambda(\Omega)$, i.e., (i) holds.

Now to show (ii) $\Rightarrow$ (iii), fix a function v satisfying the hypothesis in (iii). Set $u = \frac{|x|^2}{2N}$, $\Delta u \equiv 1$ and, as above, let $\psi^* \in H(\Omega) \cap C^1(\bar{\Omega})$ be the harmonic function such that grad $\psi^* = g^*$ in (ii). By applying Green's formula we have

$$\int_\Omega v dV = \int_\Omega [v\Delta u - u\Delta v] dV = \int_\Omega [v\Delta(u - \frac{\psi^*}{N}) - (u - \frac{\psi^*}{N})\Delta v] \, dV \tag{3.8}$$

$$= \int_{\partial\Omega} [v \frac{\partial(u - \psi^*/N)}{\partial n} - (u - \frac{\psi^*}{N}) \frac{\partial v}{\partial n}] \, d\sigma.$$

(ii) implies that

$$\frac{\partial(u - \psi^*/N)}{\partial n} = \text{grad } (u - \frac{\psi^*}{N}) \cdot n = (\frac{x}{N} - \frac{g^*}{N}) \cdot n = \frac{\lambda}{N}. \tag{3.9}$$

Since grad $(u - \frac{\psi^*}{N})|_{\partial\Omega} = \frac{\lambda}{N} n$, it follows that $u - \frac{\psi^*}{N}$ is locally a constant on $\partial\Omega$, i.e., $u - \frac{\psi^*}{N}|_{\Gamma_j} = c_j$. Hence,

$$\int_{\partial\Omega} (u - \frac{\psi^*}{N}) \frac{\partial v}{\partial n} d\sigma = \sum_{j=0}^{n} c_j \int_{\Gamma_j} \frac{\partial v}{\partial n} d\sigma = 0 \tag{3.10}$$

for all v satisfying the hypothesis in (iii). From (3.8) - (3.10) we obtain that

$$\int_{\Omega} v dV = \frac{\lambda}{N} \int_{\partial\Omega} v d\sigma. \tag{3.11}$$

Substituting $v = 1$ into (3.11), we have

$$\lambda = \frac{NV(\Omega)}{P(\partial\Omega)}. \tag{3.12}$$

(3.11) and (3.12) imply (iii).

(iii) $\Rightarrow$ (iv). Let $\varphi \in H(\Omega)$ be harmonic in a neighborhood of $\bar{\Omega}$. Denote by ω_k the "harmonic measure" of "inner" components Γ_k of the boundary $\partial\Omega$, i.e., $\Delta\omega_k = 0$ in Ω, $\omega_k = 1$ on Γ_k and $\omega_k = 0$ on $\Omega_j, k \neq j$: $k = 1, \dots, n; j = 0, 1, \dots, n$. Then we can choose $\alpha_k \in \mathbb{R}$, $k = 1, \dots n$, such that the function v

$$v := \varphi + \sum_{k=1}^{n} \alpha_k \omega_k \tag{3.13}$$

satisfies (iii). For this, we have to solve a linear system of equations

$$\int_{\Gamma_j} \frac{\partial \varphi}{\partial n} \, d\sigma + \sum_{k=1}^{n} \alpha_k \int_{\Gamma_j} \frac{\partial \omega_k}{\partial n} d\sigma = 0, \; j = 1, \dots, n$$

for $\alpha_1, \ldots, \alpha_n$. This can be done, since the matrix of coefficients here is well-known to be nonsingular. Denote by u_0 the solution of the Dirichlet problem

$$\begin{cases} \Delta u_0 = 1 & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $u = u_0 + \sum_{j=1}^n \lambda_j \omega_j$, where $\lambda_j \in \mathbb{R}$ are to be specified later. Applying (iii) and Green's formula to v in (3.13), we obtain

$$\frac{V(\Omega)}{P(\partial\Omega)} \left\{ \int_{\partial\Omega} \varphi d\sigma + \sum_{k=1}^n \alpha_k \int_{\Gamma_k} d\sigma \right\} = \frac{V}{P} \int_{\partial\Omega} v \, d\sigma$$

$$= \int_\Omega v dV = \int_\Omega [v\Delta u - u\Delta v] \, dV = \int_{\partial\Omega} [v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}] d\sigma$$

$$= \int_{\partial\Omega} v\frac{\partial u}{\partial n} \, d\sigma - \sum_{j=1}^n \lambda_j \int_{\Gamma_j} \frac{\partial v}{\partial n} \, d\sigma = \int_{\partial\Omega} v\frac{\partial u}{\partial n} \, d\sigma$$

$$= \int_{\partial\Omega} \varphi\frac{\partial u}{\partial n} \, d\sigma + \sum_{k=1}^n \alpha_k \int_{\Gamma_k} \frac{\partial u}{\partial n} \, d\sigma$$

$$= \int_{\partial\Omega} \varphi\frac{\partial u}{\partial n} \, d\sigma + \sum_{k=1}^n \alpha_k \int_{\Gamma_k} \frac{\partial u_0}{\partial n} \, d\sigma + \sum_{k=1}^n \alpha_k \sum_{j=1}^n \lambda_j \int_{\Gamma_k} \frac{\partial \omega_j}{\partial n} \, d\sigma$$

or,

(3.14)

$$\int_{\partial\Omega} \varphi(\frac{V}{P} - \frac{\partial u}{\partial n}) \, d\sigma = \sum_{k=1}^n \alpha_k \left\{ \sum_{j=1}^n \lambda_j \int_{\Gamma_k} \frac{\partial \omega_j}{\partial n} \, d\sigma - \int_{\Gamma_k} \left(\frac{V}{P} - \frac{\partial u_0}{\partial n} \right) \, d\sigma \right\}.$$

Once again, since the coefficient matrix is nonsingular we can choose $\lambda_1, \ldots, \lambda_n$ (uniquely) so that

$$\sum_{j=1}^n \lambda_j \int_{\Gamma_k} \frac{\partial \omega_j}{\partial n} \, d\sigma = \int_{\Gamma_k} \left(\frac{V}{P} - \frac{\partial u_0}{\partial n} \right) d\sigma, \ k = 1, \ldots, n$$

For this choice of λ_j we have

$$\Delta u = 1 \ \text{ in } \ \Omega$$

$$u = \lambda_j \text{ on } \Gamma_j,\ j = 1, \dots, n,\ u = 0 \text{ on } \Gamma_0$$

and

$$\int_{\partial\Omega} \varphi \left(\frac{V}{P} - \frac{\partial u}{\partial n} \right) \, d\sigma = 0$$

for all $\varphi \in H(\Omega)$. Hence $\frac{\partial u}{\partial n} = \frac{V}{P}$ on $\partial\Omega$, and (iv) is proved.

(iv) $\Rightarrow$ (ii). Let u satisfy (iv). From (iv) and Green's formula we obtain

$$\text{const } P(\partial\Omega) = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, d\sigma = \int_{\Omega} dV = V(\Omega),$$

i.e., $\frac{\partial u}{\partial n} = \frac{V}{P}$ on $\partial\Omega$. The condition that u is a local constant on $\partial\Omega$ can be rewritten as

$$\text{grad } u = \frac{V}{P} n \text{ on } \partial\Omega.$$

Define $\psi^* = \frac{1}{2}|x|^2 - Nu$. Then ψ^* is in $H(\Omega)$ and

$$x - \frac{NV}{P} n = \text{grad } \psi^* \text{ on } \partial\Omega.$$

That is (ii) holds. The proof of the theorem is now complete. □

Remarks. (i) The class of harmonic functions v in Ω satisfying

$$\int_S \frac{\partial v}{\partial n} \, d\sigma = 0 \tag{3.15}$$

for every closed oriented hypersurface S in Ω is somewhat mysterious. For $N = 2$, it is easily seen to coincide with the space of real parts of functions analytic in Ω, i.e., consists of components of all harmonic vector fields in Ω. In higher dimensions it still includes the components of harmonic vector fields. Indeed, let $g = (g_1, \dots, g_n)$ be a harmonic vector field, i.e., (cf. (1.8)):

$$\sum_{j=1}^{n} \frac{\partial g_j}{\partial x_j} = 0 \text{ and } \frac{\partial g_k}{\partial x_j} - \frac{\partial g_j}{\partial x_k} = 0 \text{ for all } j, k. \tag{3.16}$$

Now if v is a component of g, say $v = g_1$, let us show that for any closed oriented smooth hypersurface S in Ω, (3.15) holds. Indeed, from (3.16) and Stokes' theorem we have

$$\begin{aligned}
\int_S \frac{\partial v}{\partial n} d\sigma &= \int_S \sum_{j=1}^{N} (-1)^{j-1} \frac{\partial g_1}{\partial x_j} dx_1 \cdots \widehat{dx_j} \cdots dx_N \\
&= \int_S \frac{\partial g_1}{\partial x_1} dx_2 \cdots dx_N + \sum_{j=2}^{N} (-1)^{j-1} \int_S \frac{\partial g_j}{\partial x_1} dx_1 \cdots \widehat{dx_j} \cdots dx_N \\
&= \int_S \frac{\partial g_1}{\partial x_1} dx_2 \cdots dx_N + \sum_{j=2}^{N} (-1)^{j-1} \int_S d(g_j dx_2 \cdots \widehat{dx_j} \cdots dx_N) \\
&\quad - \sum_{j=2}^{N} (-1)^{j-1} \int_S (-1)^j \frac{\partial g_j}{\partial x_j} dx_2 \cdots dx_N = 0.
\end{aligned}$$

In general, however, these two classes of harmonic functions need not coincide. Yet, in [5] it was shown that for "reasonable" Ω the linear combinations of components of harmonic vector fields are dense with respect to L^1-norms in the space of harmonic functions satisfying (3.15).

(ii) *Example of an extremal domain for (3.1).* Let $\Omega = \{r < |x| < R\}$ be a spherical shell. We want to show that

$$\lambda(\Omega) = \frac{NV}{P} = \frac{R^N - r^N}{R^{N-1} + r^{N-1}}.$$

Since $n = \frac{x}{R}$ on $\Gamma_0 := \{x : |x| = R\}$ and $n = -\frac{x}{r}$ on $\Gamma_1 := \{x : |x| = r\}$, we have

$$x - \lambda n = \frac{x \quad r^{N-1}}{R(R^{N-1} + r^{N-1})}, (R + r) = \frac{x}{R^N} \frac{R^{N-1} r^{N-1} (R + r)}{R^{n-1} + r^{N-1}} \text{ on } \Gamma_0$$

and

$$x - \lambda n = \frac{x \; R^{N-1}(R + r)}{r(r^{N-1} + r^{N-1})} = \frac{x}{r^N} \frac{R^{N-1} r^{N-1} (R + r)}{R^{n-1} + r^{N-1}} \text{ on } \Gamma_1.$$

Denoting by

$$g^* = \frac{R^{N-1}r^{N-1}(R+r)}{(R^{N-1}+r^{N-1})}\frac{x}{|x|^N} = \text{grad}\left[\frac{r^{N-1}r^{N-1}(R+r)}{(2-N)(R^{N-1}+r^{N-1})}\frac{1}{|x|^{N-2}}\right].$$

we see that condition (ii) of Theorem. 3.2 is satisfied, and hence Ω is indeed an extremal domain for (3.1).

(iii) It is worth mentioning (cf. Remark at the end of §**2**) that the extremal domains, in regard to the lower bound, for $\lambda(\Omega)$ (defined with respect to $A(\Omega)$) and $\lambda_1(\Omega)$ (defined with respect to $B(\Omega)$) do in fact coincide. Indeed, since the vector field g^* in (ii) of theorem 3.2 belongs to $B(\Omega)$ (not merely to $A(\Omega)$), the inequality that occurs in the proof of (ii) $\Rightarrow$ (i) can be sharpened to yield $\lambda = \|x - g^*\|_\infty \geq \lambda_1(\Omega) \geq NV(\Omega)/P(\partial\Omega)$. Therefore, (ii) actually implies that $\lambda_1(\Omega) = NV(\Omega)/P(\partial\Omega)$ and so $\lambda_1(\Omega)$ attains this lower bound if and only if $\lambda(\Omega)$ does.

Conditions (ii) and (iv) of theorem 3.2 suggest that extremal domains for (3.1) are quite rare. In fact, if we impose an additional hypothesis on Ω and assume it to be *homeomorphic to a ball*, we have the following

Corollary 3.3. *If* $\lambda(\Omega) = \frac{NV(\Omega)}{P(\partial\Omega)}$ *and* Ω *is homeomorphic to a ball, then, in fact,* Ω *must be a ball of radius* λ.

For the proof, it suffices to notice that under our additional hypothesis the overdetermined boundary value problem (iv) in theorem 3.2 becomes

$$\begin{cases} \Delta u = 1 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \frac{V}{P} & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The celebrated theorem of Serrin (see [10], [12]) states then, that Ω must be a ball. In view of this, it is natural to formulate the following

Conjecture 3.4. *If* $\lambda(\Omega) = \frac{NV(\Omega)}{P(\partial\Omega)}$, *then* Ω *is either a ball or a spherical shell.*

Even for $N = 2$, this problem is still open (cf. [6, 7] and references cited there). As in case $N = 2$ (cf. [7]), assuming that the boundary of an extremal domain contains a piece of a spherical surface implies that Ω must either be a ball or a spherical shell. Indeed, suppose $\partial\Omega$ contains a piece Γ of a sphere of radius R centered at the origin. From (ii) of Thm. 3.2 it follows that

$$g^*(x) = x - \lambda\frac{x}{|x|} = c \ \text{grad}\ (|x|^{2-N}), \text{ where}$$
$$c = \frac{1}{2-N}(R^N - \lambda R^{N-1}). \tag{3.17}$$

Since $g^* = \text{grad}\ \psi^*$, ψ^* is harmonic in Ω, Then by the uniqueness of the solution of the Cauchy problem (Cauchy-Kovalevskaya theorem), (3.17) holds everywhere in Ω. If $c = 0$, then $g^* \equiv 0$ and $|x| = \lambda = R$ on $\partial\Omega$, i.e., Ω is a ball of radius R centered at the origin. If $c \neq 0$, then we have from Thm. 3.2 (ii) and (3.17) that everywhere on $\partial\Omega$ the normal vector n is parallel to the vector x. Hence, every boundary component of the boundary $\partial\Omega$ must be a sphere centered at the origin. So, Ω is a spherical shell.

Finally, let us mention the *regularity problem* for the free boundary of an extremal domain Ω satisfying (i)-(iv) of Thm. 3.2. Everywhere we assumed $\partial\Omega$ to be smooth, even analytic. However, having assumed $\partial\Omega$ to be merely rectifiable, it seems plausible that either the quadrature identity (iii) or condition (ii) holding *almost everywhere* on $\partial\Omega$ alone imply that $\partial\Omega$ is (a) locally real analytic and (b) consists of at most two connected components. Perhaps, one might try to approach this regularity question by considering first a case when the free boundary $\partial\Omega$ is assumed to be homeomorphic to a sphere. (For results in that direction the reader may consult the important paper [2], where a problem similar to the one in (iv), Theorem. 3.2 is treated.)

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