RUMINATIONS ON HEJHAL'S THEOREM ABOUT THE BERGMAN AND SZEGŐ KERNELS

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Abstract. We give a new proof of Dennis Hejhal's theorem on the nondegeneracy of the matrix that appears in the identity relating the Bergman and Szegő kernels of a smoothly bounded finitely connected domain in the plane. Mergelyan's theorem is at the heart of the argument. We explore connections of Hejhal's theorem to properties of the zeroes of the Szegő kernel and propose some ideas to better understand Hejhal's original theorem.

To celebrate the legacy of Harold S. Shapiro

1. Introduction

Dennis Hejhal, in a remarkable *tour de force* that filled a volume of the Memoirs of the AMS [11], proved, among many other things, that the matrix of coefficients $[\lambda_{ij}]$ that appears in the identity relating the Bergman kernel $K(z,w)$ to the Szegő kernel $S(z,w)$,

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} \lambda_{ij} F_i'(z)\overline{F_j'(w)}, \tag{1.1}$$

in a bounded n-connected smoothly bounded domain in the plane is nondegenerate and, in fact, positive definite. Hejhal's proof of this result used a great deal of machinery from analysis and geometry, including key use of theta functions on Riemann surfaces. The purpose of this paper is to give a rather short proof of the nondegeneracy of the matrix that uses only Mergelyan's theorem and basic properties of the Bergman and Szegő kernel functions. We also explore how these results are connected to properties of the zeroes of the Szegő kernel.

The authors stumbled upon this application of Mergelyan's theorem after their work on double quadrature domains [6], which turns out to be a subject closely connected to Hejhal's theorem. This work sprouted from the influential works of Harold Shapiro and his many collaborators, including [1], [14], and [15]. Both of us have greatly benefited from Harold's mentorship and generosity and so it seems fitting to offer this work in a volume in his honor. (We must also mention here that Avci's Stanford Thesis [2] also played an important role in our studies.)

We tried to give a new proof also of Hejhal's full result, that the matrix is *positive definite*, but could achieve this only in a few special cases (connectivity

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two and three). In our attempts we however learned a great deal about alternate arguments and how Hejhal's result fits in the grand scheme of things. We could not resist including some of these observations here, giving this paper an expository component. It has been a great subject to ruminate upon!

2. Preliminaries

The transformation identities for the Bergman and Szegő kernels and the harmonic measure functions yield that the nondegeneracy of the matrix $[\lambda_{ij}]$ in equation (1.1) is invariant under conformal changes of variables. Hence, we may always suppose that the domain Ω under study is a bounded domain in $\mathbb{C}$ bounded by $n > 1$ nonintersecting smooth real analytic curves.

We now set up some definitions and notation that we will use throughout the paper.

We denote the boundary of Ω by $b\Omega$ and provide it with the standard orientation. Let γ_n denote the outer boundary curve of Ω, and denote the inner boundary curves by γ_j, $j = 1, \ldots, n-1$.

The double $\widehat{\Omega}$ of Ω is a compact Riemann surface of genus $n-1$ obtained by using the Schwarz reflection principle to glue a copy $\widetilde{\Omega}$ of Ω to Ω along the boundary of Ω, using the function z as a chart on Ω and $\bar{z}$ as a chart on $\widetilde{\Omega}$.

We now define curves that go around the $n-1$ handles of $\widehat{\Omega}$. Let σ_j be a curve in Ω that starts on the outer boundary γ_n and ends on γ_j for $j = 1, \ldots, n-1$. The curves σ_j can be defined so that their closures do not intersect. Note that, in this case, $\Omega - \cup_{j=1}^{n-1} \sigma_j$ is a simply connected domain. Let β_j denote the curve on $\widehat{\Omega}$ that first follows σ_j in Ω, and then follows the copy of $-\sigma_j$ in $\widetilde{\Omega}$ to connect back to the starting point. We think of β_j as going around the j-th handle of $\widehat{\Omega}$ and we note that the $n-1$ curves γ_j, $j = 1, \ldots, n-1$, together with the $n-1$ curves β_j form a homology basis for the double.

The Bergman kernel $K(z, w)$ is the kernel for the orthogonal projection of $L^2(\Omega)$ onto its closed subspace of holomorphic functions in L^2. The Szegő kernel $S(z, w)$ is the kernel for the orthogonal projection of $L^2(b\Omega)$ onto its subspace consisting of L^2 boundary values of holomorphic functions. We refer the reader to the classic books [7, 9, 12] for the basic facts about these kernels and to [4] for a treatment of the subject very much in line with the approach of this paper. In fact, this paper fills in a missing chapter of [4].

The functions $F_j'(z)$ appearing in equation (1.1) are given by

$$F_j'(z) = 2\frac{\partial \omega_j}{\partial z}, \tag{2.1}$$

where ω_j is the harmonic function on Ω that has boundary values equal to one on γ_j and equal to zero on the other boundary curves. The notation is traditional; F_j' is locally the derivative of the holomorphic function with real part ω_j, but it is not globally the derivative of a holomorphic function on Ω.

We let $\Lambda(z,w)$ denote the complimentary kernel to the Bergman kernel which satisfies

$$K(z,w)\,dz = -\overline{\Lambda(z,w)\,dz} \tag{2.2}$$

for z in $b\Omega$ and $w \in \Omega$. (Our choice of symbols for the kernel functions follows [4]. In the literature, the kernels are often denoted by only K and L with various tildes or hats.) The identity (2.2) yields that the holomorphic one-form $K(z,w)\,dz$ on Ω extends to the double as a meromorphic one-form κ_w by setting it equal to the conjugate of $-\Lambda(z,w)\,dz$ on the backside of Ω in the double and using the identity to connect the definitions at the boundary. Let $G(z,w)$ denote the classical Green's function (with singular behavior $-\ln|z-w|$ near $z=w$). Since K and Λ are related to the Green's function via

$$\begin{aligned} K(z,w) &= -\frac{2}{\pi}\frac{\partial^2}{\partial z\partial \bar{w}}G(z,w) \\ \Lambda(z,w) &= -\frac{2}{\pi}\frac{\partial^2}{\partial z\partial w}G(z,w), \end{aligned} \tag{2.3}$$

it follows that the periods of κ_w about each β_j vanish (if w does not fall on any of the σ_j), i.e.,

$$\int_{\beta_j} \kappa_w = 0 \tag{2.4}$$

for $j = 1, \ldots, n-1$. This very important fact, due to Schiffer and Spencer [16], will be an essential ingredient in the proof in the next section. We briefly explain the result here to make this paper self contained. Using the definition $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and writing out $\int_{\sigma_j}\frac{\partial G}{\partial z}\,dz$ yields that the real part of the integral is given by $\frac{1}{2}\int_{\sigma_j}\frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy$, and so the integral is pure imaginary because G vanishes at the endpoints of σ_j, which fall on the boundary of Ω. Since G is real valued, the conjugate of $\partial G/\partial z$ is equal to $\partial G/\partial \bar{z}$, and the conjugate of the integral is equal to $\int_{\sigma_j}\frac{\partial G}{\partial \bar{z}}\,d\bar{z}$. Hence,

$$0 = \int_{\sigma_j}\frac{\partial G}{\partial z}\,dz + \int_{\sigma_j}\frac{\partial G}{\partial \bar{z}}\,d\bar{z},$$

and multiplying by $-2/\pi$, differentiating with respect to $\bar{w}$, and using (2.2) yields the result. (The three minus signs, one from the conjugate of a pure imaginary integral, one from (2.2), and one from the opposite direction of the curve, guarantee that the integrals cancel.)

We refer the reader to the standard references for the basic properties of K and Λ. We only note here that $K(z,w)$ is holomorphic in z and antiholomorphic in w, $\Lambda(z,w)$ is holomorphic in both variables off the diagonal, $K(w,z) = \overline{K(z,w)}$, $\Lambda(z,w) = \Lambda(w,z)$, $\Lambda(z,w)$ has a double pole in z at w with principal part $\frac{1}{\pi}(z-w)^{-2}$. Both $K(z,w)$ and $\Lambda(z,w)$ extend holomorphically past the boundary in z for fixed w in Ω, $K(z,w)$ is C^∞-smooth on $\overline{\Omega}\times\overline{\Omega}$ minus the diagonal $\{(z,z) : z \in b\Omega\}$ and $\Lambda(z,w)$ is in C^∞ on $\overline{\Omega}\times\overline{\Omega}$ minus the diagonal $\{(z,z) : z \in \overline{\Omega}\}$.

The Garabedian kernel $L(z,w)$ is the complimentary kernel to the Szegő kernel and satisfies the identity

$$\overline{S(z,w)}ds_z = \frac{1}{i}L(z,w)\,dz \tag{2.5}$$

for z in $b\Omega$ and $w \in \Omega$, where ds_z represents the element of arc length on the boundary. Squaring this formula yields that

$$\overline{S(z,w)^2}\,d\bar{z} = -L(z,w)^2\,dz \tag{2.6}$$

for z in $b\Omega$ and $w \in \Omega$ and this shows that the holomorphic one-form $S(z,w)^2\,dz$ extends to be a meromorphic one-form σ_w on the double by defining it to be the conjugate of $-L(z,w)^2\,dz$ on the back side of Ω in the double. The key assertion for the paper is that we can take linear combinations of σ_w that have β-periods being anything we like, and this will imply the non-degeneracy of the λ-matrix. To be precise, we have the following theorem, to be proved in Section 3.

Theorem 2.1. *The linear span of*

$$\left\{\left(\int_{\beta_1}\sigma_w,\ldots,\int_{\beta_{n-1}}\sigma_w\right) : w \in \Omega\right\}$$

is dense in $\mathbb{C}^{n-1}$. *As a consequence, the matrix* $[\lambda_{ij}]$ *is non-singular.*

To continue describing background material, we note that $L(z,w) = -L(w,z)$ and that $L(z,w)$ has a simple pole in z at w with principal part

$$\frac{1}{2\pi}\frac{1}{(z-w)}.$$

The Szegő and Garabedian kernels have extension, holomorphicity and antiholomorphicity, and smoothness properties analogous to those of K and Λ, respectively. Finally, $L(z,w)$ has the important property that $L(z,w) \neq 0$ if $z \neq w$ in $\overline{\Omega}$.

The function $4\pi L(z,w)^2$ is like $\Lambda(z,w)$ in that it has a double pole in z at w with principal part

$$\frac{1}{\pi}\frac{1}{(z-w)^2}.$$

(The vanishing of the residue term follows from the fact that $\int_{b\Omega} L(z,w)^2\,dz$ is equal to minus the conjugate of $\int_{b\Omega} S(z,w)^2\,dz$, which is zero by Cauchy's theorem.)

Standard proofs of identity (1.1) use the fact that the one-form

$$(K(z,w) - 4\pi S(z,w)^2)\,dz$$

is equal to minus the conjugate of

$$(\Lambda(z,w) - 4\pi L(z,w)^2)\,dz$$

on the boundary, which is also a holomorphic one-form because the poles cancel out, and so the given one-form extends to the double as a *holomorphic* one-form H_w. Note that we may write

$$H_w = \kappa_w - 4\pi\sigma_w,$$

where it is understood that the double poles cancel out. Such holomorphic one-forms are well-known to be generated by the $(n-1)$ holomorphic one-forms that are equal to $F_j'(z)\,dz$ on Ω and equal to minus the conjugate of $F_j'(z)\,dz$ on the back side, $j = 1, \ldots, n-1$. (See [4, p. 135] for a more elementary proof of (1.1).)

Identity (1.1) shows that the complex linear span of the functions of z given by

$$K(z, w) - 4\pi S(z, w)^2$$

as w ranges over Ω is at most an $n-1$ dimensional vector space W. We will prove Hejhal's theorem in the next section by showing that W has to be *at least* $n-1$ dimensional because the β-periods of linear combinations of H_w as w ranges over Ω can be made to be anything we like.

The motivation for the proof in the next section is that the terms $K(z, w)\,dz$ do not contribute to the value of the β-periods of the extension H_w of $(K(z, w) - 4\pi S(z, w)^2)\,dz$ to the double, and the terms $L(z, w)^2\,dz$ can be used to manipulate the value of the β-periods to be anything we like. At the heart of this result is a density theorem for the Garabedian kernel. Given a point a in Ω, let $L^0(z, a)$ denote the Garabedian kernel $L(z, a)$ and let $L^m(z, a)$ denote the derivative $\frac{\partial^m}{\partial w^m} L(z, w)$ evaluated at $w = a$. Similarly, use a superscript m to indicate differentiation of the Szegő kernel with respect to $\bar{w}$ when w is the second variable in $S(z, w)$. We will show that the "Garabedian span at a", which is the complex linear span of the functions $L^m(z, a)$ as m ranges over the natural numbers, can be used to approximate functions on the curves σ_j that will lead to elements in the linear span of H_w as w ranges over Ω with arbitrary β-periods. The "Szegő span at a" is the complex linear span of the functions $S^m(z, a)$ as m ranges over the natural numbers.

3. Proof that $[\lambda_{ij}]$ is nonsingular

We continue to assume that Ω is a bounded domain bounded by $n > 1$ nonintersecting smooth real analytic curves, and we use the notations and definitions of the previous section.

The inspiration for the new proof we are about to give comes from the proof of Lemma 5.1 in [6], and is yet another reason to view Mergelyan's theorem as the theorem that is just too good to be true.

Because the argument needed from Lemma 5.1 of [6] is short after all the machinery we have set up, we include it here for completeness. Given a small $\epsilon > 0$, let V denote the set of points in $\mathbb{C}$ that are a distance less than or equal to ϵ from $b\Omega$. We will shrink ϵ as needed in what follows; keep in mind that V depends on ϵ. For $j = 1, \ldots, n-1$, let φ_j be a continuous function on the closure of σ_j that is equal to zero on $V \cap \sigma_j$. Thus, φ_j is zero near both endpoints of σ_j. We assume that ϵ is small enough that a large open subset of each curve σ_j is not contained in V.

One version of Mergelyan's theorem states that, given a compact set K in the complex plane such that $\mathbb{C} - K$ has only finitely many components and a complex valued continuous function φ on K that is holomorphic in the interior of K, there is a rational function with possible poles only in $\mathbb{C} - K$ that is as close in the uniform norm as desired to φ on K. (See Exercise 1 of Chapter 20 in Rudin [13] or Greene and Krantz [8, p. 374].)

Let

$$K = V \cup \left(\cup_{j=1}^{n-1} \sigma_j\right),$$

and let $U = \Omega - K$. Note that U is a simply connected domain contained in Ω if ϵ is small enough. By Mergelyan's theorem, there is a rational function $r(z)$ with possible poles only in $\mathbb{C} - K$ that is as close in the uniform norm as desired to zero on V and φ_j on each σ_j. As in Stein and Shakarchi [17, p. 63] (and as in many proofs of Runge's theorem) we may slide the poles of $r(z)$ that fall in Ω to a single point a in $U \subset \Omega$. Let N denote the order of the pole of $r(z)$ at a.

The proof hinges on the following application of the residue theorem,

$$\frac{2\pi}{2\pi i} \int_{b\Omega} r(w)\, L(w, z)\, dw = r(z) - \sum_{m=0}^{N-1} c_m L^m(z, a) \tag{3.1}$$

for $z \in \Omega$ not equal to a. Note that we have used the facts that the principal part of $L(w, z)$ is

$$\frac{1}{2\pi} \frac{1}{(w - z)}$$

as a function of w and that the only pole of $r(w)$ is a pole of order N at a. The coefficients c_m only depend on the principal part of $r(z)$ at a. This identity will allow us to approximate $r(z)$ on K by functions in the Garabedian span at a. Indeed, using identity (2.5) reveals that the left hand side of the equation is equal to

$$\int_{b\Omega} S(z, w) r(w)\, ds_w,$$

where ds_w denotes arc length measure in the w-variable, and this integral is equal to the Szegő projection of $r(w)$ at the point z. Since $r(w)$ can be taken to be arbitrarily C^∞ close to the zero function on the boundary, and since the Szegő projection is a continuous operator from $C^\infty(b\Omega)$ to itself (see [4, p. 15]), the left member of (3.1) is uniformly small in z on $\overline{\Omega}$. Thus $\mathcal{L}(z) \approx r(z)$ on $\overline{\Omega}$ for the approximating element $\mathcal{L}(z) = \sum_{m=0}^{N-1} c_m L^m(z, a)$ in the Garabedian span at a. In particular, $\mathcal{L}(z) \approx 0$ on $b\Omega$, $\mathcal{L}(z) \approx \varphi_j(z)$ on σ_j.

For a fixed k we now let $\varphi_j \equiv 0$ for $j \neq k$ and let φ_k be a continuous function on σ_k that is zero on $\sigma_k \cap V$ and such that $\int_{\sigma_k} \varphi_k(z) L(z, a)\, dz = 1$. (Keep in mind that $L(z, a)$ is nonvanishing on $\Omega - \{a\}$.) We now claim that, for the element $\mathcal{L}(z)$ in the Garabedian span constructed above, $\mathcal{L}(z) L(z, a)$ has integrals with respect to dz that are close to zero along σ_j, $j \neq k$, and close to one along σ_k. Differentiating identity (2.5) m-times with respect to w and then multiplying it by (2.5) and setting w equal to a reveals that

$$\overline{S^m(z,a)S(z,a)}\, d\bar{z} = -L^m(z,a)L(z,a)\, dz.$$

Hence, there is an element $\mathcal{S}(z)$ in the Szegő span at a such that

$$\overline{\mathcal{S}(z)S(z,a)}\, d\bar{z} = \mathcal{L}(z)L(z,a)\, dz, \tag{3.2}$$

and because the function $\mathcal{L}(z)L(z,a)$ on the right hand side is small on the boundary of Ω, the function $\mathcal{S}(z)S(z,a)$ must also be small, and consequently also small on $\overline{\Omega}$. Taking the conjugate of (3.2) reveals that the one-form $\mathcal{S}(z)S(z,a)\, dz$ extends to the double as a meromorphic one-form s_a by setting it equal to the conjugate of $\mathcal{L}(z)L(z,a)\, dz$ on the back side. Our construction shows that the β_j periods of s_a are close to zero for $j \neq k$ and the β_k period is close to one.

The next step in the proof is to show that there are linear combinations of the holomorphic one-forms H_w as w ranges over Ω that have β-periods close to the β-periods of the meromorphic one-form s_a constructed above. Since the β-periods of κ_w are all zero, it will suffice to find linear combinations of the meromorphic one-forms σ_w that have β-periods close to the β-periods of s_a. This turns out to be a rather elementary exercise due to the following observations. For small complex h,

$$\begin{aligned} L^1(z,a)L(z,a) &\approx \frac{L(z,a+h)-L(z,a)}{h}\,\cdot\, L(z,a) \\ &\approx \frac{L(z,a+h)-L(z,a)}{h}\,\cdot\,\frac{L(z,a+h)+L(z,a)}{2}, \end{aligned}$$

and this last term is a linear combination of the squares $L(z,a+h)^2$ and $L(z,a)^2$. (Note that $L^0(z,a)L(z,a) = L(z,a)^2$ is a square; that's why we skipped it.) Next,

$$\begin{aligned} L^2(z,a)L(z,a) &\approx \frac{L^1(z,a+h)-L^1(z,a)}{h}\; L(z,a) \\ &\approx \frac{\frac{L(z,a+h+k)-L(z,a+h)}{k} - \frac{L(z,a+k)-L(z,a)}{k}}{h}\; L(z,a), \end{aligned}$$

which can be pulled apart into linear combinations of terms of the form

$$[L(z,a+h_1)-L(z,a+h_2)]\,\cdot\, L(z,a)$$

that can be approximated by

$$[L(z,a+h_1)-L(z,a+h_2)]\,\cdot\,\left(\frac{L(z,a+h_1)+L(z,a+h_2)}{2}\right),$$

which again is a linear combination of squares. This process can be continued to all higher order terms.

We may now state that there are linear combinations of the holomorphic one-forms H_w as w ranges over a small disc $D_\epsilon(a) \subset \Omega$ with β-periods close to any prescribed set of values. We must conclude that the linear span is $n-1$ dimensional, and that therefore, the matrix $[\lambda_{ij}]$ must be nonsingular. Notice that we above constructed linear combinations of the one-forms σ_w (for w running over Ω) which have β-periods essentially equal to those of s_a and hence generating a dense set of period vectors in $\mathbb{C}^{n-1}$. Therefore Theorem 2.1 is now proved.

By showing that $[\lambda_{ij}]$ is nonsingular, we have proved that the family of functions of z of the form

$$\sum_{i,j=1}^{n-1} \lambda_{ij} F_i'(z)\overline{F_j'(w)}$$

spans an $n-1$ dimensional vector space of functions of z as w ranges over any disc $D_\epsilon(a) \subset \Omega$. This implies also that the vectors

$$(F_1'(w), F_2'(w), \ldots, F_{n-1}'(w))$$

must span $\mathbb{C}^{n-1}$ as w ranges over $D_\epsilon(a)$.

4. Hejhal's theorem in the two-connected case

We now turn to showing directly (without using Theorem 2.1) the positivity of the matrix $[\lambda_{ij}]$ when Ω is two-connected, i.e., that $\lambda_{11} > 0$. Since there is only one function F_1' and one constant λ_{11} in (2.2), we will drop the subscripts.

It was proved in [5] (see also [4, p. 149]) that, for a in one of the boundary curves of Ω, the Szegő kernel $S(a,w)$ has exactly one zero in $\overline{\Omega} - \{a\}$ in w at a point b in the other boundary curve of Ω. Hence, (1.1) yields that

$$K(a,b) = \lambda F'(a)\,\overline{F'(b)}. \tag{4.1}$$

Multiply this equation by $T(a)\,\overline{T(b)}$ to obtain

$$T(a)K(a,b)\,\overline{T(b)} = \lambda F'(a)T(a)\,\overline{F'(b)}\,\overline{T(b)}. \tag{4.2}$$

Now the positivity of λ follows from the following two consequences of the Hopf maximium principle (Hopf lemma):

$$T(a)K(a,b)\,\overline{T(b)} < 0, \tag{4.3}$$

$$F'(a)T(a)\,\overline{F'(b)}\,\overline{T(b)} < 0. \tag{4.4}$$

In terms of the outward normal derivatives of the Green's function and of the harmonic measure ω these two inequalities express, via (2.3), (2.1), that

$$\frac{\partial^2 G(a,b)}{\partial n_a \partial n_b} > 0,$$

$$\frac{\partial \omega(a)}{\partial n_a} \cdot \frac{\partial \omega(b)}{\partial n_b} < 0.$$

The first inequality actually holds for any two $a, b \in b\Omega$, $a \neq b$, and with Ω of arbitrary connectivity. It expresses that the Poisson type kernel $p(z,a) = -\frac{1}{2\pi}\frac{\partial G(z,a)}{\partial n_a}$ ($z \in \Omega$) attains its minumum value (namely zero) at any point on the boundary (for example $z = b$) with a strictly negative slope. Similarly, the second inequality says that ω has strictly positive (negative) normal derivative on a boundary component on which it takes its maximum (minimum) value.

Using that the complex number $-iT(a)$ can be identified with the outward normal vector of $b\Omega$ at a and that $2\partial u/\partial\bar{z}$ can be identified with the gradient when u is a real-valued function, it follows that if u is constant on $b\Omega$ then

$$-2i\frac{\partial u}{\partial z}(a)T(a) \text{ is real and equals } \frac{\partial u}{\partial n_a}.$$

Using then (2.1) and (2.3) the inequalities (4.3), (4.4) follow easily.

We remark that for the Szegő kernel one has

$$T(a)S(a,b)^2\overline{T(b)} \leq 0,$$

where equality can be attained (something we have already used). The proof follows on using $S(a,b)T(a) = i\overline{L(a,b)}$ and $L(a,b) = -L(b,a)$, whereby the inequality becomes $L(a,b)\overline{L(a,b)} \geq 0$.

5. Suita's proof that $K(a,a) - 4\pi S(a,a)^2 > 0$

In the general n-connected setting, we have shown that the matrix $[\lambda_{ij}]$ is nonsingular, and we have also proved that it is positive definite in the 2-connected case. We became enthralled with the idea of setting up an induction via a homotopy argument to deduce Hejal's complete result that the matrix is positive definite in general, but we have not been able to complete the argument. In our quest to find a shorter, simpler proof of Hejhal's result, we hoped to use Suita's [18] beautiful and short proof that

$$K(a,a) - 4\pi S(a,a)^2 > 0$$

as a key step. Since Suita's result is very much in the spirit of this paper and since we have set up the tools and notation necessary to describe it, we include Suita's proof here in case our readers are inspired to someday complete the plan of our proof.

Consider the multi-valued function

$$F(z) = \exp(-G(z,a) - iG^*(z,a))$$

where $G^*(z,a)$ represents a multi-valued harmonic conjugate for the Green's function $G(z,a)$. Note that, because $G(z,a) = -\ln|z-a| + u_a(z)$, where $u_a(z)$ is the harmonic function that solves the Dirichlet problem with boundary data $\ln|z-a|$, the multi-valued function F is bounded in modulus by a constant times $|z-a|$ near $z = a$. In fact,

$$|F(z)|^2 = \exp(-2G(z,a)) = |z-a|^2 \exp(-2u_a(z))$$

is a single-valued function that is in $C^\infty(\overline{\Omega})$ and is C^∞-smooth up to the boundary and equal to one there. The Cauchy-Riemann equations yield that the complex derivative of an analytic function $u+iv$ is $u_x - iu_y = 2\partial u/\partial z$. Thus, the complex derivative of the locally defined analytic function $F(z)$ is given by

$$F'(z) = -2\frac{\partial G(z,a)}{\partial z}\exp(-G(z,a) - iG^*(z,a)) = -2\frac{\partial G(z,a)}{\partial z}F(z)$$

and so, using the shorthand notation $G = G(z, a)$,

$$|F'(z)| = 2\left|\frac{\partial G}{\partial z}\right| \exp(-G).$$

Since the complex conjugate of $\frac{\partial G}{\partial z}$ is $\frac{\partial G}{\partial \bar{z}}$, we may also write

$$|F'(z)|^2 = 4\frac{\partial G}{\partial z}\frac{\partial G}{\partial \bar{z}} \exp(-2G).$$

Now it is clear that, even though F might be multi-valued, the quotient F'/F is equal to $-2\frac{\partial G}{\partial z}$ and is a single-valued analytic function on $\Omega - \{a\}$ with a simple pole at a. Using these facts we have collected, we may compute

$$\int_\Omega |F'(z)|^2\ dxdy = 4\int_\Omega \frac{\partial G}{\partial z}\frac{\partial G}{\partial \bar{z}} e^{-2G}\ \left(\frac{1}{2i} d\bar{z} \wedge dz\right) =$$

$$= -2i\int_\Omega \frac{\partial}{\partial \bar{z}}\left(-\frac{1}{2}e^{-2G}\frac{\partial G}{\partial z}\right) d\bar{z} \wedge dz = i\int_{b\Omega} e^{-2G}\frac{\partial G}{\partial z}\, dz =$$

$$= i\int_{b\Omega} \frac{\partial G}{\partial z}\, dz = i\int_{b\Omega}\left(-\frac{1}{2(z-a)} + \text{regular}\right) dz = \pi.$$

Next, let

$$f(z) = \frac{S(z,a)}{L(z,a)}$$

be the Ahlfors map, which is an n-to-one branched covering map of Ω onto the unit disc that is C^∞-smooth up to the boundary (see [4, Chap. 13]). It is the solution of the extremal problem to maximize the modulus of the derivative at a among all analytic functions that map Ω into the unit disc, normalized so that the derivative at a is real and positive. Well known properties of f include that $f(a) = 0$, $f'(a) = 2\pi S(a, a)$, $|f| < 1$ on Ω, and $|f| = 1$ on $b\Omega$.

Suita's proof of the inequality is a fiendishly clever comparison of the Ahlfors map to the mapping F, which can be thought of as a multi-valued substitute for the Riemann map in the multiply connected setting. (The Ahlfors map can also be thought of as a non-one-to-one substitute for the Riemann map in the multiply connected case.)

Write $f(z) = (z - a)g(z)$ and note that $g(z)$ is an analytic function on Ω that is nonzero at a, has $(n - 1)$ zeroes (counted with multiplicity) in $\Omega - \{a\}$, and is C^∞-smooth up to the boundary. We now have

$$\ln\left|\frac{f(z)}{F(z)}\right|^2 = 2\ln|g(z)| + 2u_a(z)).$$

Note that $|f| = |F| = 1$ on $b\Omega$. So $\ln\left|\frac{f(z)}{F(z)}\right|^2$ is a harmonic function on Ω minus the finitely many zeroes of g in Ω, where it tends to minus infinity, and it equals zero on the boundary of Ω. Since g must have at least one zero in Ω, it follows from the maximum principle that $\ln|\frac{f(z)}{F(z)}| < 0$ in Ω. Hence

$$|f(z)| < |F(z)| \quad \text{for } z \in \Omega - \{a\}. \tag{5.1}$$

The combination

$$h(z) := f(z)\frac{F'(z)}{F(z)} = \frac{S(z,a)}{L(z,a)}\frac{F'(z)}{F(z)} = -2\frac{S(z,a)}{L(z,a)}\frac{\partial G(z,a)}{\partial z}$$

is analytic in Ω if we set $h(a) = f'(a) = 2\pi S(a,a)$. Therefore

$$2\pi S(a,a) = h(a) = \int_\Omega h(z)\overline{K(z,a)}\,dxdy,$$

and so, using (5.1),

$$\begin{aligned}4\pi^2 S(a,a)^2 &\le \int_\Omega |h(z)^2|\,dxdy \cdot \int_\Omega |K(z,a)|^2\,dxdy\\ &= \int_\Omega \left|\frac{f(z)}{F(z)}\right|^2 |F'(z)|^2\,dxdy \cdot K(a,a)\\ &< \int_\Omega |F'(z)|^2 dxdy \cdot K(a,a) = \pi K(a,a).\end{aligned}$$

Now the desired inequality is proved.

6. The three-connected case

Suita's result allows us to deduce Hejhal's theorem from rather basic facts in the case $n = 3$. The matrix $[\lambda_{ij}]$ is easily seen to be hermitian (it is in fact real and symmetric). Hence it is diagonalizable and we may introduce a new basis $U_k'(z)$ for the linear span of the functions F_j' which are linear combinations of the F_j' with real coefficients such that

$$K(z,w) - 4\pi S(z,w)^2 = \sum_{i=1}^{2} \mu_i U_i'(z)\overline{U_i'(w)}.$$

We know that the μ_i (the eigenvalues of $[\lambda_{ij}]$) are real and nonzero (because $[\lambda_{ij}]$ is nonsingular). We now consider the zeroes of the U_k'. Since $U_k'(z)T(z) = -\overline{U_k'(z)T(z)}$ on the boundary, the generalized argument principle, that allows zeroes on the boundary that are counted with a factor of one-half in front, shows that each $U_k'(z)$ has either one zero in Ω or two zeroes on the boundary.

Now, Suita's result yields that

$$\mu_1|U_1'(a)|^2 + \mu_2|U_2'(a)|^2 > 0$$

for any point a in Ω. Choosing a to be the zero of U_2', in case that zero is in Ω, shows that μ_1 is necessarily positive. If instead U_2' has two zeroes on the boundary it still follows, by sliding such a zero a little into Ω, that μ_1 cannot be strictly negative. This is because $U_1'(a) \neq 0$ at any a with $U_2'(a) = 0$ (not all holomorphic differentials in a basis can vanish at the same point). Invoking Theorem 2.1 we then actually have $\mu_1 > 0$ again. Similarly $\mu_2 > 0$. This completes the proof.

7. Wishful thinking

We have feelings and urges about Hejhal's theorem that we'd like to share here. Hejhal's proof of the positive definiteness of the lambda matrix was long and technical and used theta functions in a key way. We would like to come up with alternative ways of understanding the result, ways that might be quicker and more elementary. One potentially easier way to understand the proof might involve a homotopy argument. As a smoothly bounded n-connected domain varies in a C^∞ way, the kernel functions and lambda coefficients vary in a C^∞ way, too. Because we have shown that the matrix is nonsingular, an eigenvalue of the matrix could not pass through zero to change sign under smooth perturbations. We have shown the 1×1 matrix in the 2-connected case is positive definite. If we could show that there is just *one* n-connected domain for which the matrix is positive definite, then all n-connected domains must share that property since they are all smoothly homotopic. Our idea is to let one of the holes in an n-connected domain shrink down to a point, perhaps heading off to the outer boundary curve as a shrinking circle. If one understood the asymptotic behavior of the kernel functions in the limit under this process, then perhaps one could deduce the positive definiteness of an n-connected domain near the limiting domain from knowing the result in the $(n-1)$-connected case. The 2-connected case would start the induction process. In fact, understanding the asymptotic behavior of the Szegő kernel is all that would be needed, as we now explain.

Let dF_j denote the holomorphic 1-form that is $F_j'(z)dz$ on Ω and $-\overline{F_j'(z)dz}$ on the backside of the double. Because $F_j' = 2\partial\omega_j/dz$, it is easy to check that the β-periods of dF_j are given by two times the delta function, i.e.,

$$\int_{\beta_k} dF_j = 2\delta_{kj}.$$

Let κ_w and σ_w denote the 1-forms that we introduced in §2. Recall that the β-periods of κ_w are zero. Hence, when we take the β-periods of the identity (1.1), we get a formula that only involves the Szegő kernel, the λ coefficients, and the function F_j'. Taking this idea further, let κ denote the $(1,1)$-form gotten from extending $dz\, K(z,w)\, d\bar{w}$ to the double cross the double and let σ denote the $(1,1)$-form gotten by extending $dz\, S(z,w)^2\, d\bar{w}$. Our observations about the β-periods reveal the known (compare equation (30), p.231, in [10]) formula

$$4\pi \int_{\beta_i} \int_{\beta_j} \sigma dz\, d\bar{w} = -4\lambda_{ij}$$

when $i \neq j$. For the case $i = j$, we let $\widetilde{\beta}_i$ denote a curve obtained from sliding β_i along the i-th handle some distance. In this case, we have

$$4\pi \int_{\beta_i} \int_{\widetilde{\beta}_i} \sigma dz\, d\bar{w} = -4\lambda_{ii}.$$

This shows, strangely enough, that the λ matrix only depends on the Szegő kernel! Hence, if we understood the asymptotic behavior of the Szegő kernel as a circular hole shrinks away to nothing as it heads toward the outer boundary, we might be able to reduce Hejhal's complete result in the n-connected case to the $(n-1)$-connected case to complete our wished for induction.

Using the Kerzman-Stein integral formula for the Szegő kernel as in [4, p. 153-157] seemed like a promising way to deduce that the kernel functions of the shrinking hole domains converge in a strong sense to the kernel functions of the lower-connectivity limit domain. Assuming the shrinking hole is circular simplifies some of the arguments because the Kerzman-Stein kernel is zero for (z, w) on the same circular boundary curve. A different idea would be to use slit models for domains Ω having a hyperelliptic double and use explicit formulas, due to Barker [3], for the Szegő kernel for such domains. But alas, the needed results eluded us. We happily continue to ruminate upon it.

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