APPLICATION OF HALF-ORDER DIFFERENTIALS ON RIEMANN SURFACES TO QUADRATURE IDENTITIES FOR ARC-LENGTH

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0. Introduction and notations

The topic of this paper is so-called quadrature domains and quadrature identities for arc-length. A domain Ω in the complex plane (or sometimes in the Riemann sphere) is called a *quadrature domain* for arc-length (the phrase "for arc-length" will henceforth in this paper usually be understood) if there exist finitely many points $z_1, \ldots, z_m$ in Ω and complex numbers a_{kj} ($0 \leqq j \leqq n_k - 1$ say, $1 \leqq k \leqq m$) such that

$$\int_{\partial\Omega} f|dz| = \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} a_{kj} f^{(j)}(z_k) \tag{0.1}$$

for every f in some suitable test class $\Lambda(\Omega)$ of analytic functions in Ω. The identity (0.1) is then called a *quadrature identity* (for arc-length).

Of course, certain assumptions on Ω and $\Lambda(\Omega)$ are needed in order for (0.1) to make sense. In this paper the assumptions on Ω will generally be that $\partial\Omega$ has finitely many components each of which is a continuum of finite one-dimensional Hausdorff measure (the phrase "Ω is bounded by finitely many rectifiable continua" will be used). The most natural choice for $\Lambda(\Omega)$ turns out to be the Hardy space $E^1(\Omega)$, although it usually will be more convenient to work with the corresponding Hilbert space $E^2(\Omega)$. In these cases (0.1) will be found to make sense. (The above things are elaborated in Section 1 of the paper.)

The principal example of a quadrature domain is any disc Ω, in which case

$$\int_{\partial\Omega} f|dz| = af(z_0) \qquad \text{for all } f \in E^1(\Omega), \tag{0.2}$$

where $a = |\partial\Omega|$ (the length of $\partial\Omega$) and z_0 is the midpoint of Ω. (0.2) also holds if Ω is the exterior in the Riemann sphere of any disc, in which case $z_0 = \infty$. With the *a priori* assumptions on Ω indicated above no other domains Ω satisfy (0.2). See [3, Thm 21] and Remarks 3.4 and 6.1 in the present paper.

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One reason for investigating quadrature domains for arc-length in general is that they appear as image domains for the solutions of certain extremal problems for univalent functions. See [14]. It also turns out that the property of a domain Ω of being a quadrature domain has a hydrodynamical interpretation, namely that there exists a steady two-dimensional flow of an ideal fluid in Ω with certain (nonmovable) singularities at the points $z_1, \ldots, z_m$ in (0.1) such that $\partial\Omega$ is a free stream-line for the flow. See [14]. There is also an interpretation in two-dimensional potential theory: if $n_k = 1$ and $a_k > 0$ for all k then (0.1) implies that the gravitational field produced by the point masses a_k at z_k coincides, outside $\bar{\Omega}$, with the gravitational field of a uniform mass distribution on $\partial\Omega$. (Choose $f(z) = 1/(z-\zeta)$ for $\zeta \in \mathbf{C}\setminus\bar{\Omega}$ in (0.1).)

Quadrature identities of the kind (0.1) have earlier been considered in [3] and [14]. In [14] the simply connected quadrature domains are described, briefly as follows:

Let $g: \mathbf{D} \to \Omega$ be a Riemann mapping function ($\mathbf{D}$ the unit disc) and assume that Ω is a Smirnov domain with rectifiable boundary. Then Ω is a quadrature domain with the polynomials (or, equivalently in this case, $E^1(\Omega)$) as test class if and only if

$$g' = R^2 \tag{0.3}$$

for some rational function R.

Under the further assumption on Ω that $\partial\Omega$ has a continuously turning oriented unit tangent vector $T(z)$ (so that $dz = T(z)\,|dz|$ along $\partial\Omega$) the following characterization of the quadrature property is also found: Ω is quadrature domain for the test class $\Lambda(\Omega) = \{$functions holomorphic in a neighbourhood of $\bar{\Omega}\}$ if and only if there is a meromorphic function $H(z)$ in Ω with

$$H(z) = \overline{T(z)} \qquad \text{on } \partial\Omega. \tag{0.4}$$

Such a characterization was also obtained in [3], in the multiply connected case.

Most of the present paper is based on an idea obtained by reinterpreting (0.4) as follows. From the definition of $T(z)$ one obtains $T(z)^2 = dz^2/|dz|^2 = dz/d\bar{z}$ so that

$$\overline{T(z)} = \frac{1}{T(z)} = \sqrt{\frac{d\bar{z}}{dz}} \qquad \text{on } \partial\Omega$$

for a certain branch of the square-root. Therefore (0.4) can formally be written as

$$H(z)\sqrt{dz} = \overline{\sqrt{dz}} \qquad \text{along } \partial\Omega. \tag{0.5}$$

Let $\hat{\Omega}$ denote the Schottky double of Ω, i.e. the compact Riemann surface obtained by completing Ω with a back side $\tilde{\Omega}$ (a copy of Ω provided with the opposite conformal structure). Thus $\hat{\Omega} = \Omega \cup \partial\Omega \cup \tilde{\Omega}$. In terms of $\hat{\Omega}$ (0.5) has the following interpretation: the half-order differential $\sqrt{dz}$ on Ω extends over $\partial\Omega$ to a meromorphic half-order differential on $\hat{\Omega}$, represented on $\tilde{\Omega}$ by $H(z)\sqrt{dz}$. The concept of a half-order differential is made precise in Section 2.

The above interpretation of (0.5) gives rise to a generalization of (0.3) to multiply connected domains Ω: Let W be a standard domain of desired conformal type, let $\hat{W} = W \cup \partial W \cup \tilde{W}$ be its Schottky double and let $g : W \to \Omega$ be a conformal map. Then Ω is a quadrature domain (for the test class $E^2(\Omega)$) if and only if $\sqrt{dg}$ extends to a meromorphic half-order differential on $\hat{W}$, i.e. if and only if there exists a meromorphic half-order differential $R\sqrt{dz}$ on $\hat{W}$ such that

$$\sqrt{dg} = R\sqrt{dz} \qquad \text{in } W \tag{0.6}$$

(cf. (0.3)). This is Theorem 3.2, our most basic result.

In Section 4 we show that, given W, there always exist univalent functions g on W having the property (0.6). Thus there exist quadrature domains of all conformal types under consideration. In Section 5 we show that when a $(p + 1)$-connected domain satisfies a quadrature identity (0.1) (for the test class $\mathring{E}^2(\Omega)$) then there is in general a whole p-parameter family of $(p + 1)$-connected domains satisfying the same identity (0.1) (i.e. with the functional in the right hand side of (0.1) the same).

In Section 6 we consider quadrature domains in the Riemann sphere containing the point at infinity, and in Section 7, finally, we treat quadrature domains from a completely different point of view. To be specific, it is a simple consequence of (0.1) (with $\Lambda(\Omega) = E^1(\Omega)$) that

$$\int_{\partial\Omega} fg\,|dz| = \sum_{k=1}^{n} a_k(f)b_k(g) \tag{0.7}$$

for all $f, g \in E^2(\Omega)$ and for suitable linear functionals $a_1, \ldots, b_n$ on $E^2(\Omega)$. In Section 7 we prove that, conversely, having an identity of the kind (0.7) implies that Ω is a quadrature domain (for $E^1(\Omega)$).

It should be remarked that we will usually express the property of being a quadrature domain in a slightly different manner compared to (0.1). Let z_k and n_k be as in (0.1) and form the divisor $D = \Sigma_{k=1}^{m} n_k(z_k)$ (formal linear combination).

Assuming that the z_k are distinct we may define, for any linear space $\Lambda(\Omega)$ of holomorphic functions in Ω,

$$\Lambda_D(\Omega) = \{f \in \Lambda(\Omega) : f \text{ has a zero of order at least } n_k \text{ at } z_k \text{ for each } k = 1, \ldots, m\}.$$

Then it follows by elementary functional analysis that (0.1) holds for all $f \in \Lambda(\Omega)$, for *some* set of coefficients $\{a_{kj}\}$ if and only if

$$\int_{\partial\Omega} f|dz| = 0 \qquad \text{for all } f \in \Lambda_D(\Omega). \tag{0.8}$$

Thus Ω is a quadrature domain for the test class $\Lambda(\Omega)$ iff (0.8) holds for some positive divisor D. The identity (0.8) will also, as well as (0.1), be called a quadrature identity.

It should also be remarked that the two test classes $\Lambda(\Omega)$ used in this paper, namely $E^1(\Omega)$ and $E^2(\Omega)$, are equivalent under our assumptions on Ω. For when Ω is bounded by finitely many rectifiable continua, $E^2(\Omega) \subset E^1(\Omega)$. Thus if Ω is a quadrature domain for $E^1(\Omega)$ it is so also for $E^2(\Omega)$. On the other hand we prove (Corollary 3.1) that if Ω is a quadrature domain for $E^2(\Omega)$ then Ω is bounded by analytic curves without local singularities and in this case $E^2(\Omega)$ is dense in $E^1(\Omega)$. Hence Ω will be a quadrature domain also for $E^1(\Omega)$.

There is a slight overlap of the present paper with [3] (see the end of Section 3 for some details) but generally speaking the theory in [3] is developed along different lines than here.

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Some notations and terminology used

$\mathbf{D}(a; r) = \{z \in \mathbf{C} : |z - a| < r\}$,
$\mathbf{D} = \mathbf{D}(0; 1)$,
$\mathbf{P} = \mathbf{C} \cup \{\infty\}$ = the Riemann sphere.
domain: open connected and non-empty subset of a Riemann surface,
analytic = holomorphic (about functions etc.).
conformal map: a map between two Riemann surfaces which is analytic, one-to-one and onto.
continuum: a closed connected set consisting of more than one point.
analytic curve: the image of $\partial\mathbf{D} = \{z \in \mathbf{C} : |z| = 1\}$ under a non-constant analytic map φ defined in some neighbourhood of $\partial\mathbf{D}$ (and with values in a Riemann surface).
locally regular analytic curve: as "analytic curve" but with the additional requirement that $\varphi' \neq 0$ on $\partial\mathbf{D}$.
regular analytic curve: as "locally regular analytic curve" but with the additional requirement that φ shall be one-to-one on $\partial\mathbf{D}$ (and thus univalent in a neighbourhood of $\partial\mathbf{D}$).

$\operatorname{diam} E = \sup\{|z-\zeta| : z, \zeta \in E\}$ for $E \subset \mathbf{C}$.

If Ω is a domain in $\mathbf{C}$ or in $\mathbf{P}$ $\partial\Omega$ generally denotes the boundary of Ω in $\mathbf{P}$. Also $\bar{\Omega} = \Omega \cup \partial\Omega$.

A *divisor* D on a Riemann surface Ω is a finite formal linear ombination of the kind

$$D = \sum_{j=1}^{m} n_j \cdot (z_j) \tag{0.9}$$

with $n_j \in \mathbf{Z}$ (the integers) and $z_j \in \Omega$. Assuming henceforth that the z_j in (0.9) are distinct D is *positive* if $n_j \geqq 0$ for all j. The set of divisors in Ω form an Abelian group under addition in the obvious way. $D_1 \leqq D_2$ means that $D_2 - D_1$ is a positive divisor. If E is a subset of Ω the restriction of D in (0.9) to E, D_E, is the divisor obtained from D by deleting those terms $n_j(z_j)$ in (0.9) for which $z_j \notin E$.

The *degree* $\deg D$ of the divisor (0.9) is defined by

$$\deg D = \sum_{j=1}^{m} n_j.$$

If f is a meromorphic function in Ω not identically zero we define

$$\operatorname{Div} f = \sum_{z \in \Omega} n_z \cdot (z) \tag{0.10}$$

(formal linear combination), where $n_z \in \mathbf{Z}$ is defined by

$$f(\zeta) = a_0(\zeta - z)^{n_z} + a_1(\zeta - z)^{n_z+1} + \cdots, \qquad a_0 \neq 0$$

for ζ close to z. Similarly for meromorphic differentials etc. In general, $\operatorname{Div} f$ is not a divisor in our sense since the linear combination in (0.10) may be infinite if Ω is not compact. Nevertheless, statements such as $\operatorname{Div} f \geqq D$ etc. make obvious sense.

$\tilde{J}, \tilde{z}, \tilde{E}, \tilde{D}, \tilde{f}, \lambda$ and other notations related to symmetric Riemann surfaces and half-order differentials are defined in Section 2. See in particular the Conventions there.

The spaces $E^p(\Omega)$, $E^p_{\mathfrak{H}}(\Omega)$ are defined in Section 1 and the spaces $E(\Omega)$, $E_D(\Omega)$ both in Section 1 (1.8) and Section 2 (from different points of view).

1. Preliminaries on E^p-spaces

Lemma 1.1. *Let g be a univalent (i.e. one-to-one) meromorphic function in an arbitrary domain $\Omega \subset \mathbf{P}$. Then there exists a single-valued branch of $\sqrt{g'}$ in Ω.*

Proof. It is easy to check that $\sqrt{g'}$ exists locally everywhere, due to the local univalence of g. Since $\sqrt{g'} = \exp(\frac{1}{2}\log g')\sqrt{g'}$ is single-valued in Ω if and only if, for every simple closed oriented curve γ in Ω not passing through ∞ or the possible pole of g,

$$\int_\gamma d(\log g') \in 4\pi i \mathbf{Z}$$

(instead of just $2\pi i\mathbf{Z}$).

But a simple computation shows that $\int_\gamma d(\log g') = 0$ if g preserves the orientation (in $\mathbf{C}$) of γ, $\int_\gamma d(\log g') = \pm 4\pi i$ otherwise. Hence the lemma follows. □

Remark 1.1. It is not true that e.g. $(g')^{1/3}$ or $\log g'$ exists in general. The only powers $(g')^\alpha$ which exist in general are those with $2\alpha \in \mathbf{Z}$. However, if both Ω and $D = g(\Omega)$ contain the point $\infty \in \mathbf{P}$ and $g(\infty) = \infty$, or if none of them contain ∞ and g maps the outer component of $\partial\Omega$ onto the outer component ∂D, then $\log g'$ (and hence all powers of g') exists.

Nor is it true that the assumption of univalence for g can be replaced by that of local univalence. $g(z) = z^2$ in $\Omega = \{z \in \mathbf{C} : 1 < |z| < 2\}$ is a counterexample. □

Let $\Omega \subset \mathbf{P}$ be an arbitrary domain. There are two standard ways of generalizing the Hardy spaces H^p of analytic functions in the unit disc to Ω. The resulting spaces of analytic functions in Ω are denoted $H^p(\Omega)$ and $E^p(\Omega)$ respectively and are defined as follows (see [5], [10], [15], [17] for more details).

$f \in H^p(\Omega)$ $(1 \leqq p < \infty)$ if and only if $|f|^p$ has a harmonic majorant in Ω. $f \in E^p(\Omega)$ $(1 \leqq p < \infty)$ if and only if there exists an increasing sequence of domains D_n in Ω with $\bigcup D_n = \Omega$ and with ∂D_n consisting of finitely many rectifiable Jordan courves such that

$$\limsup_{n \to \infty} \int_{\partial D_n} |f|^p \, |dz| < \infty. \tag{1.1}$$

(The sequence $\{D_n\}$ may depend upon f.)

Sometimes (e.g. in [5]) it is required in the definition of $E^p(\Omega)$ that the lengths of ∂D_n shall be uniformly bounded. We shall however work with the definition as stated above since this is the simpler one and since it anyway is known [15], [17] that for all domains considered in this paper, namely those in Definition 1.1 below, the two definitions are equivalent.

$H^\infty(\Omega)$ consists of the bounded analytic functions in Ω and $E^\infty(\Omega)$ consists of the analytic functions in Ω which are bounded in a neighbourhood of $\partial\Omega$. Thus $H^\infty(\Omega) = E^\infty(\Omega)$.

If D is an arbitrary divisor in $\mathbf{P}$ we also have spaces $E^p_D(\Omega)$ $(\Omega \subset \mathbf{P},\ 1 \leqq p \leqq \infty)$ defined in the same way as $E^p(\Omega)$ except for that the condition on f of being holomorphic in Ω is replaced by that of being meromorphic in Ω with either $f \equiv 0$

or $f \not\equiv 0$ and $\operatorname{Div} f \geqq D_\Omega$. Thus $E^p_D(\Omega)$ is a subspace of $E^p(\Omega)$ if D is a positive divisor and $E^p_0(\Omega) = E^p(\Omega)$ (0 denoting the divisor zero).

Example: $E^2_{1\cdot(\infty)}(\Omega) = \{f \in E^2(\Omega) : f(\infty) = 0 \text{ if } \infty \in \Omega\}$.

All spaces $H^p(\Omega)$, $E^p(\Omega)$ and $E^p_D(\Omega)$ are complex linear spaces.

We now list a number of known properties of the spaces $H^p(\Omega)$ and $E^p(\Omega)$ ($\Omega \subset \mathbf{P}$, $1 \leqq p \leqq \infty$) that will be needed in the sequel.

(a) *Behaviour under conformal mappings*. Suppose $\varphi : W \to \Omega$ is a conformal map ($W, \Omega \subset \mathbf{P}$). Then $f \in H^p(\Omega)$ if and only if $f \circ \varphi \in H^p(\Omega)$. If $W, \Omega \subset \mathbf{C}$, and $\varphi'^{1/p}$ exists, (i.e. is single-valued) then $f \in E^p(\Omega)$ if and only if $(f \circ \varphi)\varphi'^{1/p} \in E^p(W)$. ($\varphi'^{1/p}$ should be interpreted as 1 if $p = \infty$.)

If W and/or Ω contains the point $\infty \in \mathbf{P}$ things become a little more complicated for $E^p(\Omega)$, unless $\varphi(\infty) = \infty$ or $p = 1, 2$ or ∞. We shall only be concerned with the cases $p = 1$ and $p = 2$ and then we have the following statements, valid for arbitrary $W, \Omega \subset \mathbf{P}$.

(i) $\varphi'^{1/p}$ always exists (Lemma 1.1),

(ii) $f \in E^1_{2\cdot(\infty)}(\Omega)$ if and only if $(f \circ \varphi)\varphi' \in E^1_{2\cdot(\infty)}(W)$.

(iii) $f \in E^2_{1\cdot(\infty)}(\Omega)$ if and only if $(f \circ \varphi)\sqrt{\varphi'} \in E^2_{1\cdot(\infty)}(W)$.

All assertions above are easily proved by just checking with the definitions.

(b) If $W \subset \mathbf{C}$ is bounded and ∂W consists of finitely many (pairwise) disjoint regular analytic curves, then $E^p(W) = H^p(W)$ ($1 \leqq p \leqq \infty$) [5], [15], [17]. Moreover, the norms on $E^p(W)$ to be defined below (1.4) are equivalent to the standard norms [5], [10] on $H^p(W)$ in this case. This follows easily e.g. from [10, (3.1.2)].

Finally, the functions analytic in a neighbourhood of $\bar{W}$ are dense in $H^p(W)$ and (hence) also in $E^p(W)$ [10, Lemma 3.4].

(c) If $W \subset \mathbf{C}$ is bounded and ∂W consists of finitely many disjoint regular analytic curves, then there is a linear map

$$\gamma : H^p(W) \to L^p(\partial W)$$

($L^p(\partial W) = L^p(\partial W$; arc-length measure), $1 \leqq p \leqq \infty$) such that every $f \in H^p(W)$ has nontangential boundary values $\gamma(f)$ almost everywhere on ∂W. Moreover, γ is injective and its range consists of those $f^* \in L^p(\partial W)$ which satisfy

$$\int_{\partial W} f^*(z)\phi(z)\,dz = 0$$

for every ϕ analytic in a neighbourhood of $\bar{W}$. In particular, the range of γ is a closed subspace of $L^p(\partial W)$. The inverse of γ (on its range) is given by $f^* \mapsto f$, where

$$f(z)=\frac{1}{2\pi i}\int_{\partial W}\frac{f^*(w)dw}{w-z}=\frac{1}{2\pi}\int_{\partial W} f^*(w)\,\frac{\partial g(w,z)}{\partial n}\,|dw|$$

and $g(w, z)$ denotes the Green's function of W. Finally, if $f \in H^p(W)$ and $\gamma(f)$ vanishes on a set of positive measure, then $f=0$. See [10] for the above matters.

(d) If Ω is bounded by finitely many continua then the domains D_n in the definition of $E^p(\Omega)$ can be taken to be independent of $f \in E^p(\Omega)$ and also independent of p ($1 \leqq p \leqq \infty$). One can e.g. take

$$D_n=\{z\in\Omega : g(z, z_0)>\delta_n\}$$

where $z_0 \in \Omega$ is fixed (though arbitrary) and $\{\delta_n\}$ is any sequence of positive numbers decreasing to zero [15], [17].

(e) *Decomposition*. Suppose $\Omega \subset \mathbf{C}$ is finitely connected, let $K_1, \ldots, K_m$ be the components of $\mathbf{P} \setminus \Omega$ and let $\Omega_j = \mathbf{P} \setminus K_j$ ($j=1, \ldots, m$). Then any function f holomorphic in Ω can be written

$$f=f_1+\cdots+f_m \qquad (\text{in } \Omega), \tag{1.2}$$

where f_j is a function holomorphic in Ω_j ($j=1, \ldots, m$). One may e.g. take

$$f_j(z)=\frac{1}{2\pi i}\int_{\gamma_j}\frac{f(w)}{w-z}\,dw, \qquad z\in\Omega_j, \tag{1.3}$$

where $\gamma_j=\gamma_j(z)$ is a contour in Ω approximating $\partial\Omega_j$ and oriented as $\partial\Omega$.

Suppose now that no component of $\partial\Omega$ degenerates to a point and that $\partial\Omega$ is rectifiable (see Remark 1.2 below). Then, in any decomposition (1.2) $f \in E^p(\Omega)$ if and only if $f_j \in E^p(\Omega_j)$ for all $j=1, \ldots, m$ ($1 \leqq p < \infty$) [15], [16], [17]. (The corresponding theorem for $H^p(\Omega)$ is also true [5], [10].) □

The following lemma may be viewed as a generalization of [9, Thm 10.11].

Lemma 1.2. *Let $W \subset \mathbf{C}$ be a bounded domain, bounded by finitely many disjoint regular analytic curves, let $\Omega \subset \mathbf{P}$ be a domain conformally equivalent to W and let $\varphi: W \to \Omega$ be a conformal map. Then, if $\infty \notin \Omega$, the following are equivalent*:

(i) *$\partial\Omega$ is rectifiable.*

(ii) *$\partial\Omega$ has finite one-dimensional Hausdorff measure.*

(iii) *$1 \in E^1(\Omega)$ (equivalently $1 \in E^p(\Omega)$ for any $1 \leqq p < \infty$).*

(iv) *$\varphi' \in E^1(W)$ (equivalently $\sqrt{\varphi'} \in E^2(W)$).*

(v) *φ extends to a continuous function on $W \cup \partial W$ with $\varphi \mid \partial W$ absolutely continuous.*

If $\infty \in \Omega$ the same is true if (iv) *is replaced by*
(iv)′ $\varphi' \in E^1_{-2\cdot(w)}(W)$ *(equivalently* $\sqrt{\varphi'} \in E^2_{-1\cdot(w)}(W)$*). Here* $w = \varphi^{-1}(\infty)$.

Remark 1.2. As a definition of "$\partial\Omega$ is rectifiable" we take: each of the finitely many components of $\partial\Omega$ is the image of [0, 1] (or $\partial\mathbf{D}$) under a function which is continuous and of bounded variation.

That $\partial\Omega$ has finite one-dimensional Hausdorff measure means, by definition: there exists a constant $M < \infty$ such that for any $\varepsilon > 0$ $\partial\Omega$ can be covered by a family $\{\Delta_j\}$ of open discs satisfying $\operatorname{diam}(\Delta_j) \leqq \varepsilon$ for all j and $\Sigma_j \operatorname{diam}(\Delta_j) \leqq M$.

Proof. (i) implies (ii): This is an elementary exercise which we omit.

(ii) implies (iii): From the definiton of "finite one-dimensional Hausdorff measure" we obtain coverings $\{\Delta_j^{(n)}\}_j$ ($n = 1, 2, \ldots$; $\Delta_j^{(n)}$ open discs) of $\partial\Omega$ with $\operatorname{diam}(\Delta_j^{(n)}) < 1/n$ (for all j) and $\Sigma \operatorname{diam}(\Delta_j^{(n)}) \leqq M < \infty$ (M independent of n). It is easy to see that $\partial\Omega$ is necessarily bounded, hence compact. Thus we may assume that each covering $\{\Delta_j^{(n)}\}_j$ is finite. Now let

$$D_n = \Omega \setminus \bigcup_j \bar{\Delta}_j^{(n)}.$$

Then $\{D_n\}$ is an exhaustion of Ω as in the definition of $E^p(\Omega)$, except that it may be necessary to pass to a subsequence to obtain $D_1 \subset D_2 \subset \cdots$ and to delete components of D_n to get D_n connected. Since

$$\int_{\partial D_n} |dz| \leqq \sum_j \int_{\partial\Delta_j^{(n)}} |dz| \leqq \pi M$$

it follows that $1 \in E^p(\Omega)$ $(1 \leqq p < \infty)$.

(iii) implies (iv): It follows immediately from the behaviour of the E^p-spaces under conformal mapping that (iii) and (iv) (or (iv)′ if $\infty \in \Omega$) are equivalent statements.

(iv) implies (v): Decompose φ according to (1.2), (1.3) so that $\varphi = \varphi_1 + \cdots + \varphi_m$ in W, where φ_j is holomorphic in $W_j = \mathbf{P}\setminus$(the jth component of $\mathbf{P}\setminus W$). It now suffices to show that, for each $j = 1, \ldots, m$, φ_j extends continuously to $W_j \cup \partial W_j \supset W \cup \partial W_j$ with $\varphi_j \,|\, \partial W_j$ absolutely continuous. (Observe that, for $k \neq j$, φ_k is even analytic in a neighbourhood of $W \cup \partial W_j$.)

From $\varphi' \in E^1(W)$ it follows by the decomposition theorem (e) above that $\varphi_j' \in E^1(W_j)$, $j = 1, \ldots, m$. Moreover, if $\infty \in W_j$, $\varphi_j(\infty) = 0$ by (1.3) and hence φ_j' has a zero of order at least two at ∞. Thus $\varphi_j' \in E^1_{2\cdot(\infty)}(W_j)$ for all j.

Since W_j is simply connected there is a Riemann mapping function $\psi_j : \mathbf{D} \to W_j$. From $\varphi_j' \in E^1_{2\cdot(\infty)}(W_j)$ it follows (by (a) (ii) before the lemma) that $(\varphi_j \circ \psi_j)' = (\varphi_j' \circ \psi_j)\psi_j' \in E^1(\mathbf{D})$. But $E^1(\mathbf{D}) = H^1(\mathbf{D}) = H^1$ (the usual Hardy space in the unit disc) and it is known [5, Thm 3.11] that $(\varphi_j \circ \psi)' \in H^1$ implies that $\varphi_j \circ \psi_j$ extends continuously to $\bar{\mathbf{D}}$ with $(\varphi_j \circ \psi_j)|_{\partial \mathbf{D}}$ absolutely continuous. From this the desired conclusion follows because ψ_j^{-1} extends analytically across ∂W_j due to the analyticity of ∂W_j.

The easy modifications of the above arguments needed to prove that (iv)′ implies (v) in the case $\infty \in \Omega$ are left to the reader.

(v) implies (i): Since absolutely continuous functions are of bounded variation this implication is obvious. □

For a domain $\Omega \subset \mathbf{P}$ to satisfy the hypotheses in Lemma 1.2 for some choice of W and φ it is necessary and sufficient that $\partial\Omega$ consists of finitely many, and at least one, continua. Therefore we shall use the following terminology.

Definition 1.1. By a *domain bounded by finitely many rectifiable continua* we mean a domain $\Omega \subset \mathbf{P}$ satisfying the hypotheses (for some choice of W and φ) and equivalent conditions in Lemma 1.2.

One should notice that if $\Omega \subset \mathbf{P}$ is bounded by finitely many rectifiable continua then either $\infty \in \Omega$ or Ω is a bounded domain in $\mathbf{C}$. In both cases $E^\infty(\Omega) \subset E^2(\Omega) \subset E^1(\Omega)$ (etc.), in particular every function holomorphic in a neighbourhood of $\bar{\Omega}$ belongs to all $E^p(\Omega)$ $(1 \leqq p \leqq \infty)$.

In the rest of this section we shall only consider domains Ω bounded by finitely many rectifiable continua.

It is time to put norms on $E^p(\Omega)$. Choose a bounded domain $W \subset \mathbf{C}$ bounded by finitely many disjoint regular analytic curves and conformally equivalent to Ω and choose also a conformal map $\varphi: W \to \Omega$. Then $\varphi' \in E^1(W)$ by Lemma 1.2. We may suppose that W and φ are chosen so that $\varphi'^{1/p}$ exists (cf. Remark 1.1.) Then $f \in E^p(\Omega)$ if and only if $(f \circ \varphi)\varphi'^{1/p} \in E^p(W)$, and we define

$$\|f\|_{E^p(\Omega)} = \|\gamma[(f \circ \varphi)\varphi'^{1/p}]\|_{L^p(\partial W)} \tag{1.4}$$

for $f \in E^p(\Omega)$, $1 \leqq p \leqq \infty$. Recall that, for $g \in E^p(W) = H^p(W)$, $\gamma(g)$ denotes the boundary function of g on ∂W.

It is immediately verified that this definition is independent of the choices of W and φ. In particular, $\|f\|_{E^p(\Omega)} = \|\gamma(f)\|_{L^p(\partial\Omega)}$ if $\partial\Omega$ is analytic.

Since the map $f \mapsto \gamma[(f \circ \varphi)\varphi'^{1/p}]$, $E^p(\Omega) \to L^p(\partial W)$ identifies $E^p(\Omega)$ with a closed subspace of $L^p(\partial W)$ (namely the range of γ) the norms (1.4) make $E^p(\Omega)$ into a Banach space. It is also clear that whenever $\varphi: \Omega_1 \to \Omega_2$ is conformal and $\varphi'^{1/p}$ exists the map $f \mapsto (f \circ \varphi)\varphi'^{1/p}$ is an isometric isomorphism of $E^p(\Omega_2)$ onto $E^p(\Omega_1)$. Finally, it is easy to see that the topology on $E^p(\Omega)$ induced by the above norms is stronger than the topology of uniform convergence on compact subsets of Ω. In particular, any functional as in the right member of (0.1) is continuous on $E^p(\Omega)$.

For $p=2$, $E^p(\Omega)=E^2(\Omega)$ becomes a Hilbert space with the inner product defined by

$$(1.5)\qquad (f,g)_{E^2(\Omega)}=(\gamma[(f\circ\varphi)\sqrt{\varphi'}],\gamma[(f\circ\varphi)\sqrt{\varphi'}])_{L^2(\partial W)}$$

($f,g\in E^2(\Omega)$), where W and φ ($\varphi: W\to\Omega$) are as in the definition of the norm. Thus

$$(f,g)_{E^2(\Omega)}=\int_{\partial\Omega}\gamma(f)\overline{\gamma(g)}\,|dz|$$

if $\partial\Omega$ is analytic. (Of course the same branch of $\sqrt{\varphi'}$ is to be chosen at both places in (1.5).)

From now on we delete the subscripts $E^p(\Omega)$ from the norms and inner products above.

It is clear that the definitions of $\|\cdot\|$ and $(\cdot,\cdot)$ make sense also for $E^p_D(\Omega)$ ($1\leqq p\leqq\infty$ and D an arbitrary divisor) and make them too into Banach spaces.

Now we are ready to define $\int_{\partial\Omega}f|dz|$ for $f\in E^1(\Omega)$ (or $f\in E^1_D(\Omega)$). Take a conformal map $\varphi: W\to\Omega$, where W is bounded by finitely many disjoint regular analytic curves. Then we have, formally,

$$\int_{\partial\Omega}f|dz|=\int_{\partial W}f(\varphi(w))|\varphi'(w)|\;|dw|=\int_{\partial W}f(\varphi(w))\varphi'(w)\cdot\frac{|\varphi'(w)|}{\varphi'(w)}|dw|.$$

If $f\in E^1(\Omega)$ then $f(\varphi(w))\varphi'(w)\in E^1(W)$, hence the boundary function $\gamma[(f\circ\varphi)\varphi']$ exists and belongs to $L^1(\partial W)$. Since $\varphi'\in E^1(W)$ and $\varphi'\not\equiv 0$ $\gamma[\varphi']\in L^1(\partial W)$ and $\gamma[\varphi']\neq 0$ almost everywhere on ∂W. Therefore $|\gamma[\varphi']|/\gamma[\varphi']$ exists almost everywhere on ∂W and belongs to $L^\infty(\partial W)$ (and has modulus one).

Thus we may define

$$(1.6)\qquad \int_{\partial\Omega}f|dz|=\int_{\partial W}\gamma[(f\circ\varphi)\varphi']\cdot\frac{|\gamma[\varphi']|}{\gamma[\varphi']}|dw|$$

for $f\in E^1(\Omega)$.

It is straightforward to check that this definition is independent of the choices of W and φ. Moreover, it is clear that the map $f\to\int_{\partial\Omega}f|dz|$ is a continuous linear functional on $E^1(\Omega)$. Finally,

$$(1.7)\qquad \int_{\partial\Omega}f|dz|=(f,1)$$

if $f\in E^2(\Omega)$. (Recall that $1\in E^2(\Omega)$ by Lemma 1.2.)

It turns out (cf. (a) (iii) before Lemma 1.2) that the space $E^2_{1\cdot(\infty)}(\Omega)$ behaves better under conformal mappings than $E^2(\Omega)$. Therefore $E^2_{1\cdot(\infty)}(\Omega)$ is the more natural space to work with when trying to do things in an invariant way, and we shall give $E^2_{1\cdot(\infty)}(\Omega)$ its own name. We define

$$E(\Omega) = E^2_{1\cdot(\infty)}(\Omega) \tag{1.8}$$

for an arbitrary domain $\Omega \subset \mathbf{P}$.

$E(\Omega)$ is a Hilbert space with the inner product (1.5). Formally, we can write

$$(f, g) = \int_{\partial\Omega} f\bar{g}\,|dz| = \int_{\partial\Omega} f\sqrt{dz}\cdot\overline{g\sqrt{dz}}$$

($f, g \in E(\Omega)$). This indicates that the elements of $E(\Omega)$ should be regarded as differentials of order one-half if one wants to have the inner product defined in an invariant way (i.e. independent of the choice of the coordinate variable z in Ω). This is what will be done in Section 2.

The above consideration also explains why $E(\Omega) = E^2_{1\cdot(\infty)}(\Omega)$ is more natural than $E^2(\Omega)$. For if f is holomorphic in Ω and $\infty \in \Omega$ then f must have a zero at ∞ in order that the half-order differential $f\sqrt{dz}$ shall be holomorphic at ∞, because $\sqrt{dz}$ has a pole of order one at $z = \infty$ (as is seen by expressing $\sqrt{dz}$ in $w = 1/z$). Thus $E(\Omega) = E^2_{1\cdot(\infty)}(\Omega)$ is the space of functions f (or half-order differentials $f\sqrt{dz}$) which are holomorphic considered as half-order differentials (and have the appropriate boundary behaviour).

If D is any divisor in $\mathbf{P}$ we define $E_D(\Omega) = E^2_{D+1\cdot(\infty)}(\Omega)$.

2. Half-order differentials

Half-order differentials on Riemann surfaces have been considered at length in [11], [12]. They are also implicit or considered in passing in many other works, e.g. [8], [6, p. 193 ff] and in fact in e.g. every work dealing with Szegö-kernels (since these are naturally regarded as half-order differentials).

In order to define in a precise fashion, the concept of a differential of order one-half we shall use the language of sheaves on Riemann surfaces as presented in [6]. Let us briefly review some notations and definitions in [6]. (For more details we refer to [6].)

Let M be an arbitrary Riemann surface and let $\mathfrak{A} = \{(U_j, z_j)\}$ (j ranging over some index set) be a holomorphic atlas on M, i.e. $\{U_j\}$ is an open cover of M, and z_j is a conformal map of U_i onto some open subset of $\mathbf{C}$ for every j. Let $\mathfrak{O}$, $\mathfrak{O}^*$ and $\mathfrak{M}$ denote, respectively, the sheaves of germs of holomorphic, nowhere-vanishing holomorphic and meromorphic functions in M. If U is an open subset of M, $\Gamma(U, \mathfrak{O})$ denotes the set of cross-sections of $\mathfrak{O}$ on U (similarly for $\mathfrak{O}^*$ and

$\mathfrak{M}$). Thus e.g. $f \in \Gamma(U, \mathfrak{O}^*)$ means that f is a nowhere-vanishing holomorphic function on U.

Suppose we have, for every ordered pair (i, j) of indices with $U_i \cap U_j \neq \emptyset$, a section $\xi_{ij} \in \Gamma(U_i \cap U_j, \mathfrak{O}^*)$ such that $\xi_{ij}\xi_{jk}\xi_{ki} = 1$ in $U_i \cap U_j \cap U_k$ whenever $U_i \cap U_j \cap U_k \neq \emptyset$. Then $\{\xi_{ij}\}$ defines an element ξ of the cohomology group $H^1(M, \mathfrak{O}^*)$. The elements of $H^1(M, \mathfrak{O}^*)$ are called (holomorphic, complex) *line bundles*. $H^1(M, \mathfrak{O}^*)$ is an Abelian group under multiplication.

If $\xi \in H^1(M, \mathfrak{O}^*)$ is defined as above and $\{f_i\}$ is a family of holomorphic functions defined in the U_i ($f_i \in \Gamma(U_i, \mathfrak{O})$) such that $f_i = \xi_{ij} f_j$ in $U_i \cap U_j$ whenever $U_i \cap U_j \neq \emptyset$ then f_i defines a *holomorphic cross-section* f of ξ. The notation for this is $f \in \Gamma(M, \mathfrak{O}(\xi))$. Meromorphic cross-sections are defined similarly (with $f_i \in \Gamma(U_i, \mathfrak{O})$ replaced by $f_i \in \Gamma(U_i, \mathfrak{M})$); notation: $f \in \Gamma(M, \mathfrak{M}(\xi))$. Finally, if D is any divisor on M, $\Gamma(M, \mathfrak{O}_D(\xi))$ consists of those $f \in \Gamma(M, \mathfrak{M}(\xi))$ for which $\operatorname{Div} f_j \geqq D_{U_j}$ for all j.

The two principal examples of line bundles are the identity bundle $\xi = 1$, defined by $\xi_{ij} = 1$ for all i, j, and the *canonical bundle* κ defined by

$$\kappa_{ij} = \frac{dz_j}{dz_i} = (z_j \circ z_i^{-1})' \circ z_i. \tag{2.1}$$

The cross-sections of $\xi = 1$ are just the functions on M and the cross-sections of κ can be identified with the differentials (one-forms) of type $(1, 0)$ (i.e. of order one in dz and order zero on $d\bar{z}$).

Any bundle $\lambda \in H^1(M, \mathfrak{O}^*)$ with $\lambda^2 = \kappa$ is called a *bundle of half-order differentials* and any section $f \in \Gamma(M, \mathfrak{O}(\lambda))$ a (holomorphic) *differential of order one-half*. If λ can be represented relative to $\mathfrak{A}$, by $\{\lambda_{ij}\}$ say, then $\lambda_{ij}^2 = \kappa_{ij}$, i.e. λ_{ij} is one branch of $\sqrt{dz_j/dz_i}$. That $f \in \Gamma(M, \mathfrak{O}(\lambda))$ then means that f is represented by $\{f_i\}$, $f_i \in \Gamma(U_i, \mathfrak{O})$, with $f_i = \lambda_{ij} f_j$ in $U_i \cap U_j$. Thus, formally,

$$f_i \sqrt{dz_i} = f_j \sqrt{dz_j} \qquad \text{in } U_i \cap U_j,$$

which explains the terminology.

Similarly the elements in $\Gamma(M, \mathfrak{M}(\lambda))$ are called meromorphic differentials of order one-half.

Assume, for a moment, that M is compact, of genus p say. For any $\xi \in H^1(M, \mathfrak{O}^*)$ the *chern-class* of ξ, $c(\xi)$, can be defined as the degree of the divisor of any meromorphic cross-section (not identically zero) of ξ:

$$c(\xi) = \deg \operatorname{Div} f,$$

$f \in \Gamma(M, \mathfrak{M}(\xi)) \setminus \{0\}$. The structure of $H^1(M, \mathfrak{O}^*)$ as an Abelian group is then given by the exact diagram of groups [6, §8]

$$0 \to P(M) \to H^1(M, \mathfrak{O}^*) \xrightarrow{c} \mathbf{Z} \to 0.$$

Here $P(M)$ is the Picard-variety of M, which as a group is isomorphic to $\mathbf{C}^p/\mathbf{Z}^{2p} \cong (\mathbf{R}/\mathbf{Z})^{2p}$. It follows that

$$H^1(M, \mathfrak{O}^*) \cong (\mathbf{R}/\mathbf{Z})^{2p} \oplus \mathbf{Z}. \tag{2.2}$$

Because the chern-class of κ is an even number, $c(\kappa) = 2(p-1)$, the isomorphism (2.2) shows that there really exist bundles λ with $\lambda^2 = \kappa$, in fact exactly 2^{2p} such λ. Clearly $c(\lambda) = p - 1$ for all those λ.

Suppose, for M an arbitrary Riemann surface, that there exists an anti-analytic map $j: M \to M$ such that $j \circ j =$ the identity map. Then the pair (M, j) is called a *symmetric Riemann surface* (cf. [7]). The typical example is the Schottky double of a plane domain: let Ω be a domain in $\mathbf{C}$ bounded by finitely many disjoint regular analytic curves. The Schottky double $\hat{\Omega}$ of Ω is the compact Riemann surface obtained by completing Ω with a "back-side" $\tilde{\Omega}$ identical with Ω as a point set but provided with the opposite conformal structure. The resulting surface $\hat{\Omega} = \Omega \cup \partial\Omega \cup \tilde{\Omega}$ becomes a Riemann surface in a natural way. If $z \in \Omega$, let $\tilde{z}$ denote the corresponding point on $\tilde{\Omega}$. Define $j: \hat{\Omega} \to \hat{\Omega}$ by

$$\begin{cases} j(z) = \tilde{z} \\ j(\tilde{z}) = z \end{cases} \quad \text{for } z \in \Omega,$$

$$j(z) = z \qquad \text{for } z \in \partial\Omega.$$

Then $(\hat{\Omega}, j)$ is a compact symmetric Riemann surface.

Any compact symmetric Riemann surface (M, j) such that $M \setminus \Gamma = M \setminus \{\zeta \in M : j(\zeta) = \zeta\}$ consists of two components each of which is conformally equivalent to a plane domain will be called *a double of a plane domain*. If W and $\tilde{W}$ are the two components of $M \setminus \Gamma$ then $\tilde{W} = j(W)$, $\Gamma = \partial W = \partial \tilde{W}$ and we will usually write $\hat{W} = W \cup \partial W \cup \tilde{W}$ instead of (M, j), the involution j being understood ($\hat{W} = M$).

If (M, j) is a symmetric Riemann surface and f a function on a subset of M we define

$$\tilde{f}(z) = \overline{f(j(z))}. \tag{2.3}$$

Thus $\tilde{f}$ is holomorphic if f is. For points $z \in M$, subsets $E \subset M$ and divisors $D = \sum_k n_k(z_k)$ in M we define

$$\begin{aligned} \tilde{z} &= j(z), \\ \tilde{E} &= j(E), \\ \tilde{D} &= \sum_k n_k(j(z_k)) = \sum_k n_k(\tilde{z}_k). \end{aligned}$$

Let $\hat{W} = W \cup \partial W \cup \tilde{W}$ be a double of a plane domain. If the connectivity of W is $p+1$ then the genus of $\hat{W}$ is p. Of the 2^{2p} bundles λ on $\hat{W}$ with $\lambda^2 = \kappa$ there is a distinguished one which we now proceed to define.

Let z be a conformal map of W onto a plane domain Ω. We may suppose that Ω is bounded by disjoint regular analytic curves and then z extends to a conformal map defined in some neighbourhood U of $W \cup \partial W$ in $\hat{W}$. We assume that U is connected. Let

$$U_1 = U, \qquad z_1 = z, \tag{2.4}$$

$$U_2 = \tilde{U} = j(U), \qquad z_2 = \tilde{z} = \overline{z \circ j}. \tag{2.5}$$

Then $\mathfrak{A} = \{(U_k, z_k) : k = 1, 2\}$ is a holomorphic atlas on $\hat{W}$.

Define

$$S = z_2 \circ z_1^{-1}. \tag{2.6}$$

Then S is a holomorphic function in $z_1(U_1 \cap U_2)$, which is a neighbourhood of $\partial\Omega = z(\partial W)$ in $\mathbf{C}$. On ∂W, $z_2 = \bar{z}_1$. Therefore

$$S(z) = \bar{z} \qquad \text{on } \partial\Omega. \tag{2.7}$$

This shows that S is the so-called Schwarz function for $\partial\Omega$. See [4]. Differentiation of (2.7) yields

$$S'(z)dz = d\bar{z} \qquad \text{along } \partial\Omega. \tag{2.8}$$

Let $T(z)$, $z \in \partial\Omega$, denote the unit tangent vector along $\partial\Omega$, oriented so that Ω lies to the left. Then $T(z) = dz/|dz|$ so that (2.8) shows that

$$S'(z) = \frac{1}{T(z)^2} \qquad \text{on } \partial\Omega. \tag{2.9}$$

This also shows that $T(z)$ extends to a holomorphic function in the neighbourhood $z_1(U_1 \cap U_2)$ of $\partial\Omega$, with (2.9) holding identically in that neighbourhood.

Now we define a line bundle $\lambda \in H^1(\hat{W}, \mathfrak{O}^*)$ by representing it by

$$\lambda_{12} = \frac{1}{T \circ z_1} \qquad \text{in } U_1 \cap U_2 \tag{2.10}$$

relative to $\mathfrak{A}$. (The remaining λ_{ij} then must be $\lambda_{21} = 1/\lambda_{12}$, $\lambda_{11} = 1$, $\lambda_{22} = 1$.) Observe that, by (2.6), the canonical bundle $\kappa \in H^1(\hat{W}, \mathfrak{O}^*)$ is represented by

$$\kappa_{12} = \frac{dz_2}{dz_1} = S' \circ z_1. \tag{2.11}$$

Therefore, by (2.9), $(\lambda_{12})^2 = \kappa_{12}$ so that λ is a bundle of half-order differentials: $\lambda^2 = \kappa$.

It should be remarked that the role of T above just is to single out a certain branch of $\sqrt{S'}$ (i.e. of $\sqrt{\kappa_{12}}$).

Example 2.1. Consider the symmetric Riemann surface of genus $p=0$ $(\mathbf{P}, j)$, where $j(z)=1/\bar{z}$. It can be viewed as representing $\hat{\mathbf{D}}$, since $\mathbf{P}=\mathbf{D}\cup\partial\mathbf{D}\cup\tilde{\mathbf{D}}$, where $\tilde{\mathbf{D}}=\{z\in\mathbf{C}\cup\{\infty\}:|z|>1\}$ and since $\partial\mathbf{D}$ is the fixed point set of j.

We may take the atlas $\mathfrak{A}=\{(U_k, z_k): k=1,2\}$ on $\mathbf{P}$ to be

$$U_1=\{z\in\mathbf{C}:|z|<r\},$$

$$U_2=\{z\in\mathbf{P}:|z|>1/r\},$$

$$z_1=z,\quad z_2=1/z$$

$(r>1)$. Then $dz_2/dz_1=-1/z^2$ in $U_1\cap U_2$ so that λ_{12} may be taken to be

$$\lambda_{12}(z)=i/z \qquad (i=\sqrt{-1}).$$

A pair $\{f_1, f_2\}$ of meromorphic functions in U_1 and U_2 respectively represents a section f in $\Gamma(\mathbf{P}, \mathfrak{M}(\lambda))$ if (and only if)

$$f_1(z)=\frac{i}{z}f_2(z) \qquad \text{for } \frac{1}{r}<|z|<r.$$

It is easy to see that such a pair $\{f_1, f_2\}$ necessarily consists of rational functions, and if

$$f_1(z)=R(z) \qquad (|z|<r),$$

R rational, then

$$f_2(z)=-izR(z) \qquad (|z|>1/r).$$

It follows in particular that

$$\operatorname{Div} f=\operatorname{Div} R-1\cdot(\infty).$$

When we speak of a half-order differential "$f(z)\sqrt{dz}$" on a subdomain Ω of $\mathbf{P}$ we will always mean that section in $\Gamma(\Omega, \mathfrak{M}(\lambda))$ which is represented by $f_1(z)=f(z)$ in $\Omega\cap U_1$ and by $f_2(z)=-izf(z)$ in $\Omega\cap U_2$. Thus $\operatorname{Div} f\sqrt{dz}=\operatorname{Div} f-1\cdot(\infty)$ if $\infty\in\Omega$ (otherwise $\operatorname{Div} f\sqrt{dz}=\operatorname{Div} f$). □

Returning now to the general case it is straightforward to show that the bundle $\lambda\in H^1(\hat{W}, \mathfrak{O}^*)$ is independent of the choices made in its definition, namely the choice of coordinate map z on W and also the choice of which one of the components of $\hat{W}\setminus\partial W$ is considered to be W, the "front side". In other words, λ is intrinsically associated with the symmetric Riemann surface $(\hat{W}, j)$.

Conventions. (a) When $\hat{W} = W \cup \partial W \cup \tilde{W}$ is the double of a plane domain and nothing else is explicitly stated λ will always denote the distinguished bundle of half-order differentials on $\hat{W}$ defined above.

(b) The expression "half-order differential" will, from now on, always refer to sections of $\mathfrak{O}(\lambda)$ or $\mathfrak{M}(\lambda)$, with λ as above.

(c) If D is a domain with $W \subset D \subset \hat{W}$ sections in $\Gamma(D, \mathfrak{M}(\lambda))$ will sometimes be denoted by symbols such as $f\sqrt{dz}$, sometimes just by symbols such as f, both to be interpreted as follows. z is a coordinate variable on W, $\mathfrak{A} = \{(U_j, z_j) : j = 1, 2\}$ is any atlas on $\hat{W}$ of the kind (2.4), (2.5) with $U_1 \supset W$ and $z_1 = z$ on W, and $f\sqrt{dz}$ (or f) is that elelment of $\Gamma(D, \mathfrak{M}(\lambda))$ which is represented by $\{f_1, f_2\}$ relative to $\mathfrak{A}$, where $f_1 = f$ in $U_1 \cap D$ and $f_2 = f/\lambda_{12}$ in $U_2 \cap D$.

(d) If g is a meromorphic function in D ($W \subset D \subset \hat{W}$) then $\sqrt{dg}$, if it exists, denotes any one of the two sections $f\sqrt{dz} \in \Gamma(D, \mathfrak{M}(\lambda))$ (interpreted as above) satisfying $dg/dz_1 = f_1^2$ in $D \cap U_1$ (and then automatically also $dg/dz_2 = f_2^2$ in $D \cap U_2$). □

Let now $\mathfrak{A} = \{(U_j, z_j) : j = 1, 2\}$ be a fixed atlas on $\hat{W} = W \cup \partial W \cup \tilde{W}$ (a double of a plane domain) and $\lambda \in H^1(\hat{W}, \mathfrak{O}^*)$ represented by λ_{12} in terms of $\mathfrak{A}$ as above. On $\Gamma(\bar{W}, \mathfrak{O}(\lambda))$, the holomorphic cross-sections of λ defined in some neighbourhood of $\bar{W}$, we define an inner product by

$$(f, g) = \int_{\partial W} f_1 \bar{g}_1 |dz_1| = \int_{\partial W} f_1 \bar{g}_1 \lambda_{12} dz_1, \tag{2.12}$$

where $f_1, g_1 \in \Gamma(\bar{W}, \mathfrak{O})$ denote the representatives of $f, g \in \Gamma(\bar{W}, \mathfrak{O}(\lambda))$ relative to U_1 in $\mathfrak{A}$.

We denote by $E(W)$ the completion of $\Gamma(\bar{W}, \mathfrak{O}(\lambda))$ with respect to the inner product (2.12). Then $E(W)$ is a Hilbert space of holomorphic differentials in W of order one-half. $E_D(W)$, where D is any divisor on W, is defined similarly in terms of $\Gamma(\bar{W}, \mathfrak{O}_D(\lambda))$.

If $W = \Omega \subset \mathbf{P}$ then, as is easily verified, the above definition of $E(W)$ agrees with the previous definition (1.8) of $E(\Omega)$ in the sense that the map $f \mapsto f\sqrt{dz}$ is an isometric isomorphism $E(\Omega) \to E(W)$.

3. Fundamental results on quadrature domains

In this section we derive our main characterization of quadrature domains, namely Theorem 3.2, and a couple of corollaries of it.

Recall that when $\hat{W} = W \cup \partial W \cup \tilde{W}$ is a double of a plane domain λ always denotes the distinguished bundle $\lambda \in H^1(\hat{W}, \mathfrak{O}^*)$ of half-order differentials defined in Section 2.

Lemma 3.1. *With $\hat{W}$ and λ as above*

$$\Gamma(\hat{W}, \mathfrak{O}(\lambda)) = 0.$$

Proof. Suppose $f \in \Gamma(\hat{W}, \mathfrak{O}(\lambda))$, represented by $\{f_1, f_2\}$ as usual. Then we have

$$(f, f) = \int_{\partial W} f_1 \bar{f}_1 \lambda_{12} dz_1 = \int_{\partial W} f_1 \tilde{f}_2 dz_1 = \int_{\partial W} f_1 \tilde{f}_2 dz_1 = 0$$

since $f_1 \tilde{f}_2 dz_1$ is holomorphic in W. Thus $f = 0$ since $(\cdot, \cdot)$ is a (non-degenerate) inner product. □

Remark 3.1. It is not generally true that $\Gamma(M, \mathfrak{O}(\lambda)) = 0$ if M is a compact Riemann surface and $\lambda \in H^1(M, \mathfrak{O}^*)$ a bundle with $\lambda^2 = \kappa$. In fact, it turns out that the 2^{2p} (p = genus for M) bundles with $\lambda^2 = \kappa$ can be classified into two groups: $2^{p-1}(2^p + 1)$ of them are *even* and the remaining $2^{p-1}(2^p - 1)$ are *odd*. It is shown in [8, Ch. VI] that $\Gamma(M, \mathfrak{O}(\lambda)) \neq 0$ for all odd λ while, unless M is a so-called exceptional Riemann surface, $\Gamma(M, \mathfrak{O}(\lambda)) = 0$ for all even λ. □

Next, let D be an arbitrary positive divisor on $\hat{W}$. We wish to compute $\dim_{\mathbf{C}} \Gamma(\hat{W}, \mathfrak{O}_{-D}(\lambda))$. The relevant version of the Riemann–Roch theorem [6] tells us that

(3.1) $$\dim \Gamma(\hat{W}, \mathfrak{O}_{-D}(\lambda)) = \dim \Gamma(\hat{W}, \mathfrak{O}_D(\kappa\lambda^{-1})) + c(\lambda) + \deg D - p + 1.$$

Since $\lambda^2 = \kappa$, $\kappa\lambda^{-1} = \lambda$ and $c(\lambda) = p - 1$. Thus

(3.2) $$\dim \Gamma(\hat{W}, \mathfrak{O}_{-D}(\lambda)) = \dim \Gamma(\hat{W}, \mathfrak{O}_D(\lambda)) + \deg D.$$

((3.1) and (3.2) are true for arbitrary, i.e. not necessarily positive, divisors D.)

Since D is positive $\Gamma(\hat{W}, \mathfrak{O}_D(\lambda)) \subset \Gamma(\hat{W}, \mathfrak{O}(\lambda))$, hence $\dim \Gamma(\hat{W}, \mathfrak{O}_D(\lambda)) = 0$ by Lemma 3.1. Thus

Lemma 3.2. *With $\hat{W}$ and λ as above and D a positive divisor on $\hat{W}$*

$$\dim_{\mathbf{C}} \Gamma(\hat{W}, \mathfrak{O}_{-D}(\lambda)) = \deg D.$$

This result means that the locations and principal parts of the poles of a section in $\Gamma(\hat{W}, \mathfrak{O}(\lambda))$ can be arbitrarily prescribed and that this determines the section uniquely.

Remark 3.2. In particular there exists, by Lemma 3.2, a uniquely determined section $\Lambda \in \Gamma(\hat{W}, \mathfrak{O}(\lambda))$ with

$$\Lambda = \Lambda_{z_0} = \Lambda(z, z_0) = \frac{\sqrt{dz}}{2\pi(z - \tilde{z}_0)} + \text{regular}$$

at a prescribed point $\tilde{z}_0 \in \tilde{W}$, and otherwise regular. The restriction of Λ to W is the classical Szegö kernel for W (if $W \subset \mathbf{C}$) and the restriction to $\tilde{W}$ (or perhaps the restriction of $\tilde{\Lambda}$ to W) is the so-called adjoint kernel. See e.g. [8] for these matters. Λ_{z_0} is the reproducing kernel for the class $E(W)$, i.e.

$$f(z_0) = (f, \Lambda_{z_0}) \qquad \text{for all } f \in E(W). \qquad \square$$

Theorem 3.1. *With $\hat{W} = W \cup \partial W \cup \tilde{W}$ a double of a plane domain and D a positive divisor in W*

$$E_D(W)^{\perp} = \Gamma(\hat{W}, \mathfrak{O}_{-\hat{D}}(\lambda)).$$

Here $E_D(W)^{\perp} = \{g \in E(W) : (f, g) = 0$ for all $f \in E_D(W)\}$. In other words, given $g \in E(W)$ we have $(f, g) = 0$ for all $f \in E_D(W)$ if and only if g extends to an element in $\Gamma(\hat{W}, \mathfrak{O}_{-\hat{D}}(\lambda))$.

Proof. By Lemma 3.2 $\dim \Gamma(\hat{W}, \mathfrak{O}_{-\hat{D}}(\lambda)) = \deg \tilde{D} = \deg D$. (Actually, we only need $\dim \Gamma(\hat{W}, \mathfrak{O}_{-\hat{D}}(\lambda)) \geqq \deg \tilde{D} = \deg D$, which is an immediate consequence of (3.2).) Also, $\dim E_D(W)^{\perp} = \dim(E(W)/E_D(W)) \leqq \deg D$ since $E_D(W)$ is defined by $\deg D$ linear conditions in $E(W)$. Thus $\dim E_D(W)^{\perp} \leqq \dim \Gamma(\hat{W}, \mathfrak{O}_{-\hat{D}}(\lambda))$. Therefore it is enough to prove that $\Gamma(\hat{W}, \mathfrak{O}_{-\hat{D}}(\lambda)) \subset E_D(W)^{\perp}$.

So take a $g \in \Gamma(\hat{W}, \mathfrak{O}_{-\hat{D}}(\lambda))$. Relative to some atlas $\mathfrak{A} = \{(U_j, z_j) : j = 1, 2\}$ of the kind (2.4), (2.5) g can be represented by $\{g_1, g_2\}$ with $g_j \in \Gamma(U_j, \mathfrak{M})$, $g_1 = g$ on W, $\operatorname{Div} g_j \geqq -\hat{D}$ and $g_1 = \lambda_{12} g_2$ in $U_1 \cap U_2$.

Observing that $|\lambda_{12}| = 1/|T \circ z_1| = 1$ on ∂W (since $|T| = 1$ on $z_1(\partial W)$), that $\bar{g}_2 = \tilde{g}_2$ on ∂W and that $\operatorname{Div} \tilde{g}_2 \geqq -D$ we obtain, for $f \in E_D(W)$,

$$(3.3) \qquad \begin{aligned} (f, g) &= \int_{\partial W} f \bar{g}_1 \bar{\lambda}_{12} dz_1 = \int_{\partial W} f \bar{g}_2 |\lambda_{12}|^2 dz_1 = \int_{\partial W} f \bar{g}_2 dz_1 \\ &= \int_{\partial W} f \tilde{g}_2 dz_1 = 2\pi i \sum \operatorname*{Res}_{W} f \tilde{g}_2 dz_1 = 0. \end{aligned}$$

Thus $g \in E_D(W)^{\perp}$, which proves the theorem. (Strictly speaking, the above computation requires that f extends continuously to ∂W but such f make up a dense subset of $E_D(W)$ so it is enough to consider such f.) $\square$

Theorem 3.2. *Let Ω be a domain in $\mathbf{C}$ bounded by finitely many rectifiable continua and let D be a positive divisor in Ω. Further, let $\hat{W} = W \cup \partial W \cup \tilde{W}$ be a double of a plane domain with W conformally equivalent to Ω, let $g : W \to \Omega$ be a conformal map and let $D_1 = g^{-1}(D)$ be the inverse image of D in W. Then*

(a) *$\sqrt{dg}$ exists as an element in $\Gamma(W, \mathfrak{O}(\lambda))$. Moreover $\sqrt{dg} \in E(W)$.*

(b)

$$\int_{\partial\Omega} f|dz| = 0 \qquad \textit{for all } f \in E_D(\Omega) \tag{3.4}$$

if and only if $\sqrt{dg}$ *extends to an element in* $\Gamma(\hat{W}, \mathfrak{O}_{-\hat{D}_1}(\lambda))$. *More concisely,* $1 \in E_D(\Omega)^\perp$ *if and only if* $\sqrt{dg} \in \Gamma(\hat{W}, \mathfrak{O}_{-\hat{D}_1}(\lambda))$.

Proof. The assertions of (a) are easily seen to be just restatements of Lemma 1.1 and (parts of) Lemma 1.2.

As to (b) we know that $g: W \to \Omega$ gives rise to an isometric isomorphism

$$g^*: E(\Omega) \to E(W),$$

namely defined by $f\sqrt{dz} \mapsto (f \circ g)\sqrt{dg}$, whenever one of the two branches of $\sqrt{dg}$ is chosen. It is clear that $1 \in E(\Omega)$ is mapped onto $\sqrt{dg} \in E(W)$. Moreover $E_D(\Omega)$ is mapped onto $E_{D_1}(W)$, hence $E_D(\Omega)^\perp$ onto $E_{D_1}(W)^\perp$. Thus $1 \in E_D(\Omega)^\perp$ if and only if $\sqrt{dg} \in E_{D_1}(W)^\perp$ and since, by Theorem 3.1, $E_{D_1}(W)^\perp = \Gamma(\hat{W}, \mathfrak{O}_{-\hat{D}_1}(\lambda))$ this proves (b). □

Remark 3.3. If W and Ω in Theorem 3.2 are identified (via g) then (b) of the theorem can be expressed: Ω is a quadrature domain if and only if $\sqrt{dz}$ extends (as a meromorphic half-order differential) to the double $\hat{\Omega} = \Omega \cup \partial\Omega \cup \tilde{\Omega}$. Moreover, the pole divisor of $\sqrt{dz}$ equals the conjugate of the divisor appearing in the quadrature identity. □

Corollary 3.1. *If* Ω *(satisfying the hypotheses of Theorem* 3.2) *is a quadrature domain, then each component of* $\partial\Omega$ *is a locally regular analytic curve.*

Proof. Choose $\hat{W}$ and g as in Theorem 3.2. Then it follows from (b) of the theorem that g extends analytically across ∂W. In is easy to check that $g(\partial W) = \partial\Omega$. Since Ω is necessarily bounded g does not have any pole on ∂W. Moreover, dg cannot have any zero on ∂W because such a zero would have to be of even order (since $\sqrt{dg}$ exists), hence of order at least two, and then g could not be univalent on W. Since each component of ∂W can be mapped biholomorphically onto $\partial \mathbf{D}$ the corollary now follows. □

Corollary 3.2. (of Corollary 3.1). *If* (3.4) *holds, with* Ω *bounded by finitely many rectifiable continua, then also*

$$\int_{\partial\Omega} f|dz| = 0 \qquad \textit{for all } f \in E_D^1(\Omega).$$

Proof. By Corollary 3.1 Ω actually is bounded by locally regular analytic curves, and therefore $E_D^2(\Omega)$ is dense in $E_D^1(\Omega)$. (This follows easily from (b) before Lemma 1.2) Since $f \mapsto \int_{\partial\Omega} f|dz|$ is a continuous functional on $E_D^1(\Omega)$ the corollary follows. □

Corollary 3.3. *If* (3.4) *holds, with* Ω *bounded finitely many rectifiable continua, then* Ω *also satisfies a quadrature identity of the following kind*:

$$\int_\Omega f dx dy = \sum_{k=1}^{m} \sum_{j=0}^{2n_k-2} a_{kj} f^{(j)}(z_k) + \sum_{k=1}^{r} b_k \int_{\gamma_k} f dz \tag{3.5}$$

for all $f \in L_a^1(\Omega)$. *Here* $L_a^p(\Omega)$ *denotes the subspace of* $L^p(\Omega;$ *area measure*) *consisting of functions holomorphic in* Ω, m, $n_k \in \Omega$ *are related to the divisor* D *in* (3.4) *through* $D = \Sigma_{k=1}^m n_k(z_k)$ (*the* z_k *assumed distinct and* $n_k \geqq 1$), $\gamma_1, \ldots, \gamma_r$ *are suitable open or closed curves in* Ω *with all their* (*possible*) *end points belonging to* $\{z_1, \ldots, z_m\}$ *and* a_{kj}, b_k *are suitable complex constants* (*with* $a_{k,2n_k-2} \neq 0$).

Proof. Let g and W be as in Theorem 3.2. Then $\sqrt{dg} \in \Gamma(\hat{W}, \mathfrak{O}_{-\tilde{D}_1}(\lambda))$ by (b) of the theorem. It follows that dg is a meromorphic differential on $\hat{W}$ with divisor $\geqq -2\tilde{D}_1$ and with no simple poles. Now [7, Thm 3 together with Remark (4) following it] shows that a quadrature identity of the kind (3.5) holds for all $f \in L_a^2(\Omega)$. Since $L_a^2(\Omega)$ is known to be dense in $L_a^1(\Omega)$ the corollary follows. □

Example 3.1. In the simply-connected case we can take W to be $\mathbf{D}$ and represent $\hat{W}$ by $\mathbf{P}$ as in Example 2.1. Theorem 3.2 combined with Example 2.1 then gives the following characterization of simply-connected quadrature domains:

Let $g: \mathbf{D} \to \Omega$ be a conformal map, where $\Omega \subset \mathbf{C}$ has rectifiable boundary, and let D be a positive divisor in Ω, say

$$D = \sum_{j=1}^{m} n_j(z_j), \qquad z_j \in \Omega, \quad n_j \geqq 0.$$

Then

$$\int_{\partial\Omega} f|dz| = 0 \qquad \text{for all } f \in E_D(\Omega) \tag{3.6}$$

if and only if $g' = R^2$ for some rational function R with $\operatorname{div} R \geqq -\tilde{D}_1 + 1\cdot(\infty)$. Here

$$\tilde{D}_1 = \sum_{j=1}^{m} n_j \cdot (1/\bar{w}_j), \qquad g(w_j) = z_j.$$

This result, essentially, was first established in [14, Theorem 2]. In [14], however, (3.6) was considered with the set of all polynomials as test class (in place of $E(\Omega)$) and they therefore had to restrict their attention to so-called Smirnov domains Ω (see [5, Ch. 10.3]). (The polynomials are dense in $E(\Omega)$ if and only if Ω is a Smirnov domain, at least if $\partial\Omega$ is a Jordan curve. See [5, Thm 10.6].)

Of course, R has no zeroes in $\mathbf{D}$. The agrument used in the proof of Corollary 3.1 shows that R also has no zeroes on $\partial\mathbf{D}$. This result was also obtained in [14].

Example 3.2. As a special case of Example 3.1 let $D = 1\cdot(z_0)$, $z_0 \in \Omega$ so that, for some $a \in \mathbf{C}$,

$$\int_{\partial\Omega} f\,|dz| = af(z_0) \qquad \text{for all } f\in E(\Omega), \tag{3.7}$$

where $\Omega \subset \mathbf{C}$ is simply connected and $\partial\Omega$ a rectifiable continuum. We may then choose $g: \mathbf{D} \to \Omega$ in Example 3.1 such that $g(0) = z_0$. Then $D = 1\cdot(z_0)$, $\tilde{D}_1 = 1\cdot(\infty)$. Thus the rational function R in Example 3.1 is not allowed to have any poles and so must be constant (necessarily non-zero). Hence $g(z) = Az + z_0$, $A \neq 0$ and we obtain: (3.7) holds for some $a \in \mathbf{C}$ if and only if Ω is a disc with center z_0. Clearly a will equal the length of the circumference of the disc. Also $a = 2\pi A$.

If (3.7) is required to hold only for all polynomials then Ω need not be a disc. See [14, Remark 2, p. 12].

Remark 3.4. If $\Omega \subset \mathbf{C}$ is not assumed simply-connected but only to be bounded by finitely many rectifiable continua then the conclusion that (3.7) implies that Ω is a disc (with center z_0 and circumference a) still holds (with $E(\Omega)$ as test class). This follows from the facts that (3.7) by Corollary 3.3 implies that

$$\int_{\Omega} f dx dy = \frac{a^2}{4\pi} f(z_0) \tag{3.8}$$

for all $f \in L_a^2(\Omega)$ which have a single-valued integral in Ω and that (3.8) implies that Ω is a disc, by [1, Theorem 7]. Compare also Remark 6.1 in the present paper.

The results in Corollaries 3.1 and 3.3, Example 3.2 and Remark 3.4 were also obtained in [3].

4. Existence of quadrature domains of arbitrary conformal type

In this section we shall use the methods of Section 3 to prove the following.

Theorem 4.1. *Among the domains in* $\mathbf{C}$ *bounded (in* $\mathbf{P}$*) by finitely many continua there exist quadrature domains of all conformal types.*

Proof. By Theorem 3.2 the quadrature domains we are looking for are produced as follows. Take a double of a plane domain $\hat{W} = W \cup \partial W \cup \tilde{W}$ such that W is of the desired conformal type and take a half-order differential $h\sqrt{dz} \in \Gamma(\hat{W}, \mathfrak{M}(\lambda))$ with poles only in $\tilde{W}$. Then $h^2 dz$ is a meromorphic differential on $\hat{W}$ which is holomorphic in W.

Suppose we can choose h so that

$$\int_{\gamma} h^2 dz = 0 \tag{4.1}$$

for every closed curve γ in W. Then

$$g(w) = \int_{w_0}^{w} h^2 dz \qquad (w \in W) \tag{4.2}$$

defines a single-valued holomorphic function in W ($w_0 \in W$ fixed, path of integration in W). If we moreover can arrange that g is univalent in W then (by Theorem 3.2) $\Omega = g(W)$ will be a quadrature domain of the desired conformal type.

By Lemma 3.2 there exist half-order differentials $h\sqrt{dz} \in \Gamma(\hat{W}, \mathfrak{M}(\lambda))$ with arbitrarily prescribed poles. To prove the theorem we need therefore only show that the condition (4.1) and that of the univalence of g can be satisfied. For this we shall make use of the following approximation theorem of Runge type, proved in [2].

Let M be an arbitrary Riemann surface, U an open proper subset of M, ξ an arbitrary holomorphic line bundle on M. Then

(4.3) $\Gamma(U, \mathfrak{O}(\xi)) \cap \Gamma(M, \mathfrak{M}(\xi))$ is dense in $\Gamma(U, \mathfrak{O}(\xi))$ in the topology of uniform convergence on compact subsets of U.

Let $\mathfrak{A} = \{(U_j, z_j) : j = 1, 2\}$ be a holomorphic atlas on $\hat{W}$ of the usual type (as at (2.4), (2.5)). We shall apply the approximation theorem with $U = U_1$ and $\xi = \lambda$ to approximate $\sqrt{dz_1}$ in $\Gamma(U_1, \mathfrak{O}(\lambda))$ by sections $h\sqrt{dz_1} \in \Gamma(U_1, \mathfrak{O}(\lambda)) \cap \Gamma(\hat{W}, \mathfrak{M}(\lambda))$. Then g defined by (4.2) will be close to z_1, so that it has a good chance to be univalent (whenever it is single-valued).

Sections h in $\Gamma(U_1, \mathfrak{O}(\lambda)) \cap \Gamma(\hat{W}, \mathfrak{M}(\lambda))$ are represented by pairs $\{h_1, h_2\}$, $h_1 \in \Gamma(U_1, \mathfrak{O})$, $h_2 \in \Gamma(U_2, \mathfrak{M})$ satisfying

$$h_1 = \lambda_{12} h_2 \qquad \text{in } U_1 \cap U_2 \tag{4.4}$$

(λ_{12} defined by (2.10)). So what the theorem says when $\sqrt{dz_1}$ is approximated is that for any compact $K \subset U_1$ and any $\varepsilon > 0$ there is a pair $\{h_1, h_2\}$ as above with

$$|h_1 - 1| \leqq \varepsilon \qquad \text{in } K. \tag{4.5}$$

Let p be the genus of $\hat{W}$, so that the connectivity of W is $p+1$, and let $\alpha_1, \ldots, \alpha_p$ be closed oriented curves in W which make up a homology basis for W. Then the requirements (4.1) can be written

$$\int_{\alpha_k} h_1^2 dz_1 = 0 \qquad \text{for } k = 1, \ldots, p. \tag{4.6}$$

In the following we shall write just $\int_{\alpha_k} h^2$ for $\int_{\alpha_k} h_1^2 dz_1$ (similarly for other expressions).

Now apply the approximation theorem with $K = \bar{W}$. Then, given $\varepsilon > 0$, we get some $h \in \Gamma(U_1, \mathfrak{O}(\lambda)) \cap \Gamma(\hat{W}, \mathfrak{M}(\lambda))$, represented by $\{h_1, h_2\}$ satisfying (4.4) and (4.5). However we cannot be sure that (4.6) holds.

Therefore we have to adjust h a little. This we do as follows. We seek $f_1, \ldots, f_p \in \Gamma(U_1, \mathfrak{O}(\lambda)) \cap \Gamma(\hat{W}, \mathfrak{M}(\lambda))$ and complex numbers $a_1, \ldots, a_p$ such that

$$\int_{\alpha_k} \left(h + \sum_{j=1}^{p} a_j f_j\right)^2 = 0 \qquad \text{for } k = 1, \ldots, p. \tag{4.7}$$

If we can find a solution of (4.7) such that $\Sigma\, a_j f_j$ is sufficiently small on W, say such that (4.5) holds with $h + \Sigma\, a_i f_i$ in place of h, then $h + \Sigma\, a_j f_j$ will have all the properties required of h if merely $\varepsilon > 0$ is small enough. In fact, $h + \Sigma\, a_j f_j \in \Gamma(U_1, \mathfrak{O}(\lambda)) \cap \Gamma(\hat{W}, \mathfrak{M}(\lambda))$ and g, defined by (4.2) with $h + \Sigma\, a_j f_j$ in place of h, will be single-valued and it is straightforward to check that (4.5) (for $h + \Sigma\, a_j f_j$) implies that g is univalent in W if $\varepsilon > 0$ is sufficiently small.

Thus consider (4.7). It can be written

$$\int_{\alpha_k} h^2 + 2 \sum_{j=1}^{p} a_j \int_{\alpha_k} f_j h + \sum_{i=1}^{p} \sum_{j=1}^{p} a_i a_j \int_{\alpha_k} f_i f_j = 0. \tag{4.8}$$

We shall in the following write just z for the parameter z_1 on U_1. z maps W onto a domain $D = z(W) \subset \mathbf{C}$ of connectivity $p+1$. Pick one point z_k $(k = 1, \ldots, p)$ in each of the p bounded components of $\mathbf{C} \setminus D$ such that $z_k \notin z(U_1)$ (this is easily seen to be possible). We may assume that the homology basis $\alpha_1, \ldots, \alpha_p$ on W is chosen so that, for each k, $z(\alpha_k)$ has winding number $+1$ with respect to z_k and winding number zero with respect to all other z_j. Thus

$$\int_{z(\alpha_k)} \frac{dz}{z - z_j} = 2\pi i \delta_{kj}.$$

Now we first choose rational functions $R_1, \ldots, R_p$ in z such that

$$\int_{z(\alpha_k)} R_j dz = \delta_{kj} \qquad (k, j = 1, \ldots, p),$$

$$\int_{z(\alpha_k)} R_i R_j dz = 0 \qquad (i, j, k = 1, \ldots, p).$$

In fact,

$$R_j(z) = \frac{1}{2\pi i(z - z_j)} + (z - z_j)Q_j(z) \qquad (j = 1, \ldots, p),$$

where $Q_1, \ldots, Q_p$ are polynomials satisfying $Q_j(z_k) = -1/2\pi i(z_k - z_j)^2$ for all $k \neq j$ (so that $R_j(z_k) = 0$ for $k \neq j$) will do that job.

The functions $R_j \circ z$ $(j = 1, \ldots, p)$ are defined and holomorphic in U_1 and therefore represent sections in $\Gamma(U_1, \mathfrak{O}(\lambda))$. We may therefore approximate them, uniformly on $\bar{W}$ by sections $f_1, \ldots, f_p$ in $\Gamma(U_1, \mathfrak{O}(\lambda)) \cap \Gamma(\hat{W}, \mathfrak{M}(\lambda))$ (f_j approximates $R_j \circ z$). This is the way $f_1, \ldots, f_p$ will be chosen in (4.7).

With $R_j \circ z$ substituted for f_j in (4.8) it takes the form

$$\text{(4.9)} \qquad \int_{\alpha_k} h^2 + 2a_k + 2\sum_{j=1}^{p} a_j \int_{\alpha_k} (R_j \circ z)\cdot(h - 1) = 0 \qquad (k = 1, \ldots, p).$$

If h approximates 1 sufficiently well this system obviously has a unique solution in $a_1, \ldots, a_p$ and this solution moreover tends to zero as $h \to 1$.

If f_j approximates $R_j \circ z$ $(j = 1, \ldots, p)$ as above the coefficients of $\{a_j\}$ and $\{a_i a_j\}$ in (4.8) will be close to the corresponding coefficients in (4.9) and it now follows from the implicit function theorem that also (4.8) will have a solution in $a_1, \ldots, a_p$, close to that of (4.9) (and unique with this property). In particular $a_j \to 0$ $(j = 1, \ldots, p)$ as $h \to 1$ and $f_j \to R_j \circ z$ $(j = 1, \ldots, p)$ and so $\Sigma\, a_j f_j \to 0$ uniformly on W as $h \to 1$ and $f_j \to R_j \circ z$. This proves the theorem. □

5. Non-uniqueness of multiply connected quadrature domains

In this section we shall study the following question: given a functional

$$\text{(5.1)} \qquad L(f) = \sum_{k=1}^{m} \sum_{j=0}^{n_k - 1} a_{kj} f^{(j)}(z_k)$$

on $E(\Omega)$, how many different domains Ω of a fixed connectivity $p + 1$ are there in general (if any) for which

$$\text{(5.2)} \qquad \int_{\partial\Omega} f|dz| = L(f)$$

holds for all $f \in E(\Omega)$? As usual we only consider domains bounded by finitely many rectifiable continua.

Results such as that in Remark 3.4 may raise the conjecture that Ω always is uniquely determined by L. This might in fact be true in the case $p = 0$ (the author does not know), but we shall find here that it is defintely false if $p > 0$. More precisely, by counting the number of parameters available when producing quadrature domains by the method of Theorem 3.2 and comparing it with the number of parameters in the quadrature functions L we shall find that $(p + 1)$-connected quadrature domains for a fixed quadrature functional L generically occur in p-parameter families.

We shall also give a kind of geometric explanation of this result by characterizing the corresponding p-dimensional space of infinitesimal boundary variations.

Recall that a quadrature domain Ω satisfying (5.2) for some L is produced as follows (Theorem 3.2). Take a double of a plane domain $\hat{W} = W \cup \partial W \cup \tilde{W}$ of genus p (if we want Ω to have connectivity $p + 1$) and a half-order meromorphic differential $h\sqrt{dz}$ on $\hat{W}$ with poles only in $\tilde{W}$. Provided that

$$\int_{\alpha_k} h^2 dz = 0 \qquad (k = 1, \ldots, p), \tag{5.3}$$

where $\alpha_1, \ldots, \alpha_p$ is a homology basis for W as at (4.6), and that the (hence well-defined) function

$$g(w) = \int_{w_0}^{w} h^2 dz \qquad (w \in W) \tag{5.4}$$

($w_0 \in W$ fixed, path of integration in W) is univalent in W, g maps W conformally onto a domain Ω of the desired kind. If h has a pole of order n at $\tilde{w} \in \tilde{W}$ then g maps the opposite point $w \in W$ to a point $z \in \Omega$ at which L evaluates derivatives up to order $n - 1$.

In what follows we will count the number of *real* parameters and conditions involved in the above construction of Ω and L. (Thus e.g. one complex equation is counted as two conditions.)

It is convenient to represent $\hat{W}$ as the double of a horizontal slit domain $W \subset \mathbf{P}$, i.e. of the kind

$$W = \mathbf{P} \setminus \bigcup_{k=0}^{p} \{w_k + tr_k : -1 \leqq t \leqq 1\}$$

for suitable $w_k \in \mathbf{C}$, $r_k > 0$. In this way all conformal equivalence classes of connectivity $p + 1$ are covered. (The fact that many of the above W are conformally equivalent will be taken care of later.) We see that the choice of W (i.e. of $\hat{W}$) depends on $3(p+1)$ parameters.

Next we choose $h\sqrt{dz} \in \Gamma(\hat{W}, \mathfrak{M}(\lambda))$, with m poles of orders $n_1, \ldots, n_m$, say. Let $n = \Sigma_{k=1}^{m} n_k$. It follows from Lemma 3.2 that the number of parameters in the choice of $h\sqrt{dz}$ is exactly $2m + 2n$.

Going from $h\sqrt{dz}$ to g gives the $2p$ conditions (5.3), but also two new free parameters, namely the choice of w_0 in (5.4). We know from Section 4 that $h\sqrt{dz}$ can be chosen so that the conditions (5.3) are fulfilled and so that the resulting function g is univalent in W, in fact even in a neighbourhood of $\bar{W}$. By considering only half-order differentials $h\sqrt{dz}$ close to a fixed one as above we achieve that the resulting functions g will be univalent whenever they are single-valued. Thus we do not have to bother about the univalency of g.

When going from the pair (W, g) to $\Omega = g(W)$ some of the free parameters collapse since we can have $g_1(W_1) = g_2(W_2)$ for many different pairs (W_j, g_j). In fact, $g_1(W_1) = g_2(W_2)$ if and only if $\varphi = g_2^{-1} \circ g_1$ maps W_1 conformally onto W_2. Keeping (W_1, g_1) fixed it follows that pairs (W_2, g_2) mapped onto the same $\Omega = g_1(W_1)$ are in bijective correspondence to conformal mappings φ on W_1 such that $\varphi(W_1)$ is also a horizontal slit domain. It is well-known that such maps depend on six real parameters.

Summarizing we have

$$3(p+1) + 2m + 2n - 2p + 2 - 6 = 2(m+n) + p - 1 \tag{5.5}$$

parameters at our disposal for producing domains Ω of connectivity $p+1$ satisfying (5.2) for different functionals L as in (5.1), where m and n are fixed and the same in (5.1) as in (5.5). The number of parameters in those L seems to be $2(m+n)$. Actually, however, this number is at most

$$2(m+n) - 1 \tag{5.6}$$

because $\Sigma_{k=1}^{m} a_{k0}$ is necessarily real (and positive) for an L satisfying (5.2), as is seen by choosing $f \equiv 1$ in (5.2).

Comparing now the numbers (5.5) and (5.6) we see that there is an overflow of at least p parameters for Ω. This means that "generically" domains Ω of connectivity $p+1$ satisfying (5.2) for a fixed L should occur in at least p-parameter families.

We shall now confirm the above result by finding the infinitesimal variations of the boundaries generating the above p-parameter families. Consider a one-parameter family $\{\Omega_t : t \in (-\varepsilon, \varepsilon)\}$ $(\varepsilon > 0)$ of $(p+1)$-connected domains Ω_t $(t \in \mathbf{R})$ with smooth boundaries and depending smoothly on t. Interpreting t as time, let $v = v_t = v_t(z)$, $z \in \partial\Omega_t$, denote the velocity by which $\partial\Omega_t$ moves, measured in the direction of the outward normal of $\partial\Omega_t$ for each t.

It is clear that if one of the domains Ω_t is a quadrature domain, satisfying (5.2) for a certain L as in (5.1), then all the other Ω_t are also quadrature domains satisfying (5.2) with the same L if and only if

$$\frac{d}{dt}\int_{\partial\Omega_t} f|dz| = 0 \tag{5.7}$$

for all f analytic in a neighbourhood of $\bar{\Omega}_t$ (say) and for all t. We shall analyze what the condition (5.7) means for v.

Writing (temporarily) ds instead of $|dz|$ for the arc-length differential, denoting the curvature of $\partial\Omega_t$ by κ ($\kappa = \kappa_t = \kappa_t(z)$, $z \in \partial\Omega_t$) and the positively oriented unit tangent vector along $\partial\Omega_t$ by T ($T = T_t = T_t(z)$, $z \in \partial\Omega_t$) we obtain, for small $\delta > 0$ (see Fig. 1),

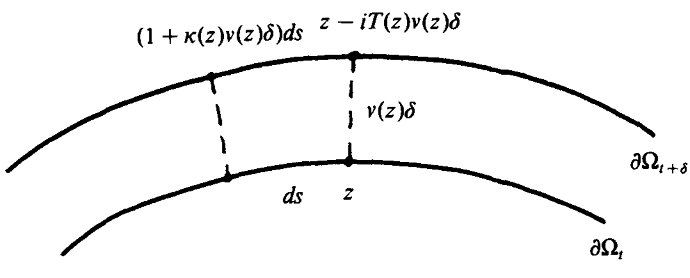

Fig. 1.

$$\begin{aligned}\int_{\partial\Omega_{t+\delta}} f ds &= \int_{\partial\Omega_t} f(z - iT(z)v(z)\delta)(1 + \kappa(z)v(z)\delta)ds + o(\delta)\\ &= \int_{\partial\Omega_t} [f(z) - iTv\delta f'(z)](ds + \kappa v\delta ds) + o(\delta)\\ &= \int_{\partial\Omega_t} fds - i\delta\int_{\partial\Omega_t} Tvf'ds + \delta\int_{\partial\Omega_t} fv\kappa ds + o(\delta).\end{aligned}$$

Thus

$$\frac{d}{dt}\int_{\partial\Omega_t} f ds = \int_{\partial\Omega_t} f v \kappa ds - i \int_{\partial\Omega_t} f' v T ds. \tag{5.8}$$

Using that $Tds = dz$ along $\partial\Omega_t$ and integrating by parts (5.8) can be written as

$$\frac{d}{dt}\int_{\partial\Omega_t} f ds = \int_{\partial\Omega_t} f v \kappa ds - i \int_{\partial\Omega_t} v df = i \int_{\partial\Omega_t} f(dv - iv\kappa ds). \tag{5.9}$$

Thus (5.7) holds if and only if $\int_{\partial\Omega_t} f(dv - iv\kappa ds) = 0$ for every f analytic in a neighbourhood of $\bar{\Omega}_t$. By a well-known theorem the latter condition is equivalent to that $dv - iv\kappa ds$ extends to a holomorphic differential in Ω_t. The most general holomorphic differential in Ω_t can be written $d(u + i{}^*u) = du + i{}^*du$, where u is a real-valued, but possibly additively multiple-valued, harmonic function in Ω_t and ${}^*u$ is its (possibly multiple-valued) harmonic conjugate. Thus (5.7) is equivalent to

$$dv - iv\kappa ds = du + i{}^*du \qquad \text{along } \partial\Omega_t \tag{5.10}$$

for some u and ${}^*u$ as above. Identifying the real parts in (5.10) gives that

$$u = v + \text{constant} \tag{5.11}$$

on each component of $\partial\Omega_t$. In particular it follows that u is actually single-valued in Ω_t.

Now continue v harmonically to Ω_t. Then (5.11) says that

$$u = v + w \qquad \text{in } \Omega_t \tag{5.12}$$

for some harmonic measure w in Ω_t. By a harmonic measure we mean a harmonic function which is constant on each component of the boundary.

The equality between the imaginary parts in (5.10) says that $-v\kappa = \partial u/\partial n$ on $\partial\Omega_t$, where $\partial/\partial n$ denotes the outward normal derivative. By taking normal derivatives of (5.12) we therefore obtain

$$\frac{\partial v}{\partial n} + \kappa v = -\frac{\partial w}{\partial n} \qquad \text{on } \partial\Omega_t. \tag{5.13}$$

Observe that u is now eliminated and that we have got a condition on v alone: if (5.7) holds then the harmonic extensions to Ω_t of v satisfy (5.13) for some harmonic measures w in Ω_t (depending on t). Conversely, it is easy to see that we can go backwards: if we have harmonic functions $v = v_t$ in Ω_t (for all t) which satisfy (5.13) for some harmonic measures w then (5.10) holds for u defined by (5.12) and hence (5.7) holds.

Thus we have characterized the boundary velocities (or the infinitesimal generators of the boundary variations) which preserve $\int_{\partial\Omega} f|dz|$ (f analytic in a neighbourhood of $\bar{\Omega}$) as those functions v on $\partial\Omega$ whose harmonic extensions to Ω satisfy (5.13).

The problem of finding v harmonic in Ω and satisfying (5.13) for a given w has a unique solution (under suitable smoothness assumptions). (Observe that κ cannot vanish on a whole component of $\partial\Omega$ since Ω is bounded.) Since the space of functions appearing in the right member of (5.13) is p-dimensional (the space of harmonic measures is $(p+1)$-dimensional, but contains the constants) we see that the space of all boundary velocities v on $\partial\Omega$ which preserve $\int_{\partial\Omega} f|dz|$ is p-dimensional, as expected.

Example 5.1. Consider the one-parameter family

$$\Omega_t = \{z \in \mathbf{C} : 1 - t < |z| < 1 + t\} \qquad (0 < t < 1)$$

of annuli with constant lengths of the boundaries. Although Ω_t are not quadrature domains we have

$$\frac{d}{dt} \int_{\partial\Omega_t} f|dz| = 0$$

for every f analytic in a neighbourhood of $\bar{\Omega}_t$. The boundary velocity $v = v_t$ of Ω_t here is identically one on $\partial\Omega_t$, hence its harmonic extension to Ω_t is identically one. The curvature κ_t of $\partial\Omega_t$ is $(1+t)^{-1}$ on the outer component and $-(1-t)^{-1}$ on the inner component. It follows that (5.13) is satisfied with $w(z) = \log|z|$ (which is a harmonic measure on each Ω_t).

6. Quadrature domains containing the point at infinity

In this section we shall generalize Theorem 3.2 to cover the case that $\Omega \subset \mathbf{P}$ contains $\infty \in \mathbf{P}$. Observe that we never can have $1 \in E(\Omega) = E^2_{1\cdot(\infty)}(\Omega)$ in this case, just $1 \in E_{-1\cdot(\infty)}(\Omega) = E^2(\Omega)$, namely if Ω is bounded by finitely many rectifiable continua (Lemma 1.2). Therefore Theorems 3.1 and 3.2 need some modifications to cover the case $\infty \in \Omega$.

Theorem 6.1. *Let $\hat{W} = W \cup \partial W \cup \tilde{W}$ be a double of a plane domain, let D_0 and D be divisors in W satisfying $D_0 \leqq D$ and let $g \in E_{D_0}(W)$. Then*

$$(f, g) = 0 \qquad \textit{for all } f \in E_D(W)$$

if and only if g extends to an element in $\Gamma(\hat{W}, \mathfrak{O}_{D_0 - \hat{D}}(\lambda))$.

Proof. With $E_D(W)^{\perp} = \{g \in E_{D_0}(W) : (f, g) = 0 \text{ for all } f \in E_D(W)\}$ the assertion of the theorem is that

$$E_D(W)^\perp = \Gamma(\hat{W}, \mathfrak{O}_{D_0-\tilde{D}}(\lambda)).$$

The inclusion $\Gamma(\hat{W}, \mathfrak{O}_{D_0-\tilde{D}}(\lambda)) \subset E_D(W)^\perp$ follows by the computation (3.3) in the proof of Theorem 3.1. Since

$$\dim \Gamma(\hat{W}, \mathfrak{O}_{D_0-\tilde{D}}(\lambda)) \geqq \deg(\tilde{D} - D_0) = \deg(D - D_0)$$

by (3.2) and

$$\dim E_D(W)^\perp = \dim(E_{D_0}(W)/E_D(W)) \leqq \deg(D - D_0)$$

the above inclusion cannot be proper, proving the theorem. □

Theorem 6.2. *Let Ω be a domain in $\mathbf{C}$ bounded by finitely many continua and with $\infty \in \Omega$, and let D be a divisor in Ω, $D \geqq -1\cdot(\infty)$. Further, let $\hat{W} = W \cup \partial W \cup \tilde{W}$ be a double of a plane domain with W conformally equivalent to Ω, let $g\colon W \to \Omega$ be a conformal map and let $D_1 = g^{-1}(D)$, $w = g^{-1}(\infty)$. Then*

(a) *$\sqrt{dg}$ exists as an element in $\Gamma(W, \mathfrak{O}_{-1\cdot(w)}(\lambda))$. Moreover $\sqrt{dg} \in E_{-1\cdot(w)}(W)$*

(b) *$\int_{\partial\Omega} f|dz| = 0$ for all $f \in E_D(\Omega) = E^2_{D+1\cdot(\infty)}(\Omega)$ if and only if $\sqrt{dg}$ extends to an element in $\Gamma(\hat{W}, \mathfrak{O}_{-\tilde{D}_1-1\cdot(w)}(\lambda))$.*

The proof is similar to that of Theorem 3.2, with Theorem 6.1 used in place of Theorem 3.1, and hence omitted.

Example 6.1. In the simply connected case we may take $\hat{W} = \mathbf{P} = \mathbf{C} \cup \{\infty\}$ with involution $j(z) = 1/\bar{z}$ and with $W = \mathbf{D}^e = \{z \in \mathbf{P} : |z| > 1\}$. (Hence $\tilde{W} = \mathbf{D}$.) Combined with Example 2.1 Theorem 6.2 then gives the following.

Suppose $\infty \in \Omega \subset \mathbf{P}$, where Ω is simply connected and has rectifiable boundary, let D be a divisor in Ω such that $D \geqq -1\cdot(\infty)$, say $D = \Sigma_{j=1}^m n_j(z_j)$, $z_j \in \Omega$. Further, let $g\colon \mathbf{D}^e \to \Omega$ be a conformal map with $g(\infty) = \infty$. Then

$$\int_{\partial\Omega} f|dz| = 0 \qquad \text{for all } f \in E_D(\Omega) = E^2_{D+1\cdot(\infty)}(\Omega)$$

if and only if $g' = R^2$ for some rational function R with $\operatorname{div} R \geqq -\tilde{D}_1 - 1\cdot(\infty) + 1\cdot(\infty) = \tilde{D}_1$, where $\tilde{D}_1 = \Sigma_{j=1}^m n_j\cdot(1/\bar{w}_j)$, $g(w_j) = z_j$.

Taking $D = 0$ e.g. we find that

$$\int_{\partial\Omega} f|dz| = 0 \qquad \text{for all } f \in E(\Omega) = E^2_{1\cdot(\infty)}(\Omega) \tag{6.1}$$

if and only if g' is constant (since a rational function R with $\operatorname{div} R \geqq 0$ necessarily in constant), hence if and only if $g(z) = Az + B$ for some $A, B \in \mathbf{C}$ with $A \neq 0$, hence if and only if Ω is the exterior of some disc in $\mathbf{C}$. □

Remark 6.1. The statement that (6.1) implies that Ω is the exterior of a disc remains true with the mere assumption on Ω that it is a domain in **P** bounded by finitely many rectifiable continua. In fact, if $\infty \notin \Omega$ then $1 \in E(\Omega)$ and (6.1) cannot hold. Thus we may assume that $\infty \in \Omega$.

Choose $\hat{W}$ and $g: W \to \Omega$ as in Theorem 6.2 with $D = 0$. Then, if (6.1) holds, $\sqrt{dg} \in \Gamma(\hat{W}, \mathfrak{O}_{-1\cdot(w)}(\lambda))$ where $w = g^{-1}(\infty) \in W$. This means that, for some constant factor $c \neq 0$, $\sqrt{dg} = c \cdot \Lambda_{\hat{w}}$ where $\Lambda_{\hat{w}}$ is the Szegö kernel (for $\tilde{W}$) as in Remark 3.2. Further,

$$dg = (\sqrt{dg})^2 \in \Gamma(W, \mathfrak{O}_{-2\cdot(w)}(\lambda^2)) = \Gamma(\hat{W}, \mathfrak{O}_{-2\cdot(w)}(\kappa)) \qquad \text{and} \quad \textstyle\int_\alpha dg = 0$$

for every closed curve α in W (or in $\tilde{W}$) since g is single-valued on W. But these properties characterize (up to a constant factor) the so-called reduced Bergman kernel $K_s = K_s(\cdot, \hat{w})$ for $\tilde{W}$, i.e. the reproducing kernel for the Hilbert space of square-integrable holomorphic differentials in $\tilde{W}$ having a single-valued integral (in $\tilde{W}$).

Thus if (6.1) holds we have $K_s = c\Lambda^2$ for some constant c (which then must be 4π). However, it is known that such a relation holds only if $\hat{W}$ has genus zero, i.e. if W is simply connected. See [8], in particular Section XII. Thus we are back in the case of Example 6.1 and Ω is the exterior of a disc, as claimed.

The above arguments may also be used to give an alternative proof of the assertion in Remark 3.4. □

7. Other aspects of quadrature identities

The following theorem contains some further aspects of quadrature identities. It has partly been suggested by Harold S. Shapiro and Jaak Peetre. For example, the condition (iii) in it is the counterpart in our context to that the so-called Friedrichs operator has finite rank (in the context of quadrature identities for area measure). See [13]. The idea to consider conditions of the kind (iv) is due to Jaak Peetre.

Theorem 7.1. *Let $\Omega \subset \mathbf{C}$ be a domain bounded by finitely many rectifiable continua and let*

$$I = \left\{ f \in E(\Omega) : \int_{\partial\Omega} fg\,|dz| = 0 \qquad \text{for all } g \in E(\Omega) \right\}.$$

Then the following conditions are equivalent.

(i) *Ω is a quadrature domain, i.e. there exists a (positive) divisor D in Ω such that*

$$\int_{\partial\Omega} f\,|dz| = 0 \qquad \textit{for all } f \in E_D(\Omega).$$

(ii) *There exists a divisor D in Ω such that*

$$E_D(\Omega) \subset I.$$

(iii) codim $I < \infty$.

(iv) *There exist continuous linear functionals $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ on $E(\Omega)$ such that*

$$\int_{\partial\Omega} fg\,|dz| = \sum_{k=1}^{m} a_k(f)b_k(g) \qquad \textit{for all } f, g \in E(\Omega).$$

Proof. (i) implies (ii): If $f \in E_D(\Omega)$ and $g \in E(\Omega)$, then, as is easily seen, $fg \in E_D^1(\Omega)$. By Corollary 3.2 (i) therefore implies that $\int_{\partial\Omega} fg\,|dz| = 0$ for $f \in E_D(\Omega)$, $g \in E(\Omega)$. But this is exactly the assertion of (ii) (with the D in (i)).

(ii) implies (iii): This is obvious, since codim $E_D(\Omega) < \infty$.

(iii) implies (iv): Assume (iii) and consider the continuous bilinear map

$$B : E(\Omega) \times E(\Omega) \to \mathbf{C},$$

$$B(f, g) = \int_{\partial\Omega} fg\,|dz|.$$

By the definition of I, B can be factorized:

$$E(\Omega) \times E(\Omega) \to E(\Omega)/I \times E(\Omega)/I \to \mathbf{C}. \tag{7.1}$$

Since I is a closed subspace of finite codimension I is the kernel of some continuous linear map $a : E(\Omega) \to \mathbf{C}^m$, $f \mapsto (a_1(f), \ldots, a_m(f))$ say, where $m =$ codim I. Hence

$$E(\Omega)/I \cong \mathbf{C}^m, \qquad [f] \mapsto (a_1(f), \ldots, a_m(f))$$

and inserting this isomorphism into (7.1) and using that the general bilinear map $\mathbf{C}^m \times \mathbf{C}^m \to \mathbf{C}$ is of the form $((z_k), (w_j)) \mapsto \sum A_{kj} z_k w_j$ the desired form of B results (with a_k as above and $b_k(g) = \sum_{j=1}^m A_{kj} a_j(g)$).

(iv) implies (iii): This is obvious, since e.g.

$$\bigcap_{k=1}^{m} \ker(a_k) \subset I.$$

(iii) implies (i): Assume (iii). Then I must contain some polynomial p, not identically zero. Write $p = qr$ where q and r are polynomials with zeroes only in Ω and Ω^c respectively. By the definition of I we have

$$\int_{\partial\Omega} qrf\,|dz| = 0 \qquad \text{for all } f \in E(\Omega). \tag{7.2}$$

Since q is bounded away from zero on $\partial\Omega$ it follows from (7.2) that

$$\int_{\partial\Omega} rf\,|dz| = 0 \qquad \text{for all } f \in E_D(\Omega), \tag{7.3}$$

where D is the divisor of q (in Ω).

Now we must get rid of the factor r in (7.3). To this end we shall prove the following.

Suppose that $0 \notin \Omega$. Then

$$zE(\Omega) \text{ is dense in } E(\Omega). \tag{7.4}$$

Suppose (7.4) is proven. Then, for each linear factor r_k in r, $r_kE(\Omega)$ will be dense in $E(\Omega)$. It is easy to see that this implies that $rE(\Omega) = r_1 \cdots r_d E(\Omega)$ is dense in $E(\Omega)$ ($d =$ the degree of r) and also, e.g. using that $E_D(\Omega) = qE(\Omega)$, that $rE_D(\Omega)$ is dense in $E_D(\Omega)$. Therefore (7.4) implies that

$$\int_{\partial\Omega} f\,|dz| = 0 \qquad \text{for all } f \in E_D(\Omega),$$

which is the desired conclusion.

It remains to prove (7.4). Let $K_1, \ldots, K_m$ be the components of $\mathbf{P} \setminus \Omega$, with $0 \in K_1$, say, and put $\Omega_j = \mathbf{P} \setminus K_j$. Let $f \in E(\Omega) = E^2(\Omega)$ ($\Omega \subset \mathbf{C}$) be the function to be approximated by functions in $zE(\Omega)$. By the decomposition theorem (1.2) we may write

$$f = f_1 + \cdots + f_m \tag{7.5}$$

where $f_j \in E^2(\Omega_j)$. This time we however do not choose f_j as in (1.3), but rather

$$f_j(z) = \frac{1}{2\pi i} \int_{\partial\Omega_j} f(w) \left(\frac{1}{w - z} - \frac{1}{w} \right) dw \qquad (z \in \Omega_j), \tag{7.6}$$

so that $f_j(0) = 0$ for $j = 2, \ldots, m$.

Since $f_j(0) = 0$ ($j \geqq 2$) we can write (7.5) as

$$f(z) = f_1(z) + z(g_2(z) + \cdots + g_m(z)) \qquad (z \in \Omega) \tag{7.7}$$

where $g_j(z) = f_j(z)/z$ ($j \geqq 2$). It is clear that $g_j \in E^2(\Omega_j) \subset E^2(\Omega)$ since $f_j \in E^2(\Omega_j)$ and $1/z$ is bounded outside a neighbourhood of $z = 0$. Hence the second term in (7.7) belongs to $zE^2(\Omega)$.

Thus it only remains to prove that $f_1 \in E^2(\Omega_1)$ in (7.7) can be approximated, in the $E^2(\Omega)$-norm, by functions in $zE^2(\Omega)$. What we shall do is to prove the stronger statement that f_1 can be approximated in the $E^2(\Omega_1)$-norm by functions in $zE^2_{1\cdot(\infty)}(\Omega_1)$. Observe that the restriction operator $E^2(\Omega_1) \to E^2(\Omega)$ is well-defined and continuous, due to the fact that $\partial\Omega \setminus \partial\Omega_1$ is rectifiable and compact in Ω_1, so the latter approximation is really stronger. More precisely, we shall prove that

$$zE^2_{1\cdot(\infty)}(\Omega_1) \text{ is dense in } E^2(\Omega_1). \tag{7.8}$$

Let $\varphi : \mathbf{D} \to \Omega_1$ be a conformal map. We first consider the case that $\infty \notin \Omega_1$. Then φ is holomorphic and bounded in $\mathbf{D}$ and $\varphi' \in H^1$ (since $1 \in E^1(\Omega_1)$; see Lemma 1.2). It follows that the map $f \mapsto (f \circ \varphi)\sqrt{\varphi'}$ is an isometric isomorphism from $E^2(\Omega_1)$ onto H^2. The statement (7.8) to be proved now takes the form: φH^2 is dense in H^2. But this is well-known to be true because every univalent functions in $\mathbf{D}$ without zeroes (such as φ) is an outer function ([5, Thm 3.17] e.g.) and then even the polynomial multiples of φ are dense in H^2.

Now suppose that $\infty \in \Omega_1$. Then φ has a pole, which we may take to be at the origin ($\varphi(0) = \infty$). By (a)(iii) in Section 1 we now get an isometric isomorphism $f \mapsto (f \circ \varphi)\sqrt{\varphi'}$ from $E^2_{1\cdot(\infty)}(\Omega_1)$ onto H^2. This isomorphism also maps $E^2(\Omega_1)$ (isometrically) onto

$$H^2_{-1\cdot(0)} = \left\{ \sum_{n=-1}^{\infty} a_n z^n : \sum |a_n|^2 < \infty \right\}.$$

Thus (7.8) takes the form: φH^2 is dense in $H^2_{-1\cdot(0)}$, or, equivalently, $z\varphi(z)H^2$ is dense in $z \cdot H^2_{-1\cdot(0)} = H^2$.

Thus it is enough to prove that $z\varphi(z)$ is an outer function in H^2 (observe that $z\varphi(z)$ is holomorphic and even bounded in $\mathbf{D}$ so that really $z\varphi(z) \in H^2$). But now $1/\varphi(z)$ is univalent and holomorphic in $\mathbf{D}$. Hence, by [5, Thm 3.16–17] $1/\varphi \in H^p$ for all $p < \frac{1}{2}$ and $1/\varphi$ has no singular inner factor. Therefore the inner factor of $1/\varphi(z)$ is just z, so that $1/z\varphi(z)$ is an outer function (in H^p, $p < \frac{1}{2}$). But this implies that also $z\varphi(z)$ is an outer function (in H^2 e.g.). This completes the proof of (iii)$\Rightarrow$(i). $\square$

References

1. D. Aharonov and H. S. Shapiro, *Domains on which analytic functions satisfy quadrature identities*, J. Analyse Math. **30** (1976), 39–73.

2. C. Auderset, *Sur le théorème d'approximation de Runge*, Enseign. Math. **26** (1980), 219–224.

3. Y. Avci, *Quadrature identities and the Schwarz function*, Doctoral Dissertation, Stanford University, 1977.

4. Pl. J. Davis, *The Schwarz Function and its Applications*, The Carus Mathematical Monographs 17, The Mathematical Association of America, 1974.

5. P. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

6. R. C. Gunning, *Lectures on Riemann Surfaces*, Princeton University Press, Princeton, 1966.

7. B. Gustafsson, *Quadrature identities and the Schottky double*, Acta Appl. Math. **1** (1983), 209–240.

8. D. Hejhal, *Theta Functions, Kernel Functions, and Abelian Integrals*, Memoirs Am. Math. Soc. no. 129, Stanford, 1972.

9. C. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

10. W. Rudin, *Analytic functions of class H_p*, Trans. Am. Math. Soc. **78** (1955), 46–66.

11. M. Schiffer, *Half-order differentials on Riemann surfaces*, SIAM J. Appl. Math. **14** (1966), 922–934.

12. M. Schiffer and N. S. Hawley, *Half-order differentials on Riemann surfaces*, Acta Math. **115** (1966), 199–236.

13. H. S. Shapiro, *Domains allowing exact quadrature identities for harmonic functions — an approach based on PDE*, in *Anniversary Volume on Approximation Theory and Functional Analysis*, P. L. Butzer, R. L. Stens and B. Sz.-Nagy (ed.), ISNM 65, Birkhäuser-Verlag, Basel, Boston, Stuttgart, 1984.

14. H. S. Shapiro and C. Ullemar, *Conformal mappings satisfying certain extremal properties, and associated quadrature identities*, Royal Institute of Technology research report TRITA—MAT-1981-6, Stockholm, 1981.

15. G. Toumarkine and S. Havinson, *On the définition of analytic functions of class E_p in multiply connected domains*, Uspehi Mat. Nauk **13** (1958), 201–206 (in Russian).

16. G. Toumarkine and S. Havinson, *On the decomposition theorem for analytic functions of class E_p in multiply connected domains*, Uspehi Mat. Nauk **13** (1958), 223–228 (in Russian).

17. G. Toumarkine and S. Havinson, *Classes de fonctions analytiques dans des domains multiplement connexes*, in *Fonctions d'une variables complexe, Problèmes contemporains*, A. I. Markouchevitch (red.), Gauthiers-Villars, Paris, 1962, pp. 37–71.

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