Nonperiodic explicit homogenization and reduction of dimension: the linear case

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Abstract

The aim of this paper is to give explicit limit expressions, for diffusion equations involving a small parameter ε, describing both nonperiodic homogenization and reduction of dimension. In other words, we give the limit behaviour, when ε tends to zero, of the diffusion equation in a thin domain, with thickness of order ε, when the coefficients of the equation also depend on ε and may present rapid, nonperiodic oscillations, provided they satisfy a suitable compensated compactness condition. We consider two kinds of reduction of dimension: the case of thin plates ($3D \to 2D$) and the case of thin cylinders ($3D \to 1D$). In particular, we give the limit diffusion equation for laminated plates. This is completely explicit and requires no special assumption, except stratification. In the case of thin cylinders, the formulae are less explicit, but we also indicate some simple applications.

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1 Introduction

In this paper, Ω^ε is a thin domain in $\mathbb{R}^N$, $N \geq 2$, representing either an horizontal plate or a thin vertical cylinder, and one considers the diffusion equation in Ω^ε, with mixed Dirichlet-Neumann boundary conditions, written in its variational form.

The generic point of $\mathbb{R}^N$ is denoted by $x = (x', x_N) = (x_1, \dots, x_{N-1}, x_N)$. The coefficients of the equation, constituting the conductivity matrix $\mathcal{A}^\varepsilon$, depend on ε and $x \in \Omega^\varepsilon$. For convenience, we introduce the matrix A^ε, defined in a fixed cylinder Ω by $A^\varepsilon(x) = \mathcal{A}^\varepsilon(x', \varepsilon x_N)$, for $(x', \varepsilon x_N)$ in the plate, or $A^\varepsilon(x) = \mathcal{A}^\varepsilon(\varepsilon x', x_N)$, for $(\varepsilon x', x_N)$ in the thin cylinder.

By classical rescaling from Ω^ε to Ω, the diffusion equation reads:

$$
\begin{gathered}
u^\varepsilon \in H^1(\Omega),\ u^\varepsilon = 0 \text{ on } \Gamma_D \text{ and } \forall v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_D,\\
\int_\Omega [A^\varepsilon(x)\nabla^\varepsilon u^\varepsilon, \nabla^\varepsilon v]\, dx = \int_\Omega f^\varepsilon v\, dx + \int_\Omega [g^\varepsilon, \nabla^\varepsilon v]\, dx + \int_{\Gamma_N} h^\varepsilon v\, d\gamma,
\end{gathered}
\tag{1.1}
$$

where $\partial\Omega = \Gamma_D \cup \Gamma_N$, the Dirichlet and Neumann conditions occuring respectively on Γ_D and Γ_N. In the above equation, $[.,.]$ denotes the scalar product in $\mathbb{R}^N$ and the operator ∇^ε has two different forms:

$$
\begin{aligned}
\nabla^\varepsilon v &= \left(\nabla' v, \tfrac{1}{\varepsilon}\tfrac{\partial v}{\partial x_N}\right) &&= \left(\tfrac{\partial v}{\partial x_1}, \dots, \tfrac{\partial v}{\partial x_{N-1}}, \tfrac{1}{\varepsilon}\tfrac{\partial v}{\partial x_N}\right) \text{ for the plate },\\
\nabla^\varepsilon v &= \left(\tfrac{1}{\varepsilon}\nabla' v, \tfrac{\partial v}{\partial x_N}\right) &&= \left(\tfrac{1}{\varepsilon}\tfrac{\partial v}{\partial x_1}, \dots, \tfrac{1}{\varepsilon}\tfrac{\partial v}{\partial x_{N-1}}, \tfrac{\partial v}{\partial x_N}\right) \text{ for the thin cylinder.}
\end{aligned}
$$

The task is to pass to the limit in (1.1), when ε tends to zero, which combines both homogenization and reduction of dimension.

It is well known that, with ∇ instead of ∇^ε in (1.1), which is the case of sole homogenization, the limit problem is

$$
\begin{gathered}
u \in H^1(\Omega),\ u = 0 \text{ on } \Gamma_D \text{ and } \forall v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_D,\\
\int_\Omega [A\nabla u, \nabla v]\, dx = \int_\Omega f v\, dx + \int_\Omega [g, \nabla v]\, dx + \int_{\Gamma_N} h v\, d\gamma,
\end{gathered}
\tag{1.2}
$$

under natural convergence assumtions on f^ε, g^ε and h^ε, with A the H-limit of A^ε (see e.g. [?], [?]). Of course, this H-limit is not explicit in general, except in the periodic case [?] or if A^ε has specific dependence upon coordinates [?], or more generally if it satisfies special compensated compactness type assumptions [?].

On the other hand, if in (1.1) $A^\varepsilon = A$ does not depend on ε, which is the case of sole reduction of dimension, the most natural expression for the limit problem of (1.1) is written on Ω and it involves two functions, u and y: $u = u(x')$ for the plate, $u = u(x_N)$ for the thin cylinder, and in any case $y = y(x', x_N)$. More precisely its variational equation takes the form

$$
\int_\Omega [A\nabla"(u, y), \nabla"(v, z)]\, dx = \int_\Omega f v\, dx + \int_\Omega [g, \nabla"(v, z)]\, dx + \int_{\Gamma_N} h v\, d\gamma,
\tag{1.3}
$$

with the operator $\nabla"$ defined by

$$
\begin{aligned}
\nabla"(v, z) &= \left(\nabla' v, \tfrac{\partial z}{\partial x_N}\right) \text{ for the plate },\\
\nabla"(v, z) &= \left(\nabla' z, \tfrac{dv}{dx_N}\right) \text{ for the thin cylinder.}
\end{aligned}
$$

However it is possible to eliminate y and to write the limit problem in terms of u only, on the limit domain of Ω^ε (the $(N-1)$-dimensional plate ω or the thread, represented by the interval (0,1)). In this reduced limit problem, the bilinear form is

$$\int_\omega [A^0 \nabla' u, \nabla' v]\, dx' \quad \text{for the plate ,}$$

$$\int_0^1 A^0 \frac{du}{dx_N} \frac{dv}{dx_N}\, dx_N \qquad \text{for the thin cylinder,}$$

with A^0 given in terms of A. Then y, which appears as a corrector, is given in terms of u and of the limits of the data. For reduction of dimension, the reader is refered e.g. to [?], [?], [?], [?], [?], [?] and [?].

If one combines homogenization and reduction of dimension, two kinds of results are known. The first one is very general (see [?] for plates and [?] for thin cylinders). Briefly speaking, it says that there exists A^0, independent of the source and boundary data, such that the reduced limit problem, written in the limit domain, has conductivity matrix A^0. The second kind of results are explicit expressions of A^0 in periodic cases (see e.g. [?], [?], [?] for plates and [?], [?] for thin cylinders).

In this paper, considering special compensated compactness assumptions on A^ε and requiring no periodicity, we prove that the limit problem of (1.1) is still (1.2), where now A is the H-limit of A^ε and is explicit, in terms of weak*-L^∞ limits of suitable combinations of coefficients of A^ε. Of course, the limit problem can be formulated in terms of u only, as above, and then A^0 is explicit in terms of the H-limit A. Our proof relies on the classical compensated compactness method, applied to a suitable decomposition of A^ε, written $A^\varepsilon = (M^\varepsilon)^{-1} P^\varepsilon$, with two different expressions of the couple $(M^\varepsilon, P^\varepsilon)$, one for the plate, one for the thin cylinder. The most striking application concerns thin laminated plates, in which case the limit problem is fully explicit.

In general it is false that the limit problem has the form (1.3). Here, the compensated compactness assumption on A^ε is indeed crucial. In Section 2, we give a counterexample for that, concerning plates.

In order to help the reader, let us comment on the organization of the paper. The case of plates is considered in Section 2, while Section 3 deals with thin cylinders. The two parts can be read independently. The outlines of both sections are very similar: the main results are presented in the beginning (three theorems each time), the remainder of the section being devoted to the proofs of these theorems, except for the examples, located in the last subsection. For completeness, some complements are given in Section 4, forming an Appendix, but we suspect they are not very original. The reader who is not interested in the technical details is recommended to skip the proofs and the Appendix, and just look at the main results and the examples.

The present results were announced in a short note [?] and some nonlinear extensions are in preparation (see [?]).

2 Nonperiodic homogenization and reduction of dimension for plates

2.1 The problem and the main results:

Let ω be a bounded domain in $\mathbb{R}^{N-1}$, $N \geq 2$, and let Ω be the cylinder $\Omega = \omega \times (-\frac{1}{2}, \frac{1}{2})$. For ε (≤ 1) running through a sequence of values tending to zero, $\Omega^\varepsilon (\subset \Omega)$ represents the horizontal plate, that is the flat cylinder $\Omega^\varepsilon = \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, with lateral boundary $\Sigma^\varepsilon = \partial\omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$.

Let $\mathcal{A}^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^{N\times N}$ be a sequence of (not necessarily symmetric) matrices with L^∞-coefficients, such that

(2.1) $\exists \alpha, \beta,\ 0 < \alpha \leq \beta,$ a.e. $x \in \Omega^\varepsilon, \forall \xi \in \mathbb{R}^N,\ [\mathcal{A}^\varepsilon(x)\xi, \xi] \geq \alpha|\xi|^2$ and $|\mathcal{A}^\varepsilon(x)\xi| \leq \beta|\xi|$,

where $[.,.]$ denotes the scalar product in $\mathbb{R}^N$ and $|.|$ denotes the Euclidian norm. Condition (2.1) means that the matrices A^ε, defined in the fixed cylinder Ω by $A^\varepsilon(x) = \mathcal{A}^\varepsilon(x', \varepsilon x_N)$ are bounded and coercive, uniformly with respect to ε and x in Ω.

We set

(2.2) $$\mathcal{V}^\varepsilon = \{V \in H^1(\Omega^\varepsilon), V = 0 \text{ on } \Sigma^\varepsilon\}$$

and we consider the variational problem, with given data $F^\varepsilon \in L^2(\Omega)$, $G^\varepsilon \in L^2(\Omega)^N$, h_+^ε and $h_-^\varepsilon \in H^{-\frac{1}{2}}(\omega)$,

(2.3) $$\begin{aligned} &U^\varepsilon \in \mathcal{V}^\varepsilon \text{ and } \forall V \in \mathcal{V}^\varepsilon, \\ &\int_{\Omega^\varepsilon} [\mathcal{A}^\varepsilon \nabla U^\varepsilon, \nabla V]\, dx = \int_{\Omega^\varepsilon} F^\varepsilon V\, dx + \int_{\Omega^\varepsilon} [G^\varepsilon, \nabla V]\, dx \\ &+ \int_\omega \varepsilon h_+^\varepsilon(x') V(x', \frac{\varepsilon}{2})\, dx' + \int_\omega \varepsilon h_-^\varepsilon(x') V(x', -\frac{\varepsilon}{2})\, dx', \end{aligned}$$

where the generic point in $\mathbb{R}^N$ is denoted by $x = (x', x_N) = (x_1, \ldots, x_{N-1}, x_N)$ and e.g. the first integral over ω denotes the duality pairing of $\varepsilon h_+^\varepsilon$, in $H^{-\frac{1}{2}}(\omega)$, and of the trace of V on $\{x_N = \frac{\varepsilon}{2}\}$, which belongs to $H^{\frac{1}{2}}(\omega)$. The factor ε before h_+^ε and h_-^ε is introduced for homogeneity reasons.

It is classical that (2.3) admits a unique solution U^ε, solving (in a weak sense) the diffusion problem

$$\begin{cases} -\text{ div } (\mathcal{A}^\varepsilon \nabla U^\varepsilon) = F^\varepsilon - \text{ div } G^\varepsilon \text{ in } \Omega^\varepsilon, \\ U^\varepsilon = 0 \text{ on } \Sigma^\varepsilon, \\ [\mathcal{A}^\varepsilon \nabla U^\varepsilon, e_N] = [G^\varepsilon, e_N] + \varepsilon h_+^\varepsilon \text{ on } \Gamma_+^\varepsilon = \omega \times \{\frac{\varepsilon}{2}\}, \\ [\mathcal{A}^\varepsilon \nabla U^\varepsilon, e_N] = [G^\varepsilon, e_N] - \varepsilon h_-^\varepsilon \text{ on } \Gamma_-^\varepsilon = \omega \times \{-\frac{\varepsilon}{2}\}, \end{cases}$$

where e_N is the unit vector of the vertical axis.

Such problems were considered by A. Damlamian and M. Vogelius in [?], with symmetric matrices A^ε. They proved that, up to extraction of a subsequence, there exists a symmetric matrix $A^0 : \omega \to \mathbb{R}^{(N-1)\times(N-1)}$, such that $U^\varepsilon(x', \varepsilon x_N)$ converges weakly in $H^1(\Omega)$ to the solution $U = U(x')$ of a $(N-1)$-dimensional problem defined in terms of A^0, provided the data converge in a natural sense. As is the case for H-convergence, the limit matrix A^0 does not depend on the source and boundary data. But of course, A^0 is not the H-limit of A^ε, since A^0 has size $(N-1)\times(N-1)$ and is defined in ω, while A^ε is a $N \times N$-matrix, defined in Ω. Moreover, no explicit expression of A^0 was known, except if A^ε is a periodic function depending on x' only (see [?], [?], [?]).

The aim of this section is to prove that A^0 is explicit in terms of the H-limit A of A^ε, under some compensated compactness condition, which requires no periodicity and generalizes the stratified case $A^\varepsilon = A^\varepsilon(x_N)$.

More precisely, we prove the following three theorems.

Theorem 1 *Assume (2.1) and define $A^\varepsilon = \left(a^\varepsilon_{ij}\right)_{i,j\le N}$ by $A^\varepsilon(x) = \mathcal{A}^\varepsilon(x', \varepsilon x_N)$. (Clearly A^ε also satisfies (2.1), with Ω in place of Ω^ε.) Then, up to extraction of a subsequence, we may suppose that we have the following weak* convergences in $L^\infty(\Omega)$:*

$$(2.4) \qquad \begin{cases} \frac{1}{a^\varepsilon_{NN}} \rightharpoonup \frac{1}{a_{NN}}, \\[2mm] \frac{a^\varepsilon_{iN}}{a^\varepsilon_{NN}} \rightharpoonup \frac{a_{iN}}{a_{NN}}, \ \textit{for all } i < N, \\[2mm] \frac{a^\varepsilon_{Nj}}{a^\varepsilon_{NN}} \rightharpoonup \frac{a_{Nj}}{a_{NN}}, \ \textit{for all } j < N, \\[2mm] a^\varepsilon_{ij} - \frac{a^\varepsilon_{iN} a^\varepsilon_{Nj}}{a^\varepsilon_{NN}} \rightharpoonup a_{ij} - \frac{a_{iN} a_{Nj}}{a_{NN}}, \ \textit{for all } i, j < N, \end{cases}$$

for some matrix $A = (a_{ij})_{i,j=1,\dots,N}$, with L^∞-coefficients.

Assume moreover that

$$(2.5) \qquad \begin{array}{l} \forall i, j < N, \ \textit{the following sequences are relatively compact in } H^{-1}(\Omega) : \\ \left\{\frac{\partial}{\partial x_j}\left(\frac{1}{a^\varepsilon_{NN}}\right)\right\}_\varepsilon, \quad \left\{\frac{\partial}{\partial x_j}\left(\frac{a^\varepsilon_{iN}}{a^\varepsilon_{NN}}\right)\right\}_\varepsilon, \quad \left\{\sum_{k<N} \frac{\partial}{\partial x_k}\left(\frac{a^\varepsilon_{Nk}}{a^\varepsilon_{NN}}\right)\right\}_\varepsilon \\[2mm] \textit{and } \left\{\sum_{k<N} \frac{\partial}{\partial x_k}\left(a^\varepsilon_{ik} - \frac{a^\varepsilon_{iN} a^\varepsilon_{Nk}}{a^\varepsilon_{NN}}\right)\right\}_\varepsilon. \end{array}$$

Then A is coercive with same constant α as $\mathcal{A}^\varepsilon$ and A^ε, and A is the H-limit of A^ε.

Here, condition (2.5) is crucial and it is a compensated compactness type assumption (see [?]). In particular it holds true if the coefficients of A^ε have special dependence upon coordinates (see also [?], [?], [?]). For example it is satisfied for laminated materials, when $A^\varepsilon = A^\varepsilon(x_N)$. In such case it is well known that the convergences (2.4) define the H-limit A of A^ε (see e.g. [?]).

Theorem 2 *Assume that the sequence of matrices A^ε satisfies (2.1), (2.5) and let A be the H-limit of A^ε, given by (2.4) . Define f^ε and g^ε from F^ε and G^ε by $f^\varepsilon(x) = F^\varepsilon(x', \varepsilon x_N)$, $g^\varepsilon(x) = G^\varepsilon(x', \varepsilon x_N)$. Assume moreover that the data converge in the following sense:*

$$\begin{cases} f^\varepsilon \rightharpoonup f, \ \text{weakly in } L^2(\Omega), \\ g^\varepsilon \rightharpoonup g, \ \text{weakly in } L^2(\Omega)^N, \\ h_+^\varepsilon \rightharpoonup h_+ \text{ and } h_-^\varepsilon \rightharpoonup h_-, \ \text{weakly in } H^{-\frac{1}{2}}(\omega) \end{cases} \tag{2.6}$$

and assume also that

$$\left\{ \frac{\partial g_N^\varepsilon}{\partial x_N} \right\}_\varepsilon \text{ is relatively compact in } H^{-1}(\Omega). \tag{2.7}$$

Let $u^\varepsilon(x) = U^\varepsilon(x', \varepsilon x_N)$, where U^ε is the solution of (2.3) and let ∇^ε be the operator defined by $\nabla^\varepsilon v = \left(\nabla' v, \frac{1}{\varepsilon}\frac{\partial v}{\partial x_N}\right) = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_{N-1}}, \frac{1}{\varepsilon}\frac{\partial v}{\partial x_N}\right)$. Then, when ε tends to zero,

$$\begin{cases} u^\varepsilon \rightharpoonup u, \ \text{weakly in } H^1(\Omega), \\ \frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial x_N} \rightharpoonup \frac{\partial y}{\partial x_N}, \ \text{weakly in } L^2(\Omega), \\ \sigma^\varepsilon = A^\varepsilon \nabla^\varepsilon u^\varepsilon \rightharpoonup \sigma = A \begin{pmatrix} \nabla' u \\ \frac{\partial y}{\partial x_N} \end{pmatrix}, \ \text{weakly in } L^2(\Omega)^N, \end{cases} \tag{2.8}$$

where $u = u(x')$ and (u, y) is the unique solution of the limit variational problem:

$$\begin{aligned} & u \in H_0^1(\omega), \ y \in L^2\left(\omega; H_m^1(-\frac{1}{2}, \frac{1}{2})\right) \text{ and} \\ & \forall v \in H_0^1(\omega), \ \forall z \in L^2\left(\omega; H_m^1(-\frac{1}{2}, \frac{1}{2})\right), \\ & \int_\Omega \left[A \begin{pmatrix} \nabla' u \\ \frac{\partial y}{\partial x_N} \end{pmatrix}, \begin{pmatrix} \nabla' v \\ \frac{\partial z}{\partial x_N} \end{pmatrix} \right] dx = \int_\Omega f v \, dx + \int_\Omega \left[g, \begin{pmatrix} \nabla' v \\ \frac{\partial z}{\partial x_N} \end{pmatrix} \right] dx \\ & \qquad + \int_\omega (h_+ + h_-) v \, dx', \end{aligned} \tag{2.9}$$

H_m^1 denoting the subset of functions of H^1, having mean value zero.

Remark 1 *That U^ε solves (2.3) is equivalent to saying that u^ε solves (2.20) below, which is the variational formulation of*

$$\begin{cases} -\ div^\varepsilon\ (A^\varepsilon \nabla^\varepsilon u^\varepsilon) = f^\varepsilon - \ div^\varepsilon\ g^\varepsilon \text{ in } \Omega, \\ u^\varepsilon = 0 \text{ on } \Sigma = \partial\omega \times (-\frac{1}{2}, \frac{1}{2}), \\ [A^\varepsilon \nabla^\varepsilon u^\varepsilon, e_N] = [g^\varepsilon, e_N] + \varepsilon h_+^\varepsilon \text{ on } \Gamma_+ = \omega \times \{\frac{1}{2}\}, \\ [A^\varepsilon \nabla^\varepsilon u^\varepsilon, e_N] = [g^\varepsilon, e_N] - \varepsilon h_-^\varepsilon \text{ on } \Gamma_- = \omega \times \{-\frac{1}{2}\}, \end{cases}$$

where div^ε *is defined by*

$$div^\varepsilon \phi = \sum_{i<N} \frac{\partial \phi_i}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial \phi_N}{\partial x_N}$$

and where e_N *is the unit vector of the vertical axis.*

Remark 2 *Note that Problem (2.9) is well posed on the space* $H_0^1(\omega) \times L^2\left(\omega; H_m^1(-\frac{1}{2},\frac{1}{2})\right)$ *and that* $H_m^1(-\frac{1}{2},\frac{1}{2})$ *can be replaced by* $H^1(-\frac{1}{2},\frac{1}{2})/\mathbb{R}$. *On the contrary,* y *is not unique if* $H_m^1(-\frac{1}{2},\frac{1}{2})$ *is replaced by* $H^1(-\frac{1}{2},\frac{1}{2})$.

Remark 3 *Theorem 2 is false in general, if assumption (2.5) does not hold. This is proved by the following counterexample in dimension 2, where* $\omega = (-1,1)$, $\varepsilon = \frac{1}{n\pi}, n \in N$, A^ε *is diagonal with* $a_{11}^\varepsilon = 2$, $a_{22}^\varepsilon = 2 + \sin\frac{x_1}{\varepsilon}$, $f^\varepsilon \equiv 0$, $h_+^\varepsilon \equiv h_-^\varepsilon \equiv 0$, $g_1^\varepsilon = 2x_2 \cos\frac{x_1}{\varepsilon}$, $g_2^\varepsilon = \sin\frac{x_1}{\varepsilon}\left(2 + \sin\frac{x_1}{\varepsilon}\right)$. *Then* $u^\varepsilon = \varepsilon x_2 \sin\frac{x_1}{\varepsilon} \rightharpoonup 0$, *weakly in* $H^1(\Omega)$, $\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial x_2} = \sin\frac{x_1}{\varepsilon} \rightharpoonup 0$, *weakly in* $L^2(\Omega)$, *but*

$$\sigma^\varepsilon = A^\varepsilon \nabla^\varepsilon u^\varepsilon = g^\varepsilon \rightharpoonup \sigma = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \text{ weakly in } L^2(\Omega)^2,$$

$$\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \neq A \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

for any matrix A, *in contradiction to (2.8). This is a counterexample, in which the limit problem is not of the form (2.9).*

The additional function y appearing at the limit plays the role of a corrector. A similar function was introduced by F. Murat and A. Sili [?] in the study of thin cylinders. Problem (2.9), involving u and y, seems to be the most natural limit problem for (2.3), since it involves the H-limit of A^ε. However, by eliminating y, u is proved to solve a **reduced limit problem** and the above result can be translated into the following one, closer to [?] (see also [?]).

Theorem 3 *The assumptions and notations are those of Theorem 2. Besides, let* $A' = (a_{ij})_{i,j<N}$, $C = (a_{iN})_{i<N}$, $L = (a_{Nj})_{j<N}$, $g' = (g_i)_{i<N}$ *and let* $B' = (b_{ij})_{i,j<N}$ *be given by*

$$B' = A' - \frac{1}{a_{NN}} CL, \qquad b_{ij} = a_{ij} - \frac{a_{iN}a_{Nj}}{a_{NN}}. \tag{2.10}$$

Then u *is the unique solution of the variational problem*

(2.11) $\quad u \in H_0^1(\omega)$ *and* $\forall v \in H_0^1(\omega)$,

$$\int_\Omega [B'\nabla' u, \nabla' v]\, dx = \int_\Omega f v\, dx + \int_\Omega \left[g' - \frac{g_N}{a_{NN}}C, \nabla' v\right] dx + \int_\omega (h_+ + h_-) v\, dx'$$

and the above equation reduces to

$$\int_\omega \left[A^0 \nabla' u, \nabla' v\right] dx' = \int_\omega f^0 v\, dx' + \int_\omega \left[g^0, \nabla' v\right] dx', \tag{2.12}$$

where $A^0 = m(B')$, $f^0 = m(f) + h_+ + h_-$, $g^0 = m(g' - \frac{g_N}{a_{NN}} C)$, *$m$ standing for the mean value over* $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Moreover

$$y = y_N - \sum_{j<N} y_j \frac{\partial u}{\partial x_j} = y_N - [y', \nabla' u]\,, \tag{2.13}$$

with y_N *and* y_j $(j < n)$ *given by*

$$y_N(x', x_N) = \int_0^{x_N} \frac{g_N}{a_{NN}}(x', s)\, ds - \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^s \frac{g_N}{a_{NN}}(x', t)\, dt\, ds, \tag{2.14}$$

$$y_j(x', x_N) = \int_0^{x_N} \frac{a_{Nj}}{a_{NN}}(x', s)\, ds - \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^s \frac{a_{Nj}}{a_{NN}}(x', t)\, dt\, ds. \tag{2.15}$$

It follows from Theorem 3 that, under conditions (2.4) and (2.5), the matrix A^0 of [?] is nothing else than $m(B')$. Let us emphasize that y_j $(j < n)$ depend on A only, and not on the source and boundary data. On the contrary, y_N depends on the source data. Notice also that (2.14) and (2.15) are respectively equivalent to the following conditions, fulfilled for a.e. $x' \in \omega$,

$$m(y_N) = 0 \quad \text{and} \quad \frac{\partial y_N}{\partial x_N} = \frac{g_N}{a_{NN}}, \tag{2.16}$$

$$m(y_j) = 0 \quad \text{and} \quad \frac{\partial y_j}{\partial x_N} = \frac{a_{Nj}}{a_{NN}}, \tag{2.17}$$

so that Theorem 2 says that $\frac{1}{\varepsilon} \frac{\partial u^\varepsilon}{\partial x_N}$ converges weakly in $L^2(\Omega)$ to

$$\frac{\partial y}{\partial x_N} = \frac{g_N}{a_{NN}} - \sum_{j<N} \frac{a_{Nj}}{a_{NN}} \frac{\partial u}{\partial x_j}.$$

Remark that $y = 0$ and $A^0 = m(A')$ if $g_N = 0$ and $L = 0$ (i.e. $a_{Nj} = 0$ for $j < N$).

Except for the last subsection, the remainder of this section is devoted to the proof of the above three theorems.

2.2 First step of the proof:

For convenience of the reader and for completeness, we briefly recall the classical arguments of reduction of dimension $3D \to 2D$ (see also [?], [?], [?], [?], [?], [?], [?], [?], [?]).

Rescaling: First, in order to study the limit behaviour of U^ε, we rescale the problem to the fixed domain $\Omega = \omega \times (-\frac{1}{2}, \frac{1}{2})$. We introduce the general notation

$$v(x) = v(x', x_N) = V(x', \varepsilon x_N) \text{ for } x = (x', x_N) \in \Omega, (x', \varepsilon x_N) \in \Omega^\varepsilon; \tag{2.18}$$

in particular this defines f^ε and g^ε from the data F^ε and G^ε. We also set

$$\nabla^\varepsilon v = (\nabla' v, \frac{1}{\varepsilon}\frac{\partial v}{\partial x_N}). \tag{2.19}$$

(These are the definitions appearing in Theorem 2). Then it is easy to check that problem (2.3) reads in the fixed domain Ω:

$$\begin{aligned} &u^\varepsilon \in \mathcal{V} = \{v \in H^1(\Omega), v = 0 \text{ on } \Sigma = \partial\omega \times (-\frac{1}{2}, \frac{1}{2})\} \text{ and } \forall v \in \mathcal{V}, \\ &\int_\Omega [A^\varepsilon(x)\nabla^\varepsilon u^\varepsilon, \nabla^\varepsilon v]\, dx = \int_\Omega f^\varepsilon v\, dx + \int_\Omega [g^\varepsilon, \nabla^\varepsilon v]\, dx \\ &\qquad + \int_\omega h_+^\varepsilon(x') v(x', \frac{1}{2})\, dx' + \int_\omega h_-^\varepsilon(x') v(x', -\frac{1}{2})\, dx'. \end{aligned} \tag{2.20}$$

A priori estimates: In the following and in the whole paper, we will write c for any constant, not depending on ε. By using (2.1) and by taking u^ε as test function in (2.20), we get

$$\begin{aligned} &\alpha \int_\Omega |\nabla u^\varepsilon|^2\, dx \le \alpha \int_\Omega |\nabla^\varepsilon u^\varepsilon|^2\, dx \le \int_\Omega [A^\varepsilon \nabla^\varepsilon u^\varepsilon, \nabla^\varepsilon u^\varepsilon]\, dx = \\ &\int_\Omega f^\varepsilon u^\varepsilon\, dx + \int_\Omega [g^\varepsilon, \nabla^\varepsilon u^\varepsilon]\, dx + \int_\omega h_+^\varepsilon(x') u^\varepsilon(x', \frac{1}{2})\, dx' + \int_\omega h_-^\varepsilon(x') u^\varepsilon(x', -\frac{1}{2})\, dx' \\ &\le ||f^\varepsilon||_{L^2(\Omega)} ||u^\varepsilon||_{L^2(\Omega)} + ||g^\varepsilon||_{L^2(\Omega)^N} ||\nabla^\varepsilon u^\varepsilon||_{L^2(\Omega)^N} \\ &+ ||h_+^\varepsilon||_{H^{-\frac{1}{2}}(\omega)} ||u^\varepsilon_{|x_N = \frac{1}{2}}||_{H^{\frac{1}{2}}(\omega)} + ||h_-^\varepsilon||_{H^{-\frac{1}{2}}(\omega)} ||u^\varepsilon_{|x_N = -\frac{1}{2}}||_{H^{\frac{1}{2}}(\omega)} \\ &\le \text{ (by Poincaré inequality and continuity of the trace mapping)} \\ &\le c\left[||f^\varepsilon||_{L^2(\Omega)} + ||g^\varepsilon||_{L^2(\Omega)^N}|| + ||h_+^\varepsilon||_{H^{-\frac{1}{2}}(\omega)} + ||h_-^\varepsilon||_{H^{-\frac{1}{2}}(\omega)}\right] \\ &\times \left[\left(\int_\Omega |\nabla u^\varepsilon|^2\, dx\right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla^\varepsilon u^\varepsilon|^2\, dx\right)^{\frac{1}{2}}\right] \\ &\le c\left[||f^\varepsilon||_{L^2(\Omega)} + ||g^\varepsilon||_{L^2(\Omega)^N}|| + ||h_+^\varepsilon||_{H^{-\frac{1}{2}}(\omega)} + ||h_-^\varepsilon||_{H^{-\frac{1}{2}}(\omega)}\right] \\ &\times \left(\int_\Omega |\nabla^\varepsilon u^\varepsilon|^2\, dx\right)^{\frac{1}{2}}. \end{aligned}$$