Shape Reconstruction From Moments: Theory, Algorithms, and Applications

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ABSTRACT

In many areas of science and engineering, it is of interest to find the shape of an object or region from indirect measurements. For instance, in geophysical prospecting, gravitational or magnetic field measurements made on the earth's surface are used to detect an oil reservoir deep inside the earth. In a different application, X-ray attenuation measurements are used in Computer Assisted Tomography (CAT) to reconstruct the shape and density of biological or inorganic materials for diagnostic and other purposes. It turns out that in these two rather disparate areas of application, among many others, the partial information can actually be distilled into moments of the underlying shapes we seek to reconstruct.

Moments of a shape convey geometric information about it. For instance, the area, center of mass, and moments of inertia of an object give a rough idea of how large it is, where it is located, how round it is, and in which direction it is elongated. For simple shapes such as an ellipse, this information is sufficient to uniquely specify the shape. However, it is well-known that, for a general shape, the infinite set of moments of the object is required to uniquely specify it. Remarkable exceptions are simple polygons, and a more general class of shapes called quadrature domains that are described by semi-algebraic curves. These exceptions are of great practical importance in that they can be used to approximate, arbitrarily closely, *any* bounded domain in the plane. In this paper, we will describe our efforts directed at developing the mathematical basis, including some stable and efficient numerical techniques for the reconstruction of (or approximation by) these classes of shapes given "measured" moments.

Keywords: Inverse problem, shape, moments, polygon, quadrature domain, algebraic curve, tomography

1. INTRODUCTION

In many areas of science and engineering it is of interest to find the shape of an object or region from indirect measurements which can actually be distilled into moments of the underlying shapes we seek to reconstruct. This paper is concerned with solving a variety of inverse problems, using tools from the method of moments. The problem of reconstructing a function and/or its domain given its moments is ubiquitous in both pure and applied mathematics. Numerous applications from diverse areas such as probability and statistics,[1] signal processing,[2] computed tomography,[3,4] and inverse potential theory[5,6] (magnetic and gravitational anomaly detection) can be cited, to name just a few. In statistical applications, time-series data may be used to estimate the moments of the underlying density, from which an estimate of this probability density may be sought. In computed tomography, the X-rays of an object can be used to estimate the moments of the underlying mass distribution, and from these the shape of the object being imaged may be estimated.[3,4] Also, in geophysical applications, the measurements of the

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exterior gravitational field of a region can be readily converted into moment information, and from these, the shape of the region may be determined.[12]

The problem of moments has appeared in many guises since it was considered by A. A. Markov beginning in 1883 (having led to his proof of the Chebyshev inequalities and consequent derivation of the law of large numbers). In all these guises, the moment problem is universally recognized as a notoriously difficult inverse problem which often leads to the solution of very ill-posed systems of equations that usually do not have a unique solution. In particular, the two-dimensional moment problem is not widely studied. The problems treated in the present paper relate to the reconstruction (or approximation) of the shape of a plane region of constant density from its moments. This problem also suffers from severe numerical instabilities and the solutions are not always unique. However, several aspects of what we shall henceforth call the shape-from-moments problem render this a rather interesting topic. The first is that, contrary to most cases, this particular manifestation of the moment problem allows a complete, closed-form solution. More remarkable still is the fact that the solutions are based on techniques of numerical linear algebra, such as generalized eigenvalue problems, which can not only yield stable and fast algorithms, but also expose a seemingly deep connection between the shape-from-moments problem and the theory of numerical quadrature over planar regions.[12,13] In fact, this connection is so fundamental that one may consider the two problems as duals. At the same time, the techniques for solving the shape reconstruction problem are intimately related to so-called array processing techniques.[8,7]

To put this problem into a larger perspective, we note that inverse problems for uniform density regions related to general elliptical equations can all be cast as moment problems which fall within the scope of application of the results of this paper. To maintain focus, however, we first approach the shape-from-moments problem directly and without reference to a particular application. We wish to emphasize that many interesting numerical problems associated with this topic remain to be studied, and it is our hope that this paper will serve as a starting place for the continued analysis of such questions.

In section 2, we discuss the problem of reconstructing polygons from their moments, and in section 3 we describe the reconstruction of the more general semi-algebraic quadrature domains from their moments. In section 4 we provide some example reconstructions and discuss some applications. Finally, in section 5, we provide some concluding remarks and briefly discuss some future prospects.

2. POLYGON RECONSTRUCTION AND QUADRATURE

During a luncheon conversation over forty-five years ago, Motzkin and Schoenberg discovered a beautiful quadrature formula over triangular regions of the complex plane.[9] Namely, given a function $f(z)$, analytic in the closure of a triangle T, they showed that the integral of the second derivative $f''(z)$ with respect to the area measure $dx\, dy$ is proportional to the second divided difference of f with respect to the vertices z_1, z_2, z_3, of the triangle, with the proportionality constant being twice the area of T. Later, Davis[10,11] generalized this result to polygonal regions:

THEOREM 2.1 (DAVIS). *Let $z_1, z_2, \cdots, z_n$ designate the vertices of a polygon P in the complex plane. Then we can find constants $a_1, \cdots, a_n$ depending upon $z_1, z_2, \cdots, z_n$, but independent of f, such that for all f analytic in the closure of P,*

$$\int\!\!\int_P f''(z)\, dx\, dy = \sum_{j=1}^{n} a_j f(z_j)\, . \tag{1}$$

When the left-hand side is being sought, the above formula is, of course, a quadrature formula. However, let us assume for a moment that the region P is unknown but that its moments with respect to some basis such as $\{z^k\}$ are given. Replacing the function $f(z)$ with the elements of this basis in (1) results in an expression proportional to the moments on the left-hand side, while the unknown vertices z_j appear on the right-hand side. The shape from moments problem then is concerned with solving for the unknown vertices and amplitudes a_j from knowledge of these moments. Hence, depending upon whether we seek to compute that left-hand side of (1), or assume that it is known, we have either a quadrature problem or a shape from moments problem. In this sense, whenever such a formula exists, numerical quadrature and shape reconstruction from moments can be regarded as dual problems.

Returning to Theorem 2.1, if we assume that the vertices z_j of P are arranged, say, in the counterclockwise direction in the order of increasing index, and extending the indexing of the z_j cyclically, so that $z_0 = z_n$, $z_1 = z_{n+1}$, the coefficients a_j can be written as (see[11]):

$$a_j = \frac{i}{2}\left(\frac{\overline{z}_{j-1} - \overline{z}_j}{z_{j-1} - z_j} - \frac{\overline{z}_j - \overline{z}_{j+1}}{z_j - z_{j+1}}\right). \tag{2}$$

The expression for a_j has a naturally intuitive interpretation. If ϕ_j denotes the angle of the side $< z_j z_{j+1} >$ with the positive real axis, then

$$\alpha_j = \frac{\overline{z}_j - \overline{z}_{j+1}}{z_j - z_{j+1}} = e^{-2i\phi_j}, \tag{3}$$

where $i = \sqrt{-1}$. In fact, α_j is in essence the complex analogue of *slope* for the line $< z_j z_{j+1} >$. Hence, the coefficients $a_j = (e^{-2i\phi_{j-1}} - e^{-2i\phi_j})\frac{i}{2}$ can be interpreted as the difference in slope of the two sides meeting at the vertex z_j. Therefore, the a_j are nonzero if, and only if, the polygon is non-degenerate. Furthermore, these coefficients can be written even more succinctly as

$$a_j = \sin(\phi_{j-1} - \phi_j)e^{-i(\phi_{j-1}+\phi_j)}, \tag{4}$$

which shows that for a non-degenerate polygon, $0 < |a_j| \leq 1$. When $|a_j|$ is unity, we have a right angle at vertex z_j, whereas when $|a_j|$ is near zero, the polygon is nearly degenerate at that vertex.

Defining the *complex analytic* moments of an n-sided polygonal region P by

$$c_k = \iint_P z^k \, dx \, dy \ , \tag{5}$$

we can compute these directly by invoking Theorem 2.1. Namely, by replacing $f(z) = z^k$, we get

$$\iint_P (z^k)'' \, dx \, dy = k(k-1)\iint_P z^{k-2} \, dx \, dy = k(k-1)c_{k-2} = \sum_{j=1}^{n} a_j z_j^k. \tag{6}$$

The related moments τ_k are then defined as

$$\tau_k \equiv k(k-1)c_{k-2} = \sum_{j=1}^{n} a_j z_j^k, \tag{7}$$

where, by definition, $\tau_0 = \tau_1 = 0$. Using Prony's method in[4] , we showed that given $c_0, c_1, \cdots, c_{2n-3}$, or equivalently, $\tau_0, \tau_1, \cdots, \tau_{2n-1}$, the vertices of the n-gon can be uniquely recovered. A much more general and stable technique for this inversion, based on the computation of the generalized eigenvalues, was derived in.[12]

More specifically, in the basis $\{z^k\}$, the moment expression (7) can be used to construct two Hankel matrices H_0 and H_1, as follows:

$$H_0 = \begin{bmatrix} \tau_0 & \tau_1 & \cdots & \tau_{n-1} \\ \tau_1 & \tau_2 & \cdots & \tau_n \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{n-1} & \tau_n & \cdots & \tau_{2n-2} \end{bmatrix}, \tag{8}$$

where H_1 has the same form as H_0, but starts with τ_1 instead of τ_0 and ends with τ_{2n-1}. As also discussed in,[8] these Hankel matrices have the following useful factorizations:

$$H_0 = VDV^T \tag{9}$$
$$H_1 = VDZV^T \ , \tag{10}$$

where $Z = diag(z_1, \cdots, z_n)$, $D = diag(a_1, \cdots, a_n)$, and V is the Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{bmatrix}. \tag{11}$$

Therefore, H_0 and H_1 are simultaneously diagonalized by V^{-1}:

$$V^{-1}H_0V^{-T} = D \tag{12}$$
$$V^{-1}H_1V^{-T} = DZ \tag{13}$$

and hence, the generalized eigenvalue problem

$$H_1\mu = zH_0\mu \tag{14}$$

has the solutions $\{z_j\}$ which are the polygon vertices we seek. The pencil problem can be more generally formulated over a different polynomial basis $\{p_k(z)\}$ constructed from a linear combination of the elements of $\{z^k\}$. As we have shown in,[12] by careful choice of the basis, this can yield an equivalent generalized eigenvalue problem, but one that can be numerically much better conditioned.

Once the vertices z_j have been determined, there exist several techniques for computing the coefficients a_j. In general, since the ordering of the vertices is not known a priori, we cannot use (2). The simplest technique is to use all the available moments (7) for $k = 0, \cdots, N-1$ and solve

$$V_N \mathbf{a} = \mathcal{T}_N \ , \tag{15}$$

where $\mathcal{T}_N = [\tau_0, \tau_1, \cdots, \tau_{N-1}]^T$. Fast Vandermonde solvers can be used for (15). These can be more accurate for some data vectors $\mathcal{T}_N$ with particular orderings of the z_j's. It is useful to note that the first two rows of (15) corresponding to $\tau_0 = \tau_1 = 0$ should be treated as linear *constraints* rather than data. Doing this yields a smaller linear problem (by two rows), with a pair of linear constraints. The magnitude of $|a_j|$, being upper bounded by unity, can also be used to further constrain and stabilize the least-squares problem.

3. RECONSTRUCTION OF QUADRATURE DOMAINS

As a major generalization to the results described above, recently discovered results identify a well-studied class of algebraic planar domains – known as quadrature domains (QD)– with a distinguished part of the extremal solutions of the truncated problem of moments in two dimensions. This important discovery establishes deep connections between the shape-from-moments problem and the fields of potential theory, complex analytic functions and even partial differential operators with analytic coefficients. Formally, a quadrature domain Ω is a bounded planar domain with the property that there exists a distribution u of finite support, contained in Ω, so that:

$$\int\int_\Omega f(z)\, dx\, dy = u(f) = \sum_{k=1}^{m}\sum_{j=0}^{\nu_k-1} \alpha_{kj} f^{(j)}(\gamma_k), \tag{16}$$

where $\gamma_k \in \Omega$ are the quadrature *nodes*, $1 \leq k \leq m$, $N = \nu_1 + \nu_2 + \ldots + \nu_m$, and f is any analytic, integrable function in Ω. This class of domains was singled out in the paper.[14] To give the simplest example, the ball of radius r centered at the origin, denoted $B_r(0)$, satisfies the following quadrature identity:

$$\int\int_{B_r(0)} f\, dx\, dy = \pi r^2 f(0), \tag{17}$$

for all analytic, integrable functions f defined in $B_r(0)$. This is also sometimes known as the mean value theorem for harmonic functions.

In general, a quadrature domain Ω has real algebraic boundary, given by a polynomial equation:

$$\Omega = \{z \in \mathrm{C};\ P(z, \overline{z}) < 0\}, \tag{18}$$

Moreover, the degree in each variable separately of the polynomial P is equal to the number N of points (counting multiplicity) in the support of the distribution u. The integer N is called the order of the quadrature domain Ω. So the ball discussed earlier is a first order QD. More details on these questions can be found in[16] and.[17]

Initially, the class of quadrature domains was introduced by Aharonov and Shapiro[14] and extensively studied by M. Sakai,[15] B. Gustafsson[16] and several other mathematicians (see the monograph[17] for details). We have proposed constructive reconstruction algorithms[13] which are exact on all quadrature domains. Since quadrature domains are dense, in the Hausdorff metric, among all domains with continuous boundary, every planar domain can therefore be approximated by a sequence of quadrature domains,[16] and such algorithms may be applied to the approximate reconstruction of general shapes from a finite set of their moments.

In the more general present context, the relevant moments of a domain Ω are the quantities

$$c_{mn} = \int\int_{\Omega} z^m \overline{z}^n dx\, dy, \qquad m, n \in \mathrm{N}, \tag{19}$$

which as the reader will note, are a more general definition of the complex moments c_k used in the polygonal case discussed in the previous section; namely, $c_k = c_{k,0}$. Also, note that these moments contain a certain amount of redundancy since $c_{mn} = \overline{c}_{nm}$.

The problem of of interest here is to see which domains Ω are determined by the finite set of moments $\{c_{mn};\ m, n \leq N\}$. As described in,[13] a unique solution to this moment problem in the plane exists if and only if the boundary of the domain Ω can be described by an algebraic equation of the type

$$P(x, y) = P(z, \overline{z}) = 0, \tag{20}$$

where P is a real polynomial of degree less than or equal to N in each variable. Contrary to the one-variable case, there has been so far no general constructive way of passing from the given moments to the unique representing polynomial P. However, as we have described in,[13] for a particular class of moment sequences, such a construction is possible.

The main mathematical tool for studying this problem is the formal *exponential transform* defined below, which maps the moments c_{mn} to a new sequence of complex numbers b_{mn} as follows:

$$1 - \exp(-\frac{1}{\pi} \sum_{m,n=0}^{\infty} c_{mn} X^{m+1} Y^{n+1}) = \sum_{m,n=0}^{\infty} b_{mn} X^{m+1} Y^{n+1}, \tag{21}$$

where it is apparent from the definition that this is a *triangular* transformation. The above transformation arises from the asymptotic expansion (Taylor expansion at infinity) of the integral transform

$$E_{\Omega}(z, \overline{z}) = \exp(-\frac{1}{\pi} \int\int_{\Omega} \frac{dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{z})}). \tag{22}$$

This itself arises from the theory of pairs of self-adjoint operators.[21] The relevance of the exponential transformation for the reconstruction problem becomes apparent from the following two results which were established in[22] and,[20] respectively.

THEOREM 3.1. *If $(c_{mn})_{m,n=0}^{\infty}$ are the complete moments of some domain Ω, then there exists a positive integer N with the property* $\det(b_{mn})_{m,n=0}^{N} = 0$ *if and only if Ω coincides, up to a set of measure zero, with a quadrature domain of order less than or equal to N. Moreover, under the assumptions of the theorem, the respective quadrature domain is determined by the moments $c_{mn},\ m, n \leq N$.*

THEOREM 3.2. *Let Ω be a bounded planar domain with exponential transform (22). Then Ω is a quadrature domain if and only if there exists a polynomial $p(z)$ with the property that the function $P(z, \overline{z}) = p(z)\overline{p(z)}E_{\Omega}(z, \overline{z})$ is polynomial at infinity. In that case, by choosing $p(z)$ of minimal degree, the equation of Ω is, up to a finite set: $\Omega = \{z \in \mathrm{C};\ \ P(z, \overline{z}) < 0\}$.*

In fact, the polynomial $p(z)$ appearing in the theorem above is precisely computed from the quadrature nodes and their multiplicities as:

$$p(z) = \prod_{k=1}^{m} (z - \gamma_k)^{\nu_k},$$

and is hence of degree N; meanwhile $P(z, \overline{z})$ has degree N in each variable. We are therefore led to the following procedure for attempting a reconstruction of a QD from a part of its moments.

ALGORITHM:

1. **Given the moments c_{mn}, $m,n \leq N$, find the polynomial $p(z)$ of minimal degree.** This can be accomplished as follows:

   (a) From the moments c_{mn} compute the numbers b_{mn} using the exponential transformation in (21). (Note that as with c_{mn}, we have $b_{mn} = \overline{b}_{nm}$.) This process will involve some symbolic computation to establish the direct relationship between these quantities.

   (b) For a quadrature domain of order N, we know via Theorem 3.1 that $\det(b_{mn})_{m,n=0}^{N} = 0$ (and that the integer N is minimal in the sense that $\det(b_{mn})_{m,n=0}^{N-1} \neq 0$). Therefore, there exist coefficients d_k, $0 \leq k \leq N-1$, with the property that, for all $0 \leq m \leq N$, we have

   $$b_{Nm} + d_{N-1}b_{N-1,m} + \ldots + d_0 b_{0m} = 0, \tag{23}$$

   so that the minimal polynomial $p(z)$ vanishing at the quadrature nodes is precisely:

   $$p(z) = z^N + d_{N-1}z^{N-1} + \ldots + d_0.$$

   To obtain the coefficients d_k, we may solve the linear system of equations:

   $$B_0 \mathbf{d} = -b, \tag{24}$$

   where B_0 is a positive-definite Hermitian matrix formed using (23), keeping in mind that $b_{mn} = \overline{b}_{nm}$:

   $$B_0 = \begin{bmatrix} b_{00} & b_{10} & \cdots & b_{N-1,0} \\ \overline{b}_{10} & b_{11} & \cdots & b_{N-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b}_{N-1,0} & \overline{b}_{N-1,1} & \cdots & b_{N-1,N-1} \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{N-1} \end{bmatrix}, \qquad b = \begin{bmatrix} b_{N,0} \\ b_{N,1} \\ \vdots \\ b_{N,N-1} \end{bmatrix}.$$

2. **Form the product**

   $$R(z,\overline{z}) = p(z)\overline{p(z)} \left(1 - \sum_{m,n=0}^{N-1} b_{mn} \frac{1}{z^{m+1}\overline{z}^{n+1}}\right) \tag{25}$$

   and identify $P(z,\overline{z})$ as the part of $R(z,\overline{z})$ that does not contain negative powers of z or $\overline{z}$:

   $$R(z,\overline{z}) = P(z,\overline{z}) + O(z^{-1},\overline{z}^{-1})$$

   The minimal defining equation for the boundary $\partial\Omega$ will be

   $$P(z,\overline{z}) = 0$$

It is worth mentioning here that the procedure in step 1.b of the above algorithm for finding the coefficients of the polynomial is basically the same as the first step of Prony's method in linear prediction theory – an idea that also surfaced in the polygon reconstruction case, as we discussed in[12] and earlier in.[4] The next step in Prony's method would compute the roots of the polynomial, which in this case would yield the quadrature nodes (and in the polygonal case, we recall, yielded the vertices). It is important to note that this step is, strictly speaking, not necessary here since all we need is the polynomial $p(z)$ itself.

If for some reason we were, however, interested in finding the nodes of the quadrature domain, we could find the roots of $p(z)$ directly. But a better numerical approach to finding these nodes, which avoids the computation of the inverse B_0^{-1}, would be to use a generalized eigenvalue framework like that in section 2 (and in[12]). In such an approach, the quadrature nodes γ_k are the solutions of the generalized eigenvalue problem

$$B_1 u = \gamma B_0 u, \tag{26}$$

where B_1 has exactly the same structure as B_0, except that it begins with b_{10} at the upper left corner, and ends with $b_{N,N-1}$ at the lower right corner. Many of the results derived in[12] on improving the numerical solution of such poorly conditioned generalized eigenvalue problems also apply here. Along these lines, it is also important to mention here an alternate matrix formulation of the above as described in[22] and.[20] Namely, there exists an $N \times N$ complex matrix U, so that U^H admits a cyclic vector ξ with complex elements such that for asymptotic values of $|z| \gg 0$,

$$E_\Omega(z, \bar{z}) = 1 - \| \left(U^H - \bar{z}I\right)^{-1} \xi\|^2.$$

where we recognize the quantity inside the norm as the resolvent of the operator U^H. Recalling (21), we can see that the transformed moments b_{mn} can be written in terms of U and ξ as

$$b_{mn} = \xi^H U^m (U^H)^n \xi, \tag{27}$$

and the quadrature formula (16) can be written as

$$\int\int_\Omega f(z)\, dx\, dy = \xi^H f(U)\xi. \tag{28}$$

The polynomial $p(z)$ is then the minimal polynomial for the matrix U, or equivalently, U is the companion matrix for the polynomial $p(z)$ so that the quadrature nodes γ_k are the eigenvalues of the matrix U, and we can write

$$U = B_0^{-1} B_1,$$

where B_0 and B_1 were defined earlier. We finally note that the boundary of the domain is given by $\partial\Omega = \{z;\ \|(U^H - \bar{z}I)^{-1}\xi\|^2 = 1\}$

Clearly, the procedure outlined above will be approximate for an arbitrary underlying domain that is not a QD. However, the approximation error can be studied,[13] and the method tends to work well in practice on general convex domains as we will demonstrate in the next section.

4. SOME APPLICATIONS AND EXAMPLES

4.1. Applications

In this section we briefly present two applications of the shape-from-moments inverse problem, to tomography and geophysics, and present a pair of representative examples. Further details about the connection between reconstruction from moments, tomography, and gravimetry, can be found in[4] and.[12]

In this section we work with real variables x, y, so that $z = x + iy$. Let Ω be a bounded planar domain and let:

$$\mu_{mn} = \int\int_\Omega x^m y^n dxdy,$$

be its moment sequence. Let t be a positive integer and let $\theta \in [0, \pi)$. The *Radon transform* of the domain Ω is the function:

$$g_\Omega(t, \theta) = \int\int_\Omega \delta(t - x\cos\theta - y\sin\theta)dxdy,$$

where δ denotes the Dirac delta function. The function g_Ω is the projection of Ω at the angle θ. Let T be a positive constant and let $F \in L^2([-T, T], dt)$. Accordingly, the definition of g_Ω yields:

$$\int_{-T}^{T} g_\Omega(t, \theta)F(t)dt = \int\int_\Omega F(x\cos\theta + y\sin\theta)dxdy.$$

By taking $F(t) = t^n$ and expanding the second integral by the binomial formula we obtain:

$$\int_{-T}^{T} g_\Omega(t, \theta)t^n dt = \sum_{k+l=n} \binom{n}{k} \cos^k(\theta)\ \sin^l(\theta)\ \mu_{kl}.$$

Thus, knowing the projection g_Ω at the angle θ, and hence its moments, one knows the above linear combinations among the moments μ_{kl}. Since the determinant of these linear combinations for different angles θ is non-zero, the following observation holds (as noted in,[4] Proposition 3): *given line integral projections of the domain* Ω *at* $N+1$ *distinct angles, one can determine all the moments* $\mu_{k,l}$ *of order* $k+l \leq N$. As a consequence, taken with our earlier results and proposed algorithms, we have

THEOREM 4.1. *Let* Ω *be a quadrature domain of order* N. *The line integral projections* $g_\Omega(t,\theta_j)$ *at* $2N+1$ *distinct angles* θ_j, $0 \leq j \leq 2N$, *uniquely determines* Ω.

This also gives a simple way of deciding from tomographic data when a domain Ω is a quadrature domain of order N. Thus (as is well-known) a disk or ellipse of uniform mass is determined by exactly three line integral projections; more interestingly, a cardioid or a lemniscate (which are quadrature domains of order 2) need 5 projections, etc.

Also of great interest is the related inverse problem for the logarithmic potential, or the Cauchy transform, of a planar domain (See for instance[23,5,12]). In this case, it is of interest to find the shape of a region by measuring the external gravitational field induced by it. This is an exceedingly rich area of research. To impart the flavor of the problem we mention that, knowing the Cauchy transform $F(\zeta)$ of a bounded domain Ω in the complex plane:

$$F(\zeta) = -\frac{1}{\pi}\int\int_\Omega \frac{dA(z)}{z-\zeta}, \qquad \zeta \in \mathbf{C}\setminus\overline{\Omega},$$

a part of the exponential transform E_Ω, is known. More precisely, $F(\zeta) = \lim_{z\to\infty} \overline{z}(1-E_\Omega(\zeta,\overline{z}))$.

Thus, knowing $F(\zeta)$ for ζ large (sufficiently far from the boundary of the region), gives important information about E_Ω and consequently about Ω. On the other hand there are simple examples of continuous families of quadrature domains with the same Cauchy transform at infinity, see for instance.[16]

Let us discuss the logarithmic potential application in a bit more detail. The *exterior* logarithmic potential $\phi(x,y)$ due to a planar object of unit density is a harmonic function which behaves as $c\log(x^2+y^2)^{1/2}$ for some constant c (see[5]). This class of potential functions are limiting cases of the standard Newtonian potentials for (cylindrical) objects of infinite extent.

The (vector) gravitational field $G(x,y) = \nabla\phi$ can be embedded in the complex plane by defining the variable $\zeta = x+iy$ and writing

$$G(\zeta) = \frac{\partial\phi}{\partial x} + i\,\frac{\partial\phi}{\partial y}, \tag{29}$$

where $G(\zeta)$ is now, by construction, an analytic function outside of Ω and admits an integral representation:

$$G(\zeta) = -2ig\int\int_\Omega \frac{dA(z)}{z-\zeta}, \tag{30}$$

where g is the universal gravitational constant. For values of ζ outside of Ω, we can expand the field into an asymptotic series as follows:

$$G(\zeta) = 2ig\sum_{k=0}^{\infty} c_k\,\zeta^{-(k+1)}, \tag{31}$$

where the coefficients $c_k = c_{k0}$ are the complex analytic moments of the domain. Therefore, we observe that if the field $G(\zeta)$ is known, then the moments c_k are determined and hence the inverse potential problem is equivalent to the reconstruction of the region Ω from its moments.

In practice, we must consider a truncated asymptotic expansion of G:

$$G(\zeta) \approx 2ig\sum_{k=0}^{K-1} c_k\,\zeta^{-(k+1)}, \tag{32}$$

and assume that at least K measurements $G(\zeta_1)$, $G(\zeta_2)$, $\cdots$, $G(\zeta_K)$ are given. Collecting these in vector form and rewriting, we have the Vandermonde system

$$\begin{bmatrix} \zeta_1 G(\zeta_1) \\ \zeta_2 G(\zeta_2) \\ \vdots \\ \zeta_K G(\zeta_K) \end{bmatrix} = 2ig \begin{bmatrix} 1 & \zeta_1^{-1} & \cdots & \zeta_1^{-(K-1)} \\ 1 & \zeta_2^{-1} & \cdots & \zeta_2^{-(K-1)} \\ \vdots & \vdots & & \vdots \\ 1 & \zeta_K^{-1} & \cdots & \zeta_K^{-(K-1)} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{K-1} \end{bmatrix} = \Xi_K\, C_K. \tag{33}$$

Once we have solved (33) for the moments c_k, we can proceed as before with the approximate reconstruction of Ω as a polygon.

4.2. Examples

EXAMPLE 1: In this example, we demonstrate the application of the algorithm to the problem of shape reconstruction from gravitational field measurements. Specifically, we produce simulated measurements of the gravitational field due to the solid polygonal object shown in Figure 1. For convenience we choose to simulate these measurements at equally-spaced points (roots of unity) on the unit circle as again shown in Figure 1.

A total of 20 measurements of the gravitational field were simulated in the clockwise direction at roots of unity starting at $\zeta = 1$. The magnitude and phase of the simulated measurements $G(\xi)$ are shown in Figure 2. These values were then corrupted by (complex) Gaussian white noise with standard deviation $\sigma = 2 \times 10^{-3}$. From these noisy data, the first 20 complex moments c_k of the underlying shape were computed, allowing polygonal reconstruction with up to 10 vertices. The reconstruction using 4 vertices is shown as the dashed polygon in Figure 1.

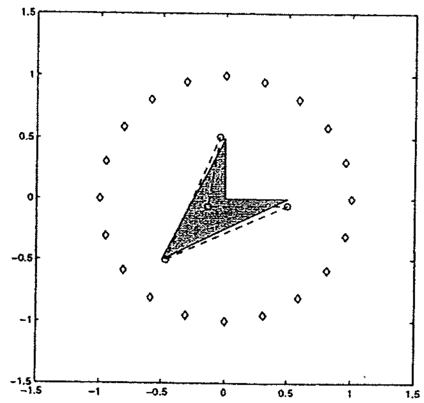

Figure 1. The underlying polygon (-), the reconstructed polygon (- -) and the locations of the gravity probes (diamonds) for Example 1

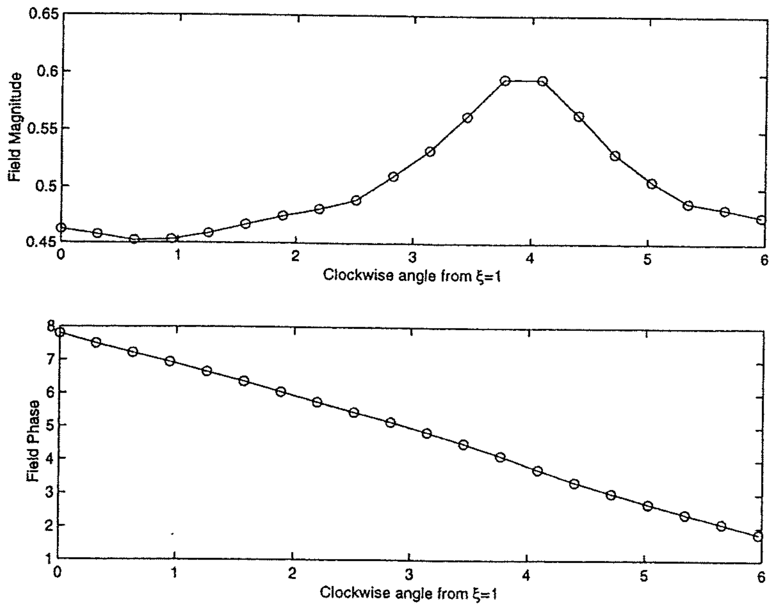

Figure 2. The magnitude and phase of the complex gravity field measurements for Example 1

EXAMPLE 2: In this example we demonstrate the reconstruction of a square from its moments using the framework of semi-algebraic quadrature domains. We intentionally picked a non-smooth curve to demonstrate the convergence of the approach. We computed the first N moments of the square in Figure 3, and used these data to reconstruct the shape for various choices of N. Note the behavior of the reconstructions at the corners of the square. We believe this may be related to the fact that the exponential transform E_Ω does not extend analytically at these corners. This is in exact correspondence to similar phenomena in Padé approximation theory.

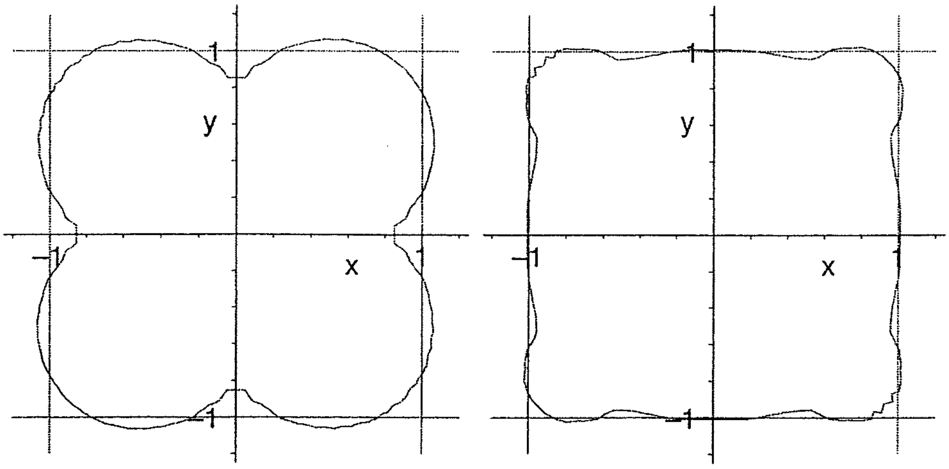

Figure 3. Left: N = 4 Right: N = 8

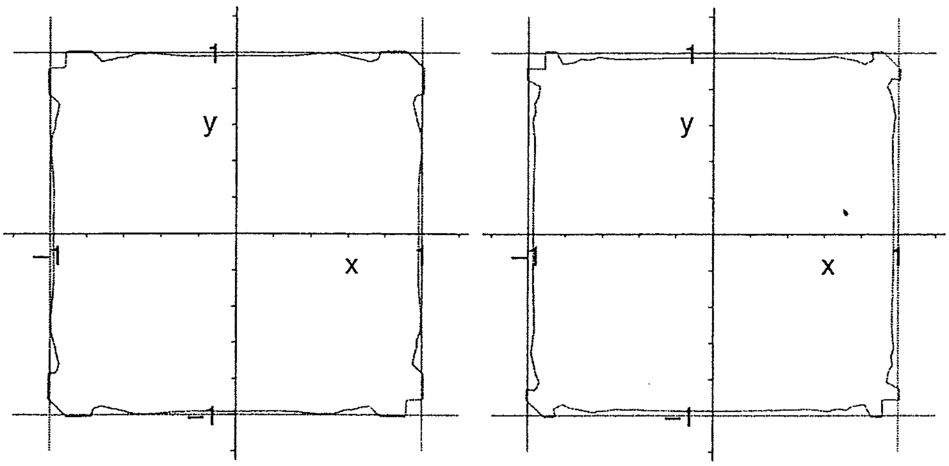

Figure 4. Left: N = 12 Right: N = 16

5. CONCLUSIONS

In this paper we presented a unified framework for solving the problem of shape reconstruction from moments. This problem has many applications including tomographic reconstruction and geophysical inversion. A rather remarkable feature of this moment problem is that it can be thought of as the dual of the problem of quadrature in two dimensions. Clearly, many questions, both theoretical, and practical, remain to be studied. Chief among these is the numerical analysis of the algorithms employed to solve for the shapes. An important area of work, which remains to be studied, is the regularization of the shape reconstruction problem using geometrically-motivated constraints. We are at present actively pursuing both avenues of research. It is our hope that the results of this paper along with the above observations will stimulate further work in this area both in terms of new theoretical and numerical techniques, and also in terms of applications of these techniques to solving physical problems.

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