Hadamard's variational formula in terms of stress and strain tensors

Harold S. Shapiro, in memoriam

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Abstract

Starting from a Lagrangian action functional for two scalar fields we construct, by variational methods, the Laplacian Green function for a bounded domain and an appropriate stress tensor. By a further variation, imposed by a given vector field, we arrive at an interior version of the Hadamard variational formula, previously considered by P. Garabedian. It gives the variation of the Green function in terms of a pairing between the stress tensor and a strain tensor in the interior of the domain, this contrasting the classical Hadamard formula which is expressed as a pure boundary variation.

Keywords: Green function, Hadamard formula, stress tensor, strain tensor, energy momentum tensor, Lie derivative.
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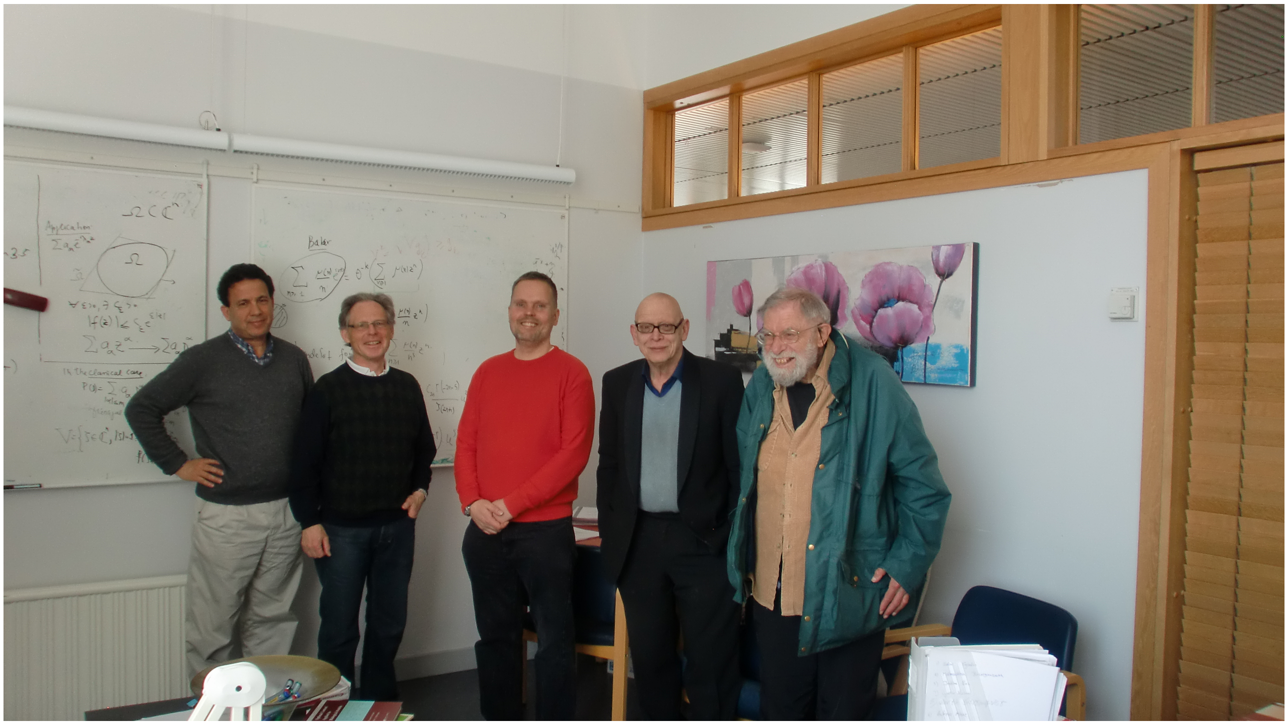

Figure 0.1: Harold S. Shapiro with some friends (colleagues and visitors) at the mathematical department of KTH, probably in April 2012. From left: Ahmed Sebbar, Björn Gustafsson, Håkan Hedenmalm, Jan-Erik Björk, Harold S. Shapiro. Photo taken by Henrik Shahgholian (in his office).

In memoriam: We believe that this paper is much in spirit of the interests and work of Harold S. Shapiro. In fact, during his long carrier at KTH, Harold S. Shapiro gave several doctoral courses, and also gave topics for doctoral theses, inspired by that book of P. Garabedian out of which this paper grew.

1 Introduction

The Hadamard variational formula expresses how the Green function for a domain changes under an infinitesimal variation of the boundary of the domain. It is usually formulated in terms of a boundary integral, like in (2.2) below. However, in his book [2], Paul Garabedian formulated the principle instead in terms of an area integral (in two dimensions) containing a generalization of the (Maxwell) stress tensor, a kind of energy-momentum tensor for the electromagnetic field (see [3]). The present paper grew out from attempts to understand Garabedian's point of view from a more general perspective.

We elaborate the subject in a setting of subdomains of a Riemannian manifold of arbitrary dimension using tools of differential geometry and tensor analysis. The main result of the paper, Theorem 6.1, expresses the Hadamard principle in terms of a bulk integral containing a stress tensor and a strain tensor. The Green function takes the role of being a physical scalar field (or potential), and it is the main ingredient in a Lagrangian action functional representing a polarized energy. In addition, the Green function turns out to coincide (except for a sign) with the value of the action at extremum. The stress tensor is obtained by varying the action with respect to the underlying Riemannian metric, while the strain tensor represents the information of how an imposed vector field deforms the domain.

2 Hadamard formula in Euclidean setting

The (Laplacian) Green function G_a for a (bounded) domain $\Omega \subset \mathbb{R}^n$ is defined by the properties

$$\begin{aligned} -\Delta G_a &= \delta_a \quad \text{in } \Omega, \\ G_a &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Writing $G(x,a) = G_a(x)$, $G(x,a)$ is symmetric with respect to x and a. This is most clearly seen by using standard Green's formulas to express the Green function as a mutual energy:

$$G(a,b) = \int_\Omega (\nabla G_a \cdot \nabla G_b)\, dx. \tag{2.1}$$

The Green function certainly depends on the domain, $G = G_\Omega$, and Hadamard's classical formula [2] expresses how $G_\Omega(x,a)$ changes under small variations of the boundary of this domain. In traditional notations it reads

$$\delta G(a,b) = \int_{\partial\Omega} \frac{\partial G(\cdot,a)}{\partial n}\frac{\partial G(\cdot,b)}{\partial n}\,\delta n\,d\sigma, \tag{2.2}$$

where δn represents an infinitesimal deformation of $\partial\Omega$ in the outward normal direction. To express this more accurately one may divide by an infinitesimal time interval δt so that $v_n = \delta n/\delta t$ represents the velocity of the boundary in normal direction under an evolution $\Omega(t)$ with respect to t. Then the formula becomes

$$\frac{d}{dt}G_{\Omega(t)}(a,b) = \int_{\partial\Omega} \frac{\partial G(\cdot,a)}{\partial n}\frac{\partial G(\cdot,b)}{\partial n}\,v_n\,d\sigma. \tag{2.3}$$

For the derivation of (2.3) it is useful to think of v_n as the normal component of a vector field $\mathbf{v}$ which is defined everywhere, and then let all of $\Omega \cup \partial\Omega$ move with $\mathbf{v}$. In particular the boundary points move, and since we shall not keep track of individual points on the boundary the effective meaning simply becomes that the speed of the boundary $\partial\Omega$ in the normal direction equals the normal component $v_n = \mathbf{v}\cdot\mathbf{n}$ of $\mathbf{v}$ on $\partial\Omega$.

Thus the tangential component of $\mathbf{v}$ on $\partial\Omega$ is insignificant for (2.3). The same is true for the interior points of Ω: the restriction of $\mathbf{v}$ to Ω never enters the formula. This is exactly what marks the difference between the formula (2.3) and the formula given in [2]. The latter formula is based on the strain on the points in Ω caused by $\mathbf{v}$. This strain makes up a *strain tensor* D_{ij}, and together with a certain *stress tensor* T^{ij} the variational formula becomes

$$\frac{d}{dt}G_{\Omega(t)}(a,b) = \int_{\Omega} T^{ij}D_{ij}\,dx + \text{source terms}.$$

See more precisely Theorem 6.1 below. Garabedian's formula appears as equation (15.20) in [2], and the stress tensor there is also given in our Example 3.1.

3 Several variations of an action functional

We shall put the variational formula (2.2) in a context of field theory, where we vary a Lagrangian action functional with respect to all fields involved.

The action is

$$S = \int_\Omega \nabla\psi_a \cdot \nabla\psi_b - \psi_a(b) - \psi_b(a), \tag{3.1}$$

a polarized energy for two real-valued scalar fields and provided with source terms (point sources at a and b). The fields are to vanish on the boundary:

$$\psi_a = \psi_b = 0 \quad \text{on } \partial\Omega.$$

Variation of S with respect to ψ_a, ψ_b and requiring it to be stationary to the first order (i.e. setting $\delta S = 0$ in traditional notation) gives

$$-\Delta\psi_a = \delta_a, \quad -\Delta\psi_b = \delta_b, \tag{3.2}$$

hence that ψ_a, ψ_b are actually the Green functions at a and b:

$$\psi_a = G_a = G(\cdot, a), \quad \psi_b = G_b = G(\cdot, b). \tag{3.3}$$

From this, together with (2.1), we see that "on-shell" (i.e. with (3.3) inserted) the action equals the Green function itself, modulo a sign:

$$S = G(a,b) - G(b,a) - G(a,b) = -G(a,b). \tag{3.4}$$

This is a negative number, which is natural since setting $\delta S = 0$ should mean that the action is minimized. Notice that $\psi_a = \psi_b = 0$ are allowed test functions.

In relativistic field theory one often introduces energy-momentum tensors by varying an action with respect to the underlying Minkowskian metric. In our case there is no time variable present, and it is more appropriate to speak of just a stress tensor (or possibly stress-energy tensor), and this can then be introduced on an abstract basis by varying the underlying Riemannian metric. So far we have not seen any metric, but the Euclidean metric $ds^2 = dx_1^2 + \cdots + dx_n^2$ actually is there, implicit in the scalar product and the nabla operator. When varying this Euclidean metric we get more general Riemannian metrics. Therefore it is natural that we, from outset, let Ω be a subdomain of a Riemannian manifold M.

We shall need notations from differential geometry, in particular those of tensor analysis and differential forms. We shall then write coordinates with upper indices, like $x^1, \dots, x^n$, and we write the metric as

$$ds^2 = g_{ij}(x)dx^i \otimes dx^j,$$

with (g_{ij}) symmetric and positive definite at each point. Summation over repeated indices (when one is up, the other down) is implied.

In this setting the action functional (3.1) becomes

$$S = \int_\Omega \frac{\partial \psi_a}{\partial x^i} \frac{\partial \psi_b}{\partial x^j} \, g^{ij} \sqrt{g} \, dx - \psi_a(b) - \psi_b(a). \tag{3.5}$$

In (3.5) we have also used the metric tensor with upper indices. By definition, (g^{ij}) represents the inverse of (g_{ij}) when these tensors are viewed as a matrices:

$$g_{ij} g^{jk} = g_i^k = \begin{cases} 1 & (i = k), \\ 0 & (i \neq k). \end{cases} \tag{3.6}$$

Moreover, $g = \det(g_{ij})$ denotes the determinant of (g_{ij}), and

$$dx = dx^1 \wedge \cdots \wedge dx^n.$$

This n-form depends on the choice of coordinates, while there is an invariant version, namely the volume form given by

$$\mathrm{vol}^n = \sqrt{g} \, dx.$$

The manifold is assumed to be oriented, and of course we choose $\sqrt{g} > 0$.

If we want to spell out all dependencies for S we may write

$$S = S[\psi_a, \psi_b; (g_{ij}); \Omega]. \tag{3.7}$$

We have already varied S with respect to ψ_a and ψ_b, this was elementary in the Euclidean setting and it extends directly to the Riemannian case. See (4.2) and thereafter for some details and coordinate expressions.

The variation with respect to (g_{ij}) is also standard, but for the sake of completeness we shall give the details. In fact, this is what makes the stress tensor pop up. When varying S with respect to (g_{ij}) it is convenient to think of (g_{ij}) as depending on a real parameter, say t, and write $g_{ij} = g_{ij}(t)$. We shall only make variations which keep (g_{ij}) symmetric. Thus, on denoting t-derivatives by a dot whenever convenient we have $\dot{g}_{ij} = \dot{g}_{ji}$.

Now $\frac{d}{dt}(g_{ij} g^{jk}) = 0$, hence $\dot{g}_{ij} g^{jk} + g_{ij} \dot{g}^{jk} = 0$. Therefore,

$$\dot{g}^{k\ell} = -g^{ki} \dot{g}_{ij} g^{j\ell}, \quad \dot{g}_{ij} = -g_{ik} \dot{g}^{k\ell} g_{\ell j}. \tag{3.8}$$

In general, if A is an $n \times n$ matrix, assumed here to be symmetric with positive eigenvalues (just for simplicity), then

$$\frac{d}{dt}(\det A) = \frac{d}{dt}(e^{\log \det A}) = e^{\log \det A} \cdot \frac{d}{dt}(\log \det A) =$$

$$= \det M \cdot \frac{d}{dt}(\operatorname{tr} \log A) = \det A \cdot \operatorname{tr} \frac{d}{dt}(\log A) = \det A \cdot \operatorname{tr}(A^{-1}\dot{A}).$$

With $A = (g_{ij})$, $g = \det(g_{ij})$ this gives

$$\dot{g} = g g^{ij} \dot{g}_{ij}.$$

It also follows that

$$\frac{d}{dt}\sqrt{g} = \frac{\dot{g}}{2\sqrt{g}} = \frac{1}{2}\sqrt{g} g^{ij} \dot{g}_{ij}. \tag{3.9}$$

Starting from (3.5) and using (3.8), (3.9) we now have

$$\frac{d}{dt} S[\psi_a, \psi_b; (g_{ij}(t)); \Omega] = \frac{d}{dt} \int_\Omega \frac{\partial \psi_a}{\partial x^k} \frac{\partial \psi_b}{\partial x^\ell} \, g^{k\ell} \sqrt{g} \, dx =$$

$$= \int_\Omega \big(- \frac{\partial \psi_a}{\partial x^k} \frac{\partial \psi_b}{\partial x^\ell} \, g^{ki} \dot{g}_{ij} g^{j\ell} \big) \sqrt{g} \, dx + \frac{1}{2} \int_\Omega \frac{\partial \psi_a}{\partial x^k} \frac{\partial \psi_b}{\partial x^\ell} \, g^{k\ell} \sqrt{g} g^{ij} \dot{g}_{ij} \, dx =$$

$$= \frac{1}{2} \int_\Omega \Big(- \frac{\partial \psi_a}{\partial x^k} \frac{\partial \psi_b}{\partial x^\ell} \, g^{ki} g^{j\ell} - \frac{\partial \psi_a}{\partial x^k} \frac{\partial \psi_b}{\partial x^\ell} \, g^{kj} g^{i\ell} + \frac{\partial \psi_a}{\partial x^k} \frac{\partial \psi_b}{\partial x^\ell} \, g^{k\ell} g^{ij} \Big) \dot{g}_{ij} \sqrt{g} \, dx =$$

$$= -\frac{1}{2} \int_\Omega T^{ij} \dot{g}_{ij} \sqrt{g} \, dx.$$

Here we have defined the *stress tensor* in contravariant form (upper indices) as that tensor $T^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ which has components

$$T^{ij} = \frac{\partial \psi_a}{\partial x^k} \frac{\partial \psi_b}{\partial x^\ell} g^{ki} g^{j\ell} + \frac{\partial \psi_a}{\partial x^k} \frac{\partial \psi_b}{\partial x^\ell} g^{kj} g^{i\ell} - \frac{\partial \psi_a}{\partial x^k} \frac{\partial \psi_b}{\partial x^\ell} g^{k\ell} g^{ij}. \tag{3.10}$$

In covariant form (lowered indices) it is the tensor $T_{ij} dx^i \otimes dx^j$ with

$$T_{ij} = \frac{\partial \psi_a}{\partial x^i} \frac{\partial \psi_b}{\partial x^j} + \frac{\partial \psi_a}{\partial x^j} \frac{\partial \psi_b}{\partial x^i} - \frac{\partial \psi_a}{\partial x^k} \frac{\partial \psi_b}{\partial x^\ell} \, g^{k\ell} g_{ij}. \tag{3.11}$$

Example 3.1. When $n = 2$ and $g_{ij} = \delta_{ij}$ we get, on setting $\psi_a = G_a$, $\psi_b = G_b$, $x = x^1$, $y = x^2$,

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial G_a}{\partial x}\frac{\partial G_b}{\partial x} - \frac{\partial G_a}{\partial y}\frac{\partial G_b}{\partial y} & \frac{\partial G_a}{\partial x}\frac{\partial G_b}{\partial y} + \frac{\partial G_a}{\partial y}\frac{\partial G_b}{\partial x} \\ \frac{\partial G_a}{\partial x}\frac{\partial G_b}{\partial y} + \frac{\partial G_a}{\partial y}\frac{\partial G_b}{\partial x} & \frac{\partial G_a}{\partial y}\frac{\partial G_b}{\partial y} - \frac{\partial G_a}{\partial x}\frac{\partial G_b}{\partial x} \end{pmatrix}.$$

This is exactly the expression given in Garabedian [2].

4 Classical Hadamard by Lie derivatives

The final step now is to vary the action (3.5), (3.7) with respect to the domain Ω, and so obtain the Hadamard formula. This step becomes more elegant when expressed in the language of differential forms.

We let the smoothly bounded domain $\Omega = \Omega(t) \subset M$ move in the flow of a vector field $\mathbf{v} = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}$, and denote by $\mathcal{L}_{\mathbf{v}}$ the Lie derivative, and by $i(\mathbf{v})$ interior derivation ("contraction"), with respect to $\mathbf{v}$. See in general Frankel [1] for differential geometric concepts and notations.

In terms of differential forms the representation (2.1) of the Green function becomes

$$G(a, b) = \int_\Omega dG(\cdot, a) \wedge *dG(\cdot, b), \tag{4.1}$$

the star being the Hodge star operator. When acting on a one-form $\nu = v_j dx^j$, the Hodge star is related to interior derivation with the corresponding vector field $\mathbf{v} = v^k \frac{\partial}{\partial x^k}$ by $*\nu = i(\mathbf{v})\mathrm{vol}^n$. This makes the n-form

$$d * \nu = (\sqrt{g}\, v^k)_{,k}\, dx = v^k_{;k}\sqrt{g}\, dx = (\operatorname{div} \mathbf{v})\mathrm{vol}^n \tag{4.2}$$

have the role of being the "divergence" of ν. Above we have used some standard tensor analysis notations, like

$$v^k_{,j} = \frac{\partial v^k}{\partial x^j}, \quad v^k_{;j} = v^k_{,j} + \Gamma^k_{j\ell} v^\ell$$

for ordinary and covariant derivatives (respectively), with

$$\Gamma^k_{j\ell} = \frac{1}{2} g^{ki}(g_{\ell i,j} + g_{ij,\ell} - g_{j\ell,i})$$

denoting the "Christoffel symbols" (connection coefficients). Implicit in (4.2) is the crucial identity $(v^k\sqrt{g})_{,k} = v^k_{;k}\sqrt{g}$. For the Laplacian of a function ϕ we have, similarly,

$$d * d\phi = \frac{\partial}{\partial x^k}(\sqrt{g}\, g^{kj}\frac{\partial \phi}{\partial x^j})dx = \phi_{;kj}g^{kj}\sqrt{g}\, dx = \Delta\phi \,\mathrm{vol}^n.$$

We remark also that $\mathrm{vol}^n = *1$.

Now the action functional becomes

$$S = \int_\Omega d\psi_a \wedge *d\psi_b - \psi_a(b) - \psi_b(a). \tag{4.3}$$

Here the Riemannian metric is not visible, but it is built into the Hodge operator. Varying S in (4.3) with respect to ψ_a (for example) gives, by partial integration, that

$$-d * d\psi_b = \varepsilon_b,$$

where ε_b denotes the Dirac measure at b regarded as a n-form current. The relation to δ_b as a Dirac "function" (or distribution) is

$$\varepsilon_b = \delta_b \,\mathrm{vol}^n = \delta_b\sqrt{g}\, dx.$$

In view of (3.4) it is a matter of taste whether one performs the variation with respect to Ω in the equation (4.1) for $G(a, b)$ or in the expression (4.3) for S, but it is slightly more elegant and general to work directly with (4.3) as far as possible. To simplify notation we set

$$\alpha = \alpha_j dx^j = d\psi_a, \quad \beta = \beta_j dx^j = d\psi_b, \quad \Phi = \alpha_k\beta^k. \tag{4.4}$$

Thus $\alpha_j = \partial\psi_a/\partial x^j$, $\beta_j = \partial\psi_b/\partial x^j$. Clearly $d\alpha = d\beta = 0$ in Ω, and ψ_a, ψ_b being constant (zero) on the boundary give that $\alpha = \beta = 0$ along $\partial\Omega$.

Let

$$d\sigma = \mathrm{vol}^{n-1} = i(\mathbf{n})\mathrm{vol}^n$$

denote the surface area element on $\partial\Omega$ when this is regarded as a manifold in itself, and where $\mathbf{n}$ denotes the outward unit normal vector on $\partial\Omega$. We then interprete

$$i(\mathbf{v})\alpha = \frac{\partial\psi_a}{\partial n}\, v_n \quad \text{on } \partial\Omega,$$

$$*\beta = \frac{\partial\psi_b}{\partial n}\, d\sigma \quad \text{along } \partial\Omega.$$

Two basic properties of the Lie derivative are the "homotopy" formula

$$\mathcal{L}_{\mathbf{v}} = d \circ i(\mathbf{v}) + i(\mathbf{v}) \circ d$$

(when acting on differential forms) and the fact that for a domain $\Omega(t)$ (or chain of integration of any sort) moving in the flow of $\mathbf{v}$,

$$\frac{d}{dt} \int_{\Omega(t)} (\dots) = \int_{\Omega(t)} \mathcal{L}_{\mathbf{v}}(\dots).$$

In the notations (4.4) the action S takes the form

$$S = \int_{\Omega} \alpha \wedge *\beta - \psi_a(b) - \psi_b(a) = \int_{\Omega} \Phi \, \mathrm{vol}^n - \psi_a(b) - \psi_b(a).$$

From this it follows that

$$\frac{dS}{dt} = \int_{\Omega} \mathcal{L}_{\mathbf{v}}(\alpha \wedge *\beta) = \int_{\Omega} (d \circ i(\mathbf{v}) + i(\mathbf{v}) \circ d)(\alpha \wedge *\beta) =$$

$$= \int_{\Omega} d(i(\mathbf{v})(\alpha \wedge *\beta)) + 0 = \int_{\partial\Omega} i(\mathbf{v})(\alpha \wedge *\beta) =$$

$$= \int_{\partial\Omega} (i(\mathbf{v})\alpha) \wedge *\beta - \int_{\partial\Omega} \alpha \wedge i(\mathbf{v})(\beta) = \int_{\partial\Omega} \frac{\partial\psi_a}{\partial n} \frac{\partial\psi_b}{\partial n} \, v_n d\sigma - 0.$$

This result can be rewritten as

$$\frac{d}{dt} \int_{\Omega(t)} d\psi_a \wedge *d\psi_b = \int_{\partial\Omega} \frac{\partial\psi_a}{\partial n} \frac{\partial\psi_b}{\partial n} \, v_n d\sigma.$$

We never used that ψ_a, ψ_b eventually are to be the Green functions of Ω, but inserting finally (3.3) and using (4.1) gives the Hadamard formula in its classical form (2.3).

5 Divergence of stress tensor

In terms of α, β, Φ in (4.4) the covariant version (3.11) of the stress tensor can be written

$$T_{ij} = \alpha_i \beta_j + \alpha_j \beta_i - g_{ij} \alpha_k \beta^k = \alpha_i \beta_j + \alpha_j \beta_i - \Phi \, g_{ij}. \tag{5.1}$$

The trace of T is

$$\operatorname{tr} T = T_{ij} g^{ij} = (2-n)\Phi, \tag{5.2}$$

and the Green function becomes, with (3.3) in force,

$$G(a,b) = \int_\Omega \alpha \wedge *\beta = \int_\Omega \Phi \operatorname{vol}^n. \tag{5.3}$$

The contravariant version (3.10) of the stress tensor has components

$$T^{ij} = T_{rs}\, g^{ri} g^{sj} = \alpha^i \beta^j + \alpha^j \beta^i - \Phi\, g^{ij}. \tag{5.4}$$

The divergence of this tensor is obtained by contracting the second index with the covariant derivative:

$$\operatorname{div} T = T^{ij}_{;j} \frac{\partial}{\partial x^i}.$$

Using the fact that all covariant derivatives of the metric tensor vanish and that, by (4.4), $\alpha_{i;j} = \alpha_{j;i}$, $\beta_{i;j} = \beta_{j;i}$ we have

$$T^{ij}_{;j} = \alpha^i_{;j}\beta^j + \alpha^i \beta^j_{;j} + \alpha^j_{;j}\beta^i + \alpha^j \beta^i_{;j} - \Phi_{;j}\, g^{ij} = \alpha^i \beta^j_{;j} + \beta^i \alpha^j_{;j}\,. \tag{5.5}$$

When $\psi_a = G_a$, $\psi_b = G_b$, so that $\alpha^j_{;j} = \Delta\psi_a = -\delta_a$, $\beta^j_{;j} = \Delta\psi_b = -\delta_b$, equation (5.5) becomes

$$T^{ij}_{;j} = -\alpha^i \delta_b - \beta^i \delta_a. \tag{5.6}$$

The right member in (5.6) can be interpreted as a source concentrated at the points a and b. More precisely, it is a vector field with distributional coefficients (a vector current) composed by the isolated vector $-\nabla G_a$ sitting (like a point charge) at the point b and the vector $-\nabla G_b$ sitting at a. In summary we have

Lemma 5.1. *When $\psi_a = G_a$, $\psi_b = G_b$ the divergence of the stress tensor vanishes except for the two point source field given in (5.6). Expressed in vector notation:*

$$\operatorname{div} T = -(\nabla G_a)\, \delta_b - (\nabla G_b)\, \delta_a. \tag{5.7}$$

6 Hadamard in terms of stress and strain tensors

In the final computation in Section 4 we had an integral over Ω involving a Lie derivative, and this integral was pushed to the boundary. But there is also the possibility not to go to the boundary. Then also the vector field $\mathbf{v}$ becomes differentiated, and one may arrange matters so that the derivatives of $\mathbf{v}$ appear only in a certain strain tensor D. This is to be paired with the stress tensor T discussed in Sections 3 and 5.

The stress and strain tensors are the main actors in linear elasticity theory, where the basic result, Hooke's law (first formulated in 1678), expresses that the stress and strain tensors are proportional (more exactly, linearly related) to each other for an elastic material (see in general [4]).

Given a vector field $\mathbf{v}$, thought of as representing an infinitesimal deformation of some material, the corresponding *strain tensor* D is the symmetric covariant tensor defined by

$$2D = 2D_{ij}(x)\,dx^i \otimes dx^j = \mathcal{L}_{\mathbf{v}}(g_{ij}dx^i \otimes dx^j). \tag{6.1}$$

The components of D are given by

$$2D_{ij} = g_{ik}v^k_{;j} + g_{kj}v^k_{;i} = v_{i;j} + v_{j;i}.$$

Using this we can now formulate the following generalization of equation (15.20) in [2].

Theorem 6.1. *The variation of the Green function $G_\Omega(a,b)$ due to a deformation of $\Omega \subset M$ driven by a smooth vector field $\mathbf{v}$ is, in terms of the stress tensor $T = T(a,b)$ and the strain tensor $D = D(\mathbf{v})$, given by*

$$\frac{d}{dt}G_{\Omega(t)}(a,b) = \int_\Omega T^{ij}D_{ij}\,\mathrm{vol}^n - \mathbf{v}(G_a)(b) - \mathbf{v}(G_b)(a). \tag{6.2}$$

In the right member $\mathbf{v} = v^j\frac{\partial}{\partial x^j}$ is regarded as a derivation (directional derivative).

Proof. Using (5.3) and (4.2) (essentially) we first have

$$\frac{d}{dt}G_{\Omega(t)}(a,b) = \int_\Omega \mathcal{L}_{\mathbf{v}}(\alpha \wedge *\beta) = \int_\Omega \mathcal{L}_{\mathbf{v}}(\Phi\,\mathrm{vol}^n) =$$

$$= \int_\Omega d(i(\mathbf{v})\Phi\,\mathrm{vol}^n) = \int_\Omega d(i(\Phi\mathbf{v})\,\mathrm{vol}^n) = \int_\Omega (\Phi\, v^j)_{;j}\,\mathrm{vol}^n. \tag{6.3}$$

The next step is to show that

$$\int_\Omega (\Phi\, v^j)_{;j}\, \mathrm{vol}^n = \int_\Omega (T^{ij} v_i)_{;j}\, \mathrm{vol}^n. \tag{6.4}$$

This will be achieved by pushing the difference between the two members to the boundary, after which cancellations will make it disappear.

Let $\mathbf{n}$ denote the outward unit normal vector on $\partial\Omega$ and let $n_j dx^j$ be the corresponding one-form. It may be realized as $n_j dx^j = du$ for a function u which is defined near $\partial\Omega$, vanishes on $\partial\Omega$, and increases away from Ω with $|\nabla u| = 1$ on $\partial\Omega$. (One may take $u = -G_a/|\nabla G_a|$, for example.) Then inserting (4.4), (5.4) and using Stokes' formula we have

$$\int_\Omega (T^{ij}\, v_i)_{;j}\, \mathrm{vol}^n - \int_\Omega (\Phi\, v^j)_{;j}\, \mathrm{vol}^n =$$

$$= \int_\Omega \big((\alpha^i\beta^j + \alpha^j\beta^i - \alpha_k\beta^k g^{ij} - \alpha^k\beta_k g^{ij})\, v_i \big)_{;j}\, \mathrm{vol}^n =$$

$$= \int_\Omega \big((\alpha^i\beta^j - \alpha_k\beta^k g^{ij}) v_i + (\alpha^j\beta^i - \alpha^k\beta_k g^{ij})\, v_i \big)_{;j}\, \mathrm{vol}^n =$$

$$= \int_{\partial\Omega} \big((\alpha^i\beta^j - \alpha_k\beta^k g^{ij}) v_i + (\alpha^j\beta^i - \alpha^k\beta_k g^{ij})\, v_i \big)\, n_j d\sigma =$$

$$= \int_{\partial\Omega} \beta^j (\alpha_i n_j - \alpha_j n_i)\, v^i d\sigma + \int_{\partial\Omega} \alpha^j (\beta_i n_j - \beta_j n_i)\, v^i d\sigma = 0.$$

In the last step we used that, along the boundary $\partial\Omega$,

$$\alpha_i dx^i = \beta_i dx^i = n_i dx^i = 0,$$

hence that the covectors with components α_i, β_i, n_i are proportional at each point of $\partial\Omega$. From this it follows that

$$\alpha_i n_j = \alpha_j n_i, \quad \beta_i n_j = \beta_j n_i \quad \text{for all } \ i, j.$$

Thus (6.4) is now established.

Finally, using the symmetries of T and D together with (5.6), (5.7) we can continue the right member of (6.4) by

$$\int_\Omega (T^{ij} v_i)_{;j}\, \mathrm{vol}^n = \int_\Omega (T^{ij}_{;j} v_i + T^{ij} v_{i;j})\, \mathrm{vol}^n =$$

$$= \int_\Omega (-\alpha^i \delta_b - \beta^i \delta_a) v_i \, \mathrm{vol}^n + \int_\Omega T^{ij} D_{ij} \, \mathrm{vol}^n =$$

$$= -\mathbf{v}(G_a)(b) - \mathbf{v}(G_b)(a) + \int_\Omega T^{ij} D_{ij} \, \mathrm{vol}^n.$$

Combing this with (6.3), (6.4) completes the proof.

□

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