Complex Variables, 1990, Vol. 13, pp. 239–250
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Printed in the United States of America

Hankel Forms on Multiply Connected Plane Domains. Part Two. The Case of Higher Connectivity

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The boundedness criterion for Hankel forms over a plane domain of connectivity two, derived in a previous paper (*Complex Variables* **10** (1988), 123–139), is extended to the case of higher connectivity. The main result is however that the relevant reproducing kernel extends meromorphically to the Schottky double of the domain, and this again implies the desired weak factorization.

AMS No. 30C40, 47B99
Communicated: R. P. Gilbert
(Received November 23, 1988)

0. INTRODUCTION

In [8] (see notably Appendix 1) a general theory of Hankel forms over a domain in $\mathbb{C}^d$ $(d \geqslant 1)$ was developed. The present paper belongs to a series of papers [11], [12], [7], where concrete illustrations to this theory are given. In particular, [12] ("Part One") was devoted to the case of plane domains $(d=1)$ of connectivity two. Now we turn to the case of higher connectivity. Whereas in [12] heavy use could be made of elliptic functions, the present treatment is less explicit, as no such marvelous tool is any more available.

Our basic idea is to invoke the Schottky $\hat{\Omega}$ of the domain in question, Ω. In particular, we prove that the reproducing kernels of the Hilbert spaces of interest to us, viz. the Dzhrbashyan (or weighted Bergman) spaces $A^{\alpha,2}(\Omega)$ (α integer $\geqslant 0$), regarded as differential forms, have meromorphic extensions to $\hat{\Omega}$. This allows us to prove the weak factorization of the kernels, needed in order to make work the general scheme in [8]. Our main result is the expected boundedness criterion for Hankel forms H_b with analytic symbol b in $A^{\alpha,2}(\Omega) \times A^{\alpha,2}(\Omega)$:

$$H_b \text{ is bounded} \Leftrightarrow \omega(z)^{\alpha+2}|b(z)| \leqslant C$$

where $ds = |dz|/\omega(z)$ is the Poincaré metric on Ω.

Probably, this result also extends with no great difficulty to the more general case of open Riemann surfaces bounded by finitely many analytic curves, not necessarily conformally equivalent to a bounded domain. But for simplicity we have presently restricted attention to the planar situation, because then we have also at our disposal the artifice of Schwarz functions (cf. [1]), making things appear more "global".

The plan of the paper is roughly the following. Sections 1 and 2 contain preparatory material about the Schottky double and the Schwarz function. The bulk of the paper constitutes Section 3 devoted to the aforementioned meromorphic continuation of the reproducing kernel (Theorem 1). In Section 4 the weak factorization of the kernel (Theorem 2) is effectuated. The application to the boundedness of Hankel forms (Theorem 3) is then given in the short Section 5. Finally, in Section 6 some open questions are mentioned. For example, the corresponding S_p-result ("trace ideal criterion") for Hankel form is still unproved.

1. SOME NOTATION

Let Ω be a bounded domain in $\mathbb{C}$ whose boundary consists of m disjoint closed analytic curves ($1 \leqslant m < \infty$).

Remark (on conformal invariance) As the theory we are interested in is conformally invariant, this assumption could have been relaxed considerably. On the other hand, various canonical conformal models exist for our domain, for instance, models where all the bounding curves are circles (see e.g. [5, pp. 481–488]). However, we have not been able to exploit this possibility in any essential way.

Let $\hat{\Omega}$ be the Schottky double of Ω. This is a compact Riemann surface of genus $m-1$. Set-theoretically, it is $\Omega \cup \partial\Omega \cup \tilde{\Omega}$, where $\tilde{\Omega}$ is a copy of Ω. We denote by j the map which to a point z in Ω assigns its counterpart in $\tilde{\Omega}$, often written $\tilde{z}$. The analytic structure is obtained as follows: First, let us agree that Ω has the analytic structure given by its embedding in $\mathbb{C}$ and that $\tilde{\Omega}$ is given the opposite analytic structure. Let now z_0 be any point of $\partial\Omega$. By the analyticity of the boundary $\partial\Omega$, we can represent $\partial\Omega$ near z_0 in the form $z = \phi(t)$ (with t real), where ϕ is an analytic function defined in a neighbourhood V of the origin 0, symmetric about the real axis, with $z_0 = \phi(0)$. We may assume that $\gamma(t) \in \Omega$ if $t \in V \cap U$; here and in the sequel $U = \{t \in \mathbb{C}\colon \operatorname{Im} t > 0\}$ stands for the *upper* (or *Poincaré*) *halfplane*. A local coordinate $\hat{t}$ on $\hat{\Omega}$ near z_0 can be defined by putting $\hat{t}(z) = t$ if z is in Ω and $\hat{t}(\tilde{z}) = \bar{t}$ if $\tilde{z} = j(z)$ is in $\tilde{\Omega}$, where in both cases $z = \phi(t)$.

It is clear that the map j extends to an antianalytic involution of Ω, $j^2 = \mathrm{id}$.

The universal covering surface of Ω is isomorphic to U. The Poincaré metric $ds = |dz|/\omega(z)$ on Ω is thereby induced from the corresponding metric $ds = |dt|/2 \operatorname{Im} t$ on U.

If $\psi\colon U \to \Omega$ is any universal covering map then ψ extends by reflexion to a covering map

$$\psi\colon U \cup (\text{an open subset of } \mathbb{R} \cup \{\infty\}) \cup \bar{U} \to \hat{\Omega}.$$

This gives rise to a so-called projective structure on $\hat{\Omega}$. Namely, we take as projective coordinates local inverses of the map ψ. Then changes of coordinates are always effectuated by projective transformations (i.e. Möbius transformations), which is exactly what having a projective structure amounts to (cf. [2], [4]).

Let κ be the canonical sheaf on $\hat{\Omega}$, i.e. (local) sections of κ are locally of the form $f = f(t)\,dt$, where t is any local coordinate on $\hat{\Omega}$. Similarly, sections of any power κ^n of κ are of the form $f = f(t)(dt)^n$ (forms of degree n or n-forms). Below (Section 2) we shall see that one can define in a canonical way a "square root" of κ, i.e. a holomorphic line bundle λ over $\hat{\Omega}$ such that $\lambda^2 = \kappa$. Then one can also speak of forms of halfinteger order. Especially (see [4]) one can define for each integer $\mu \geqslant 0$ the "Bol operator"

$$L_\mu \colon \lambda^{1-\mu} \to \lambda^{1+\mu},$$

mapping "integrals" into "differentials". If $f(t)$ is the coefficient of a section f of $\lambda^{1-\mu}$ in a projective coordinate t then the corresponding coefficient of $L_\mu f$ in $\lambda^{1+\mu}$ is simply $f^{(\mu)}(t)$.

Remark As will become clear from the definition of λ, the sheaf λ and all its powers are trivial over Ω. Therefore, sections defined over Ω may be identified with functions, using the variable z as a local coordinate.

Let α be an integer $\geqslant 0$. We define a metric for analytic functions defined over Ω by putting

$$(f, g)_\alpha = (\alpha + 1) \int_\Omega f\bar{g}\omega^\alpha \,dx\,dy/\pi,$$

$$\|f\|_\alpha^2 = (f, f)_\alpha.$$

We also set

$$(f, g)_{-1} = \int_{\partial\Omega} f\bar{g}|dz|/2\pi,$$

$$\|f\|_{-1}^2 = (f, f)_{-1}.$$

$A^{\alpha,2}(\Omega)$ $(\alpha \geqslant -1)$ is the Hilbert space of all analytic functions over Ω with $\|f\|_\alpha^2 < \infty$, known as the weighted Bergman (or Dzhrbazhyan) space if $\alpha \geqslant 0$ and as the Hardy space if $\alpha = -1$; $A^{-1,2}(\Omega) \equiv H^2(\Omega)$ in the habitual notation.

Elements in $A^{\alpha,2}(\Omega)$ will be viewed as forms of degree $\dfrac{\alpha+2}{2}$.

$A_e^{\alpha,2}(\Omega)$ is the set of all functions f in $A^{\alpha,2}(\Omega)$ which are of the form $L_\mu F = f$ for some analytic function F ("exact differentials"). It is easy to see that

$$A^{\alpha,2}(\Omega) = A_e^{\alpha,2}(\Omega) \oplus B(\Omega)$$

where $B(\Omega)$, the orthogonal complement of $A_e^{\alpha,2}(\Omega)$ in $A^{\alpha,2}(\Omega)$, consists of those elements in $A^{\alpha,2}(\Omega)$ which are restrictions to Ω of elements in $\Gamma(\hat{\Omega}, \mathcal{O}(\lambda^{\alpha+2}))$, the holomorphic forms on $\hat{\Omega}$ of degree $\dfrac{\alpha+2}{2}$.

Remark Using the Riemann–Roch theorem, the simple fact that $\Gamma(\hat{\Omega}, \mathcal{O}(\lambda)) = 0$ (see [3]) and the fact that $\mathcal{O}(\lambda^2) = \mathcal{O}(\kappa)$ has a nowhere-vanishing section when $m = 2$

the dimension of $B(\Omega)$ can easily be computed to be

$$\dim B(\Omega)=\begin{cases}0 & \text{if } m=1,\\ 0 & \text{if } m=2,\ \alpha \text{ odd},\\ 1 & \text{if } m=2,\ \alpha \text{ even},\\ \mathrm{m}-1 & \text{if } m\geqslant 3,\ \alpha=0,\\ (\alpha+1)(m-2) & \text{if } m\geqslant 3,\ \alpha\neq 0\end{cases}$$

$(\alpha=-1,0,1,\ldots)$.

2. MORE PRELIMINARIES

In the definition of the analytic structure on $\hat{\Omega}$ we can actually do with an "atlas" consisting of only two coordinate "charts", of the form (V_1, ϕ_1) and (V_2, ϕ_2), where V_1 is a neighbourhood of $\Omega \cup \partial\Omega$ in $\hat{\Omega}$ and V_2 a neighbourhood of $\tilde{\Omega} \cup \partial\Omega$ in $\hat{\Omega}$.

We set $\phi_1(z)=z$ if $z\in\Omega$; this map extends analytically to V_1, provided V_1 is sufficiently small. We then take $V_2=j(V_1)$ and $\phi_2=\overline{\phi_1\circ j}$; thus $\phi_2(\tilde{z})=\bar{z}$ if $\tilde{z}=j(z)$ is in $\tilde{\Omega}$.

The coordinate transition function

$$S=\phi_2\circ\phi_1^{-1}$$

is defined and holomorphic in the neighbourhood $\phi_1(V_1\cap V_2)$ of (the image of) $\partial\Omega$ in $\mathbb{C}$, and on $\partial\Omega$ it satisfies

(1) $$S(z)=\bar{z} \qquad (z\in\partial\Omega).$$

Thus $S(z)$ is the so-called Schwarz function for $\partial\Omega$ [1].

Example If $\partial\Omega$ contains a piece of a circle then near this piece one has

$$S(z)=\frac{r^2}{z-z_0}+\bar{z}_0$$

where z_0 is the center of the circle and r its radius. As a limiting case one obtains

$$S(z)=z$$

for the real axis.

Alternatively we may argue as follows. If f is any analytic function on Ω then f admits an analytic continuation to a neighbourhood of $\Omega\cup\partial\Omega$ in $\hat{\Omega}$ iff there exists an analytic function g defined and continuous in a "one sided" neighbourhood of $\partial\Omega$ such that

(2) $$f(z)=\overline{g(z)} \qquad \text{for} \quad z\in\partial\Omega.$$

Taking here $f(z)=z$ we clearly obtain (1).

Let $T(z)$, $z\in\partial\Omega$, denote the unit tangent vector to $\partial\Omega$ at z, oriented so that Ω lies to the left of it. Differentiation of (1) gives

(3) $$S'(z)=\frac{1}{T(z)^2}$$

for $z \in \partial\Omega$. This relation then also gives a holomorphic extension of $T(z)$ to a neighbourhood of $\partial\Omega$.

We can now also give a handy description of certain holomorphic line bundles (or "invertible" sheafs) over $\hat{\Omega}$. In particular we can, as promised (Section 1), define the square root λ of the canonical bundle (or sheaf) κ.

Any nowhere vanishing holomorphic function $m(z)$ defined in a neighbourhood (in $\mathbb{C}$) of $\partial\Omega$ defines a (holomorphic) line bundle ξ on $\hat{\Omega}$. A meromorphic section f of ξ is represented by a pair of meromorphic functions f_1 and f_2, f_j defined in $\phi_j(V_j)$, such that

(4) $$f_1(z) = f_2(S(z))m(z)$$

in a neighbourhood of $\partial\Omega$. For example, $m(z) \equiv 1$ gives the identity (trivial) bundle $\xi = 1$, whose sections are functions. Note that in this case (4) essentially reduces to (2) (cf. (5) below).

$$m(z) = S'(z)$$

gives the canonical bundle $\xi = \kappa$ whose sections are differentials (1-forms). Choosing

$$m(z) = \frac{1}{T(z)}$$

we get by (3) a bundle $\xi = \lambda$ satisfying $\lambda^2 = \kappa$. Sections of λ will be called $\frac{1}{2}$-forms or half-order differentials. More generally, sections of the bundle λ^s ($z \in \mathbb{Z}$), corresponding to $m(z) = T(z)^{-s}$, will be called $\frac{s}{2}$-forms.

Since both members of (4) are analytic it is enough that (4) holds on $\partial\Omega$. This gives rise to a slightly more convenient way to represent ξ-sections: a meromorphic ξ-section corresponds to a pair f and g of meromorphic functions in Ω, continuous up to $\partial\Omega$ and there satisfying (generalizing (2))

(5) $$f(z) = \overline{g(z)}m(z).$$

The relation to f_1 and f_2 is:

$$f(z) = f_1(z),$$
$$g(z) = \overline{f_2(\bar{z})}.$$

In the case $\xi = \lambda^s$ ($s \in \mathbb{Z}$) the relation (5) can also be written

(6) $$f(z) = \overline{g(z)}\,\overline{T(z)}^s \qquad (z \in \partial\Omega),$$

since $\overline{T(z)} = T(z)^{-1}$. A more suggestive way of writing (6) is

$$f(z)(dz)^{s/2} = \overline{g(z)}(\overline{dz})^{s/2} \qquad \text{along } \partial\Omega.$$

Remark Note that $\{(V_1, \phi_1), (V_2, \phi_2)\}$ still is an atlas for $\hat{\Omega}$ if e.g. V_1 is shrinked to $V_1 = \Omega$. Sometimes this kind of non-symmetric atlas is preferable because the functions $m(z)$ and $f_1(z)$ in the representation (4) then do not have to be defined outside Ω (and (4) is required to hold only in a one-sided neighbourhood (namely in Ω) of $\partial\Omega$).

3. THE KERNEL FUNCTION

Our study of the reproducing kernel of the space $A^{\alpha,2}(\Omega)$ will be based on the following result, of independent interest.

PROPOSITION 1 *Let F and g be holomorphic in Ω, smooth up to $\partial\Omega$. Then the following formula holds for $\mu \geqslant 1$*

$$\int_\Omega L_\mu F\bar{g}\omega^{\mu-1}\,dz\,d\bar{z} = (\mu-1)!\,(-1)^{\mu-1}\int_{\partial\Omega} F\bar{g}\left(\frac{\partial\omega}{\partial z}\right)^{\mu-1} d\bar{z} \tag{7}$$

$$= (\mu-1)!\,i^{\mu-1}\int_{\partial\Omega} F\bar{g}(dz)^{\frac{1-\mu}{2}}(d\bar{z})^{\frac{1+\mu}{2}}.$$

Here $(dz)^{\frac{1-\mu}{2}}(d\bar{z})^{\frac{1+\mu}{2}}$ shall be interpreted as $\bar{T}^{\mu-1}\,d\bar{z} = \bar{T}^{\mu}|dz| = \bar{T}^{\mu+1}\,dz$. (7) can also be written as

$$(L_\mu F, g)_\alpha = \mu!\,i^\mu \int_{\partial\Omega} F\bar{g}(dz)^{\frac{1-\mu}{2}}(d\bar{z})^{\frac{1+\mu}{2}}/2\pi, \tag{7'}$$

where $\alpha = \mu - 1$ and is then formally valid also for $\mu = 0$ ($\alpha = -1$).

Proof We first remark that both sides of (7) have an invariant meaning even if F is not holomorphic; as ω transforms as a form of bidegree $(-\frac{1}{2}, -\frac{1}{2})$, we have only to make sure that F transforms as a form of bidegree $\left(\dfrac{1-\mu}{2}, 0\right)$. Therefore we can allow F to be smooth on $\Omega \cup \partial\Omega$ with its support contained in a small neighbourhood of a boundary point, and then we can work in terms of a projective coordinate t. We then have to verify the formula

$$\int_U \frac{\partial^\mu F}{\partial t^\mu}\bar{g}(2\,\mathrm{Im}\,t)^{\mu-1}\,dt\,d\bar{t} = (\mu-1)!\,i^{\mu-1}\int_{\mathbb{R}} F\bar{g}\,dt, \tag{7''}$$

which is easily done by μth fold partial integration. The general case is proved readily using a partition of unity. ∎

Remark Another proof of this proposition can be found in [4].

We mention also the following comparison result, which will not be used in this paper but may be of independent interest.

PROPOSITION 2 *Let F and G be holomorphic in Ω, smooth up to $\partial\Omega$. Then the integral*

$$\int_\Omega F\bar{G}\omega^\alpha\,dx\,dy/\pi \overset{\text{fed}}{=} ((F, G))_\alpha$$

is an analytic function of $\alpha \in \mathbb{C}$ for $\mathrm{Re}\,\alpha > -1$ and it has a meromorphic extension to $\mathbb{C}\backslash\{-1, -2, \ldots\}$ with

$$\operatorname*{Res}_{\alpha=-\mu-1}((F, G))_\alpha = \frac{(-i)^\mu}{\mu!}\int_{\partial\Omega} F\overline{L_\mu G}(dz)^{\frac{1-\mu}{2}}(d\bar{z})^{\frac{1+\mu}{2}}/2\pi \qquad (\mu = 0, 1, 2, \ldots). \quad ∎$$

We are now ready for the following theorem, which is, maybe, the main result of the whole paper, not only of this section.

THEOREM 1 *Let α be an integer $\geqslant -1$ and consider any subspace H of $A^{\alpha,2}(\Omega)$ contained in the interval*

$$A_e^{\alpha,2}(\Omega) \subset H \subset A^{\alpha,2}(\Omega),$$

with the reproducing kernel $k(z,\zeta)=k_\zeta(z)$. Then $k(z,\zeta)$ extends as a meromorphic $\frac{\alpha+2}{2}$-form in z and an anti-meromorphic $\frac{\alpha+2}{2}$-form in ζ to all of $\hat{\Omega}\times\hat{\Omega}$, with a pole of order (exactly) $\alpha+2$, the polar division being defined by the equation $z=j(\zeta)$.

Proof Set $\mu=\alpha+1$ and assume first that $k(z,\zeta)$ is smooth up to $\partial\Omega$ (in z). Then by Proposition 1 and the fact that $k(z,\zeta)$ reproduces on $A_e^{\alpha,2}(\Omega)$

$$L_\mu F(\zeta) = \mu!\, i^\mu \int_{\partial\Omega} F(z)\overline{k(z,\zeta)}\,\overline{T(z)}^{\mu+1}\, dz/2\pi \tag{8}$$

for every F holomorphic in Ω and smooth up to $\partial\Omega$. On the other hand, the Cauchy formula gives for those F

$$L_\mu F(\zeta) = \frac{1}{2\pi i}\int_{\partial\Omega} F(z) L_\mu\left(\frac{1}{z-\dot{\zeta}}\right) dz. \tag{9}$$

(The dot indicates the variable on which the operator L_μ acts; this notation will be employed in the sequel too.) Subtracting (8) and (9) it follows that the function $g(z)=g(z,\zeta)$ defined on $\partial\Omega$ by

$$g(z,\zeta)=\overline{k(z,\zeta)}\,\overline{T(z)}^{\mu+1}-\frac{(-1)^\mu i^{\mu+1}}{\mu!}L_\mu\left(\frac{1}{z-\dot{\zeta}}\right) \tag{10}$$

satisfies $\int_\Omega Fg\,dz=0$ for all F as above and hence extends to a holomorphic function (in z) in Ω. Set

$$h(z,\zeta)=\frac{(-1)^\mu i^{\mu+1}}{\mu!}L_\mu\left(\frac{1}{z-\dot{\zeta}}\right)+g(z,\zeta) \tag{11}$$

$(z,\zeta\in\Omega)$. Writing $\zeta=\psi(t)$, where t is a projective coordinate, we have

$$L_\mu\left(\frac{1}{z-\dot{\zeta}}\right)=\psi'(t)^{-\frac{1+\mu}{2}}\frac{d^\mu}{dt^\mu}\left(\frac{\psi'(t)^{\frac{1-\mu}{2}}}{z-\psi(t)}\right).$$

It follows that $h(z,\zeta)$ is meromorphic in z with the singular part of the form

$$\sum_{k=0}^{\mu}\frac{a_k(\zeta)}{(z-\zeta)^{k+1}}$$

where $a_k(\zeta)$ are functions holomorphic in a neighbourhood of $\bar{\Omega}$ and $a_\mu(\zeta)=-(-i)^{\mu+1}=\text{constant}\neq 0$.

By (10)

$$h(z,\zeta)=\overline{k(z,\zeta)}\,\overline{T(z)}^{\mu+1}\qquad(z\in\partial\Omega). \tag{12}$$

Thus $h(z,\zeta)$, the "adjoint kernel" of $k(z,\zeta)$, represents the extension (in z) of $k(z,\zeta)$ as a meromorphic $\frac{\mu+1}{2}$-form to the Schottky double $\hat{\Omega}$. We can express this by writing

$$k(\cdot,\zeta)\in\Gamma(\hat{\Omega},\mathcal{O}(\lambda^{\mu+1}-(\mu+1)\tilde{\zeta}))\qquad(\zeta\in\Omega).$$

Let us now indicate how the assumption that $k(z,\zeta)$ is smooth up to the boundary can be removed. It is easy to prove that $\mathcal{O}(\bar{\Omega})$, the set of functions holomorphic in a neighbourhood of $\bar{\Omega}$, is dense in $A^{\alpha,2}(\Omega)$ (when $\partial\Omega$ is analytic). (Cf. [13] for the case $\alpha=-1$.) Hence there is a sequence $k_n(z)=k_n(z,\zeta)$ in $\mathcal{O}(\bar{\Omega})$ converging to $k(z,\zeta)$ in $A^{\alpha,2}(\Omega)$. For each k_n Proposition 1 is valid, so that

$$(L_\mu F,k_n)_\alpha=\mu!\,i^\mu\int_{\partial\Omega}F\bar{k}_n(dz)^{\frac{1-\mu}{2}}(d\bar{z})^{\frac{1+\mu}{2}}/2\pi.$$

For V a sufficiently small neighbourhood of $\bar{\Omega}$ and $F\in\mathcal{O}(\bar{V})$ the integral on the right-hand side above can be written

$$\int_{\partial\Omega}F(z)\overline{k_n(z)}T(z)^{-(\mu+1)}\,dz=\int_{\partial\Omega}F(z)\overline{k_n(\overline{S(z)})}T(z)^{-(\mu+1)}\,dz$$

$$=\int_{\partial V}F(z)\overline{k_n(\overline{S(z)})}T(z)^{-(\mu+1)}\,dz.$$

When $z\in\partial V$ then $\overline{S(z)}$ belongs to a compact subset of Ω. It follows that as $n\to\infty$ $k_n(\overline{S(z)})$ converges uniformly on ∂V to $k(\overline{S(z)})$. Since on the other hand $(L_\mu F,k_n)_\alpha\to(L_\mu F,k)_\alpha$ we obtain

$$L_\mu F(\zeta)=\mu!\,i^\mu\int_{\partial V}F(z)\overline{k(\overline{S(z)},\zeta)}T(z)^{-(\mu+1)}\,dz/2\pi, \tag{8'}$$

valid for all $F\in\mathcal{O}(\bar{V})$.

Comparing with (9) with $\partial\Omega$ replaced by ∂V, it follows that the holomorphic function $g(z)=g(z,\zeta)$ defined in some neighbourhood of ∂V by

$$g(z)=\overline{k(\overline{S(z)},\zeta)}T(z)^{-(\mu+1)}-\frac{(-1)^\mu i^{\mu+1}}{\mu!}L_\mu\left(\frac{1}{z-\zeta}\right) \tag{10'}$$

satisfies $\int_{\partial V}Fg\,dz=0$ for all $F\in\mathcal{O}(\bar{V})$. This shows that g extends holomorphically to all V and the extension of k to $\hat{\Omega}$ now follows as in the Remark at the end of Section 2. (Taking the complex conjugate of (10') and using $\overline{S(z)}$ as independent variable (10') takes the form (4).)

Now $k(z,\zeta)$ is defined on $\hat{\Omega}\times\Omega$. By the elementary symmetry

$$k(z,\zeta)=\overline{k(\zeta,z)}$$

$k(z, \zeta)$ then also extends, as an anti-meromorphic $\frac{\mu+1}{2}$-form in ζ, to $\Omega \times \hat{\Omega}$. The remaining extension to all of $\hat{\Omega} \times \hat{\Omega}$ follows now automatically from Hartog's theorem (see e.g. [6, Thm. 2.3.2]).

In fact, the missing part of $\hat{\Omega} \times \hat{\Omega}$ is just $K = (\tilde{\Omega} \cup \partial\Omega) \times (\tilde{\Omega} \cup \partial\Omega)$, which is a compact subset of $V_2 \times V_2 \subset \hat{\Omega} \times \hat{\Omega}$ with $(V_2 \times V_2)\backslash K$ connected, and $V_2 \times V_2$ can be (holomorphically) identified with an open subset of $\mathbb{C}^2$ (e.g. the set $\phi_2(V_2) \times \phi_2(V_2)$). By Hartog's theorem every holomorphic function in $(V_2 \times V_2)\backslash K$ extends holomorphically to all $V_2 \times V_2$. The fact that $k(z, \zeta)$ is anti-analytic instead of analytic in ζ of course does not make any difference (just consider the "opposite" analytic structure on the second component); also, the singular part has to be subtracted off before applying the theorem, just observing that this singular part itself is analytic-antianalytic on $V_2 \times V_2$. ■

Theorem 1 has the following immediate corollary.

COROLLARY 1 *Near $\partial\Omega$ we have*

$$k(z, z) \sim \delta(z)^{-(\alpha+2)} \qquad (z \in \Omega) \tag{14}$$

(*i.e.* $c_1\delta(z)^{-(\alpha+2)} \leqslant k(z, z) \leqslant c_2\delta(z)^{-(\alpha+2)}$, $\quad 0 < c_1 < c_2 < \infty$) *where* $\delta(z) = \operatorname{dist}(z, \partial\Omega)$.

Proof We consider $k(z, \zeta)$ in the neighbourhood $\phi_1(V_1) \times \phi_1(V_1)$ of $(\Omega \cup \partial\Omega) \times (\Omega \cup \partial\Omega)$ in $\mathbb{C}^2$. For z and ζ close to $\partial\Omega$ $k(z, \zeta)$ has, by Theorem 1, a singularity with the leading term

$$\frac{c(z)}{(S(z) - \bar{\zeta})^{\alpha+2}} \tag{15}$$

where $c(z)$ is holomorphic and non-vanishing.

In fact, if e.g. $\zeta \in \Omega$ then (12) shows that the extension of $k(z, \zeta)$ from Ω to $\phi_1(V_1) \subset \mathbb{C}$ (in z) is given by

$$k(z, \zeta) = \overline{h(\overline{S(z)}, \zeta)}T(z)^{-(\mu+1)} \qquad (z \in \phi_1(V_1)\backslash\Omega). \tag{16}$$

(Note that both members of (16) are holomorphic in z and that (16) reduces to (12) when $z \in \partial\Omega$.) By (11), (16) we get

$$\begin{aligned} k(z, \zeta) &= \overline{\frac{(-1)^\mu i^{\mu+1}}{\mu!} L_\mu\left(\frac{1}{\overline{S(z)} - \zeta}\right)} T(z)^{-(\mu+1)} + \text{function holomorphic in } z \\ &= -i^{\mu+1} \frac{T(z)^{-(\mu+1)}}{(S(z) - \bar{\zeta})^{\mu+1}} + \text{lower order terms} \end{aligned}$$

for $\zeta \in \Omega$, z in a neighbourhood of $\partial\Omega$.

This expression, which necessarily is valid in a full neighbourhood of $\partial\Omega \times \partial\Omega$ gives the exact form of the singularity.

Clearly the corollary follows from (15), since $\frac{1}{2}(S(z) - \bar{z}) \sim \delta(z)$ near $\partial\Omega$ (observe that map $z \to \overline{S(z)}$ is the anti-analytic reflection in $\partial\Omega$). ■

4. WEAK FACTORIZATION

Let now k and l be the reproducing kernels in $A^{\alpha,2}(\Omega)$ and $A^{\beta,2}(\Omega)$ respectively, where $\beta = 2\alpha + 2$.

THEOREM 2 *Let* $\zeta \in \Omega$. *There exist finitely many holomorphic functions* u_ν *and* v_ν $(\nu = 1, \ldots, N)$ *such that*

$$l(z,\zeta) = \sum_{\nu=1}^{N} u_\nu(z) v_\nu(z) \tag{17}$$

and, uniformly in ζ,

$$\sum_{\nu=1}^{N} \|u_\nu\|_{A^{\alpha,2}(\Omega)} \|v_\nu\|_{A^{\alpha,2}(\Omega)} = O(k(\zeta,\zeta)) = O\left(\frac{1}{\omega(\zeta)^{\alpha+2}}\right). \tag{18}$$

Proof It suffices to prove the theorem when ζ is sufficiently close to the boundary $\partial\Omega$. Applying Theorem 1 (with α replaced by β) to the function l we see that we can write

$$l(z,\zeta) = \sum_{\nu=1}^{\beta+2} \frac{c_\nu}{(z - \overline{S(\zeta)})^\nu} + r(z)$$

where the c_k are constants and $r(z)$ is a holomorphic function in some neighbourhood of Ω, all uniformly bounded in ζ. Therefore we obtain formula (17) with $N = \beta + 3$ if we take, say

$$u_\nu = \frac{1}{(z - \overline{S}(\zeta))^{[\nu/2]}}, \qquad v_\nu = \frac{c_\nu}{(z - \overline{S}(\zeta))^{\nu - [\nu/2]}}, \qquad (\nu = 1, \ldots, \beta + 2),$$

$$u_\nu = r, \qquad v_\nu = 1 \qquad (\nu = \beta + 3).$$

It is easy to check that

$$\left\| \frac{1}{(z - \overline{S(\zeta)})^k} \right\|^2_{A^{\alpha,2}(\Omega)} \sim \delta(\zeta)^{\alpha+2-2k}$$

and using Corollary 1 (18) now follows. (Observe that $\omega(\zeta) \sim \delta(\zeta)$ near the boundary.) ■

Remark At an early stage of this investigation we throught that one could here (as in the case $m = 2$ [12]) do with factors u_ν and v_ν which, considered as forms, extend to meromorphic objects in the whole Schottky double $\hat{\Omega}$ of Ω, not only a small neighbourhood of $\Omega \cup \partial\Omega$, this utilizing the standard facts (Riemann–Roch theorem etc.) about compact Riemann surfaces. However, we have not been able to prove this except in some additional assumptions. Nor do we know what the deeper implications, if any, of such an improved factorization might be.

Remark Let us remark that Ewa Ligocka has established, on one of the authors' request [10], the weak factorization of the corresponding kernel in the case of strictly pseudoconvex domain in $\mathbb{C}^d$. This follows from standard facts about the Bergman kernel for such domain (Boutet de Monvel–Sjöstrand etc.) combined with the so-called Rudin–Forelli construction, which with the aid of an ascent in dimension allows one to reduce to the case of the Bergman kernel ($\alpha = 0$) (see e.g. [14, Chap. 7]).

5. APPLICATION TO HANKEL FORMS

Finally we are ready to give the standard application to boundedness of Hankel forms.

By the Hankel form Γ_b over $A^{\alpha,2}(\Omega)$ with symbol b, an analytic function, we mean the bilinear form

$$\Gamma_b(f, g) = \int_\Omega \overline{b(z)} f(z) g(z) \omega(z)^\beta \, dx \, dy/\pi .$$

$$(\beta = 2\alpha + 2,\ f, g \in A^{\alpha,2}(\Omega).)$$

THEOREM 3 *Γ_b is bounded iff $b(z) = O(\omega(z)^{-(\alpha+2)})$.*

Proof This is entirely standard from Theorem 2, so we omit the details referring instead to [12] or, in fuller generality, [8], especiall Appendix 1. ■

6. SOME OPEN QUESTIONS

We conclude by stating some open questions, more or less lightly connected with the topics of this paper; some of them have already been alluded to in the text.

1. The S_p-theory. To obtain an S_p-criterion, rather than a mere boundedness result (Theorem 3), for Hankel forms one requires (cf. [8, Appendix 1]) an estimate for the inverse of the linear operator whose kernel is $(k(z, \zeta))^2$ is requested. Perhaps our weak factorization (Theorem 2) could be helpful in this context, because it also single out a leading term in $(k(z, \zeta))^2$ corresponding to an invertible operator ($=$ the identity).
2. The case when α is not an integer. In this case we expect k to have an essential singularity of the type $(z - \zeta)^{-(\alpha+2)}$ on the Schottky double (and hence will not be single-valued).
3. Already in Section 4 we mentioned the question whether it was not possible to have a weak factorization $l(z, \zeta) = \sum u_\nu(z) v_\nu(z)$ where all the factors u_ν and v_ν are meromorphic on the double, not only in a neighbourhood of the boundary.
4. The extension to open Riemann surfaces bounded by finitely many analytic curves. This we alluded to already in the Introduction.
5. Extending Pekarskiĭ's theory of best rational approximation (see e.g. [9]). In the present context it is natural to try to approximate analytic functions $f(z)$ in a domain Ω by finite linear combinations $\sum_{\nu=1}^{n} a_\nu k(z, \zeta_\nu)$ ($\zeta_\nu \in \Omega$) of reproducing kernels. We conjecture that one has results analogous to Pekarskiĭ's ($\Omega =$ unit disk) in this case too.
6. The theory of Hankel forms (operators) has formal analogies with "quantization" (operator calculi). In particular, this suggests that there is perhaps a possibility to construct a kind of analogue of the operator calculus of A and S. Unterberger

(see e.g. [15] and the references given there) in the present context of multiply connected domains.

7. Comparison with the case $m=2$ where an explicit computation is possible (cf. [12]) suggests that if $m>2$ the reproducing kernel in a $A_e^{\alpha,2}(\Omega)$, say $k_e=k_e(z,\zeta)$, can be written in the form

$$k_e(z,\zeta)=L_\mu Z(z,\zeta)$$

where $Z=Z(z,\zeta)$ is a function holomorphic (in z) over Ω which (for ζ fixed) has a meromorphic continuation across the boundary $\partial\Omega$; Z may thus be viewed as an analogue of the Weierstrass ζ-function in the theory of elliptic functions. But as it has a single pole at $\bar{\zeta}$ the continuation cannot in general be single-valued on $\hat{\Omega}$. Therefore Z should rather be considered on a suitable covering surface of Ω. Can Z be characterized in a more intrinsic way?

8. Observe that (14) also can be written

$$k(z,z)\sim\omega(z)^{-(\alpha+2)} \tag{14'}$$

and then has an invariant meaning (hence holds without the assumption of analyticity of $\partial\Omega$). Is there a more direct proof of (14′) which works for more general weight functions $\omega(z)$ (and in particular for non-integer values of α)?

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