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Homogenization of Foliated Annuli (*).

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Abstract. - *Let $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ be a regular annulus in $\boldsymbol{R}^N$ and $\phi: \overline{\Omega} \to \boldsymbol{R}$ be a regular function such that $\phi = 0$ on $\partial\Omega_0$, $\phi = 1$ on $\partial\Omega_1$ and $\nabla\phi \neq 0$. Let K_n be the subset of functions $v \in W^{1,p}(\Omega)$ such that $v = 0$ on $\partial\Omega_0$, $v = 1$ on $\partial\Omega_1$, $v =$ (unprescribed) constant on n given level surfaces of ϕ. We study the convergence of sequences of minimization problems of the type*

$$\operatorname{Inf}\left\{ \int_\Omega \frac{1}{a_n \circ \phi} G(x, (a_n \circ \phi)\nabla v)\, dx;\ v \in K_n \right\},$$

where $a_n \in L^\infty(0,1)$ and $G: (x, \zeta) \in \Omega \times \boldsymbol{R}^N \to G(x, \zeta) \in \boldsymbol{R}$ is convex with respect to ξ and verifies some standard growth conditions.

1. - Introduction.

Let $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ where Ω_0 and Ω_1 are two regular open subsets of $\boldsymbol{R}^N (N \geq 2)$, $\Omega_0 \supset \overline{\Omega}_1$ and let $\phi \in C^1(\overline{\Omega}; \boldsymbol{R})$ be such that $\phi = 0$ on $\partial\Omega_0$, $\phi = 1$ on $\partial\Omega_1$ and $\nabla\phi \neq 0$ on $\overline{\Omega}$. Hence $0 < \phi < 1$ in Ω and the level surfaces of ϕ

$$\Gamma_t = \{x \in \overline{\Omega};\ \phi(x) = t\} \quad \text{where } 0 \leq t \leq 1 \ (\Gamma_0 = \partial\Omega_0, \Gamma_1 = \partial\Omega_1)$$

form a nested family separating Γ_0 from Γ_1.

For $n \in \boldsymbol{N}$, let $T_n = \{t_{i,n};\ 0 \leq i \leq n\}$ where $(t_{i,n})_i$ is a sequence of $(n+1)$ real numbers in the interval $[0,1]$ such that $0 = t_{0,n} < t_{1,n} < \ldots < t_{n-1,n} < t_{n,n} = 1$.

Hence the annulus Ω with the $n-1$ leaves $\Gamma_{i,n} = \{\phi = t_{i,n}\}$ may represent a kind of foliated structure.

For every $n \in \boldsymbol{N}$, let us now consider convex minimization problems in the Sobolev

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space $W^{1,p}(\Omega)$ $(1<p<\infty)$ of the form

$$(\mathcal{P}_n) \qquad \operatorname{Inf}\left\{\int_\Omega \frac{1}{a_n(x)} G(x, a_n(x)\nabla v(x))\,dx;\ v\in K_n\right\},$$

where

— $K_n=\{v\in W^{1,p}(\Omega);\ v=0 \text{ on } \Gamma_0,\ v=1 \text{ on } \Gamma_1;\ \forall t\in T_n,\ v=\text{(unprescribed) constant on } \Gamma_t\}$,

— $a_n=\underline{a}_n\circ\phi$ where $\underline{a}_n\in L^\infty(0,1)$ and verify

$$(1.1) \qquad \exists C>0\colon \forall n\in \boldsymbol{N},\quad \text{a.e. } t\in]0,1[,\ \underline{a}_n(t)\geq C,$$

— $G\colon (x,\zeta)\in\Omega\times\boldsymbol{R}^N\to G(x,\zeta)\in\boldsymbol{R}$ is a Carathéodory function (that is measurable with respect to x, continuous with respect to ζ) such that

— for almost every $x\in\Omega$, $G(x,\cdot)$ is a strictly convex functional which admits a gradient denoted by $g(x,\cdot)$,

— there exist c_1, c_2, $c_4>0$ and $c_3\in L^1(\Omega)$ such that, for almost every $x\in\Omega$ and for every $\zeta\in\boldsymbol{R}^N$

$$(1.2) \qquad c_1|\zeta|^p\leq G(x,\zeta)\leq c_2|\zeta|^p+c_3(x),$$

$$(1.3) \qquad |g(x,\zeta)|\leq c_4(1+|\zeta|^{p-1}).$$

In this article we study the convergence of the sequence of minimization problems $(\mathcal{P}_n)$ as n goes to infinity. We shall prove the convergence of the solutions u_n (resp. of the minima) of $(\mathcal{P}_n)$ to the solution (resp. to the minimum) of a minimization problem $(\mathcal{P})$ of the same form as $(\mathcal{P}_n)$ under suitable assumptions on the limit behavior of the sequences $(\underline{a}_n)$ and (T_n). Actually the limit problem $(\mathcal{P})$ is more simple as it reduces to a one dimensional problem. Most of the following results were announced for the particular case $p=2$ and $G(x,\zeta)=|\zeta|^2$ in [6]. In the more particular case where Ω is an interval of $\boldsymbol{R}$, say $\Omega=]0,1[$, $(\mathcal{P}_n)$ becomes

$$\operatorname{Inf}\left\{\int_0^1 a_n(t)\,v'(t)^2\,dt;\ v\in H^1(0,1),\ v(0)=0,\ v(1)=1\right\};$$

its convergence has been considered as an interesting example by S. SPAGNOLO (CF. [12]).

ONE PHYSICAL INTERPRETATION OF $(\mathcal{P}_n)$ for $p=2$ and $G(x,\zeta)=|\zeta|^2$ is—following [10], p. 51-54—that the quantity $\operatorname{Inf}(\mathcal{P}_n)$ represents the total conductance (one over the resistance) from Γ_0 to Γ_1, when $\overline{\Omega}$ is considered as a resistor with (nonconstant) conductivity coefficients a_n, in which a family $\{\Gamma_{i,n},\ i=1,\dots,n-1\}$ of infinitely thin sheets of perfect conductors are inserted (Γ_0 and Γ_1 should also be perfect conductors). Thus $\overline{\Omega}$ can be regarded as a compound resistor consisting of n smaller resistors coupled in series, the i-th one being the part $\Omega_{i,n}$ of Ω lying between $\Gamma_{i-1,n}$ and

$\Gamma_{i,n}$. If an electric voltage of unit strengh is applied to $\bar{\Omega}$, namely so that Γ_0 is given potential zero, and Γ_1 potential one, then u_n represents the electric potential in Ω. Moreover, the minimum of $(\mathscr{P}_n)$ then represents not only the total conductance, but also the total electric effect produced in Ω, and the total electric current through each surface Γ_t (see section 3*a* below).

In a subsequent paper [7], we will replace the infinitely thin sheets of perfect conductors $\Gamma_{i,n}$ by thin layers $\Sigma_{i,n,\varepsilon} = \{t_{i,n} - \varepsilon < \phi < t_{i,n} + \varepsilon\}$ of a highly conducting material, that is a_n is replaced by some constant $\lambda \gg a_n$ on $\Sigma_{i,n,\varepsilon}$. We will study the limit problem when $\varepsilon \to 0$, $\lambda \to \infty$, $n \to \infty$.

2. – Convergence of $(\mathscr{P}_n)$ to $(\mathscr{P})$.

THEOREM 2.1. – *Under the previous assumptions, let us suppose that*

(2.1) *there exists* $\underline{a} \in L^\infty(0,1)$ *such that* $\dfrac{1}{\underline{a}_n} \to \dfrac{1}{\underline{a}}$ *weakly* in* $L^\infty(0,1)$, *as* $n \to \infty$.

(2.2) $|T_n| = \max\{t_{i,n} - t_{i-1,n};\ 1 \leqslant i \leqslant n\} \to 0$, *as* $n \to \infty$.

Then the solution u_n *of*

$$(\mathscr{P}_n) \qquad \operatorname{Inf}\left\{\int_\Omega \frac{1}{a_n} G(x, a_n \nabla v)\, dx;\ v \in K_n\right\}$$

converges weakly in $W^{1,p}(\Omega)$ *to the solution* u *of*

$$(\mathscr{P}) \qquad \operatorname{Inf}\left\{\int_\Omega \frac{1}{a} G(x, a \nabla v)\, dx;\ v \in K\right\}$$

where $a = \underline{a} \circ \phi$ *and*

$$K = \{v \in W^{1,p}(\Omega);\ v = 0 \text{ on } \Gamma_0,\ v = 1 \text{ on } \Gamma_1,\ \forall t \in]0,1[,\ v = \text{constant on } \Gamma_t\}.$$

Moreover

$$\int_\Omega \frac{1}{a_n} G(x, a_n \nabla u_n)\, dx \to \int_\Omega \frac{1}{a} G(x, a \nabla u)\, dx.$$

2*a*) *Some comments on the hypothesis* (2.1) *and* (2.2).

Since $0 < 1/\underline{a}_n(t) \leqslant 1/C$ a.e. $t \in]0,1[$, the sequence $(1/\underline{a}_n)$ is bounded in $L^\infty(0,1)$ and then a subsequence converges in w^*-$L^\infty(0,1)$. Assumption (2.1) means that all the sequence $(1/\underline{a}_n)$ converges in w^*-$L^\infty(0,1)$ to $1/\underline{a}$ with $\underline{a} \in L^\infty(0,1)$. We shall prove that this convergence is equivalent to the convergence of the sequence $(1/a_n)$ in w^*-

$L^\infty(\Omega)$. Notice that assumption (1.1) can be weakened to

$$(1.1)' \qquad \forall n \in N,\ \exists C_n > 0, \text{a.e. } t \in]0,1[,\ \underline{a}_n(t) \geq C_n \ (\text{i.e. } \underline{a}_n > 0 \text{ and } 1/\underline{a}_n \in L^\infty(0,1)).$$

In fact, by the uniform boundedness theorem, (2.1) implies that the sequence $(1/\underline{a}_n)$ is bounded in $L^\infty(0,1)$, hence (1.1).

Assumption (2.2) is related to the convergence of the sequence of convex K_n to the convex K in the Mosco sense (cf. [9]). Actually K can be identified with a subspace of $W^{1,p}(0,1)$.

Those comments are the object of three following lemmas:

LEMMA 2.1. – *Let $(\underline{a}_n)$ such that $(1/\underline{a}_n) \in L^\infty(0,1)$. The two following properties are equivalent*

(i) $\dfrac{1}{\underline{a}_n} \to \dfrac{1}{\underline{a}}$ *weakly* in* $L^\infty(0,1)$.

(ii) $\dfrac{1}{a_n} \to \dfrac{1}{a}$ *weakly* in* $L^\infty(\Omega)$ *where* $a_n = \underline{a}_n \circ \phi$ *and* $a = \underline{a} \circ \phi$.

PROOF. – Let us first assume (i). Applying the coarea formula (cf. [5] p. 249 or [11] p. 697), we get, for every $f \in L^1(\Omega)$,

$$\int_{\Gamma_t} \frac{f}{|\nabla \phi|} d\gamma \in L^1(0,1) \quad \text{and} \quad \int_\Omega \frac{f}{a_n} dx = \int_0^1 \frac{1}{\underline{a}_n}\left(\int_{\Gamma_t} \frac{f}{|\nabla\phi|} d\gamma\right) dt,$$

where $d\gamma$ denotes the $(N-1)$ dimensional Hausdorff measure (sometimes denoted by dH^{N-1}). It follows

$$\int_0^1 \frac{1}{\underline{a}_n}\left(\int_{\Gamma_t} \frac{f}{|\nabla\phi|} d\gamma\right) dt \to \int_0^1 \frac{1}{\underline{a}}\left(\int_{\Gamma_t} \frac{f}{|\nabla\phi|} d\gamma\right) dt = \int_\Omega \frac{f}{a} dx.$$

Hence we have

$$\int_\Omega \frac{f}{a_n} dx \to \int_\Omega \frac{f}{a} dx, \qquad \text{for every } f \in L^1(\Omega),$$

that is (ii).

Conversely, assume now (ii). Let $\underline{f} \in L^1(0,1)$ and define f by

$$\text{a.e. } x \in \Omega,\ f(x) = \underline{f}(\phi(x))\left(\int_{\Gamma_{\phi(x)}} \frac{d\gamma}{|\nabla\phi|}\right)^{-1}.$$

From the coarea formula, we get $f \in L^1(\Omega)$ and

$$\int_\Omega \frac{f(x)}{a_n(x)}\,dx = \int_0^1 \left(\int_{\Gamma_t} \frac{\underline{f}(\phi(x))}{\underline{a}_n(\phi(x))|\nabla\phi|}\,d\gamma \right) \left(\int_{\Gamma_t} \frac{d\gamma}{|\nabla\phi|} \right)^{-1} dt = \int_0^1 \frac{\underline{f}(t)}{\underline{a}_n(t)}\,dt.$$

Consequently, (ii) implies (i).

LEMMA 2.2. – *Let $\underline{K} = \{\underline{v} \in W^{1,p}(0,1);\ \underline{v}(0) = 0, \underline{v}(1) = 1\}$. Then the linear mapping: $\underline{v} \to v = \underline{v} \circ \phi$ is an isomorphism from $W^{1,p}(0,1)$ onto $E = \{v \in W^{1,p}(\Omega);$ $\forall t \in [0,1],\ v = \text{constant on } \Gamma_t\}$ mapping $\underline{K}$ onto K.*

PROOF. – Clearly $\underline{v}_1 \circ \phi = \underline{v}_2 \circ \phi$ implies $\underline{v}_1 = \underline{v}_2$. For $v \in E$, we define $\underline{v}$ as follows: for all $t \in [0,1]$, $\underline{v}(t) =$ the trace of v on Γ_t. Then $\underline{v} \in W^{1,p}(0,1)$, $v = \underline{v} \circ \phi$, and $\underline{v}(0) = 0$, $\underline{v}(1) = 1$ if and only if $v \in K$. It remains to prove that there exists a constant $C > 0$ such that: $\forall v \in E$,

$$\|\underline{v}\|_{W^{1,p}(0,1)} \leq C \|v\|_{W^{1,p}(\Omega)},$$

for then it will follow from the Banach theorem that $\underline{v} \to \underline{v} \circ \phi$ is an isomorphism from $W^{1,p}(0,1)$ onto E. In order to prove this inequality, we use the coarea formula and we obtain

$$\int_\Omega |v|^p\,dx = \int_0^1 \left(\int_{\Gamma_t} \frac{|v|^p}{|\nabla\phi|}\,d\gamma \right) dt = \int_0^1 |\underline{v}|^p \left(\int_{\Gamma_t} \frac{d\gamma}{|\nabla\phi|} \right) dt \geq C \int_0^1 |\underline{v}|^p\,dt$$

and

$$\int_\Omega |\nabla v|^p\,dx = \int_0^1 |\underline{v}'(t)|^p \left(\int_{\Gamma_t} |\nabla\phi|^{p-1}\,d\gamma \right) dt \geq C \int_0^1 |\underline{v}'|^p\,dt.$$

Lemma 2.2 is proved.

LEMMA 2.3. – *Assume $|T_n| \to 0$ as $n \to \infty$. Then, if $v_n \in K_n$ and $v_n \to v$ in $w\text{-}W^{1,p}(\Omega)$ as $n \to \infty$, we have $v \in K$.*

PROOF. – Let $v_n \to v$ in $w\text{-}W^{1,p}(\Omega)$ such that $v_n \in K_n$. Let $\underline{w}_n \in \underline{K}$ be defined by

$$\begin{cases} \underline{w}_n(t_{i,n}) = v_{i,n}, \text{ the trace of } v_n \text{ on } \Gamma_{i,n}, & 0 \leq i \leq n, \\ \underline{w}_n \text{ is an affine function on each interval } [t_{i-1,n}, t_{i,n}]. \end{cases}$$

Let $w_n = \underline{w}_n \circ \phi$. Then $w_n \in K$ and since K is weakly closed in $W^{1,p}(\Omega)$, the lemma follows if one can prove that a subsequence of w_n converges to v in $w\text{-}W^{1,p}(\Omega)$. For this, we will use the following assertion: there exists a C^1-diffeomorphism D from $\overline{\Omega}$ onto $[0,1]\times\Gamma_0$ such that

$$D\colon x\in\overline{\Omega}\to D(x)=(\phi(x),\psi(x))=(t,y)\in[0,1]\times\Gamma_0,$$

and the image measure $D(dx)$ is equivalent to $dt\times dH^{N-1}(y)$ denoted $dt\times d\gamma(y)$ (cf. Appendix). Let $V_n=v_n\circ D^{-1}$ and $W_n=w_n\circ D^{-1}$. For $t_{i,n}\leqslant t\leqslant t_{i+1,n}$ and $y\in\Gamma_0$, we have

$$W_n(t,y)-V_n(t,y)=\underline{w}_n(t)-V_n(t,y)=$$

$$=(v_{i,n}-V_n(t,y))\frac{t_{i+1,n}-t}{t_{i+1,n}-t_{i,n}}+(v_{i+1,n}-V_n(t,y))\frac{t-t_{i,n}}{t_{i+1,n}-t_{i,n}}.$$

But

$$v_{i,n}-V_n(t,y)=V_n(t_{i,n},y)-V_n(t,y)=\int_t^{t_{i,n}}\frac{\partial V_n}{\partial t}(\tau,y)\,d\tau.$$

We get

$$|W_n(t,y)-V_n(t,y)|\leqslant\int_{t_{i,n}}^{t_{i+1,n}}\left|\frac{\partial V_n}{\partial t}(\tau,y)\right|d\tau\leqslant\left\|\frac{\partial V_n}{\partial t}(\cdot,y)\right\|_{L^p(0,1)}|T_n|^{1/p'},$$

by Hölder's inequality $(1/p+1/p'=1)$. It follows that

$$\int_{\Gamma_0}|W_n(t,y)-V_n(t,y)|^p\,d\gamma(y)\leqslant\left\|\frac{\partial V_n}{\partial t}\right\|^p_{L^p((0,1)\times\Gamma_0)}|T_n|^{p-1}\leqslant$$

$$\leqslant C\|v_n\|^p_{W^{1,p}(\Omega)}|T_n|^{p-1}\leqslant C|T_n|^{p-1}.$$

Consequently, $\|w_n-v_n\|_{L^p(\Omega)}\leqslant C|T_n|^{1/p'}$ and then a subsequence of w_n converges to v in $L^p(\Omega)$.

Let us now prove that (w_n) is bounded in $W^{1,p}(\Omega)$. From this, it will follow that a subsequence of (w_n) converges to v in $w\text{-}W^{1,p}(\Omega)$ and thus $v\in K$. Since

$$\underline{w}_n'(t)=\frac{v_{i+1}-v_{i,n}}{t_{i+1,n}-t_{i,n}}\qquad\text{on }[t_{i,n},t_{i+1,n}]$$

we have

$$\int_0^1|\underline{w}_n'|^p\,dt=\sum_{i=0}^{n-1}\frac{|v_{i+1,n}-v_{i,n}|^p}{(t_{i+1,n}-t_{i,n})^{p-1}}\,;$$

But

$$v_{i+1,n} - v_{i,n} = \int_{t_{i,n}}^{t_{i+1,n}} \frac{\partial V_n}{\partial t}(\tau, y)\, d\tau, \qquad \text{a.e. } y \in \Gamma_0,$$

and from Hölder's inequality,

$$|v_{i+1,n} - v_{i,n}|^p \leqslant (t_{i+1,n} - t_{i,n})^{p-1} \int_{t_{i,n}}^{t_{i+1,n}} \left| \frac{\partial V_n}{\partial t}(\tau, y) \right|^p d\tau, \qquad \text{a.e. } y \in \Gamma_0 .$$

We obtain

$$\int_0^1 |\underline{w}_n'|^p\, dt \leqslant \int_0^1 \left| \frac{\partial V_n}{\partial t}(t, y) \right|^p dt, \qquad \text{a.e. } y \in \Gamma_0$$

which gives, by integration with respect to y,

$$\int_0^1 |\underline{w}_n'|^p\, dt \leqslant \frac{1}{|\Gamma_0|} \int_{]0,1[\times\Gamma_0} \left| \frac{\partial V_n}{\partial t}(t, y) \right|^p dt\, d\gamma(y) \leqslant C \|V_n\|^p_{W^{1,p}(]0,1[\times\Gamma_0)} \leqslant C \|v_n\|^p_{W^{1,p}(\Omega)} .$$

Thanks to the boundary conditions

$$\|\underline{w}_n\|^p_{W^{1,p}(]0,1[)} \leqslant C \|v_n\|^p_{W^{1,p}(\Omega)},$$

and from Lemma 2.2,

$$\|w_n\|^p_{W^{1,p}(\Omega)} \leqslant C \|v_n\|^p_{W^{1,p}(\Omega)} .$$

Consequently, (w_n) is bounded in $W^{1,p}(\Omega)$. The proof is complete.

2*b*) *Proof of Theorem* 2.1.

The proof consists of three steps:

LEMMA 2.4. – *For every* $v \in K$, *there exist* $v_n \in K_n$ *such that* $v_n \to v$ *in* w-$W^{1,p}(\Omega)$ *and*

$$\int_\Omega \frac{1}{a_n} G(x, a_n \nabla v_n)\, dx \to \int_\Omega \frac{1}{a} G(x, a \nabla v)\, dx .$$

PROOF. – Let $v \in K$. By Lemma 2.2, $v = \underline{v} \circ \phi$ with $\underline{v} \in \underline{K}$. Define $\underline{v}_n$ by

$$\underline{v}_n(t) = \frac{1}{\delta_n} \int_0^t \frac{a}{\underline{a}_n} \underline{v}'\, ds, \qquad \text{where } \delta_n = \int_0^1 \frac{a}{\underline{a}_n} \underline{v}'\, ds.$$

Hence $\underline{v}_n \in \underline{K}$ and, from (2.1), $\delta_n \to 1$ and $\underline{v}_n(t) \to \underline{v}(t)$ for each t. Let $v_n = \underline{v}_n \circ \phi$. Then $v_n \in K$ and, by Lebesgue's convergence theorem, $v_n \to v$ in $L^p(\Omega)$. Moreover $\nabla v_n = (a/(a_n \delta_n))\nabla v \to \nabla v$ in $w\text{-}L^p(\Omega)^N$ from (2.1) and Lemma 2.1, and

$$\int_\Omega \frac{1}{a_n} G(x, a_n \nabla v_n)\, dx = \int_\Omega \frac{1}{a_n} G\left(x, \frac{a}{\delta_n} \nabla v\right) dx.$$

But $G(x, (a/\delta_n)\nabla v) \to G(x, a\nabla v)$ in $L^1(\Omega)$, by (1.2), since $(a/\delta_n)\nabla v \to a\nabla v$ in $L^p(\Omega)^N$ (see [8] p. 22). Applying Lemma 2.1, we obtain

$$\int_\Omega \frac{1}{a_n} G(x, a_n \nabla v_n)\, dx \to \int_\Omega \frac{1}{a} G(x, a\nabla v)\, dx.$$

LEMMA 2.5. – *If $v_n \to v$ in $w\text{-}W^{1,p}(\Omega)$, then*

$$\liminf \int_\Omega \frac{1}{a_n} G(x, a_n \nabla v_n)\, dx \geq \int_\Omega \frac{1}{a} G(x, a\nabla v)\, dx.$$

PROOF. – Since $G(x, \cdot)$ is convex and admits a gradient denoted $g(x, \cdot)$, we have

$$G(x, a_n \nabla v_n) \geq G(x, a\nabla v) + g(x, a\nabla v)(a_n \nabla v_n - a\nabla v).$$

Thus

$$\int_\Omega \frac{1}{a_n} G(x, a_n \nabla v_n)\, dx \geq \int_\Omega \frac{1}{a_n} G(x, a\nabla v)\, dx + \int_\Omega g(x, a\nabla v)\left(\nabla v_n - \frac{a}{a_n}\nabla v\right) dx.$$

But $g(x, a\nabla v) \in L^{p'}(\Omega)^N$ by (1.3), $\nabla v^n \to \nabla v$ in $w\text{-}(L^p(\Omega))^N$ and, from Lemma 2.1,

$$\int_\Omega g(x, a\nabla v) \frac{a}{a_n} \nabla v\, dx \to \int_\Omega g(x, a\nabla v)\nabla v\, dx.$$

Consequently $\liminf \int_\Omega \frac{1}{a_n} G(x, a_n \nabla v_n)\, dx \geq \int_\Omega \frac{1}{a} G(x, a\nabla v)\, dx.$

END OF THE PROOF OF THEOREM 2.1. – From Lemma 2.4, there exists $v_n \in K_n$ such that $v_n \to u$ in $w - W^{1,p}(\Omega)$ and

$$\int_\Omega \frac{1}{a_n} G(x, a_n \nabla v_n)\, dx \to \int_\Omega \frac{1}{a} G(x, a \nabla u)\, dx .$$

Since (u_n) is the solution of $(\mathcal{P}_n)$, we have

$$\int_\Omega \frac{1}{a_n} G(x, a_n \nabla u_n)\, dx \leqslant \int_\Omega \frac{1}{a_n} G(x, a_n \nabla v_n)\, dx .$$

It follows, by (1.1) and (1.2), that (u_n) is bounded in $W^{1,p}(\Omega)$ and so there exist a subsequence of (u_n) (still denoted (u_n) for simplicity) and $u^* \in W^{1,p}(\Omega)$ such that $u_n \to u^*$ in $w - W^{1,p}(\Omega)$. From Lemma 2.3, $u^* \in K$. Applying Lemma 2.5, we get

$$\int_\Omega \frac{1}{a} G(x, a \nabla u^*)\, dx \leqslant \liminf \int_\Omega \frac{1}{a_n} G(x, a_n \nabla u_n)\, dx \leqslant \limsup \int_\Omega \int_\Omega \frac{1}{a_n} G(x, a_n \nabla u_n)\, dx \leqslant$$

$$\leqslant \limsup \int_\Omega \frac{1}{a_n} G(x, a_n \nabla v_n)\, dx = \int_\Omega \frac{1}{a} G(x, a \nabla u)\, dx .$$

Consequently $u^* = u$, the unique solution of $(\mathcal{P})$, all the sequence (u_n) converges to u in $w - W^{1,p}(\Omega)$ and

$$\int_\Omega \frac{1}{a} G(x, a \nabla u)\, dx = \lim \int_\Omega \frac{1}{a_n} G(x, a_n \nabla u_n)\, dx .$$

REMARK. – The previous method used in order to prove the convergence of the minimization problems $(\mathcal{P}_n)$ is, in fact, a method introduced by De Giorgi which consists of first proving the Γ-convergence of the functionals one wants to minimize and then to deduce the convergence of related minimization problems (cf. [4] and also [2]).

3. – Some further results.

3a) *Some properties of* $(\mathcal{P}_n)$.

In this paragraph, we give Euler's equation of $(\mathcal{P}_n)$ and some properties of the solution and the minimum of $(\mathcal{P}_n)$ for the case where G is positively homogeneous.

PROPOSITION 3.1. – *The solution u_n of $(\mathcal{P}_n)$ is characterized by*

$$
(3.1) \qquad \begin{cases} \operatorname{div}\,(g(x, a_n \nabla u_n)) = 0 \quad \text{in } \Omega_{i,n}, \ 1 \leq i \leq n, \\ u_n = 0 \ \text{on } \Gamma_0, \quad u_n = 1 \ \text{on } \Gamma_1, \\ u_n = c_{i,n} \ (\text{constant}) \ \text{on } \Gamma_{i,n}, \quad 1 \leq i \leq n-1, \\ \displaystyle\int_{\Gamma_{i,n}} g(x, a_n \nabla u_n)\cdot \nu \, d\gamma \ \text{is independent of } i \ (1 \leq i \leq n), \end{cases}
$$

where $\nu(x) = \nabla\phi(x)/|\nabla\phi(x)|$ is the unit normal to the hypersurface $\{\phi = \phi(x)\}$ at the point x and pointing towards Γ_1.

Moreover

$$
\int_{\Gamma_{i,n}} g(x, a_n \nabla u_n)\cdot \nu \, d\gamma = \frac{1}{c_{i,n} - c_{i-1,n}} \int_{\Omega_{i,n}} g(x, a_n \nabla u_n)\cdot \nabla u_n \, dx
$$

and

$$
\int_{\Gamma_t} g(x, a_n \nabla u_n)\cdot \nu \, d\gamma \ \text{is independent of } t \in [0,1].
$$

PROOF. – The proof is classical and we omit it.

COROLLARY 3.2. – *Assume that G is positively homogeneous of degree p with respect to ζ, that is*

$$
(3.2) \qquad \text{a.e. } x \in \Omega, \ \forall \zeta \in \boldsymbol{R}^N, \ \forall \lambda \in \boldsymbol{R}_+, \ G(x, \lambda\zeta) = \lambda^p G(x, \zeta).
$$

Then

$$
\int_{\Omega} \frac{1}{a_n} G(x, a_n \nabla u_n)\, dx = \frac{1}{p} \int_{\Gamma_t} g(x, a_n \nabla u_n)\cdot \nu \, d\gamma .
$$

PROOF. – Since G is positively homogeneous, we have $G(x, \zeta) = (1/p)\, g(x, \zeta)\cdot\zeta$. Applying also Proposition 3.1, we get

$$
\int_{\Omega} \frac{1}{a_n} G(x, a_n \nabla u_n)\, dx = \sum_{i=1}^{n} \int_{\Omega_{i,n}} \frac{1}{p} g(x, a_n \nabla u_n)\cdot \nabla u_n \, dx =
$$

$$
= \frac{1}{p} \sum_{i=1}^{n} (c_{i,n} - c_{i-1,n}) \int_{\Gamma_{i,n}} g(x, a_n \nabla u_n)\cdot \nu \, d\gamma = \frac{1}{p} \int_{\Gamma_t} g(x, a_n \nabla u_n)\cdot \nu \, d\gamma .
$$

REMARK 3.3. – Assume that G is positively homogeneous. We can make $(\mathcal{P}_n)$ explicit in terms of the solution $v_{i,n}$ of

$$\begin{cases} \operatorname{div}\,(g(x, a_n \nabla v_{i,n})) = 0 & \text{in } \Omega_{i,n}, \\ v_{i,n} = 0 \text{ on } \Gamma_{i-1},\ v_{i,n} = 1 \text{ on } \Gamma_{i,n} \end{cases}$$

as follows:

$$u_n = (c_{i,n} - c_{i-1,n})\, v_{i,n} + c_{i-1,n} \qquad \text{on } \Omega_{i,n},$$

$$\int_\Omega \frac{1}{a_n} G(x, a_n \nabla u_n)\, dx = \frac{1}{R_n}$$

where

$$M_{i,n} = \int_{\Omega_{i,n}} \frac{1}{a_n} G(x, a_n \nabla v_{i,n})\, dx,$$

$$R_n = \left(\sum_{i=1}^{n} M_{i,n}^{-1/(p-1)}\right)^{p-1},$$

$$c_{i,n} = R_n^{-1/(p-1)} \sum_{j=1}^{i} M_{j,n}^{-1/(p-1)}.$$

3*b*) *Explicit resolution of* $(\mathcal{P})$.

In the remark just above, we gave an explicit resolution of $(\mathcal{P}_n)$, in the case when (3.2) holds true. The explicit resolution of $(\mathcal{P})$ is quite general and much simpler. It is intuitively clear that $(\mathcal{P})$ is effectively a one dimensional problem.

To be precise, consider the Carathéodory function $\underline{G}$: $]0,1[\times \boldsymbol{R} \to \boldsymbol{R}$ defined by

$$\underline{G}(t, z) = \int_{\Gamma_t} \frac{G(x, z\nabla\phi)}{|\nabla\phi|}\, d\gamma. \tag{3.3}$$

Then, $\underline{G}(t,\cdot)$ admits a gradient denoted by $\underline{g}(t,\cdot)$ and

$$\underline{g}(t, z) = \int_{\Gamma_t} g(x, z\nabla\phi)\cdot\nu\, d\gamma.$$

Clearly $\underline{G}$ is strictly convex as a function of z and $\underline{G}$ has the same properties as G in (1.2) and (1.3):

$$\underline{c}_1 |z|^p \le \underline{G}(t, z) \le \underline{c}_2 |z|^p + \underline{c}_3(t), \tag{$\underline{1.2}$}$$

$$|\underline{g}(t, z)| \le \underline{c}_4 (1 + |z|^{p-1}), \tag{$\underline{1.3}$}$$

with

$$\underline{c}_1 = c_1 \operatorname*{Min}_t \int_{\Gamma_t} |\nabla\phi|^{p-1} d\gamma, \qquad \underline{c}_2 = c_2 \operatorname*{Max}_t \int_{\Gamma_t} |\nabla\phi|^{p-1} d\gamma, \qquad \underline{c}_3(t) = \int_{\Gamma_t} \frac{c_3}{|\nabla\phi|} d\gamma,$$

$\underline{c}_3 \in L^1(0,1)$ by the coarea theorem, $\underline{c}_4 = c_4 \max\left(\max_t |\Gamma_t|, \max_t \int_{\Gamma_t} |\nabla\phi|^{p-1} d\gamma\right)$.

Let us define

$$(\underline{\mathcal{P}}) \qquad \operatorname{Inf}\left\{\int_0^1 \frac{1}{\underline{a}} \underline{G}(t, \underline{a}\underline{v}')\, dt;\ \underline{v} \in \underline{K}\right\}.$$

The problem $(\underline{\mathcal{P}})$ has a unique solution $\underline{u}$.

THEOREM 3.4. – *We have*

a) $\operatorname{Inf}(\mathcal{P}) = \operatorname{Inf}(\underline{\mathcal{P}})$.

b) *The solution of* $(\mathcal{P})$ *is* $u = \underline{u} \circ \phi$, *where* $\underline{u}$ *is the solution of* $(\underline{\mathcal{P}})$.

PROOF. – By definition, u is solution of $(\mathcal{P})$ if and only if $u \in K$ and

$$\forall v \in K, \quad \int_\Omega \frac{1}{a} G(x, a\nabla u)\, dx \leqslant \int_\Omega \frac{1}{a} G(x, a\nabla v)\, dx,$$

that is, by Lemma 2.2, the function $\underline{u}$ defined by «$\underline{u}(t)$ = the trace of u on Γ_t» belongs to $\underline{K}$ and from the coarea formula, for every $\underline{v} \in \underline{K}$, $v = \underline{v} \circ \phi$,

$$\int_0^1 \frac{1}{\underline{a}(t)}\left(\int_{\Gamma_t} \frac{G(x, a\nabla u)}{|\nabla\phi|} d\gamma\right) dt \leqslant \int_0^1 \frac{1}{\underline{a}(t)}\left(\int_{\Gamma_t} \frac{G(x, a\nabla v)}{|\nabla\phi|} d\gamma\right) dt.$$

Since $\nabla v = \underline{v}'(\phi)\nabla\phi$, (resp. $\nabla u = u'(\phi)\nabla\phi$) we get the necessary and sufficient condition

$$\underline{u} \in \underline{K},$$

$$\forall \underline{v} \in \underline{K}, \quad \int_0^1 \frac{1}{\underline{a}} \underline{G}(t, \underline{a}\underline{u}')\, dt \leqslant \int_0^1 \frac{1}{\underline{a}} \underline{G}(t, \underline{a}\underline{v}')\, dt,$$

which gives that $\underline{u}$ is the solution of $(\underline{\mathcal{P}})$.

REMARK 3.5. – The solution $\underline{u}$ of $(\underline{\mathcal{P}})$ is characterized by the Euler equation

$$\frac{d}{dt}(\underline{g}(t, \underline{a}\underline{u}')) = 0 \quad \text{in }]0,1[,$$

$$\underline{u}(0) = 0, \qquad \underline{u}(1) = 1.$$

In general, this equation cannot be solved explicitly.

However, assume that G is homogeneous, that is verifies (3.2). Then the Euler equation of $(\underline{\mathcal{P}})$ can be solved explicitly, and we get:

$$\underline{u}(t) = \frac{\int_0^t \underline{a}(\tau)^{-1}\underline{G}(\tau, 1)^{-1/(p-1)}\,d\tau}{\int_0^1 \underline{a}(\tau)^{-1}\underline{G}(\tau, 1)^{-1/(p-1)}\,d\tau}$$

and $\operatorname{Inf}(\underline{\mathcal{P}}) = \left(\int_0^1 \underline{a}(\tau)^{-1}\underline{G}(\tau, 1)^{-1/(p-1)}\,d\tau\right)^{1-p}$.

3c) *A particular case when the constraints do not play any role.*

In general, the problems $(\mathcal{P}_n)$ do not reduce to one-dimensional problems (see *e.g.* Proposition 3.1), because the level surfaces of a_n and u_n do not necessarily coincide between the Γ_t, $t \in T_n$. However this does occur in *e.g.* complete spherical symmetry, or, more generally, in the situation described below. In this case the constraint $v =$ constant on Γ_t can be dropped from $(\mathcal{P}_n)$ and $(\mathcal{P})$. In fact, we have the

PROPOSITION 3.6. – *Consider the minimization problem*

$$(\mathcal{Q}) \qquad \operatorname{Inf}\left\{\int_\Omega \frac{1}{a}G(x, a\nabla v)\,dx;\ v \in W^{1,p}(\Omega),\ v = 0 \text{ on } \Gamma_0,\ v = 1 \text{ on } \Gamma_1\right\}.$$

Then the solution of $(\mathcal{Q})$ *is constant on the level surfaces of* ϕ, *that is* $(\mathcal{Q})$ *has the same solution and minimum as* $(\mathcal{P})$, *if and only if there exists* $\underline{k} \in L^p(0,1)$ *such that*

$$(C) \qquad \int_0^1 \frac{\underline{k}}{\underline{a}}\,dt = 1 \quad \textit{and} \quad \operatorname{div} g(x, (\underline{k}\circ\phi)\nabla\phi) = 0 \ \textit{in}\ \Omega.$$

Actually $\underline{k} = \underline{a}\underline{u}'$ *where* $\underline{u}$ *is the solution of* $(\underline{\mathcal{P}})$.

The solution and minimum are explicit:

— $u = \underline{u}\circ\phi$ *where* $\underline{u}(t) = \int_0^t \frac{\underline{k}}{\underline{a}}\,d\tau$,

— $\int_\Omega \frac{1}{a}G(x, a\nabla u)\,dx = \int_0^1 \frac{1}{\underline{a}}\underline{G}(t, \underline{k})\,dt$.

PROOF. – The solution u of $(\mathcal{Q})$ is constant on the level surfaces of ϕ if and only if u is the solution of $(\mathcal{P})$, which is equivalent (from Theorem 3.4) to $u = \underline{u}\circ\phi$ where $\underline{u}$ is the solution of $(\underline{\mathscr{P}})$. In this case, from the Euler equation of $(\mathscr{Q})$, we have

$$\operatorname{div} g(x, a\nabla u) = 0 \quad \text{in } \Omega, \text{ that is } \operatorname{div} g(x, (\underline{a}\circ\phi)(\underline{u}'\circ\phi)\nabla\phi) = 0\,.$$

Hence (C) is verified with $\underline{k} = \underline{a}\underline{u}'$. Conversely, assume (C). Let $\underline{u}$ be defined by

$$\begin{cases} \underline{u}' = \dfrac{\underline{k}}{\underline{a}} & \text{in }]0,1[\\ \underline{u}(0) = 0\,. \end{cases}$$

We have $\underline{u}(1) = \int_0^1 (\underline{k}/\underline{a})\,dt = 1$ and $\operatorname{div} g(x, (\underline{a}\circ\phi)(\underline{u}'\circ\phi)\nabla\phi) = 0$ in Ω, that is $\operatorname{div} g(x, a\nabla u) = 0$ in Ω where $u = \underline{u}\circ\phi$. Hence u is the solution of $(\mathscr{Q})$. Notice that in this case the Euler equation of $(\underline{\mathscr{P}})$ can be solved explicitly.

COROLLARY 3.7. – *Assume G is homogeneous, that is verifies* (3.2), *and assume* $\operatorname{div} g(x, \nabla\phi) = 0$ *in* Ω. *Then the solutions* u_n *of* $(\mathscr{P}_n)$ *are*

$$u_n = \underline{u}_n\circ\phi \quad \textit{with } \underline{u}_n(t) = \underline{k}_n\int_0^t \frac{1}{\underline{a}_n}\,d\tau \textit{ and } \frac{1}{\underline{k}_n} = \int_0^1 \frac{1}{\underline{a}_n}\,d\tau$$

and the minima of $(\mathscr{P}_n)$ *are*

$$\int_\Omega \frac{1}{a_n} G(x, a_n\nabla u_n)\,dx = \int_0^1 \frac{1}{\underline{a}_n}\underline{G}(t, \underline{k}_n)\,dt\,.$$

The proof is quite easy.

REMARK 3.8. – The problems $(\mathscr{P}_n)$ and $(\mathscr{P})$ do not depend on the particular choices of $\underline{a}_n$, $\underline{a}$ and ϕ such that $a_n = \underline{a}_n\circ\phi$, $a = \underline{a}\circ\phi$. That is $(\mathscr{P}_n)$ and $(\mathscr{P})$ do not change if ϕ is replaced by $\psi = f\circ\phi$, $\underline{a}$ by $\underline{b} = \underline{a}\circ f^{-1}$, $\underline{a}_n$ by $\underline{b}_n = \underline{a}_n\circ f^{-1}$, T_n by $f(T_n)$, where $f\colon[0,1]\to[0,1]$ is a C^1 strictly increasing function such that $f(0) = 0$, $f(1) = 1$. On the contrary, $(\underline{\mathscr{P}})$ depends on the particular choices of $\underline{a}$ and ϕ: with the previous changes, $(\underline{\mathscr{P}})$ becomes $(\underline{\mathscr{Q}})$ corresponding to $\underline{b}$ and ψ, $(\underline{\mathscr{Q}})$ has the same minimum as $(\underline{\mathscr{P}})$ (and $(\mathscr{P})$) and its solution is $\underline{w} = \underline{v}\circ f^{-1}$. A convenient choice of $\underline{a}$ and ϕ may simplify the resolution of $(\underline{\mathscr{P}})$, as well is the verification of the assumption (2.1). This will be illustrated in section 4.

3*d) Generalization to* a_n *«asymptotically constant» on the level surfaces of* ϕ.

Our main result of Theorem 2.1 is still valid if the a_n are only assumed to be «asymptotically constant» (instead of being constant) on the level surfaces of ϕ, that is if we assume that

$$\|a_n - \underline{\alpha}_n\circ\phi\|_{L^\infty(\Omega)} \to 0 \quad \text{when } n\to\infty\,, \tag{3.4}$$

where

$$\underline{\alpha}_n(t) = \frac{\displaystyle\int_{\Gamma_t} \frac{d\gamma}{|\nabla\phi|}}{\displaystyle\int_{\Gamma_t} \frac{d\gamma}{a_n|\nabla\phi|}}.$$

(Note that, by the coarea theorem, for almost every t, $1/a_n \in L^1(\Gamma_t)$ and $\underline{\alpha}_n \in L^\infty(0,1)$).

The assumption (2.1) is then replaced by

$$\frac{1}{a_n} \to \frac{1}{a} \quad \text{weakly* in } L^\infty(\Omega), \text{ with } a \in L^\infty(\Omega).$$

We note that this implies (with (3.4))

$$a = \underline{\alpha} \circ \phi \tag{3.5}$$

(where $\underline{\alpha}$ is defined as $\underline{\alpha}_n$ with a_n replaced by a) that is a is constant on the level surfaces of ϕ. In fact

$$\left\| \frac{1}{a_n} - \frac{1}{\underline{\alpha}_n \circ \phi} \right\|_{L^\infty(\Omega)} = \left\| \frac{\underline{\alpha}_n \circ \phi - a_n}{a_n(\underline{\alpha} \circ \phi)} \right\|_{L^\infty(\Omega)} \leqslant \frac{1}{C^2} \|\underline{\alpha}_n \circ \phi - a_n\|_{L^\infty(\Omega)}$$

which tends to zero when n tends to infinity. It is easy to prove (as in Lemma 2.1) that

$$\frac{1}{\underline{\alpha}_n} \to \frac{1}{\underline{\alpha}} \quad \text{weakly * in } L^\infty(0,1),$$

which, by Lemma 2.1, is equivalent to

$$\frac{1}{\underline{\alpha}_n \circ \phi} \to \frac{1}{\underline{\alpha} \circ \phi} \quad \text{weakly * in } L^\infty(\Omega).$$

Since $1/a_n \to 1/a$ weakly * in $L^\infty(\Omega)$, the previous inequality implies (3.5).

With these new assumptions, we can adapt the proof of Lemma 2.4 as follows, everything else being unchanged. We define $\underline{v}_n$ by

$$\underline{v}_n(t) = \frac{1}{\delta_n} \int_0^t \frac{\underline{\alpha}}{\underline{\alpha}_n} \underline{v}'\, ds, \qquad \text{where } \delta_n = \int_0^1 \frac{\underline{\alpha}}{\underline{\alpha}_n} \underline{v}'\, ds.$$

As previously, $\delta_n \to 1$ and $v_n = \underline{v}_n \circ \phi \to v$ in $L^p(\Omega)$. Moreover

$$\nabla v_n = \frac{1}{\delta_n} \frac{\underline{\alpha} \circ \phi}{\underline{\alpha}_n \circ \phi} \nabla v \to \nabla v \quad \text{weakly in } L^p(\Omega)^N.$$

On the contrary, $a_n \nabla v_n \to a \nabla v$ strongly in $L^P(\Omega)^N$, as previously, since

$$a_n \nabla v_n = \frac{1}{\delta_n} \frac{a_n}{\underline{a}_n \circ \phi} a \nabla v, \qquad \delta_n \to 1, \qquad \frac{a_n}{\underline{a}_n \circ \phi} \to 1 \text{ in } L^\infty(\Omega) \text{ since}$$

$$\left\| \frac{a_n}{\underline{a}_n \circ \phi} - 1 \right\|_{L^\infty(\Omega)} = \left\| \frac{\underline{a}_n \circ \phi - a_n}{\underline{a}_n \circ \phi} \right\|_{L^\infty(\Omega)} \leqslant \frac{1}{C} \|\underline{a}_n \circ \phi - a_n\|_{L^\infty(\Omega)} \to 0, \qquad \text{as } n \to \infty.$$

4. – An example.

A typical situation (see *e.g.* [2], [3]) which the theory of homogenization is supposed to handle is that of having an object (the resistor in our case) composed of two different materials alternating with each other in thin layers. In our case, this could correspond to having the functions $\underline{a}_n$ of the form

$$\underline{a}_n = \begin{cases} \alpha & \text{on } [t_{i,n}, t_{i+1,n}] \text{ for even } i, \\ \beta & \text{on } [t_{i,n}, t_{i+1,n}] \text{ for odd } i, \end{cases} \tag{4.1}$$

where α, $\beta > 0$, $\alpha \neq \beta$.

The condition (2.2) is simple enough to understand, so let us try to interprete (2.1) when the $\underline{a}_n$ are given by (4.1) and, for simplicity, when $\underline{a}$ is a constant. Set, for every interval I included in $[0, 1]$, $I_{\alpha,n} = \{t \in I, \underline{a}_n(t) = \alpha\}$, $I_{\beta,n} = \{t \in I, \underline{a}_n(t) = \beta\}$. It is easy to check that condition (i) in Lemma 2.1 is equivalent to

$$\int_I \frac{dt}{\underline{a}_n} \to \int_I \frac{dt}{\underline{a}} \quad \text{for every interval } I.$$

It follows that, setting $\eta = (1/\underline{a} - 1/\beta)(1/\alpha - 1/\underline{a})^{-1}$, (2.1) holds if and only if $|I_{\alpha,n}|/|I_{\beta,n}| \to \eta$ for every interval I included in $[0, 1]$. (Note that this is true independently of whether (2.2) is satisfied or not).

If $t_{i,n} = i/n$, then clearly $|I_{\alpha,n}|/|I_{\beta,n}| \to 1$, so that we get the weak*-convergence of $1/\underline{a}_n$ to

$$\frac{1}{\underline{a}} = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right). \tag{4.2}$$

More generally, if $t_{i,n} = i/n$ for even values of i and $t_{i,n} = (1/n)(i + (\eta - 1)/(\eta + 1))$ (for some $\eta > 0$) for odd values of $i < n$, then $|I_{\alpha,n}|/|I_{\beta,n}| \to \eta$, so that $1/\underline{a}_n$ converges weakly* to

$$\frac{1}{\underline{a}} = \frac{1}{\eta + 1}\left(\frac{\eta}{\alpha} + \frac{1}{\beta}\right). \tag{4.3}$$

APPLICATION. – We give below the limit problems, for three simple given configurations of $\{\Gamma_{i,n}\}$. By convenient choices of ϕ, they all correspond to $t_{i,n} = i/n$.

a) The volume of $\Omega_{i,n}$ *is independent of* i. Assume that the $\Gamma_{i,n}$ are level surfaces of some smooth function ψ satisfying the usual assumption on ϕ and assume that the volume of $\Omega_{i,n}$ is independent of i. Then the distribution function of ψ, $\mu(t) = |\psi > t|$ is smooth, and the $\Gamma_{i,n}$ have equation $\phi(x) = i/n$ for the choice $\phi(x) = (v_0 - v(x))/(v_0 - v_1)$ where $v_0 = |\Omega_0|$, $v_1 = |\Omega_1|$, $v(x) = v_1 + \mu(\psi(x))$. For the limit problem, $\underline{a}$ is given by (4.2) and

$$K = \{v \in W^{1,p}(\Omega);\ v = 0 \text{ on } \Gamma_0,$$
$$v = 1 \text{ on } \Gamma_1,\ v \text{ is constant on the level surfaces of } \psi\}.$$

b) The width of $\Omega_{i,n}$ *is independent of* i. Assume that Γ_0 and Γ_1 are parallel, at distance d from each other, and that the $\Gamma_{i,n}$ are parallel and equidistant. By choosing $\phi(x) = (1/d)\, d(x, \Gamma_0)$ where $d(x, \Gamma_0)$ is the distance from x to Γ_0, we get, for the limit problem, $\underline{a}$ given by (4.2) and

$$K = \{v \in W^{1,p}(\Omega);\ v = 0 \text{ on } \Gamma_0,$$
$$v = 1 \text{ on } \Gamma_1,\ v \text{ is constant on all hypersurfaces parallel to } \Gamma_0 \text{ and } \Gamma_1\}.$$

c) $\Gamma_{i+1,n}$ *is homothetic from* $\Gamma_{i,n}$ *with ratio independent of* i. Assume that Γ_0, Γ_1 and $\Gamma_{i,n}$ are homothetic and such that

$$\Gamma_1 = \rho\Gamma_0 \quad (0 < \rho < 1), \qquad \Gamma_{i+1,n} = \rho^{1/n}\Gamma_{i,n} = \rho^{(i+1)/n}\Gamma_0.$$

Let us denote by $\rho(x)$ the ratio of the homothecy which transforms Γ_0 into the hypersurface containing x. By choosing $\phi(x) = \operatorname{Log} \rho(x)/\operatorname{Log} \rho$, we easily obtain that $\underline{a}$ is again given by (4.2) and

$$K = \{v \in W^{1,p}(\Omega);\ v = 0 \text{ on } \Gamma_0,\ v = 1 \text{ on } \Gamma_1,$$
$$v \text{ is constant on all hypersurfaces homothetic to } \Gamma_0 \text{ and } \Gamma_1\}.$$

Appendix.

LEMMA. – *$\overline{\Omega}$ is C^1-diffeomorphic to* $[0,1] \times \Gamma_0$.

PROOF. – Let $D\colon \overline{\Omega} \to [0,1] \times \Gamma_0$, $D(x) = (\phi(x), \psi(x))$, where $\psi\colon \overline{\Omega} \to \Gamma_0$ is defined as follows. Consider the smooth vector field $\nabla\phi$ in $\overline{\Omega}$. For each $x \in \overline{\Omega}$, there exists a unique integral curve to $\nabla\phi$ which passes through x and this integral curve hits Γ_0 at the point $y = \psi(x)$.

If one wants to do the above more explicitly, one gets simpler formulas by using the vector field $\nabla\phi/|\nabla\phi|^2$ (instead of $\nabla\phi$). Then the integral curve through $x \in \overline{\Omega}$ is given by the solution $z(t) = z(t, x)$ of the differential equation:

$$\begin{cases} z'(t) = \dfrac{\nabla\phi(z(t))}{|\nabla\phi(z(t))|^2} \\ z(\phi(x)) = x\,. \end{cases}$$

The solution $z(t)$ reaches Γ_0 for $t = t_0$ such that $\phi(z(t_0)) = 0$. We have

$$\begin{cases} \dfrac{d}{dt}\phi(z(t)) = \nabla\phi(z(t))\, z'(t) = 1\,, \\ \phi(z(t)) = t\,. \end{cases}$$

Thus $t_0 = 0$, $y = z(0, x) = \psi(x)$. It is clear that z: $[0,1] \times \overline{\Omega} \to \overline{\Omega}$ is of class C^1. Hence ψ: $\overline{\Omega} \to \Gamma_0$ defined by $\psi(x) = z(0, x)$ belongs to $C^1(\overline{\Omega})$. It is easy to prove that D is one to one and $D^{-1}(t, y) = z(t, y)$.

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