[9] B. Kawohl: When are Solutions of Nonlinear Elliptic Boundary Value Problems Convex? Comm. in PDE 10(1985), 1213-1225.

[10] D. Kinderlehrer, L. Nirenberg, J. Spruck: Regularity in Elliptic Free Boundary Problems. J. D'Analyse Math. 34(1978), 86-119.

[11] N. Korevaar: Convex Solutions to Nonlinear Elliptic and Parabolic Boundary Value Problems. Indiana U. Math. J. 32(1983), 603-614.

[12] P. Laurence, E. Stredulinsky: Existence of Regular Solutions with Convex Level Sets for Semilinear Elliptic Equations with Nonmonotone L^1-Nonlinearities. Part I: An Approximating Free Boundary Problem. Part II: Passage to the Limit. Indiana U. Math J. (to appear).

[13] M.A. Lavrentiev, B.W. Shabat: Methoden der Komplexen Funktionentheorie. VEB Deutscher Verlag der Wissenschaften, Berlin, 1967.

Andrew Acker
Department of Mathematics and Statistics
The Wichita State University

B GUSTAFSSON AND M SAKAI

Some geometric properties of solutions of a Hele–Shaw flow moving boundary problem

1. Introduction.

Our results are most naturally stated in terms of a certain operator F. For B a sufficiently large ball in $\mathbb{R}^N$

$$F : H^{-1}(B) \longrightarrow H^{-1}(B)$$

is the orthogonal projection onto the closed convex set $K = \{\nu \in H^{-1}(B) : \nu \leq 1\}$. Thus $\nu = F(\mu)$ minimizes $\|\mu - \nu\|^2 =$ (the energy of $\mu - \nu$) under the constraint $\nu \leq 1\,(\mu, \nu \in H^{-1}(B))$. Expressed in another way

$$F(\mu) = \mu + \Delta u \tag{1}$$

where $u \in H_0^1(B)$ is the solution of the variational inequality (in complementarity form)

$$\mu + \Delta u \leq 1, \tag{2}$$

$$u \geq 0, \tag{3}$$

$$< 1 - \mu - \Delta u, u >= 0. \tag{4}$$

($< \cdot, \cdot >$ denotes the dual pairing between $H^{-1}(B)$ and $H_0^1(B)$.) B is supposed to be so large that μ and $F(\mu)$ (and hence u) have compact support in B, and then $F(\mu)$ does not depend on B. When acting on measures F can be regarded as a kind of balayage operator (cf. (9)). The definition of F easily extends to arbitrary measures of compact support.

Under mild assumptions on μ $F(\mu)$ has the form

$$F(\mu) = \chi_\Omega + \mu\chi_{B\setminus\Omega} \tag{5}$$

where $\Omega =$ (the largest open set in which $F(\mu) = 1$).

Typically, Ω simply coincides with the non-coincidence set $\{u > 0\}$ for (2), (3), (4).

If $\mu \geq 0$ and satisfies suitable additional conditions, e.g. that μ (as a measure) is singular with respect to Lebesgue measure or that there exists an open set D such that $\mu \geq 1$ on $D, \mu = 0$ outside D, then the second term in (5) drops off and one simply has

$$F(\mu) = \chi_\Omega. \tag{6}$$

The situation (6) occurs in a number of free (and moving) boundary problems. One example is the Hele-Shaw flow moving boundary problem in which one starts with an initial domain (blob of fluid) Ω_0 and asks for the (increasing) family of domains $\{\Omega(t) : t \geq 0\}$ satisfying

$$\begin{aligned} &\text{(i)} \quad \Omega(0) = \Omega_0; \\ &\text{(ii)} \quad \partial\Omega(t) \text{ moves with velocity } -(\nabla p)\Big|_{\partial\Omega(t)}, \end{aligned} \tag{7}$$

where, for each t, $p = p(x,t)$ denotes the solution of

$$\begin{cases} -\Delta p = f & \text{in} \quad \Omega(t) \\ \quad p = 0 & \text{on} \quad \partial\Omega(t). \end{cases}$$

Here $f(x,t) \geq 0$ is a (given) source term with supp $f(\cdot,t) \subset \Omega_0$ for each t.

It is well-known that problem (7) always has a unique (weak) solution $\{\Omega(t) : t \geq 0\}$, and this is given by

$$F(\mu(t)) = \chi_{\Omega(t)} \tag{8}$$

where

$$\mu(t) = \chi_{\Omega_0} + \int_0^t f(\cdot,\tau)d\tau.$$

Another application of F is to so-called quadrature domains: if (6) holds then (roughly, and if μ is a measure)

$$\int_\Omega \varphi\, d\mu = \int_\Omega \varphi\, dx \tag{9}$$

for all integrable harmonic functions φ in Ω and one says that Ω is a quadrature domain for μ with respect to harmonic functions. (Actually (9) holds,

with = replaced by $\leq$, for all integrable subharmonic φ.) The term equipotential domain could also have been used because (9) essentially means that the Newtonian potentials of Ω and μ coincide outside Ω, if Ω is regarded as a body of density one.

2. Main results.

THEOREM 1. *Suppose $\mu \geq 0$ and supp $\mu \subset \overline{D}$ where D is an open halfspace, say $D = \{x \in \mathbf{R}^N : x_N < 0\}$. Then*

$$F(\mu)|_{D^e} = \chi_\Omega$$

where Ω is an open set of the form

$$\Omega = \{(x', x_N) \in \mathbf{R}^N : x' \in \omega, 0 < x_N < g(x')\}$$

for some open $\omega \subset \mathbf{R}^{N-1}$ and some real analytic $g : \omega \to \mathbf{R}$. ($D^e = \mathbf{R}^N \setminus \overline{D}$.)

SKETCH OF PROOF.: Referring to (1)–(4), let $\tilde{u}$ denote the reflection of u in the hypersurface $x_N = 0$, i.e.

$$\tilde{u}(x', x_N) = u(x', -x_N)$$

and define

$$v = \begin{cases} \inf(u, \tilde{u}) & \text{in} \quad D^e \\ u & \text{on} \quad \overline{D}. \end{cases}$$

Clearly $v \geq 0$ everywhere and it is easy to check that $\Delta v \leq 1 - \mu$ (and that $v \in H_0^1(B)$). From this it follows that

$$u \leq v \tag{10}$$

because it is well-known that the solution u of (2), (3), (4) is the smallest of all functions satisfying (2), (3) alone.

(10) shows that $u \leq \tilde{u}$ in D^c and this gives that

$$\frac{\partial u}{\partial x_N} \leq 0 \quad \text{on} \quad \partial D.$$

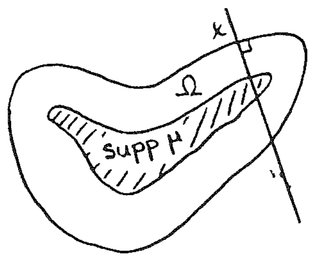
Now the maximum principle can be applied to $\partial u/\partial x_N$ in $\{u > 0\} \cap D^e$ and one obtains that $\partial u/\partial x_N \leq 0$ in all D^c. From this the statements of the theorem follows easily (for the regularity part one has to apply the regularity theory of Caffarelli and others.) □

Applying Theorem 1 to all half-spaces containing supp μ gives

COROLLARY 1. *Suppose $\mu \geq 0$ and let K denote the closed convex hull of supp μ. Then the restriction of $F(\mu)$ to K^c is of the form χ_Ω where Ω is an open set with $\partial\Omega \backslash K$ consisting of real analytic hypersurfaces (without singularities). Moreover $(\Omega \cup K)^c$ is connected.*

A particularly nice and concrete consequence of Theorem 1 is the following.

COROLLARY 2. *With assumptions and notations as in Corollary 1, for any $x \in \partial\Omega \backslash K$ the normal of $\partial\Omega$ at x intersects K. If $N = 2$ and supp μ is connected the normal even has to intersect supp μ itself.*

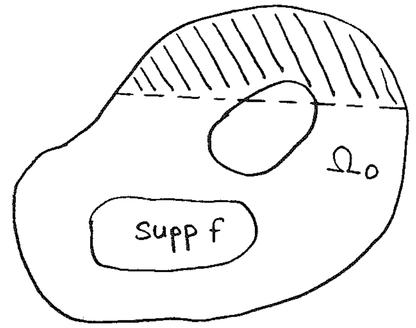


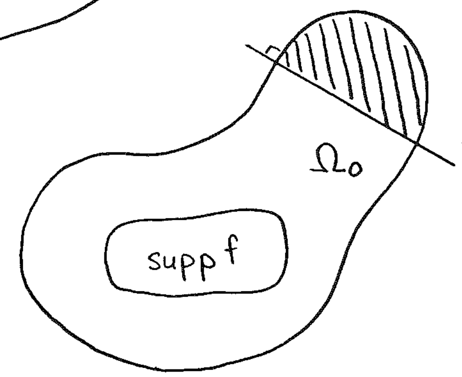
Clearly (by (8)) Corollary 2 says a lot about the geometry of the solution $\Omega(t)$ of the Hele-Shaw problem (7). By reversion of the time variable t one also gets interesting information for the corresponding suction problem, i.e. the problem (7) with $f \leq 0$. In fact, if $f \leq 0$ and $\Omega(t)$ solves (7) one has

$$F\left(\chi_{\Omega(t)} - \int_0^t f(\cdot,\tau)d\tau\right) = \chi_{\Omega_0} \, (t > 0).$$

(Now $\Omega(t)$ shrinks as t increases and one has to assume that supp $f \subset \Omega(t)$ for all t under consideration.)

Theorem 1 and Corollary 2 then show that the shaded areas in the figures below never can be completely emptied by $\Omega(t)$.





Finally we state without proof another result about F, which to a part can be viewed as a generalization of Theorem 1.

THEOREM 2. *Suppose $\mu \geq 0$ and let D be an open set with smooth boundary and with supp $\mu \subset \overline{D}$. Then there exists a $\nu \in H^{-1}(B)$ with $\nu \geq 0$ and supp $\nu \subset \partial D$ such that*

$$F(\mu)\Big|_{D^c} = F(\nu)\Big|_{D^c}.$$

Björn Gustafsson
Matematiska institutionen
Kungl. Tekniska Högskolan
S–100 44 Stockholm
Sweden

Makoto Sakai
Department of Mathematics
Tokyo Metropolitan University
Minamiōsawa, Hachiōji-shi
Tokyo, 192–03
Japan