

Exponential transforms, resultants and moments

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Abstract

We give an overview of some recent developments concerning harmonic and other moments of plane domains, their relationship to the Cauchy and exponential transforms, and to the meromorphic resultant and elimination function. The paper also connects to certain topics in mathematical physics, for example domain deformations generated by harmonic gradients (Laplacian growth) and related integrable structures.

Keywords: Cauchy transform, exponential transform, resultant, elimination function, moment, quadrature domain, Laplacian growth, string equation, Polubarinova-Galin equation.

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1 Introduction

In the present article we will focus on developments concerning harmonic and other moments of a domain in the complex plane, the exponential transform

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and, in the case of a quadrature domain, the relation between these objects and the resultant and elimination function. Much of the material is based on joint work with Vladimir Tkachev, Mihai Putinar, Ahmed Sebbar and is in addition inspired by ideas of Mark Mineev-Weinstein, Paul Wiegmann, Anton Zabrodin and others.

The organization of the paper is as follows. Section 2 gives the basic definitions of Cauchy and exponential transforms, including extended versions in four complex variables. In Section 3 it is shown how these generate harmonic and exponential moments. In the case of quadrature domains (equivalently, algebraic domains or finitely determined domains), there are strong algebraic relationships between the various transforms and moments, and this is exposed in Section 4. Most of the material in Sections 2–4 is by now relatively classical, being based on developments in the period 1970–2000 within complex analysis and operator theory. For example, the exponential transform came out as a byproduct from the theory of hyponormal operators.

Central for the paper is Section 5 on the meromorphic resultant, which was introduced in [19]. One main message is that, in the case of quadrature domains, the elimination function, which is defined by means of the meromorphic resultant (and hence is a purely algebraic object), turns out to be the same as the exponential transform (which is an analytic object). In Section 6, some potential theoretic interpretations of the resultant are given.

In the last three sections, Section 7–9, we connect the previously discussed material to some quite exciting and relatively recent developments in mathematical physics. It much concerns deformations (or variations) of domains when the harmonic moments are used as coordinates (in the simply connected case). Such deformations fit into the framework of Laplacian growth processes, or moving boundaries for Hele-Shaw flows. More exactly, they can naturally be thought of as evolutions driven by harmonic gradients. We mostly restrict to finite dimensional subclasses of domains, namely polynomial (of a fixed degree) conformal images of the unit disk, this in order to keep the presentation transparent and rigorous. A nice feature then is that the Jacobian determinant for the transition between harmonic moment coordinates and the coordinates provided by the coefficients of the polynomial mapping functions can be made fully explicit, (7.5), in terms of a resultant involving the mapping function. This result is due to O. Kuznetsova and V. Tkachev [25], [46].

The further results concern the string equation (7.8) (which is equivalent to the Polubarinova-Galin equation for a Hele-shaw blob), integrability

properties of the harmonic moments (3.6), and a corresponding prepotential (3.8), which is the logarithm of a τ -function and which in some sense explains the mentioned integrability properties. Finally, we give Hamiltonian formulations of the evolution equation for the conformal map under variation of the moments (9.8). All these matters are due to M. Mineev-Weinstein, P. Wiegmann, A. Zabrodin and others, see for example [53], [29], [27], [24], but we try to explain everything in our own words and using the particular settings of the present paper.

1.1 List of notations

Below is a list of general notations that will be used.

- $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, $\mathbb{D}(a, r) = \{\zeta \in \mathbb{C} : |\zeta - a| < r\}$.
- $\mathbb{P} = \mathbb{C} \cup \{\infty\}$, the Riemann sphere.
- $dm = dm(z) = dx \wedge dy = \frac{1}{2i} d\bar{z} \wedge dz$. ($z = x + iy$).
- $f^*(\zeta) = \overline{f(1/\bar{\zeta})}$.

2 The Cauchy and exponential transforms

The *logarithmic potential* of a measure μ with compact support in \mathbb{C} is

$$U_\mu(z) = \int \log \frac{1}{|z - \zeta|} d\mu(\zeta), \quad (2.1)$$

satisfying $-\Delta U_\mu = 2\pi\mu$ in the sense of distributions. The gradient of U_μ can, modulo a constant factor and a complex conjugation, be identified with the *Cauchy transform*

$$C_\mu(z) = -\frac{2}{\pi} \frac{\partial U_\mu(z)}{\partial z} = -\frac{1}{\pi} \int \frac{d\mu(\zeta)}{\zeta - z},$$

which may be naturally viewed as a differential: $C_\mu(z)dz$. From these potentials, or fields, μ can be recovered by

$$\mu = \frac{\partial C_\mu}{\partial \bar{z}} = -\frac{1}{2\pi} \Delta U_\mu. \quad (2.2)$$

We shall mainly deal with measures of the form $\mu = \chi_\Omega m$, where m denotes Lebesgue measure in \mathbb{C} and $\Omega \subset \mathbb{C}$ is a bounded open set. Then we write U_Ω, C_Ω in place of U_μ and C_μ . On writing C_Ω as

$$C_\Omega(z) = \frac{1}{2\pi i} \int_\Omega \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z} = \frac{1}{2\pi i} \int_\Omega \frac{d\zeta}{\zeta - z} \wedge d\bar{\zeta}.$$

one is naturally led to consider the more symmetric “double Cauchy transform”,

$$C_\Omega(z, w) = \frac{1}{2\pi i} \int_\Omega \frac{d\zeta}{\zeta - z} \wedge \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}}, \quad (2.3)$$

which is much richer than the original transform. It may be viewed as a double differential: $C_\mu(z, w) dz d\bar{w}$. After exponentiation it gives the by now quite well studied [5], [34], [36], [37], [13] *exponential transform* of Ω :

$$E_\Omega(z, w) = \exp C_\Omega(z, w).$$

The original Cauchy transform reappears when specializing one variable at infinity:

$$C_\Omega(z) = \operatorname{Res}_{w=\infty} C_\Omega(z, w) d\bar{w} = - \lim_{w \rightarrow \infty} \bar{w} C_\Omega(z, w).$$

As a substitute for (2.2) we have, for the double transform,

$$\frac{\partial^2 C_\Omega(z, w)}{\partial \bar{z} \partial w} = -\pi \delta(z - w) \chi_\Omega(z) \chi_\Omega(w).$$

Even the double Cauchy transform is not fully complete. It contains the Cauchy kernel $\frac{d\zeta}{\zeta - z}$, which is a meromorphic differential on the Riemann sphere with a pole at $\zeta = z$, but it has also a pole at $\zeta = \infty$. It is natural to make the latter pole visible and movable. This has the additional advantage that one can avoid the two Cauchy kernels which appear in the definitions of the double Cauchy transform, and the exponential transform, to have coinciding poles (namely at infinity). Thus one arrives naturally at the extended (or four variable) Cauchy and exponential transforms:

$$C_\Omega(z, w; a, b) = \frac{1}{2\pi i} \int_\Omega \left(\frac{d\zeta}{\zeta - z} - \frac{d\zeta}{\zeta - a} \right) \wedge \left(\frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} - \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{b}} \right), \quad (2.4)$$

$$E_\Omega(z, w; a, b) = \exp C_\Omega(z, w; a, b) = \frac{E_\Omega(z, w) E_\Omega(a, b)}{E_\Omega(z, b) E_\Omega(a, w)}. \quad (2.5)$$

Clearly $C_\Omega(z, w) = C_\Omega(z, w; \infty, \infty)$, $E_\Omega(z, w) = E_\Omega(z, w; \infty, \infty)$. If the points z, w, a, b are taken to be all distinct, then both transforms are well defined and finite for any open set Ω in the Riemann sphere \mathbb{P} . For example, $E_{\mathbb{P}}(z, w; a, b)$ turns out to be the modulus squared of the cross-ratio:

$$E_{\mathbb{P}}(z, w; a, b) = |(z : a : w : b)|^2, \quad (2.6)$$

where, according to the classical definition [2],

$$(z : a : w : b) = \frac{(z - w)(a - b)}{(z - b)(a - w)} \quad (2.7)$$

with the variables in this order. To prove (2.6), one may use the formula for the two-variable exponential transform for the disk $\Omega = \mathbb{D}(0, R)$ when both variables are inside, namely (see [13])

$$E_{\mathbb{D}(0, R)}(z, w) = \frac{|z - w|^2}{R^2 - z\bar{w}} \quad (z, w \in \mathbb{D}(0, R)),$$

insert this into the last member of (2.5) and let $R \rightarrow \infty$. A different proof of (2.6) will be indicated in Section 6.

We record also the formula for the exponential transform for an arbitrary disk $\mathbb{D}(a, R)$ when both variables are outside ($z, w \in \mathbb{C} \setminus \mathbb{D}(a, R)$):

$$E_{\mathbb{D}(a, R)}(z, w) = 1 - \frac{R^2}{(z - a)(\bar{w} - \bar{a})}. \quad (2.8)$$

See again [13].

From (2.6) it is immediate that for any domain $\Omega \subset \mathbb{P}$,

$$E_\Omega(z, w; a, b) E_{\mathbb{P} \setminus \Omega}(z, w; a, b) = |(z : a : w : b)|^2 \quad (2.9)$$

(clearly E_D makes sense even if the set D is not open). The two-variable exponential transform $E_{\mathbb{P} \setminus \Omega}(z, w)$ is identically zero if Ω is bounded, but there is still a counterpart of (2.9) in this case. It is

$$E_\Omega(z, w) \cdot \frac{1}{H_\Omega(z, w)} = |z - w|^2, \quad (2.10)$$

where $H_\Omega(z, w)$ is the *interior exponential transform*, which is a renormalized version of one over the exponential transform of the exterior domain, more precisely

$$H_\Omega(z, w) = \lim_{R \rightarrow \infty} \frac{1}{R^2 E_{\mathbb{D}(0, R) \setminus \Omega}}.$$

We consider this function only for $z, w \in \Omega$, and then it is analytic in z and antianalytic in w .

When the integral in the definition of the exponential transform is transformed into a boundary integral it turns out that the formulas for $E_\Omega(z, w)$ (with the variables outside) and $H_\Omega(z, w)$ become exactly the same:

$$E_\Omega(z, w) = \exp \left[\frac{1}{i\pi} \int_{\partial\Omega} \log |\zeta - z| \overline{d\log(\zeta - w)} \right] \quad (z, w \in \mathbb{C} \setminus \Omega),$$

$$H_\Omega(z, w) = \exp \left[\frac{1}{i\pi} \int_{\partial\Omega} \log |\zeta - z| \overline{d\log(\zeta - w)} \right] \quad (z, w \in \Omega).$$

Here one can easily identify the modulus and the angular part, for example for $E_\Omega(z, w)$:

$$|E_\Omega(z, w)| = \exp \left[-\frac{1}{\pi} \int_{\partial\Omega} \log |\zeta - z| d\arg(\zeta - w) \right],$$

$$\arg E_\Omega(z, w) = -\frac{1}{\pi} \int_{\partial\Omega} \log |\zeta - z| d\log |\zeta - w|$$

($z, w \in \mathbb{C} \setminus \Omega$). These formulas open up for geometrical interpretations.

The exponential transform originally arose as a side product in the theory of hyponormal operators on a Hilbert space, see [5], [28], [30], and the inner product in the Hilbert space then automatically forces the transform to have certain positive definiteness properties, namely the two variable transforms $1/E_\Omega$ and H_Ω are positive definite (as kernels) in all \mathbb{C} , and $1 - E_\Omega$ is positive semidefinite outside Ω . See [15] for direct proofs. For the four variable exponential transform we similarly have, for example, that $1/E$ is positive definite in the sense that

$$\sum_{k,j} \frac{\lambda_k \overline{\lambda_j}}{E_\Omega(z_k, z_j; a_k, a_j)} \geq 0$$

for any finite tuples $(z_j, a_j) \in \mathbb{C}^2$, $\lambda_j \in \mathbb{C}$, and with strict inequality unless all λ_j are zero.

Aside from operator theory, the ideas of the exponential transform appear implicitly as a tool in the theory of boundaries of analytic varieties, see [21], [3] (Chapter 19).

3 Harmonic, complex and exponential moments

The Cauchy and exponential transforms are generating functions for series of moments of a bounded domain $\Omega \subset \mathbb{C}$. Specifically we have the *complex moments* $M_{kj} = M_{kj}(\Omega)$,

$$M_{kj} = \frac{1}{\pi} \int_{\Omega} z^k \bar{z}^j dm(z) = \frac{1}{2\pi i} \int_{\Omega} z^k \bar{z}^j d\bar{z}dz \quad (k, j \geq 0),$$

which in view of the Weierstrass approximation theorem completely determine Ω up to nullsets, and the more restricted *harmonic moments* $M_k = M_k(\Omega)$:

$$M_k = M_{k0} = \frac{1}{\pi} \int_{\Omega} z^k dm(z) = \frac{1}{2\pi i} \int_{\Omega} z^k d\bar{z}dz \quad (k \geq 0). \quad (3.1)$$

The latter do not completely determine the domain, not even in the simply connected case (see [41]). However, for simply connected domains with analytic boundaries they are at least sensible for local variations of the domain, *i.e.*, we have a local one-to-one correspondence

$$\Omega \leftrightarrow (M_0, M_1, M_2, \dots). \quad (3.2)$$

A precise statement in this respect is the following: there exists a compact subset $K \subset \Omega$ such that any Jordan domain $D \supset K$ with $M_k(D) = M_k(\Omega)$ for all $k \geq 0$ necessarily agrees with Ω . See [12], Corollary 3.10, for a proof.

It is actually not *a priori* obvious that the moments M_k ($k \geq 0$) are independent of each other (for example, the complex moments M_{kj} are definitely not independent, given that they are moments of domains), but as will be discussed in Sections 7 and 8, it is indeed possible to define deformations of Ω which change one of the M_k without changing the other ones. See also [6] and [40] for further discussions and results.

The generating properties for the harmonic and complex moments are

$$C_{\Omega}(z) = \sum_{k=0}^{\infty} \frac{M_k}{z^{k+1}}, \quad (3.3)$$

$$C_{\Omega}(z, w) = \sum_{k, j \geq 0} \frac{M_{kj}}{z^{k+1} \bar{w}^{j+1}}.$$

Finally we introduce the *exponential moments* $B_{kj} = B_{kj}(\Omega)$ having $1 - E(z, w)$ as generating function, hence being defined by

$$1 - \exp\left[-\sum_{k,j \geq 0} \frac{M_{kj}}{z^{k+1}\bar{w}^{j+1}}\right] = \sum_{k,j \geq 0} \frac{B_{kj}}{z^{k+1}\bar{w}^{j+1}}. \quad (3.4)$$

The positive semidefiniteness of $1 - E(z, w)$ implies that also the infinite matrix (B_{kj}) is positive semidefinite. The extended (four variable) exponential transform similarly generate some four index moments, see [32], [33], [36], [37], [28].

In the remaining part of this section we shall assume that Ω is a simply connected domain with $0 \in \Omega$ and such that $\partial\Omega$ is an analytic curve (without singularities). By writing (3.1) as a boundary integral,

$$M_k = \frac{1}{2\pi i} \int_{\partial\Omega} z^k \bar{z} dz, \quad (3.5)$$

the harmonic moments become meaningful also for $k < 0$, and these can be considered as functions of the moments for $k \geq 0$, as far as local variations are concerned. Thus we may write

$$M_{-k} = M_{-k}(\Omega) = M_{-k}(M_0, M_1, M_2, \dots)$$

for $k > 0$.

In a series of papers by I. Krichever, A. Marshakov, M. Mineev-Weinstein, P. Wiegmann and A. Zabrodin, for example [53], [29], [27], [24], the so extended moment sequence has been shown to enjoy remarkable integrability properties, for example (in present notation)

$$\frac{1}{k} \frac{\partial M_{-k}}{\partial M_j} = \frac{1}{j} \frac{\partial M_{-j}}{\partial M_k} \quad (k, j \geq 1). \quad (3.6)$$

This can be explained in terms of the presence of a prepotential $\mathcal{F}(\Omega)$ such that

$$\frac{1}{k} M_{-k}(\Omega) = \frac{\partial \mathcal{F}(\Omega)}{\partial M_k}. \quad (3.7)$$

In fact, for any sufficiently large $R > 0$ (it is enough that $\bar{\Omega} \subset \mathbb{D}(0, R)$), the energy functional

$$\mathcal{F}(\Omega) = \mathcal{F}_R(\Omega) = \frac{1}{\pi^2} \int_{\mathbb{D}(0,R) \setminus \Omega} \int_{\mathbb{D}(0,R) \setminus \Omega} \log \frac{1}{|z - \zeta|} dm(z) dm(\zeta) \quad (3.8)$$

serves as a prepotential. The exponential of $\mathcal{F}(\Omega)$ can be identified with a certain τ -function which appears as a partition function in mathematical models in statistical mechanics, see [23]. An indication of the proof of (3.7), and hence (3.6), will be given in Section 8.

Remark 3.1. The dependence of M_{-k} and $\mathcal{F}(\Omega)$ on the M_j is actually not analytic, so the above partial derivatives are really Wirtinger derivatives:

$$\begin{aligned}\frac{\partial}{\partial M_j} &= \frac{1}{2} \left(\frac{\partial}{\partial \operatorname{Re} M_j} - i \frac{\partial}{\partial \operatorname{Im} M_j} \right), \\ \frac{\partial}{\partial \bar{M}_j} &= \frac{1}{2} \left(\frac{\partial}{\partial \operatorname{Re} M_j} + i \frac{\partial}{\partial \operatorname{Im} M_j} \right).\end{aligned}$$

On extending the power series (3.3) for the Cauchy transform to a full Laurent series by means of the negative moments one gets, at least formally, the *Schwarz function* of $\partial\Omega$:

$$S(z) = \sum_{k=-\infty}^{\infty} \frac{M_k}{z^{k+1}}. \quad (3.9)$$

Even though the full series here need not converge anywhere it can be given the following precise meaning: the negative part of the series defines a germ of an analytic function at the origin and the positive part a germ of an analytic function at infinity. When $\partial\Omega$ is analytic the domains of analyticity of these two analytic functions overlap, and the overlap region contains $\partial\Omega$. Thus $S(z)$, as written above, makes sense and is analytic in a neighborhood of $\partial\Omega$. Moreover, it satisfies

$$S(z) = \bar{z} \quad \text{for } z \in \partial\Omega. \quad (3.10)$$

This can be seen by realizing that the positive part of the power series (3.9) defines the Cauchy transform $C_\Omega(z)$, which can be written, for large z ,

$$\sum_{k=0}^{\infty} \frac{M_k}{z^{k+1}} = -\frac{1}{2\pi i} \int_{\partial\Omega} \frac{\bar{\zeta} d\zeta}{\zeta - z} = C_\Omega(z).$$

Similarly, the negative part of the series agrees with the power series expansion of a boundary integral which can be interpreted as the Cauchy transform of the exterior domain:

$$\sum_{k=-\infty}^{-1} \frac{M_k}{z^{k+1}} = -\frac{1}{2\pi i} \int_{\partial(\mathbb{P}\setminus\Omega)} \frac{\bar{\zeta} d\zeta}{\zeta - z} = C_{\mathbb{P}\setminus\Omega}(z)$$

for z small. Now (3.10) follows from a well-known jump formula for Cauchy integrals.

4 Finitely determined domains

In order to exhibit transparent links between some of the previously defined objects we shall now work in a basically algebraic framework. This will mean that we shall work with classes of domains involving only finitely many parameters, specifically *finitely determined domains* in the terminology of [34], in other words *quadrature domains* [1], [42], [44], [17] or *algebraic domains* [48]. In this section our domains need not be simply connected, they will from outset be just bounded domains in the complex plane (bounded open set would work equally well).

There are many definitions or equivalent characterizations of quadrature domains, the most straightforward in the context of the present paper perhaps being the following: a bounded domain Ω in the complex plane is a (classical) quadrature domain (or algebraic domain), if the exterior Cauchy transform is a rational function, *i.e.*, if there exists a rational function $R(z)$ such that

$$C_{\Omega}(z) = R(z) \quad \text{for all } z \in \mathbb{C} \setminus \Omega. \quad (4.1)$$

Below is a list of equivalent requirements on a bounded domain $\Omega \subset \mathbb{C}$. Strictly speaking, the last three items, c)-e), are insensitive for changes of Ω by nullsets, but one may achieve equivalence in the pointwise sense by requiring that the domain Ω considered is complete with respect to area measure m , *i.e.*, that all points $a \in \mathbb{C}$ such that $m(\mathbb{D}(a, \varepsilon) \setminus \Omega) = 0$ for some $\varepsilon > 0$ have been adjoined to Ω .

- a) There exist finitely many points $a_k \in \Omega$ and coefficients $c_{kj} \in \mathbb{C}$ such that

$$\int_{\Omega} h \, dm = \sum_{k=1}^m \sum_{j=0}^{n_k-1} c_{kj} h^{(j)}(a_k) \quad (4.2)$$

for every function h which is integrable and analytic in Ω . This is the original definition of quadrature domain, used in [1] for example.

- b) There exists a meromorphic function $S(z)$ in Ω , extending continuously to $\partial\Omega$ with

$$S(z) = \bar{z} \quad \text{for } z \in \partial\Omega. \quad (4.3)$$

This $S(z)$ will be the Schwarz function of $\partial\Omega$ [7], [44].

- c) The exponential transform $E_\Omega(z, w)$ is, for z, w large, a rational function of the form

$$E_\Omega(z, w) = \frac{Q(z, \bar{w})}{P(z)\overline{P(w)}}, \quad (4.4)$$

where P and Q are polynomials in one and two variables, respectively.

- d) Ω is determined by a finite sequence of complex moments $(M_{kj})_{0 \leq k, j \leq N}$ (see Example 4.1 below for a clarification of the meaning of this).
- e) For some positive integer N there holds

$$\det(B_{kj})_{0 \leq k, j \leq N} = 0.$$

Basic references for c)-e) are [34], [35], [36], [13].

When the above conditions hold then the minimum possible number N in d) and e), the degree of P in c) and the number of poles (counting multiplicities) of $S(z)$ in Ω , all coincide with the *order* of the quadrature domain, *i.e.*, the number $N = \sum_{k=1}^m n_k$ in (4.2). For Q , see more precisely below.

If Ω is simply connected, the above conditions a)-e) are also equivalent to that any conformal map $f : \mathbb{D} \rightarrow \Omega$ is a rational function. If Ω is multiply connected, then it is necessarily finitely connected (in fact, the boundary will, up to finitely many points, be exactly the algebraic curve $Q(z, \bar{z}) = 0$ with Q as in c) above), and there is a beautiful extension, due to D. Yakubovich, of the aforementioned statement: $\partial\Omega$ is traced out (on $\partial\mathbb{D}$) by the eigenvalues of a rational normal matrix function $F : \mathbb{D} \rightarrow \mathbb{C}^{N \times N}$ which is holomorphic in \mathbb{D} ; see [54] for details.

The positive definiteness properties of the exponential transform (see end of Section 2) imply that when Ω is a quadrature domain of order N then $Q(z, w)$ admits the following representation [14]:

$$Q(z, \bar{w}) = P(z)\overline{P(w)} - \sum_{k=0}^{N-1} P_k(z)\overline{P_k(w)}.$$

Here each $P_k(z)$ is a polynomial of degree k exactly. Recall that $P(z) = P_N(z)$ has degree N , and one usually normalizes it to be monic (thereby also

making $Q(z, \bar{w})$ uniquely determined). Thus the rational form (4.4) of the exponential transform can be expressed as

$$1 - E_{\Omega}(z, w) = \sum_{k=0}^{N-1} \frac{P_k(z)\overline{P_k(w)}}{P(z)\overline{P(w)}}.$$

With the same polynomials, the rational form (4.1) of the Cauchy transform is precisely

$$C_{\Omega}(z) = \frac{P_{N-1}(z)}{P(z)}.$$

Note however that the double Cauchy transform $C_{\Omega}(z, w)$ will not be rational, only its exponential will be.

Example 4.1. The following example should clarify the meaning of a domain Ω being finitely determined (determined by finitely many moments M_{kj}), as in the equivalent condition d) above. Suppose that, by some measurements, we happen to know the moments M_{00} , M_{10} , M_{01} , M_{11} , but we do not know from which domain they come. Generally speaking, there will be infinitely many domains whose moment sequence starts out with just these four moments, unless of course the data are contradictory so that no domain has them. However, for exceptional choices of the data it happens that these four moments (or, in general, finitely many moments) determine the domain uniquely. This happens exactly for quadrature domains.

Assume for example that

$$M_{00} = M_{10} = M_{01} = 4, \quad M_{11} = 12, \quad (4.5)$$

and let us try to compute the exponential moments B_{kj} as far as possible. We have

$$\begin{aligned} 1 - \exp \left[- \sum_{k,j \geq 0} \frac{M_{kj}}{z^{k+1}\bar{w}^{j+1}} \right] &= 1 - \exp \left[- \frac{4}{z\bar{w}} - \frac{4}{z^2\bar{w}} - \frac{4}{z\bar{w}^2} - \frac{12}{z^2\bar{w}^2} - \dots \right] \\ &= \frac{4}{z\bar{w}} + \frac{4}{z^2\bar{w}} + \frac{4}{z\bar{w}^2} + \frac{12}{z^2\bar{w}^2} + \dots - \frac{1}{2!} \left(-\frac{4}{z\bar{w}} - \dots \right)^2 - \frac{1}{3!} \left(-\frac{4}{z\bar{w}} - \dots \right)^3 - \dots \\ &= \frac{4}{z\bar{w}} + \frac{4}{z^2\bar{w}} + \frac{4}{z\bar{w}^2} + \frac{4}{z^2\bar{w}^2} + \text{higher order terms.} \end{aligned}$$

Thus we see that

$$B_{00} = B_{10} = B_{01} = B_{11} = 4.$$

Clearly the determinant $\det(B_{kj})_{0 \leq k, j \leq N}$ vanishes for $N = 1$. The theory (equivalent condition e) above) then tells that the moments come from a quadrature domain of order one, *i.e.*, a disk. To determine which disk one uses (3.4) together with (4.4). Arguing in general one first realizes that

$$P(z)\overline{P(w)} \sum_{k, j \geq 0} \frac{B_{kj}}{z^{k+1}\bar{w}^{j+1}} = P(z)\overline{P(w)} - Q(z, \bar{w}), \quad (4.6)$$

in particular the left member is a polynomial. Setting $P(z) = \sum_{j=0}^N \alpha_j z^j$ and choosing $\alpha_N = 1$ (normalization), the vanishing of the coefficient of $1/z\bar{w}$ in the left member gives

$$\sum_{k, j=0}^N \alpha_j \bar{\alpha}_k B_{kj} = 0.$$

This, by the way, explains the condition $\det(B_{kj})_{0 \leq k, j \leq N} = 0$. It also gives the equation

$$\sum_{j=0}^N B_{kj} \alpha_j = 0 \quad (k = 0, \dots, N)$$

for the coefficients of $P(z)$. In our particular example this equation gives $\alpha_0 = -\alpha_1 = -1$, hence

$$P(z) = z - 1.$$

Next, $Q(z, \bar{w})$ is easily obtained from (4.6):

$$\begin{aligned} Q(z, \bar{w}) &= P(z)\overline{P(w)} - P(z)\overline{P(w)} \sum_{k, j \geq 0} \frac{B_{kj}}{z^{k+1}\bar{w}^{j+1}} \\ &= (z\bar{w} - z - \bar{w} + 1) - (z\bar{w} - z - \bar{w} + 1) \left(\frac{4}{z\bar{w}} + \frac{4}{z^2\bar{w}} + \frac{4}{z\bar{w}^2} + \frac{4}{z^2\bar{w}^2} + \dots \right) \\ &= z\bar{w} - z - \bar{w} + 1 - 4 + \text{terms containing negative powers of } z \text{ or } \bar{w} \\ &= z\bar{w} - z - \bar{w} - 3 = (z - 1)(\bar{w} - 1) - 4. \end{aligned}$$

Thus we have identified Ω as the disk

$$\Omega = \{z \in \mathbb{C} : |z - 1|^2 < 4\}.$$

No other domain (or open set) has the first four moments given by (4.5).

5 The meromorphic resultant and the elimination function

In this section we review the definitions of the meromorphic resultant and the elimination function, as introduced in [19], referring to that paper for any details. If f is a meromorphic function on any compact Riemann surface M we denote by (f) its divisor of zeros and poles, symbolically $(f) = f^{-1}(0) - f^{-1}(\infty)$ (written in additive form). If D is any divisor and g is any function we denote by $\langle D, g \rangle$ the additive action of D on g , and by $g(D)$ the multiplicative action of g on D . See the example below for clarification.

Now the *meromorphic resultant* between f and g is, by definition,

$$\mathcal{R}(f, g) = g((f)) = e^{\langle (f), \log g \rangle}$$

whenever this makes sense (the resultant is undefined if expressions like $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ appear when the expression in the middle member is evaluated). In the last expression, $\log g$ refers to arbitrarily chosen local branches of the logarithm. When defined, $\mathcal{R}(f, g)$ is either a complex number or ∞ . As a consequence of the Weil reciprocity theorem [51] the resultant is symmetric:

$$\mathcal{R}(f, g) = \mathcal{R}(g, f).$$

Example 5.1. To illustrate and explain the above definitions we spell out what they look like for the sample divisor

$$D = 1 \cdot (a) + 1 \cdot (b) - 2 \cdot (c).$$

Here $a, b, c \in M$ (any compact Riemann surface) and g is a function on M (typically a meromorphic function, but this is not absolutely necessary as for the definitions). It is possible to think of D as a 0-chain and of g as a 0-form. Then the above definitions amount to

$$\langle D, g \rangle = 1 \cdot g(a) + 1 \cdot g(b) - 2 \cdot g(c) = \int_D g, \quad (5.1)$$

$$g(D) = e^{\langle D, \log g \rangle} = \frac{g(a)g(b)}{g(c)^2}. \quad (5.2)$$

We can associate to D the 2-form current

$$\delta_D dx \wedge dy = \delta_a dx \wedge dy + \delta_b dx \wedge dy - 2\delta_c dx \wedge dy, \quad (5.3)$$

where $\delta_a dx \wedge dy$ denotes the Dirac measure (point mass) at a considered as a 2-form (so that it acts on functions by integration over M). Then (5.1) can be augmented to

$$\langle D, g \rangle = \int_D g = \int_M g \delta_D dx \wedge dy.$$

If f is a meromorphic function with divisor $(f) = D$, then the above $g(D)$ (in (5.2)) is also the resultant $\mathcal{R}(f, g) = g((f))$. For example, if $M = \mathbb{P}$ and

$$f(z) = \frac{(z-a)(z-b)}{(z-c)^2},$$

then

$$\mathcal{R}(f, g) = g((f)) = \frac{g(a)g(b)}{g(c)^2}.$$

It is also possible to let functions of two (or more) variables act on divisors. For example, if $g(z, w)$ is such a function and D_1, D_2 are two divisors, say $D_1 = 1 \cdot (a) + 1 \cdot (b) - 2 \cdot (c)$, $D_2 = 3 \cdot (p) - 3 \cdot (q)$, then (by definition)

$$g(D_1, D_2) = \exp \langle D_1 \otimes D_2, \log g \rangle = \frac{g(a, p)^3 g(b, p)^3 g(c, q)^6}{g(a, q)^3 g(b, q)^3 g(c, p)^6}. \quad (5.4)$$

Next the *elimination function* between two meromorphic functions, f and g , on M is defined by

$$\mathcal{E}_{f,g}(z, w) = \mathcal{R}(f - z, g - w),$$

where $z, w \in \mathbb{C}$ are parameters. It is always a rational function in z and w , more precisely of the form

$$\mathcal{E}_{f,g}(z, w) = \frac{Q(z, w)}{P(z)R(w)}, \quad (5.5)$$

where Q, P and R are polynomials, and it embodies the necessary (since M is compact) polynomial relationship between f and g :

$$\mathcal{E}_{f,g}(f(\zeta), g(\zeta)) = 0 \quad (\zeta \in M).$$

There is also an extended elimination function, defined by

$$\mathcal{E}_{f,g}(z, w; a, b) = \mathcal{R}\left(\frac{f-z}{f-a}, \frac{g-w}{g-b}\right).$$

To relate the elimination function to the exponential transform one need integral formulas for the elimination function. For this purpose we shall make some computations within the framework of currents (distributional differential forms) on M . As a first issue, if f is a meromorphic function in M we need to make its logarithm a single-valued function almost everywhere on M . This is done as usual by making suitable ‘cuts’ on M , and by choosing a branch of the logarithm, call it $\text{Log } f$, on the remaining set $M \setminus (\text{cuts})$. Its differential in the sense of currents will then be of the form

$$d\text{Log } f = \frac{df}{f} + \text{distributional contributions on the cuts.} \quad (5.6)$$

If there is a cut along the x -axis, for example, the distributional contribution will be dy times a measure on the cut, that measure being arc length measure times the size of the jump (an integer multiple of $2\pi i$) over the cut, between the two branches of the logarithm.

The next issue is the identity

$$\frac{1}{2\pi i} d\left(\frac{df}{f}\right) = \delta_{(f)} dx \wedge dy.$$

See [19] for the simple proof.

Now, proceeding formally we transform the resultant to an integral in the following series of steps:

$$\begin{aligned} \mathcal{R}(f, g) &= g((f)) = e^{\langle (f), \text{Log } g \rangle} = \exp \int_{(f)} \text{Log } g \\ &= \exp \left[\int_M \text{Log } g \delta_{(f)} dx \wedge dy \right] = \exp \left[\frac{1}{2\pi i} \int_M \text{Log } g d\left(\frac{df}{f}\right) \right] \\ &= \exp \left[\frac{1}{2\pi i} \int_M \frac{df}{f} \wedge d\text{Log } g \right]. \end{aligned}$$

It should be remarked that, in the final integral, the only contributions come from the jumps of $\text{Log } g$ (the last term in (5.6) when stated for $\text{Log } g$) because outside this set of discontinuities the integrand contains $d\zeta \wedge d\zeta = 0$ as a factor. Similarly one gets, for the elimination function,

$$\mathcal{E}_{f,g}(z, w) = \exp \left[\frac{1}{2\pi i} \int_M \frac{df}{f - z} \wedge d\text{Log}(g - w) \right]. \quad (5.7)$$

Now we shall apply (5.7) in the case that M is the Schottky double of a plane domain Ω , and connect it to the exponential transform of Ω . So let Ω be a finitely connected plane domain with analytic boundary or, more generally, a bordered Riemann surface and let

$$M = \widehat{\Omega} = \Omega \cup \partial\Omega \cup \widetilde{\Omega}$$

be the *Schottky double* of Ω , *i.e.*, the compact Riemann surface obtained by completing Ω with a backside with the opposite conformal structure, the two surfaces glued together along $\partial\Omega$ (see [8], for example). On $\widehat{\Omega}$ there is a natural anticonformal involution $J : \widehat{\Omega} \rightarrow \widehat{\Omega}$ exchanging corresponding points on Ω and $\widetilde{\Omega}$ and having $\partial\Omega$ as fixed points.

Let f and g be two meromorphic functions on $\widehat{\Omega}$. Then

$$f^* = \overline{(f \circ J)}, \quad g^* = \overline{(g \circ J)}$$

are also meromorphic on $\widehat{\Omega}$.

Theorem 5.1. *With Ω , $\widehat{\Omega}$, f , g as above, assume in addition that f has no poles in $\Omega \cup \partial\Omega$ and that g has no poles in $\widetilde{\Omega} \cup \partial\Omega$. Then, for large z, w ,*

$$\mathcal{E}_{f,g}(z, \bar{w}) = \exp \left[\frac{1}{2\pi i} \int_{\Omega} \frac{df}{f-z} \wedge \frac{d\bar{g}^*}{\bar{g}^* - \bar{w}} \right].$$

In particular,

$$\mathcal{E}_{f,f^*}(z, \bar{w}) = \exp \left[\frac{1}{2\pi i} \int_{\Omega} \frac{df}{f-z} \wedge \frac{d\bar{f}}{\bar{f} - \bar{w}} \right].$$

Proof. For the divisors of $f - z$ and $g - w$ we have, if z, w are large enough, $\text{supp}(f - z) \subset \widetilde{\Omega}$, $\text{supp}(g - w) \subset \Omega$. Moreover, $\log(g - w)$ has a single-valued branch, $\text{Log}(g - w)$, in $\widetilde{\Omega}$. Using that $g = \bar{g}^*$ on $\partial\Omega$ we therefore get, starting with (5.7),

$$\begin{aligned} \mathcal{E}_{f,g}(z, \bar{w}) &= \exp \left[\frac{1}{2\pi i} \int_{\widehat{\Omega}} \frac{df}{f-z} \wedge d \text{Log}(g - \bar{w}) \right] \\ &= \exp \left[\frac{1}{2\pi i} \int_{\Omega} \frac{df}{f-z} \wedge d \text{Log}(g - \bar{w}) \right] \\ &= \exp \left[-\frac{1}{2\pi i} \int_{\partial\Omega} \frac{df}{f-z} \wedge \text{Log}(g - \bar{w}) \right] \\ &= \exp \left[-\frac{1}{2\pi i} \int_{\partial\Omega} \frac{df}{f-z} \wedge \text{Log}(\bar{g}^* - \bar{w}) \right] \\ &= \exp \left[\frac{1}{2\pi i} \int_{\Omega} \frac{df}{f-z} \wedge \frac{d\bar{g}^*}{\bar{g}^* - \bar{w}} \right]. \end{aligned}$$

as claimed. □

We next specialize to the case that Ω is a quadrature domain. Let $S(\zeta)$ be the Schwarz function of Ω . Then the relation (4.3) can be interpreted as saying that the pair of functions $S(\zeta)$ and $\bar{\zeta}$ on Ω combines into a meromorphic function on the Schottky double $\widehat{\Omega}$ of Ω , namely that function g which equals $S(\zeta)$ on Ω , and equals $\bar{\zeta}$ on $\widetilde{\Omega}$.

The function $f = g^* = \overline{g \circ J}$ is then represented by the opposite pair: ζ on Ω , $\overline{S(\zeta)}$ on $\widetilde{\Omega}$. Therefore, on Ω we have $f(\zeta) = g^*(\zeta) = \zeta$, and Theorem 5.1 immediately gives

Corollary 5.1. *For any quadrature domain Ω ,*

$$E_{\Omega}(z, w) = \mathcal{E}_{f, f^*}(z, \bar{w}) \quad (|z|, |w| \gg 1), \quad (5.8)$$

where f, f^* are the two meromorphic functions on $\widehat{\Omega}$ given on Ω by $f(\zeta) = \zeta$, $f^*(\zeta) = S(\zeta)$.

An alternative way of conceiving the corollary is to think of f being defined on an independent surface W , so that $f : W \rightarrow \Omega$ is a conformal map. Then Ω is a quadrature domain if and only if f extends to a meromorphic function of the Schottky double \widehat{W} , and the assertion of the corollary then is that the exponential transform of Ω is given by (5.8), with the elimination function in the right member now taken in \widehat{W} .

If Ω is simply connected we may choose $W = \mathbb{D}$, so that $\widehat{W} = \mathbb{P}$ with involution $J : \zeta \mapsto 1/\bar{\zeta}$. Then $f : \mathbb{D} \rightarrow \Omega$ is a rational function when (and only when) Ω is a quadrature domain, hence we conclude that $E_{\Omega}(z, w)$ in this case is the elimination function for two rational functions, $f(\zeta)$ and $f^*(\zeta) = \overline{f(1/\bar{\zeta})}$.

Finally we want to illustrate the effectiveness of the notation (5.4) by giving a formula for how the exponential transforms under rational conformal maps.

Theorem 5.2. *Let Ω be bounded domain in the complex plane and let F be a rational function which is bounded and one-to-one on Ω . Then for $z, w \in \mathbb{C} \setminus \overline{F(\Omega)}$ we have*

$$E_{F(\Omega)}(z, w) = E_{\Omega}((F - z), (F - w)). \quad (5.9)$$

Proof. We have

$$\begin{aligned}
E_{F(\Omega)}(z, w) &= \exp \left[\frac{1}{2\pi i} \int_{F(\Omega)} \frac{d\zeta \wedge d\bar{\zeta}}{(\zeta - z)(\bar{\zeta} - \bar{w})} \right] \\
&= \exp \left[\frac{1}{2\pi i} \int_{F(\Omega)} d \log(\zeta - z) \wedge \overline{d \log(\zeta - w)} \right] \\
&= \exp \left[\frac{1}{2\pi i} \int_{\Omega} d \log(F(\zeta) - z) \wedge \overline{d \log(F(\zeta) - w)} \right].
\end{aligned}$$

Let $D_z = (F - z)$ denote the divisor of $F(\zeta) - z$, and consider it as a function $D_z : \mathbb{P} \rightarrow \mathbb{Z}$, namely that function which at each point evaluates the order of the divisor (hence is zero at all but finitely many points). Then

$$\begin{aligned}
d \log(F(\zeta) - z) &= \sum_{\alpha \in \mathbb{P}} D_z(\alpha) d \log(\zeta - \alpha), \\
\overline{d \log(F(\zeta) - w)} &= \sum_{\beta \in \mathbb{P}} D_w(\beta) d \log(\bar{\zeta} - \bar{\beta}).
\end{aligned}$$

With these observations we can continue the above series of equalities:

$$\begin{aligned}
E_{F(\Omega)}(z, w) &= \exp \left[\frac{1}{2\pi i} \int_{\Omega} d \log(F(\zeta) - z) \wedge \overline{d \log(F(\zeta) - w)} \right] \\
&= \exp \left[\frac{1}{2\pi i} \sum_{\alpha \in \mathbb{P}} \sum_{\beta \in \mathbb{P}} D_z(\alpha) D_w(\beta) \int_{\Omega} \frac{d\zeta}{\zeta - \alpha} \wedge \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{\beta}} \right] \\
&= \prod_{\alpha, \beta \in \mathbb{P}} E_{\Omega}(\alpha, \beta)^{D_z(\alpha) D_w(\beta)} = E_{\Omega}(D_z, D_w),
\end{aligned}$$

which is (5.9). □

6 Potential theoretic remarks

Recall the definition (2.1) of logarithmic potential of a signed measure μ with compact support in \mathbb{C} . If μ has not zero net mass, then the behavior of the potential at infinity is such that it automatically puts an extra point mass at infinity, to the effect that the potential globally (on \mathbb{P}) becomes the potential of a measure (again denoted μ) having vanishing net mass. If μ is given by the divisor of a meromorphic (rational) function f , *i.e.*, if

$$d\mu = \delta_{(f)} dx \wedge dy$$

in the notation (5.3), then

$$U_\mu = -\log |f|.$$

The point we wish to make here is that the resultant $\mathcal{R}(f, g)$ between two rational functions, f and g , has a corresponding interpretation, namely in terms of the mutual energy. In general, the mutual energy between two signed measure μ and ν of zero net mass (on \mathbb{P}) is given by

$$\begin{aligned} I(\mu, \nu) &= \iint \log \frac{1}{|z - \zeta|} d\mu(z) d\nu(\zeta) = \int U_\mu d\nu = \int U_\nu d\mu \\ &= \frac{1}{2\pi} \int dU_\mu \wedge *dU_\nu. \end{aligned}$$

Some care is needed here since the above is not always well-defined (for example if μ and ν have common point masses). But this is just the same situation as for the resultant. Now, with f related as above to μ , and similarly g related to ν by $d\nu = \delta_{(g)} dx \wedge dy$, then we have the relationship

$$I(\mu, \nu) = -\log |\mathcal{R}(f, g)|.$$

The proof is just a straight-forward computation:

$$\begin{aligned} |\mathcal{R}(f, g)|^2 &= \exp [\langle (f), \log g \rangle + \langle (f), \overline{\log g} \rangle] \\ &= \exp [2\langle (f), \log |g| \rangle] = \exp \left[2 \int_{(f)} \log |g| \right] \\ &= \exp \left[-2 \int U^\nu d\mu \right] = \exp [-2I(\mu, \nu)]. \end{aligned}$$

Similarly, one can relate the discriminant of f with a renormalized self-energy of μ . We refer to [19] for some details concerning this.

All of the above generalizes well from \mathbb{P} to an arbitrary compact Riemann surface M . The potential will then be less explicit, but it follows from classical theory (for example Hodge theory) that given any signed measure μ on M with $\int_M d\mu = 0$ there is potential U_μ , uniquely defined up to an additive constant, such that, considering μ as a 2-form current,

$$-d * dU_\mu = 2\pi\mu. \quad (6.1)$$

Choosing in particular $d\mu = \delta_a dx \wedge dy - \delta_b dx \wedge dy$ for two points $a, b \in M$ gives a fundamental potential that we shall denote $V(z) = V(z, w; a, b)$. Besides depending on a and b it depends on a parameter w , for normalization. The potential is characterized by the singularity structure

$$V(z) = V(z, w; a, b) = -\log |z - a| + \log |z - b| + \text{harmonic}, \quad (6.2)$$

together with the normalization $V(w, w; a, b) = 0$ of the additive level. It has a few symmetries,

$$V(z, w; a, b) = V(a, b; z, w) = -V(z, w; b, a), \quad (6.3)$$

and the transitivity property

$$V(z, w; a, b) + V(z, w; b, c) = V(z, w; a, c). \quad (6.4)$$

If the Riemann surface happens to be symmetric (like the Schottky double), with involution J , then we have in addition

$$V(z, w; a, b) = V(J(z), J(w); J(a), J(b)). \quad (6.5)$$

As an application, the ordinary Green's function $g(z, \zeta)$ for a plane domain (or bordered Riemann surface) Ω can be obtained from the potential V for $M = \widehat{\Omega}$ by

$$g(z, \zeta) = \frac{1}{2}V(z, J(z); \zeta, J(\zeta)) = V(z, w; \zeta, J(\zeta)), \quad (6.6)$$

where w is an arbitrary point on $\partial\Omega$ and where the second equality follows from (6.3), (6.4), (6.5). Cf. [9], p. 125f. See also [16] for some more details.

The potential V appears implicitly or explicitly in classical texts, e.g., [52], [43]. Here we discuss it briefly because it connects to the exponential transform. To start, it has a certain self-reproducing property, expressed in the identity

$$V(z, w; a, b) = \frac{1}{2\pi} \int_M dV(\cdot, c; z, w) \wedge^* dV(\cdot, c; a, b), \quad (6.7)$$

valid for arbitrary non-coinciding points $z, w, a, b, c \in M$. Alternatively,

$$V(z, w; a, b) = \frac{i}{\pi} \int_M \partial V(\cdot, c; z, w) \wedge \bar{\partial} V(\cdot, c; a, b), \quad (6.8)$$

where $\partial V = \frac{\partial V}{\partial z} dz$, $\bar{\partial} V = \frac{\partial V}{\partial \bar{z}} d\bar{z}$. The proof of (6.7) is straight-forward: the right member becomes, after partial integration, an application of (6.1) and by using the symmetries (6.3) and (6.4),

$$\begin{aligned} & -\frac{1}{2\pi} \int_M V(\cdot, c; z, w) \wedge d * dV(\cdot, c; a, b) \\ & = V(a, c; z, w) - V(b, c; z, w) = V(z, w; a, b), \end{aligned}$$

which is the left member.

In the case of the Riemann sphere, V relates to the cross-ratio (2.7 by

$$V(z, w; a, b) = -\log |(z : w : a : b)| = -\log \left| \frac{(z-a)(w-b)}{(z-b)(w-a)} \right|. \quad (6.9)$$

Therefore the extended exponential transform for any domain $\Omega \subset \mathbb{P}$ can be conveniently expressed in terms of V for \mathbb{P} , for example

$$E_\Omega(z, w; a, b) = \exp \left[\frac{2}{i\pi} \int_\Omega \partial V(\cdot, c; z, a) \wedge \bar{\partial} V(\cdot, c; w, b) \right],$$

and for the modulus,

$$|E_\Omega(z, w; a, b)| = \exp \left[-\frac{1}{\pi} \int_\Omega dV(\cdot, c; z, a) \wedge *dV(\cdot, c; w, b) \right].$$

In the case of a compact Riemann surface, $\int_M \partial V(\cdot, c; z, a) \wedge \bar{\partial} V(\cdot, c; w, b)$ is purely imaginary, by (6.8). In particular we get, for $\Omega = M = \mathbb{P}$, and by using (6.8) and (6.9), that

$$E_{\mathbb{P}}(z, w; a, b) = \exp \left[\frac{2}{i\pi} \int_{\mathbb{P}} \partial V(\cdot, c; z, a) \wedge \bar{\partial} V(\cdot, c; w, b) \right]$$

$$= \exp \left[-2V(z, a; w, b) \right] = |(z : a : w : b)|^2.$$

Thus formula (2.6), which was asserted and proved in another way in Section 2, has been proved again.

7 Moment coordinates

In this section we restrict again to the case that $\Omega \subset \mathbb{C}$ is a simply connected domain with analytic boundary, and with $0 \in \Omega$. In order to keep the presentation reasonably rigorous we shall, much of the time, actually restrict even more, namely by considering only those domains which are conformal images of the unit disk under polynomial conformal maps,

$$f(\zeta) = \sum_{k=0}^N a_k \zeta^{k+1}, \quad (7.1)$$

with $a_0 > 0$ (normalization) and N fixed. This will be a manifold, call it \mathcal{M}_{2N+1} , of real dimension $2N + 1$, in fact it is an open subset of $\mathbb{R} \times \mathbb{C}^N \cong \mathbb{R}^{2N+1}$ with $a_0 \in \mathbb{R}$ and $a_j \in \mathbb{C}$ ($1 \leq j \leq N$) as coordinates.

Conformal images of the unit disk under polynomial conformal maps are exactly the simply connected quadrature domains with the origin as the sole quadrature node, *i.e.*, with $m = 1$, $a_1 = 0$ in the notation of (4.2), and the order of the quadrature domain then equals the degree of the polynomial ($N + 1$ in the above notation). The class of domains we are considering here is slightly smaller than this class of quadrature domains because the assumption of analytic boundary require our polynomials f to be univalent in a full neighborhood of the closed unit disk. But it has the advantage that \mathcal{M}_{2N+1} will be an ordinary open manifold (without boundary). For algebraic purposes one could equally well work with the just locally univalent polynomials, *i.e.*, those polynomials f in (7.1) for which $f' \neq 0$ on $\overline{\mathbb{D}}$.

Remark 7.1. The restriction to polynomial f does not change anything in principle, all formulas will look basically the same as in the general case. In the other extreme, without changing much in practice one may pass from the infinite dimensional case to the formal level (with no bothering about convergence) of transfinite functions, see [18].

As real coordinates on \mathcal{M}_{2N+1} we may of course use the real and imaginary parts of the coefficients appearing in (7.1), namely a_0 , $\operatorname{Re} a_j$, $\operatorname{Im} a_j$

($1 \leq j \leq N$), but much of what we are going to discuss concern what happens when we switch to $M_0, \operatorname{Re} M_j, \operatorname{Im} M_j$ ($1 \leq j \leq N$) as local coordinates.

The map (or change of coordinates) $(a_0, a_1, a_2, \dots) \mapsto (M_0, M_1, M_2, \dots)$ is explicitly given by Richardson's formula [39],

$$M_k = \sum (j_0 + 1) a_{j_0} \cdots a_{j_k} \bar{a}_{j_0 + \dots + j_k + k}. \quad (7.2)$$

The summation here goes over all multi-indices $(j_0, \dots, j_k) \geq (0, \dots, 0)$ for which $j_0 + \dots + j_k + k \leq N$. It is convenient to set $a_j = 0$ for $j > N$, and then the upper bound is not needed. (And with $a_j = 0$ for $j < 0$ the lower bound is not needed either.) The formula (7.2) is proved by pulling (3.5) back to the unit circle by the map (7.1) and then using the residue theorem (after having replaced \bar{f} by f^*).

It is clear from (7.2) that $M_{N+1} = M_{N+2} = \dots = 0$ when $a_{N+1} = a_{N+2} = \dots = 0$, and it follows from basic facts about quadrature domains that the converse statement holds as well (when f is univalent in a full neighborhood of the closed unit disk). It is virtually impossible to invert (7.2) in any explicit way. Even for $N = 1$ one would need to solve a third degree algebraic equation in order to express a_0, a_1 in terms of M_0, M_1 . And the map $(a_0, a_1, a_2, \dots) \mapsto (M_0, M_1, M_2, \dots)$ is (generally speaking) not one-to-one, as the example [41] (and several similar examples, cf. [55]) shows. The author does not know, however, of any such example in the present setting of polynomial conformal maps. C. Ullemar [47] gives an example of two polynomials of degree three, one univalent and one only locally univalent, which give rise to the same moments.

Remark 7.2. Even when $M_{N+1} = M_{N+2} = \dots = 0$, the negative moments M_{-1}, M_{-2}, \dots generally make up a full infinite sequence (of nonzero numbers). This is related to the fact that when Ω is a quadrature domain, then its complement is almost never a quadrature domain (one can easily make sense to the notion of an unbounded quadrature domain, or a quadrature domain in \mathbb{P}).

A deformation of a domain Ω as above corresponds to a smooth curve $t \mapsto \Omega(t)$ with $\Omega(0) = \Omega$ (equivalently, $t \mapsto f(\zeta, t)$ in \mathcal{M}_{2N+1} , and its velocity at $t = 0$ is a vector in the real tangent space of \mathcal{M}_{2N+1} at the point in question. This tangent space has, in terms of the coordinates introduced

above, two natural bases, namely

$$\begin{aligned} & \frac{\partial}{\partial a_0}, \quad \frac{\partial}{\partial \operatorname{Re} a_j}, \quad \frac{\partial}{\partial \operatorname{Im} a_j} \\ & \frac{\partial}{\partial M_0}, \quad \frac{\partial}{\partial \operatorname{Re} M_j}, \quad \frac{\partial}{\partial \operatorname{Im} M_j} \end{aligned}$$

($1 \leq j \leq N$), respectively.

It is natural to consider also the corresponding complexified tangent space, obtained by allowing complex coefficients in front of the above basis vectors. For this tangent space we also have the bases

$$\begin{aligned} & \frac{\partial}{\partial a_0}, \quad \frac{\partial}{\partial a_j}, \quad \frac{\partial}{\partial \bar{a}_j} \\ & \frac{\partial}{\partial M_0}, \quad \frac{\partial}{\partial M_j}, \quad \frac{\partial}{\partial \bar{M}_j}. \end{aligned}$$

Similar considerations apply for the cotangent spaces, *i.e.*, we have the two natural bases, $da_0, da_j, d\bar{a}_j$ and $dM_0, dM_j, d\bar{M}_j$, etc. It should be noted, however, that complex tangent vectors such as $\frac{\partial}{\partial \bar{M}_j}$ ($j \geq 1$) do not really correspond to velocity vectors for deformations of domains, only vectors in the real tangent space do.

The dependence of f on the coefficients a_0, \dots, a_N is certainly analytic by (7.1), but as can be understood from the appearance of conjugations in (7.2), the dependence on M_0, \dots, M_N is no longer analytic. Therefore we prefer to write this dependence as

$$f(\zeta) = f(\zeta; \bar{M}_N, \dots, \bar{M}_1, M_0, M_1, \dots, M_N), \quad (7.3)$$

or just $f(\zeta; M)$, with M shorthand for the list of moments:

$$M = (\bar{M}_N, \dots, \bar{M}_1, M_0, M_1, \dots, M_N). \quad (7.4)$$

Despite the implicit nature of the dependence of f on the moments, several interesting general statements can be made, and we want to highlight a couple of them. The first is due to O. Kuznetsova and V. Tkachev [25], [46], who were able to compute the Jacobian determinant for the transition between the above sets of coordinates, thereby confirming a conjecture of

C. Ullemar [47]. The beautiful result can be expressed in terms of the ratio between the volume forms as follows:

$$d\bar{M}_N \wedge \dots \wedge d\bar{M}_1 \wedge dM_0 \wedge dM_1 \wedge \dots \wedge dM_N = \quad (7.5)$$

$$= 2a_0^{N^2+3N+1} \mathcal{R}(f', f'^*) \cdot d\bar{a}_N \wedge \dots \wedge d\bar{a}_1 \wedge da_0 \wedge da_1 \wedge \dots \wedge da_N. \quad (7.6)$$

Here $\mathcal{R}(f', f'^*)$ is the meromorphic resultant between f' and f'^* , as discussed in Section 5. When f is univalent (or locally univalent), then $\mathcal{R}(f', f'^*) \neq 0$, hence the coordinate transition is locally one-to-one.

The second general statement is actually a series of results obtained by M. Mineev-Weinstein, P. Wiegmann, A. Zabrodin, with later contributions also from other authors, [53], [29], [27], [24]. We shall below, and in the next two sections, discuss some of these results, namely the string equation (7.8), the prepotential (3.8) and integrability of moments (3.6) (we will provide a proof of (3.7)) and, finally, Hamiltonian formulations of the evolution equations for the mapping functions (9.8).

The results are partly formulated in terms of a *Poisson bracket*, which for any two functions $f = f(\zeta; M)$, $g = g(\zeta; M)$, analytic for ζ in a neighborhood of the unit circle and also depending on the moments M , is defined by

$$\{f, g\} = \zeta \frac{\partial f}{\partial \zeta} \frac{\partial g}{\partial M_0} - \zeta \frac{\partial g}{\partial \zeta} \frac{\partial f}{\partial M_0}. \quad (7.7)$$

As a first choice here we take f to be the conformal map (7.1) and $g = f^*$, the reflection of f in the unit circle: $f^*(\zeta; M) = \overline{f(1/\bar{\zeta}; M)}$. Since we are assuming that Ω has analytic boundary, f and f^* are both analytic in some neighborhood of $\partial\mathbb{D}$, hence so is $\{f, f^*\}$. Now it turns out that it is much better than so, in fact $\{f, f^*\}$ is analytic in the whole Riemann sphere, hence is constant. This remarkable fact is sometimes referred to as the *string equation*, which more precisely reads

$$\{f, f^*\} = 1. \quad (7.8)$$

As for the proof (which follows [53] etc.) one first notices that $f^* = S \circ f$ (S is the Schwarz function), since this holds on $\partial\mathbb{D}$. Since also $S = S(z; M)$ depends on the moments this spells out to

$$f^*(\zeta; M) = S(f(\zeta; M); M).$$

Using hence the chain rule when computing $\frac{\partial f^*}{\partial M_0}$ one arrives at

$$\{f, f^*\} = \zeta \frac{\partial f}{\partial \zeta} \cdot \left(\frac{\partial S}{\partial M_0} \circ f \right).$$

Now we notice from (3.9) that

$$\frac{\partial S}{\partial M_0}(z; M) = \frac{1}{z} + \text{positive powers of } z,$$

since the coefficients M_1, M_2, \dots of the remaining negative powers of z in (3.9) are independent of M_0 . From this one sees that $\{f, f^*\}$ is holomorphic in \mathbb{D} . Similarly, it is holomorphic in $\mathbb{P} \setminus \overline{\mathbb{D}}$, hence it must in fact be constant, and one easily checks that this constant is one, proving (7.8).

The string equation (7.8) may appear to be quite remarkable identity for conformal maps, but actually it can be identified with an equation which was discovered much earlier, namely the *Polubarinova-Galin equation* [10], [31], [50], [20], [49] for the evolution of a Hele-Shaw blob with a source at the origin (Laplacian growth), this taken in combination with the Richardson moment conservation law [39] for such a flow. These considerations will be elaborated in the next section.

8 Domain deformations driven by harmonic gradients

Continuing on the theme of the previous section, we now discuss general domain variations from the point of view of Laplacian growth. We first consider an, in principle arbitrary, deformation $t \mapsto \Omega(t)$ of a simply connected domain $\Omega = \Omega(0)$. Let V_n denote the speed of $\Gamma(t) = \partial\Omega(t)$ measured in the outward normal direction. We need to assume that all data are real analytic. This can for example be expressed by saying that the Schwarz function $S(z, t)$ of $\Gamma(t)$ is real analytic in the parameter t (besides being analytic in z , in a neighborhood of $\Gamma(t)$). Then, by the Cauchy-Kovalevskaya theorem, there exists, for each t , a harmonic function $g(\cdot, t)$ defined in some neighborhood of $\Gamma(t)$ and satisfying

$$\begin{cases} g(\cdot, t) & = 0 & \text{on } \Gamma(t), \\ -\frac{\partial g(\cdot, t)}{\partial n} & = V_n & \text{on } \Gamma(t). \end{cases} \quad (8.1)$$

This function g can actually also be defined directly. In fact, it is easy to see that (8.1) can be expressed as

$$\frac{\partial S(z, t)}{\partial t} = -4 \frac{\partial g(z, t)}{\partial z}$$

holding on $\Gamma(t)$, and hence identically (both members being analytic in z). To be precise, the latter equation actually contains (8.1) only with the first condition weakened to $g(\cdot, t)$ being constant on $\Gamma(t)$. From the above one gets the formula

$$g(z, t) = -\frac{1}{2} \operatorname{Re} \int_{z_0(t)}^z \frac{\partial S}{\partial t}(\zeta, t) d\zeta,$$

where $z_0(t)$ is an arbitrary point on $\Gamma(t)$. Certainly, there will be exist neighborhood of $\Gamma(0)$ in which all the functions $g(\cdot, t)$, for $|t|$ sufficiently small, are defined.

Near $\Gamma(0)$ we have the usual cartesian coordinates (x, y) ($z = x + iy$), but from g we also get another pair, (t, θ) , namely

$$\begin{cases} t = t(z) & \text{determined by } z \in \Gamma_{t(z)}, \\ \theta = - * g(z, t(z)). \end{cases}$$

Here $*g$ is the conjugate harmonic function of g . We shall also write $d\theta = - * dg$, so this star is actually the Hodge star. Thus $t = t(z)$ is the time when $\Gamma(t)$ arrives at the point z , and θ is a coordinate along each $\Gamma(t)$. In order for $t(z)$ to be well-defined, at least locally, we need to assume that $V_n \neq 0$.

Fix a value of t and introduce auxiliary coordinates (n, s) in a neighborhood of $\Gamma(t)$ such that, on $\Gamma(t)$, $\frac{\partial}{\partial s}$ is the tangent vector and $\frac{\partial}{\partial n}$ is the outward normal vector (thus s is arc length along $\Gamma(t)$). Then we have (on $\Gamma(t)$)

$$*dg = -\frac{\partial g}{\partial s} dn + \frac{\partial g}{\partial n} ds = \frac{\partial g}{\partial n} ds,$$

and so

$$d\theta = - * dg(z, t) = V_n ds.$$

On the other hand (still on $\Gamma(t)$),

$$\frac{\partial t}{\partial s} = 0, \quad \frac{\partial t}{\partial n} = \frac{1}{V_n},$$

so that

$$dt = \frac{1}{V_n} dn.$$

It follows that $dt \wedge d\theta = dn \wedge ds$ on $\Gamma(t)$. But since $(\frac{\partial}{\partial n}, \frac{\partial}{\partial s})$ is just a rotation of $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, we also have $dn \wedge ds = dx \wedge dy$. This gives $dt \wedge d\theta = dx \wedge dy$ on $\Gamma(t)$, which does not involve the auxiliary coordinates (n, s) . Hence the latter equation actually holds in a full neighborhood of $\Gamma(t)$. Equivalently,

$$\frac{\partial(x, y)}{\partial(t, \theta)} = 1. \quad (8.2)$$

We summarize.

Proposition 8.1. *Any real analytic deformation $t \mapsto \Gamma(t)$ can be described, locally in regions where $V_n \neq 0$ and in terms of local coordinates (t, θ) introduced as above, by the “string equation”*

$$dt \wedge d\theta = dx \wedge dy. \quad (8.3)$$

Compare discussions in, for example, [22]. The reason for calling (8.3) a string equation will become clear below.

Now we change the point of view. In the above considerations, the normal velocity V_n of the boundary was the given data (essentially arbitrary, except for regularity requirements) and the harmonic function g was introduced as a secondary object. *Laplacian growth*, on the other hand, starts with a rule for prescribing a harmonic function $g = g_\Omega$, associated to any given domain Ω , defined in the domain (or at least near the boundary of it) and vanishing on the boundary. Then one asks the boundary to move with the normal velocity

$$V_n = -\frac{\partial g}{\partial n}.$$

Thus g now works as the generating function which forces the motion of $\partial\Omega$, and setting again $\theta = - * g$, (8.3) remains valid in regions where $V_n \neq 0$. Typically, g is specified by suitable source terms in Ω or by boundary conditions on some fixed boundaries.

The classical case is that $0 \in \Omega$ and that $g = g_\Omega$ is the Green’s function of Ω with pole at $z = 0$:

$$\begin{cases} g(z, t) = -\log |z| + \text{harmonic} & (z \in \Omega), \\ g(z, t) = 0 & (z \in \partial\Omega). \end{cases} \quad (8.4)$$

When pulled back to the unit disk by $f : \mathbb{D} \rightarrow \Omega$ (normalized conformal map) one then gets $g(f(\zeta), t) = -\log |\zeta|$, and hence $\theta = -*g(f(\zeta), t) = \arg \zeta$. Thus θ will simply be the angular variable on the unit circle. In addition, one sees that $d\theta$ can be interpreted as harmonic measure (on $\partial\mathbb{D}$, and on $\partial\Omega$). This classical version of Laplacian growth is connected to the harmonic moments in that it is characterized by the moments M_k , $k \geq 1$, being constants of motions ('integrals'), while M_0 evolves linearly in time. Precisely, with g defined by (8.4) we will have (as functions of t)

$$\begin{cases} M_0 = 2t + \text{constant}, \\ M_k = \text{constant} \quad (k \geq 1). \end{cases} \quad (8.5)$$

These conservation laws were discovered by S. Richardson [39], and they are easy to verify directly. In fact, one shows that

$$\frac{d}{dt} \int_{\Omega(t)} h dm = 2\pi h(0)$$

for any function h which is harmonic in a region containing the closure of $\Omega(t)$, and then (8.5) is obtained by choosing $h(z) = z^k$ ($k \geq 0$).

Now we can compare (7.8) with (8.2) or (8.3): since $\{f, f^*\}$ is an analytic function, the restriction of (7.8) to $\partial\mathbb{D}$ contains all information. And on $\partial\mathbb{D}$ we have, on setting $\zeta = e^{i\theta}$ and writing $f(e^{i\theta}, t) = x + iy$,

$$\{f, f^*\} = 2 \operatorname{Im} \left[\frac{\partial f}{\partial \theta} \frac{\partial \bar{f}}{\partial M_0} \right] = 2 \frac{\partial(x, y)}{\partial(M_0, \theta)}.$$

Since $2 \frac{\partial}{\partial M_0} = \frac{\partial}{\partial t}$ by (8.5) it follows that (7.8) is the same as (8.2), or (8.3). This was the reason for calling (8.3) a string equation.

Remark 8.1. A common variant of the above is that Ω is a domain in the Riemann sphere, $\infty \in \Omega$, and that g is the Green's function with pole at infinity.

Remark 8.2. Most of the above considerations also apply in higher dimensions, the main difference being that, in n real dimensions, $d\theta = -*dg$ will be an $(n-1)$ -form and that (8.3) will take the form $dt \wedge d\theta = dx_1 \wedge \cdots \wedge dx_n$ (the volume form). Another generalization, elliptic growth, is discussed in [22].

Now we shall connect Laplacian growth evolutions to the derivatives $\frac{\partial}{\partial M_j}$, $\frac{\partial}{\partial M_j}$. These evolutions will not be monotonic, so we must allow now V_n to change sign, and hence to be zero at some points of the boundaries. Let L denote a real-valued differential operator with constant coefficients acting on functions in \mathbb{C} , for example $L = \frac{\partial^k}{\partial x^k}$. Then, given $\Omega(0)$ with analytic boundary, there exists locally an evolution $t \mapsto \Omega(t)$ such that

$$\frac{d}{dt} \int_{\Omega(t)} h \, dm = 2\pi(Lh)(0) \quad (8.6)$$

for every function h which is harmonic in a region containing the closure of $\Omega(t)$. This is a version of Laplacian growth, in fact it is the evolution is driven by the harmonic function

$$g_L(z) = \{L_a g(z, a)\}_{a=0},$$

where $g(z, a) = -\log |z - a| + \text{harmonic}$ is the ordinary Green's function with pole at a , and the differential operator $L_a = L$ acts on that variable (the location of the pole). Thus, g_L is harmonic in Ω , has a multipole singularity at the origin and vanishes on $\partial\Omega$.

It is actually not easy to prove that evolutions $\Omega(t)$ driven by a given harmonic gradient really exist, but it can be done under the right assumptions (analyticity of the initial boundary is usually necessary, for example). An example of such a proof is given in [38]. Other approaches are presented in [45] and [26]. In the algebraic framework of the previous section, *i.e.*, with $\Omega \in \mathcal{M}_{2N+1}$, the full evolution will stay in \mathcal{M}_{2N+1} provided the order k of the differential operator L is at most N , and the existence proof is much easier, see [11]. In fact, the existence and uniqueness can even be read off from (7.5).

For the rest of this section, and also for the next section, even if it does not change anything in practice it is useful to think of being in the finite dimensional setting of \mathcal{M}_{2N+1} , hence assuming that $k \leq N$.

To prove the assertion that g_L really achieves (8.6), recall that the evolution $t \mapsto \Omega(t)$ is defined by the outward normal velocity of the boundary being $V_n = -\frac{\partial g_L(z)}{\partial n}$. Since $-\Delta g(\cdot, a) = 2\pi\delta_a$ we have

$$-\Delta g_L = -\{\Delta L_a g(\cdot, a)\}_{a=0} = -\{L_a \Delta g(\cdot, a)\}_{a=0} = 2\pi L\delta_0.$$

With h harmonic this gives

$$\frac{d}{dt} \int_{\Omega(t)} h \, dm = \int_{\partial\Omega(t)} V_n h \, ds = - \int_{\partial\Omega(t)} h \frac{\partial g_L}{\partial n} \, ds$$

$$= - \int_{\Omega(t)} h \Delta g_L dm = 2\pi Lh(0),$$

proving (8.6).

Useful particular choices of L above are, for $k \geq 1$,

$$L_1 = \frac{\partial^k}{\partial x^k}, \quad L_2 = \frac{\partial^k}{\partial x^{k-1} \partial y}.$$

We may allow h in (8.6) to be complex-valued, and choosing $h(z) = z^j$ and evaluating at $z = 0$ one gets

$$\{L_1(z^k)\}_{z=0} = k!, \quad \{L_2(z^k)\}_{z=0} = i \cdot k!, \quad (8.7)$$

and $\{L_i(z^j)\}_{z=0} = 0$ ($i = 1, 2$) in all remaining cases. Similarly, with $h(z) = \bar{z}^j$ we have

$$\{L_1(\bar{z}^k)\}_{z=0} = k!, \quad \{L_2(\bar{z}^k)\}_{z=0} = -i \cdot k!, \quad (8.8)$$

and $\{L_i(\bar{z}^j)\}_{z=0} = 0$ in the remaining cases.

In what follows, we will denote the time coordinates for the two evolutions corresponding to L_1 and L_2 by different letters, namely t_1 and t_2 , respectively. It is a consequence of (8.6) and (8.7), (8.8) that the evolution $t_1 \mapsto \Omega(t_1)$ generated by g_{L_1} has the property that

$$\frac{d}{dt_1} M_k = \frac{d}{dt_1} \bar{M}_k = 2k!, \quad \frac{d}{dt_1} M_j = \frac{d}{dt_1} \bar{M}_j = 0 \quad (j \neq k).$$

Similarly, for the evolution generated by g_{L_2} :

$$\frac{d}{dt_2} M_k = -\frac{d}{dt_2} \bar{M}_k = i \cdot 2k!, \quad \frac{d}{dt_2} M_j = \frac{d}{dt_2} \bar{M}_j = 0 \quad (j \neq k),$$

In other words, the tangent vector giving the velocity for the evolution driven by g_{L_1} is

$$\frac{d}{dt_1} = 2k! \left(\frac{\partial}{\partial M_k} + \frac{\partial}{\partial \bar{M}_k} \right) = 2k! \frac{\partial}{\partial \operatorname{Re} M_k},$$

and that for g_{L_2} is

$$\frac{d}{dt_2} = 2ik! \left(\frac{\partial}{\partial M_k} - \frac{\partial}{\partial \bar{M}_k} \right) = 2k! \frac{\partial}{\partial \operatorname{Im} M_k}.$$

Now we wish to extract $\frac{\partial}{\partial M_k}$ and $\frac{\partial}{\partial \bar{M}_k}$ from the above relations. We then see that these partial derivatives will correspond to linear combinations

with complex (and non-real) coefficients of the two different evolutions, or tangent vectors, hence they will not in themselves correspond to evolutions of domains. Still they make sense, of course, as vectors in the complexified tangent space of \mathcal{M}_{2N+1} . Precisely, we get

$$\begin{aligned}\frac{\partial}{\partial M_k} &= \frac{1}{4k!} \left(\frac{d}{dt_1} - i \frac{d}{dt_2} \right), \\ \frac{\partial}{\partial \bar{M}_k} &= \frac{1}{4k!} \left(\frac{d}{dt_1} + i \frac{d}{dt_2} \right).\end{aligned}$$

This is what we need in order to prove (3.7). Using (3.8) we get

$$\begin{aligned}\frac{\partial \mathcal{F}(\Omega)}{\partial M_k} &= \frac{\partial}{\partial M_k} \frac{1}{\pi^2} \int_{\mathbb{D}(0,R) \setminus \Omega} \int_{\mathbb{D}(0,R) \setminus \Omega} \log \frac{1}{|z - \zeta|} dm(z) dm(\zeta) \\ &= \frac{1}{4\pi^2 k!} \left(\frac{d}{dt_1} - i \frac{d}{dt_2} \right) \int_{\mathbb{D}(0,R) \setminus \Omega} \int_{\mathbb{D}(0,R) \setminus \Omega} \log \frac{1}{|z - \zeta|} dm(z) dm(\zeta).\end{aligned}$$

When computing the $\frac{d}{dt_1}$, for example, Ω is to be replaced with the evolution $\Omega(t_1)$, and we evaluate the derivative at $t_1 = 0$ ($\Omega(0) = \Omega$). There are two occurrences of Ω in the expression for $\mathcal{F}(\Omega)$, and $\mathcal{F}(\Omega)$ symmetric in them, so it is enough to differentiate one of the occurrences and then multiply the result by two. Writing $2 \log |z - \zeta| = \log(\zeta - z) + \log(\bar{\zeta} - \bar{z})$, and using (8.6), (3.5) this gives

$$\begin{aligned}& \frac{d}{dt_1} \Big|_{t_1=0} \int_{\mathbb{D}(0,R) \setminus \Omega(t_1)} \int_{\mathbb{D}(0,R) \setminus \Omega(t_1)} \log |z - \zeta| dm(z) dm(\zeta) \\ &= \int_{\mathbb{D}(0,R) \setminus \Omega} \frac{d}{dt_1} \Big|_{t_1=0} \left(\int_{\mathbb{D}(0,R)} - \int_{\Omega(t_1)} (\log(\zeta - z) + \log(\bar{\zeta} - \bar{z})) dm(z) \right) dm(\zeta) \\ &= 2\pi \int_{\mathbb{D}(0,R) \setminus \Omega} \left(\frac{(k-1)!}{\zeta^k} + \frac{(k-1)!}{\bar{\zeta}^k} \right) dm(\zeta) \\ &= i\pi(k-1)! \int_{\partial\Omega} \left(\frac{1}{\zeta^k} + \frac{1}{\bar{\zeta}^k} \right) \bar{\zeta} d\zeta = -2\pi^2 k! \left(\frac{1}{k} M_{-k} + \frac{1}{k} \bar{M}_{-k} \right).\end{aligned}$$

Similarly, the corresponding derivative with respect to t_2 gives

$$\frac{d}{dt_2} \Big|_{t_2=0} \int_{\mathbb{D}(0,R) \setminus \Omega(t_1)} \int_{\mathbb{D}(0,R) \setminus \Omega(t_1)} \log |z - \zeta| dm(z) dm(\zeta)$$

$$= -2i\pi^2 k! \left(\frac{1}{k} M_{-k} - \frac{1}{k} \bar{M}_{-k} \right).$$

Hence, in the combination $\frac{d}{dt_1} - i\frac{d}{dt_2}$, \bar{M}_{-k} will cancel out, and we get

$$\frac{\partial \mathcal{F}(\Omega)}{\partial M_k} = \frac{1}{k} M_{-k},$$

as desired.

9 Hamiltonians

As a final topic, we wish to explain, in our notations and settings, the Hamiltonian descriptions of the evolution of the mapping function f presented in [53] and related articles. Recall that we work within \mathcal{M}_{2N+1} , and we use M as a shorthand notation for the moments, which serve as coordinates on \mathcal{M}_{2N+1} , see (7.4).

Let $W = W(z; M)$ be a primitive function of the Schwarz function $S = S(z; M)$. By (3.9), the power series expansion of W will be

$$W(z; M) = - \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{M_k}{k z^k} + M_0 \log z + C(M), \quad (9.1)$$

where $C(M)$ is a constant. Decomposing W into real and imaginary parts, $W = w + i * w$, the real part can be made perfectly well-defined. In fact, $S(z) = 2\frac{\partial w}{\partial z}$, and since $S(z) = \bar{z}$ on $\partial\Omega$ we see that the real-valued function

$$u(z) = \frac{1}{4}|z|^2 - \frac{1}{2}w(z)$$

satisfies $\frac{\partial u}{\partial z} = 0$ on $\partial\Omega$, hence u is constant on $\partial\Omega$. We shall fix the free additive constant in u , and hence that in w , by requiring that $u = 0$ on $\partial\Omega$. It follows that u then satisfies

$$\begin{cases} \Delta u = 1 & \text{in a neighborhood of } \partial\Omega, \\ u = \frac{\partial u}{\partial z} = 0 & \text{on } \partial\Omega. \end{cases}$$

Next we note from (9.1) that

$$\frac{\partial W}{\partial M_0} = \log z + \sum_{k>0} \frac{1}{k} \frac{\partial M_{-k}}{\partial M_0} z^k + \frac{\partial C}{\partial M_0}.$$

Thus, as for the real part, $\frac{\partial w}{\partial M_0} = \log |z| + \text{harmonic}$ in Ω . On $\partial\Omega$ we have $\frac{\partial w}{\partial M_0} = -2\frac{\partial u}{\partial M_0} = 0$, because u vanishes to the second order on $\partial\Omega$. Hence it follows that $\frac{\partial w}{\partial M_0}$ is minus the Green's function $g = g_\Omega$ of Ω with pole at the origin:

$$\frac{\partial w}{\partial M_0} = -g. \quad (9.2)$$

Let $G = g + i * g$ be the analytic completion of g . This Green's function $G = G(z; M)$ is (like g) a function of $z \in \Omega$ and the moments. By (9.2),

$$G(z; M) = -\frac{\partial W(z; M)}{\partial M_0}. \quad (9.3)$$

Recall that $f = f(\zeta; M)$ denotes the conformal map (7.1) from \mathbb{D} to Ω . Define the (complex-valued) *Hamiltonian function* $H_0 = H_0(\zeta; M)$ associated to M_0 by

$$H_0(\zeta; M) = G(f(\zeta; M); M) = -\log \zeta, \quad (9.4)$$

the last equality because $G \circ f$ is the Green's function of \mathbb{D} . We see that H_0 actually is independent of the moments. Therefore, as $\frac{\partial H_0}{\partial M_0} = 0$, $\zeta \frac{\partial H_0}{\partial \zeta} = -1$ by (9.4), we trivially arrive at the evolution equation

$$\frac{\partial \varphi}{\partial M_0} = \{\varphi, H_0\}, \quad (9.5)$$

valid for any smooth function $\varphi = \varphi(\zeta; M)$.

For the higher order moments, there are corresponding identities, and they are more interesting and more selective: they hold only for $\varphi = f$. We first define, for $k \geq 1$,

$$H_k(\zeta; M) = -\frac{\partial W(z; M)}{\partial M_k}, \quad \text{where } z = f(\zeta; M). \quad (9.6)$$

The above means that the derivative $\frac{\partial}{\partial M_k}$ only acts on the M_k which appears in $W(z; M)$, not that in $f(\zeta; M)$. Always when we write z in place of $f(\zeta; M)$ it shall have this implication. Now, $G(f(\zeta; M); M) = -\log \zeta$ gives again, by differentiating with respect to ζ and M_k ,

$$\frac{\partial G}{\partial z}(z; M) \frac{\partial f}{\partial \zeta}(\zeta; M) = -\frac{1}{\zeta},$$

$$\frac{\partial G}{\partial z}(z; M) \frac{\partial f}{\partial M_k}(\zeta; M) + \frac{\partial G}{\partial M_k}(z; M) = 0.$$

Thus, multiplying the latter equation with $\zeta \frac{\partial f}{\partial \zeta}$ and using the first equation gives

$$\frac{\partial f}{\partial M_k}(\zeta; M) = \zeta \frac{\partial f}{\partial \zeta}(\zeta; M) \frac{\partial G}{\partial M_k}(z; M). \quad (9.7)$$

Here we wish to remove G from the right member, in favor of H_k . From (9.6) we get

$$\zeta \frac{\partial H_k}{\partial \zeta}(\zeta; M) = -\frac{\partial^2 W}{\partial z \partial M_k}(z; M) \cdot \zeta \frac{\partial f}{\partial \zeta}(\zeta; M)$$

and, using also (9.3),

$$\begin{aligned} & \zeta \frac{\partial f}{\partial \zeta}(\zeta; M) \frac{\partial H_k}{\partial M_0}(\zeta; M) \\ &= -\zeta \frac{\partial f}{\partial \zeta}(\zeta; M) \frac{\partial^2 W}{\partial z \partial M_k}(z; M) \frac{\partial f}{\partial M_0}(\zeta; M) - \zeta \frac{\partial f}{\partial \zeta}(\zeta; M) \frac{\partial^2 W}{\partial M_0 \partial M_k}(z; M) \\ &= \zeta \frac{\partial H_k}{\partial \zeta}(\zeta; M) \frac{\partial f}{\partial M_0}(\zeta; M) + \zeta \frac{\partial f}{\partial \zeta}(\zeta; M) \frac{\partial G}{\partial M_k}(z; M). \end{aligned}$$

The last term coincides with the right member in (9.7). Thus substituting into (9.7) we get

$$\frac{\partial f}{\partial M_k} = \{f, H_k\}, \quad (9.8)$$

which is the evolution equation for f we wanted to arrive at.

As for the classical Laplacian growth evolution, even though the situation is essentially trivial in the moment coordinates (in view of the explicit solution (8.5)), it may be nice to put it all in a traditional Hamiltonian framework (with real-valued Hamiltonian function). One possibility then is to choose the phase space to be $\partial\mathbb{D} \times \mathcal{M}_{2N+1} \subset \partial\mathbb{D} \times \mathbb{R}^{2N+1}$, with real coordinates $(\theta, M_0, \operatorname{Re} M_1, \operatorname{Im} M_1, \dots, \operatorname{Re} M_N, \operatorname{Im} M_N)$, symplectic form

$$\begin{aligned} \omega &= \frac{1}{2} d\theta \wedge dM_0 + d\operatorname{Re} M_1 \wedge d\operatorname{Im} M_1 + \dots + d\operatorname{Re} M_N \wedge d\operatorname{Im} M_N \\ &= \frac{1}{2i} [id\theta \wedge dM_0 + d\bar{M}_1 \wedge dM_1 + \dots + d\bar{M}_N \wedge dM_N] \end{aligned}$$

and Hamiltonian function

$$H(\theta, M_0, \operatorname{Re} M_1, \dots, \operatorname{Im} M_N) = \theta.$$

The Hamilton equations are, generally speaking (see [4]),

$$-dH = \omega(\xi, \cdot), \quad (9.9)$$

where $\xi = \frac{d}{dt}$ is the velocity vector for the evolution, a vector in the tangent space of the phase space. In our case we have (using dot for time derivative)

$$\xi = \dot{\theta} \frac{\partial}{\partial \theta} + \dot{M}_0 \frac{\partial}{\partial M_0} + \operatorname{Re} \dot{M}_1 \frac{\partial}{\partial \operatorname{Re} M_1} + \cdots + \operatorname{Im} \dot{M}_N \frac{\partial}{\partial \operatorname{Im} M_N},$$

giving in (9.9)

$$\begin{aligned} \dot{\theta} &= 0, & \dot{M}_0 &= 2, \\ \operatorname{Re} \dot{M}_j &= \operatorname{Im} \dot{M}_j = 0 & (1 \leq j \leq N), \end{aligned}$$

as expected (cf. (8.5)).

Note that the first term in ω , with $dM_0 = 2dt$, can be identified with $-dx \wedge dy$, in the notation of Proposition 8.1.

References

- [1] D. AHARONOV AND H. S. SHAPIRO, *Domains on which analytic functions satisfy quadrature identities*, J. Analyse Math., 30 (1976), pp. 39–73.
- [2] L. V. AHLFORS, *Complex analysis: An introduction of the theory of analytic functions of one complex variable*, Second edition, McGraw-Hill Book Co., New York, 1966.
- [3] H. ALEXANDER AND J. WERMER, *Several complex variables and Banach algebras*, vol. 35 of Graduate Texts in Mathematics, Springer-Verlag, New York, third ed., 1998.
- [4] V. I. ARNOLD, *Mathematical methods of classical mechanics*, Springer-Verlag, New York, 1978. Translated from the Russian by K. Vogtmann and A. Weinstein, Graduate Texts in Mathematics, 60.
- [5] R. W. CAREY AND J. D. PINCUS, *An exponential formula for determining functions*, Indiana Univ. Math. J., 23 (1973/74), pp. 1031–1042.
- [6] V. G. CHEREDNICHENKO, *Inverse logarithmic potential problem*, Inverse and Ill-posed Problems Series, VSP, Utrecht, 1996.

- [7] P. J. DAVIS, *The Schwarz function and its applications*, The Mathematical Association of America, Buffalo, N. Y., 1974. The Carus Mathematical Monographs, No. 17.
- [8] H. M. FARKAS AND I. KRA, *Riemann surfaces*, vol. 71 of Graduate Texts in Mathematics, Springer-Verlag, New York, second ed., 1992.
- [9] J. D. FAY, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics, Vol. 352, Springer-Verlag, Berlin, 1973.
- [10] L. A. GALIN, *Unsteady filtration with a free surface*, C. R. (Doklady) Acad. Sci. URSS (N.S.), 47 (1945), pp. 246–249.
- [11] B. GUSTAFSSON, *On a differential equation arising in a Hele-Shaw flow moving boundary problem*, Ark. Mat., 22 (1984), pp. 251–268.
- [12] ———, *On quadrature domains and an inverse problem in potential theory*, J. Analyse Math., 55 (1990), pp. 172–216.
- [13] B. GUSTAFSSON AND M. PUTINAR, *An exponential transform and regularity of free boundaries in two dimensions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 26 (1998), pp. 507–543.
- [14] ———, *Linear analysis of quadrature domains. II*, Israel J. Math., 119 (2000), pp. 187–216.
- [15] ———, *Linear analysis of quadrature domains. IV*, in Quadrature domains and their applications, vol. 156 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 2005, pp. 173–194.
- [16] B. GUSTAFSSON AND A. SEBBAR, *Critical points of Green's function and geometric function theory*, Indiana University Journal, to appear, (2012).
- [17] B. GUSTAFSSON AND H. S. SHAPIRO, *What is a quadrature domain?*, in Quadrature domains and their applications, vol. 156 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 2005, pp. 1–25.
- [18] B. GUSTAFSSON AND V. TKACHEV, *On the Jacobian of the harmonic moment map*, Complex Anal. Oper. Theory, 3 (2009), pp. 399–417.

- [19] B. GUSTAFSSON AND V. G. TKACHEV, *The resultant on compact Riemann surfaces*, Comm. Math. Phys., 286 (2009), pp. 313–358.
- [20] B. GUSTAFSSON AND A. VASIL'EV, *Conformal and potential analysis in Hele-Shaw cells*, Advances in Mathematical Fluid Mechanics, Birkhäuser Verlag, Basel, 2006.
- [21] F. R. HARVEY AND H. B. LAWSON, JR., *On boundaries of complex analytic varieties. I*, Ann. of Math. (2), 102 (1975), pp. 223–290.
- [22] D. KHAVINSON, M. MINEEV-WEINSTEIN, AND M. PUTINAR, *Planar elliptic growth*, Complex Anal. Oper. Theory, 3 (2009), pp. 425–451.
- [23] I. K. KOSTOV, I. KRICHEVER, M. MINEEV-WEINSTEIN, P. B. WIEGMANN, AND A. ZABRODIN, *The τ -function for analytic curves*, in Random matrix models and their applications, vol. 40 of Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, Cambridge, 2001, pp. 285–299.
- [24] I. KRICHEVER, A. MARSHAKOV, AND A. ZABRODIN, *Integrable structure of the Dirichlet boundary problem in multiply-connected domains*, Comm. Math. Phys., 259 (2005), pp. 1–44.
- [25] O. S. KUZNETSOVA AND V. G. TKACHEV, *Ullemar's formula for the Jacobian of the complex moment mapping*, Complex Var. Theory Appl., 49 (2004), pp. 55–72.
- [26] Y.-L. LIN, *Perturbation theorems for Hele-Shaw flows and their applications*, Ark. Mat., 49 (2011), pp. 357–382.
- [27] A. MARSHAKOV, P. WIEGMANN, AND A. ZABRODIN, *Integrable structure of the Dirichlet boundary problem in two dimensions*, Comm. Math. Phys., 227 (2002), pp. 131–153.
- [28] M. MARTIN AND M. PUTINAR, *Lectures on hyponormal operators*, vol. 39 of Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 1989.
- [29] M. MINEEV-WEINSTEIN AND A. ZABRODIN, *Whitham-Toda hierarchy in the Laplacian growth problem*, J. Nonlinear Math. Phys., 8 (2001), pp. 212–218. Nonlinear evolution equations and dynamical systems (Kolimbary, 1999).

- [30] J. D. PINCUS, D. XIA, AND J. B. XIA, *The analytic model of a hyponormal operator with rank one self-commutator*, Integral Equations Operator Theory, 7 (1984), pp. 516–535.
- [31] P. Y. POLUBARINOVA-KOCHINA, *On a problem of the motion of the contour of a petroleum shell*, Dokl. Akad. Nauk USSR, 47 (1945), pp. 254–257.
- [32] M. PUTINAR, *A two-dimensional moment problem*, J. Funct. Anal., 80 (1988), pp. 1–8.
- [33] ———, *The L problem of moments in two dimensions*, J. Funct. Anal., 94 (1990), pp. 288–307.
- [34] ———, *On a class of finitely determined planar domains*, Math. Res. Lett., 1 (1994), pp. 389–398.
- [35] ———, *Linear analysis of quadrature domains*, Ark. Mat., 33 (1995), pp. 357–376.
- [36] ———, *Extremal solutions of the two-dimensional L -problem of moments*, J. Funct. Anal., 136 (1996), pp. 331–364.
- [37] ———, *Extremal solutions of the two-dimensional L -problem of moments. II*, J. Approx. Theory, 92 (1998), pp. 38–58.
- [38] M. REISSIG AND L. VON WOLFERSDORF, *A simplified proof for a moving boundary problem for Hele-Shaw flows in the plane*, Ark. Mat., 31 (1993), pp. 101–116.
- [39] S. RICHARDSON, *Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel*, J. Fluid Mech., 56 (1972), pp. 609–618.
- [40] J. ROSS AND D. WITT NYSTRÖM, *The Hele-Shaw flow and moduli of holomorphic discs*, (2012).
- [41] M. SAKAI, *A moment problem on Jordan domains*, Proc. Amer. Math. Soc., 70 (1978), pp. 35–38.
- [42] ———, *Quadrature domains*, vol. 934 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1982.

- [43] M. SCHIFFER AND D. C. SPENCER, *Functionals of finite Riemann surfaces*, Princeton University Press, Princeton, N. J., 1954.
- [44] H. S. SHAPIRO, *The Schwarz function and its generalization to higher dimensions*, University of Arkansas Lecture Notes in the Mathematical Sciences, 9, John Wiley & Sons Inc., New York, 1992. A Wiley-Interscience Publication.
- [45] F. R. TIAN, *A Cauchy integral approach to Hele-Shaw problems with a free boundary: the case of zero surface tension*, Arch. Rational Mech. Anal., 135 (1996), pp. 175–196.
- [46] V. G. TKACHEV, *Ullemar’s formula for the moment map. II*, Linear Algebra Appl., 404 (2005), pp. 380–388.
- [47] C. ULLEMAR, *Uniqueness theorem for domains satisfying a quadrature identity for analytic functions*, Research Bulletin TRITA-MAT-1980-37, Royal Institute of Technology, Department of Mathematics, Stockholm, 1980.
- [48] A. N. VARCHENKO AND P. I. ETINGOF, *Why the Boundary of a Round Drop Becomes a Curve of Order Four*, American Mathematical Society AMS University Lecture Series, Providence, Rhode Island, third ed., 1992.
- [49] A. VASIL’EV, *From the Hele-Shaw experiment to integrable systems: a historical overview*, Complex Anal. Oper. Theory, 3 (2009), pp. 551–585.
- [50] Y. P. VINOGRADOV AND P. P. KUFAREV, *On a problem of filtration*, Akad. Nauk SSSR. Prikl. Mat. Meh., 12 (1948), pp. 181–198.
- [51] A. WEIL, *Scientific works. Collected papers. Vol. I (1926–1951)*, Springer-Verlag, New York, 1979.
- [52] H. WEYL, *Die Idee der Riemannschen Fläche*, Vierte Auflage. Unveränderter Nachdruck der dritten, Vollständig umgearbeiteten Auflage, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1964.
- [53] P. B. WIEGMANN AND A. ZABRODIN, *Conformal maps and integrable hierarchies*, Comm. Math. Phys., 213 (2000), pp. 523–538.

- [54] D. V. YAKUBOVICH, *Real separated algebraic curves, quadrature domains, Ahlfors type functions and operator theory*, J. Funct. Anal., 236 (2006), pp. 25–58.
- [55] L. ZALCMAN, *Some inverse problems of potential theory*, in Integral geometry (Brunswick, Maine, 1984), vol. 63 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1987, pp. 337–350.