Vortex motion in two-dimensional hydrodynamics

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Abstract

We study the motion of point vortices in a two dimensional flow region of arbitrary geometry. The fluid is supposed to be incompressible and non-viscous. The total kinetic energy of the flow is infinite and no information can be derived from its conservation. Nevertheless it is possible to divide it into an infinite core energy and a finite part, the renormalized kinetic energy. Conservation of the renormalized kinetic energy gives a great deal of information on the dynamics of the vortex centers. The interaction of vortex pairs is given by the hydrodynamic Green's function; the interaction of each vortex with the boundary by the hydrodynamic Robin function. Our proof of long time existence for vortex pairs involves a careful analysis of the boundary behaviour of these functions. The difficult step is to exclude collisions of equally oriented vortices at the boundary. The result is optimal in the sense that three vortices can collide in finite time. Finally we estimate the energy exchange rate between large vortex clusters. This is a first step towards stability estimates for vortex clusters; which is one of the outstanding question in vortex dynamics.

Keywords: ideal fluid dynamics, vortex motion, energy renormalization, Robin function

Mathematical Subjects Classification: 35J65, 76C05, 76M15

1 Introduction

We ask the following questions concerning the dynamics of point vortices in two dimensional ideal fluid dynamics.

1. What is the equation of motion for the vortex centers?

2. What can be said about the qualitative behaviour of this dynamical system like existence of periodic orbits, stationary constellations, possibility of collisions, behaviour of vortices near the boundary?

3. How do we numerically solve the initial value problem for the vortex centers?

Regarding the first question we present two natural methods to derive the equation of motion for vortex centers. The energy renormalization or core energy method is well known in physics but less so amongst mathematicians. Roughly speaking it is a rescaling technique for dynamical systems of infinite energy to obtain a finite conserved quantity. The idea is to separate the energy into two parts each of which is conserved separately. The infinite part (the core energy) is the same for all geometries, while the remaining part (the renormalized energy) is the interesting one that carries geometrical information. The second derivation is a novel method based on the calculus of residues. In the resulting equation of motion the interaction between different vortices is described by the hydrodynamic Green's function while the interaction of each vortex with the boundary is governed by the hydrodynamic Robin function. A detailed analysis of the properties of the hydrodynamic Green's and Robin function provides information on the qualitative behaviour of isolated vortices and that of vortex clusters. It turns out that a single vortex or a highly concentrated vortex patch moves along the level lines of the hydrodynamic Robin function. In a simply connected domain these lines are easily computed by solving Liouville's equation. For multiply connected flow regions we present a variant of the boundary element method.

2 Planar hydrodynamics

The flow region is represented by a domain $\Omega \subset I\!R^2$. We suppose that it is bounded and finitely connected with Lipschitz boundary

$$\partial\Omega = \bigcup_{k=1}^{K} \Gamma_k$$

which satisfies a uniform exterior ball condition. Each of the boundary components Γ_k is supposed to be topologically equivalent to a circle. The exterior unit normal is denoted by ν. The velocity of the fluid at time τ is denoted by $v(\tau)$. First we investigate the time evolution of the scalar vorticity

$$\omega := \operatorname{curl} v := \partial_1 v^2 - \partial_2 v^1$$

in the smooth case. The velocity field of an incompressible fluid in two dimensions admits a stream function

$$v = J\nabla\psi$$

where J denotes the symplectic matrix

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

corresponding to clockwise rotation by $\pi/2$. By definition of the vorticity the stream function is a solution of the Poisson equation

$$\begin{aligned} -\Delta\psi &= \omega \text{ in } \Omega, \\ \partial_{J\nu}\psi &= 0 \text{ on } \partial\Omega \end{aligned}$$

where $\partial_{J\nu}$ denotes the tangential derivative. Each boundary component is a stream line. In particular the stream function is defined globally. The basic equation of non-viscous fluid motion are the Euler equations of ideal fluid dynamics

$$\partial_\tau v + (v \cdot \nabla)\, v + \frac{\nabla p}{\rho} = 0. \tag{1}$$

Here p denotes the pressure and ρ the (constant) density of the fluid. The convection term can be written as

$$(v \cdot \nabla)\, v = \frac{1}{2}\nabla |v|^2 - \omega J v.$$

In the stationary irrotational case we obtain Bernoulli's law

$$\frac{1}{2}|v|^2 + \frac{p}{\rho} = \text{const.} \tag{2}$$

Taking another curl of the Euler equations yields the transport equation for the vorticity. Using the incompressibility of the fluid one finds that the material derivative vanishes

$$D_\tau\omega := \partial_\tau\omega + (v \cdot \nabla)\,\omega = 0. \tag{3}$$

This means that the vorticity is convected along the flow. A particular consequence of this and the incompressibility is that the vorticity distribution at an arbitrary instant τ is an equimeasurable rearrangement of the initial vorticity distribution, i.e. $\partial_\tau |\{z \in \Omega : \omega(z,\tau) > \lambda\}| = 0$ for every λ. Moreover the *circulation*

$$c_k := \int_{\Gamma_k} v \cdot dz = \int_{\Gamma_k} \partial_\nu\psi$$

around each of the boundary components is conserved by

$$\dot{c}_k = \int_{\Gamma_k} \partial_\tau v \cdot dz = -\int_{\Gamma_k} \nabla\left(\frac{1}{2}|v|^2 + \frac{p}{\rho}\right) \cdot dz + \int_{\Gamma_k} \omega J v \cdot dz = 0.$$

This is Kelvin's theorem (Figure 1).

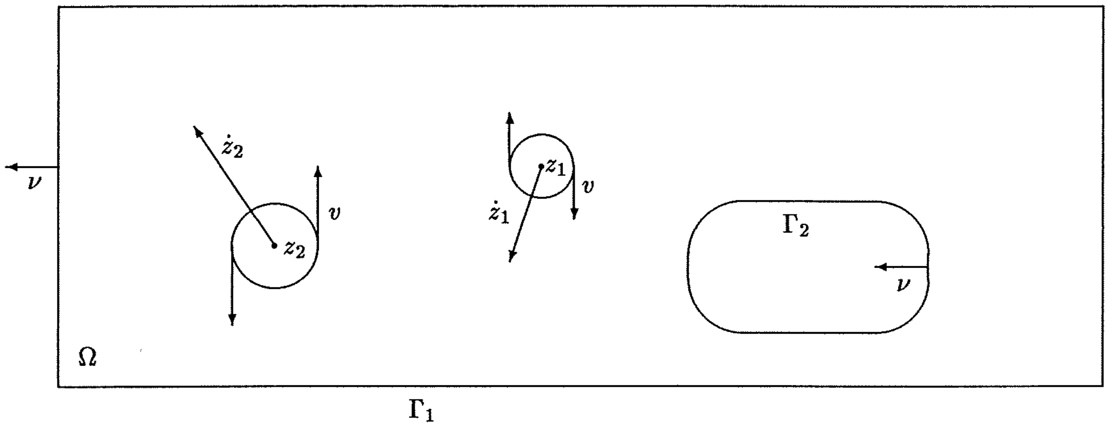

Figure 1: Flow region and point vortices.

Passing to the point vortex limit means to consider an initial vorticity distribution concentrated at finitely many points represented by a collection of Dirac masses

$$\omega = \sum_{p=1}^{P} \omega_p \delta_{z_p}, \quad z_p \in \Omega.$$

The strength of the p'th vortex is ω_p and may be positive or negative according to its orientation. Strictly speaking the product term $\omega J v$ is not defined for such vorticity distributions. Even the weak form of the Euler equations is meaningless. Nevertheless $\omega J v$ vanishes except at the vortex centers. Thus the Euler equations simplify to

$$(4) \qquad \partial_\tau v = -\nabla\left(\frac{1}{2}|v|^2 + \frac{p}{\rho}\right) \text{ in } \Omega \setminus \bigcup_p \{z_p(\tau)\}.$$

As a limit form of the equimeasurability property for smooth vortex distributions we postulate that the vorticity at any time $\tau \geq 0$ remains of the form

$$(5) \qquad \omega(\tau) = \sum_p \omega_p \delta_{z_p(\tau)}.$$

3 Hydrodynamic Green's and Robin function

The properties of the stream function in the presence of point vortices suggests the definition of a special Green's function. It corresponds to the stream function of a single point vortex of unit strength.

Definition 1 (Hydrodynamic Green's function) *The hydrodynamic Green's function with periods $\gamma_1, \ldots, \gamma_K$ subject to $\sum \gamma_k = -1$ is defined as the solution G_z of the problem*

$$\begin{aligned} -\Delta G_z &= \delta_z \quad in \ \Omega, \\ \partial_{J\nu} G_z &= 0 \quad on \ \partial\Omega, \\ \int_{\Gamma_k} \partial_\nu G_z &= \gamma_k \quad for\ every\ k, \\ \int_{\partial\Omega} G_z \partial_\nu G_\zeta &= 0 \quad for\ every\ z, \zeta \in \Omega. \end{aligned}$$

The hydrodynamic Green's function is constant on each boundary component. Its values will be denoted by $g_z^k := G_z|_{\Gamma_k}$. Strictly speaking the above definition only applies to smooth domains. Nevertheless a general domain satisfying the assumptions of Section 2 is conformally equivalent to a smooth domain. The hydrodynamic Green's function, the circulations and the normalization are conformally invariant. This permits to define the hydrodynamic Green's function for non-smooth domains. The last requirement is a normalization condition. It selects a unique solution (Proposition 7). The singularity at z represents a source of unit strength. Therefore necessarily $\sum \gamma_k = -1$. The first, second and third requirement specify G_z up to an additive constant. The periods are independent of the normalization. Furthermore the hydrodynamic Green's function is symmetric. Integration by parts yields

$$G_z(\zeta) \;=\; \int_\Omega G_z\,(-\Delta G_\zeta) \;=\; -\int_\Omega \nabla G_z \cdot \nabla G_\zeta + \int_{\partial\Omega} G_z \partial_\nu G_\zeta$$

which is symmetric by the normalization. Uniqueness follows from the maximum principle. Indeed, the difference of two solutions $u = G_z - \widetilde{G}_z$ is harmonic in Ω. It is locally constant at the boundary and the periods vanish. By the strong maximum principle $\max u$ is attained on some Γ_k and either u is constant in Ω or $\partial_\nu u > 0$ on Γ_k. In the former case $u = 0$ by the normalization condition. In the latter case the period $\int_{\Gamma_k} \partial_\nu u$ would be non-zero which is a contradiction. Details will be given in Section 8.

In order to define the hydrodynamic Robin function we decompose the hydrodynamic Green's function into a radially symmetric singular and a regular part

$$G_z \;=\; F_z - H_z .$$

The singular part is the fundamental solution

$$F_z(\zeta) \;=\; -\frac{1}{2\pi}\log|\zeta - z|$$

while the regular part is harmonic in Ω with

$$\begin{aligned} H_z \;&=\; F_z - g_z^k \quad \text{on } \Gamma_k, \\ \int_{\Gamma_k} \partial_\nu H_z \;&=\; \int_{\Gamma_k} \partial_\nu F_z - \gamma_k . \end{aligned}$$

In particular we can evaluate the regular part at the singularity itself.

Definition 2 (Hydrodynamic Robin function) *The value of the regular part of the hydrodynamic Green's function at the singularity*

$$t(z) \;:=\; H_z(z)$$

is called hydrodynamic Robin function.

This function is thus associated to the Laplacian acting on the class of functions which are locally constant on the boundary with periods γ_k. Near the singularity the hydrodynamic Green's function can be expanded as

$$G_z(\zeta) \;=\; -\frac{1}{2\pi}\log|\zeta - z| - t(z) + O\left(|\zeta - z|\right).$$

At the boundary the hydrodynamic Robin function tends to $+\infty$ (Proposition 8). This is essential for long time existence of one and two vortices (Sections 9 and 10.1).

4 Point vortex model

Now we can formulate the equation of motion for the vortex centers. For $\Omega = \mathbb{R}^2$ it goes back to the work of Kirchhoff in 1876. In the presence of solid boundaries an additional self interaction term appears. This was observed by Routh in 1881. The renormalized kinetic energy is also called *Kirchhoff-Routh path function.* Finally in 1943 C.C. Lin gave the equation of motion in the most general case.

THEOREM 3 (Kirchhoff [11], Routh [16], Lin [12]) *Let Ω be a domain satisfying the general assumptions of Section 2 and suppose that the initial vorticity distribution is of the form $\omega(0) = \sum_p \omega_p \delta_{z_p(0)}$ with total vorticity $\sum_p \omega_p \neq 0$. Given the circulations c_k satisfying the consistency relation $\sum_k c_k + \sum_p \omega_p = 0$ define the periods*

$$\gamma_k := \frac{c_k}{\sum_p \omega_p}$$

and the corresponding hydrodynamic Green's function G_z and Robin function t as in the Definitions 1 and 2. Then:

1. *The speed of the vortex centers is given by*

$$\dot{z}_p = \sum_{q \neq p} \omega_q J \nabla G_{z_q}(z_p) - \frac{\omega_p}{2} J \nabla t(z_p). \tag{6}$$

2. *The* renormalized kinetic energy

$$E(z_1, \ldots, z_P) := \sum_{\{p,q\,:\,q>p\}} \omega_p \omega_q G_{z_q}(z_p) - \frac{1}{2} \sum_p \omega_p^2 t(z_p) \tag{7}$$

is an integral of motion.

3. *The equations of motion (6) has a Hamiltonian structure with Hamiltonian E and symplectic form $d\Omega = \sum_p \omega_p dx_p dy_p$. I.e.*

$$\dot{z}_p = \frac{1}{\omega_p} J \nabla_{z_p} E.$$

Proof. The stream function is given by

$$\psi = \sum_p \omega_p G_{z_p}.$$

By construction it has the prescribed circulations c_k. Regarding the velocity field we single out the contribution of the p'th vortex and decompose it into singular and regular part

$$v = \omega_p J \nabla F_{z_p} - \omega_p J \nabla H_{z_p} + \sum_{q \neq p} \omega_q J \nabla G_{z_q}.$$

The term $J \nabla F_{z_p}$ describes a pure rotation around the center z_p which does not contribute to the motion of z_p itself. Only the regular part contributes to its drift

$$\dot{z}_p = v_{reg}(z_p) = \sum_{q \neq p} \omega_q J \nabla G_{z_q}(z_p) - \omega_p J \nabla H_{z_p}(z_p). \tag{8}$$

The equation of motion (6) follows from the relation $\nabla t(z) = 2 \nabla H_z(z)$ which holds by symmetry of the hydrodynamic Green's function. Conservation of E is a consequence of the equation of motion and antisymmetry of the symplectic form. □

The argument of equation (8) is heuristic. In Section 7 we make up for this by giving a rigorous derivation of the point vortex model. Moreover the calculation proving conservation of the renormalized kinetic energy is simple but lengthy and it does not explain how the expression for the renormalized kinetic energy was found. In Section 5 it will be derived by means of the core energy method leading to a short and natural derivation. Physical justifications of the point vortex model will be given in Section 5. In fact it is the limit of several more realistic models.

Example 4 For $\Omega = \mathbb{R}^2$ Theorem 3 does not apply directly. Nevertheless the conclusions hold under rest conditions at infinity with $G_z = F_z$ and $t = 0$. The equation of motion for two vortices in the plane is

$$\dot{z}_1 = \frac{\omega_2}{2\pi} J \frac{z_2 - z_1}{|z_2 - z_1|^2}, \quad \dot{z}_2 = \frac{\omega_1}{2\pi} J \frac{z_1 - z_2}{|z_1 - z_2|^2}, \quad E = -\frac{\omega_1 \omega_2}{2\pi} \log |z_1 - z_2| .$$

In particular the distance of the vortex centers remains constant and the center of mass $\omega_1 z_1 + \omega_2 z_2$ is at rest. The motion of an arbitrary number of point vortices in the plane admits three independent integrals which are in involution with respect to the symplectic form $d\Omega$, namely

$$E, \left| \sum_p \omega_p z_p \right|^2, \sum_p \omega_p |z_p|^2 .$$

The corresponding conserved quantities (symmetries) are energy (time shift), moment of inertia (translation) and angular momentum (rotation). Thus the motion of up to three vortices is integrable while the motion of 4 and more vortices may be chaotic depending on the initial data (Aref and Pomphrey [2], Aref [1]).

In the exceptional case of vanishing total vorticity $\sum_p \omega_p = 0$ Theorem 3 is modified as follows. Let G_z be the hydrodynamic Green's function with periods γ_k chosen arbitrarily. If all circulations c_k vanish the stream function is $\psi = \sum_p \omega_p G_{z_p}$ and the theorem remains valid as it stands. Otherwise

$$\psi = \sum_p \omega_p G_{z_p} + \psi_0$$

where ψ_0 is the stream function for a stationary circulating flow, i.e. ψ_0 is harmonic in Ω and constant on each Γ_k. These constants are chosen such that $\int_{\Gamma_k} \partial_\nu \psi_0 = c_k$ which is possible since $\sum_k c_k = 0$. In the law of motion (6) the additional term $J\nabla\psi_0(z_p)$ appears on the right side. The renormalized energy contains the additional term $\sum_p \omega_p \psi_0(z_p)$. With these modifications the theorem also covers the case of vanishing total vorticity. In the same way unbounded domains with prescribed stationary flow conditions at infinity are treated.

Example 5 For the upper half plane $\{z = x + iy \,:\, y > 0\}$ one has

$$G_z(\zeta) = -\frac{1}{2\pi} \log \left| \frac{\zeta - z}{\zeta - \bar{z}} \right|, \quad t(z) = -\frac{1}{2\pi} \log (2y) .$$

A single vortex moves parallel to the boundary. Also the motion of two vortices of opposite orientation at equal distance from the boundary $z_{1,2} = \pm x + iy$ can be computed easily. We suppose no flux conditions at infinity. According to [3] the Robin function of the quarter plane $\{x + iy \,:\, x > 0,\ y > 0\}$ is $t(z) = -\frac{1}{4\pi} \log \frac{2x^2y^2}{x^2+y^2}$. Reflection at the y-axis gives the renormalized energy

$$E(z_1, z_2) = \frac{\omega^2}{4\pi} \log \frac{2x^2y^2}{x^2 + y^2} .$$

The vortex trajectories are of the form $\{x^2y^2 = c^2 (x^2 + y^2)\}$ (Fig. 2). The parameter c gives the asymptotic distance from both axes.

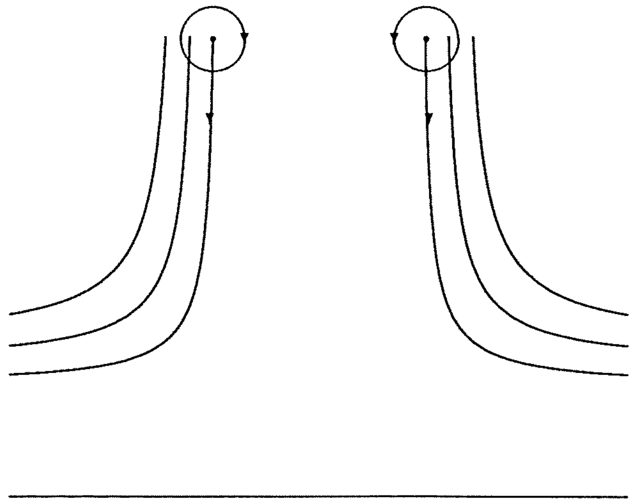

Figure 2: Motion of two vortices of opposite orientation in a half plane.

5 Core energy method

The core energy method is a heuristic way of deriving a finite conserved quantity from an infinite energy. The guess obtained in this way can be verified a posteriori using the equations of motion. For simplicity we restrict ourselves to the case of a single vortex of unit strength, i.e. $\psi = G_z$. The kinetic energy of the flow is

$$E_{kin} = \frac{1}{2}\int_\Omega \rho\,|v|^2 = \frac{\rho}{2}\int_\Omega |\nabla G_z|^2 = \infty.$$

The unbounded contribution comes from the core of the vortex. This suggests the excision of a small ball B_z^ρ around the center, dividing the kinetic energy into a finite and an infinite part. Since the fluid is incompressible and circulates around the center only a small amount of kinetic energy can cross the artificial boundary ∂B_z^ρ in finite time. This means that the energy diffusion between the finite and the infinite part of the energy can be neglected in the limit as the radius of the ball tends to zero. The finite part of the energy is

$$\begin{aligned}
\int_{\Omega\setminus B_z^\rho} |\nabla G_z|^2 &= \int_{\partial(\Omega\setminus B_z^\rho)} G_z \partial_\nu G_z = -\int_{\partial B_z^\rho} G_z \partial_\nu G_z \\
&= -\int_{\partial B_z^\rho} F_z\partial_\nu F_z + \int_{\partial B_z^\rho} F_z\partial_\nu H_z + \int_{\partial B_z^\rho} H_z\partial_\nu F_z - \int_{\partial B_z^\rho} H_z\partial_\nu H_z \\
&= -\frac{1}{2\pi}\log(\rho) + 0 - t(z) - \int_{B_z^\rho} |\nabla H_z|^2 \\
&= -\frac{1}{2\pi}\log(\rho) - t(z) + O\left(\rho^2\right).
\end{aligned}$$

The gradient of this is

$$\begin{aligned}
\nabla_z \int_{\Omega\setminus B_z^\rho} |\nabla G_z|^2 &= -\nabla t(z) - \nabla_z \int_{B_z^\rho} |\nabla H_z|^2 \\
&= -\nabla t(z) - \int_{\partial B_z^\rho} |\nabla H_z|^2 \nu - \int_{B_z^\rho} \nabla_z |\nabla H_z|^2 \\
&= -\nabla t(z) + O(\rho).
\end{aligned}$$

Thus the time derivative of $\int_{\Omega\setminus B_z^\rho} |\nabla G_z|^2$ equals that of $t(z)$ up to an error term which tends to zero as $\rho \to 0$. We conclude that $E = -\frac{1}{2}t$ is conserved along the flow. In the case of multiple vortices the only difference is an additional contribution to the regular part due to the other vortices obtained from the relation $\int_\Omega \nabla G_z \cdot \nabla G_\zeta = G_z(\zeta)$.

6 Regularizations of point vortices

Motion of point vortices can be seen as the limit of several more realistic moving boundary problems. Each of them justifies the core energy method as a method for the derivation of the point vortex model. A popular regularization of the point vortex model is obtained by smearing out the vorticity. Turkington [18] and Marchioro and Pagani [13] (cf. also Marchioro and Pulvirenti [14]) have justified this view by showing that the solution of the regularized problem converges to that of the point vortex problem locally uniformly in time. In the stationary case a special type of vortex patches was studied by Turkington [17]. He showed that for a simply connected domain and given $\overline{\omega} > 0$

$$\sup\left\{E_{kin}(\omega)\ :\ \omega \in L^\infty(\Omega),\ 0 \le \omega \le \overline{\omega},\ \int_\Omega \omega = 1\right\}$$

is achieved by a stationary vortex patch of constant vorticity $\overline{\omega}$. Moreover as $\overline{\omega} \to \infty$ the vortex patch concentrates at a minimum point of the Robin function. Existence of stationary vortex patches close to stable constellations of point vortices (see Corollary 11) has been proved by Elcrat and Miller [7].

A further feasible regularization is to replace each vortex center z_p by a rigid particle $B^\rho_{z_p}$ moving freely with the fluid, i.e. $\int_{\partial B^\rho_{z_p}} p\,\nu = 0$. However, the most realistic regularization of the point vortex model is to replace each vortex center by a vortex core consisting of a small bubble of air. At the free surface the pressure drop is proportional to the mean curvature h of the free interface. This yields the free boundary condition for this problem which we call *dynamic Bernoulli problem.*

$$\begin{aligned} \partial_\tau v + \frac{1}{2}\nabla |v|^2 + \frac{\nabla p}{\rho} &= 0 \text{ in } \Omega \setminus A(\tau), \\ \operatorname{div} v &= 0 \text{ in } \Omega \setminus A(\tau), \\ p - p_0 &= 2\sigma h \text{ on } \partial A(\tau) \end{aligned}$$

where $\sigma \ge 0$ denotes the surface tension and p_0 the pressure of the air. Since Ω is fixed and $|\Omega \setminus A(\tau)|$ is constant so is the volume of the bubble. In the stationary case without surface tension the free boundary condition is $|v| = \partial_\nu u =$ const. on ∂A. A complete discussion of this case can be found in [8]. In the limit as the pressure of the air drops to $-\infty$ the free boundary shrinks to a point and we expect convergence to the point vortex model. In this sense the study of the point vortex model is a first step towards the solution of the dynamic Bernoulli problem.

7 Complex derivation of the point vortex model

We present a novel derivation of the point vortex model using the calculus of residues. In terms of the stream function Euler's equation (1) can be written as

$$J\nabla(\partial_\tau\psi) + \nabla\left(\frac{1}{2}|\nabla\psi|^2 + \frac{p}{\rho}\right)\Delta\psi\nabla\psi = 0.$$

Using complex variable notation replacing ∇ by $2\partial_{\overline{z}}$, J by $-i$, Δ by $4\,\partial_z\partial_{\overline{z}}$, $\Delta\psi\nabla\psi$ by $4\,\partial_z\left(\partial_{\overline{z}}\psi\right)^2$ and

$$f(z,\tau) := 2\partial_z\psi\partial_{\overline{z}}\psi + \frac{p}{\rho} - i\partial_\tau\psi$$

Euler's equation can be written as

$$\partial_{\overline{z}} f = 2\partial_z\left(\partial_{\overline{z}}\psi\right)^2.$$

In particular f is analytic in regions without vorticity. This statement can be regarded as a generalized form of Bernoulli's law (2). Indeed, in the stationary case f is real valued and hence must be constant, which is the same as saying that (2) holds. Another useful form of Euler's equation is

$$d\left(f\,dz + 2\left(\partial_{\overline{z}}\psi\right)^2 d\overline{z}\right) = 0.$$

Integration of one of these forms over an arbitrary subdomain $\Omega' \subset \Omega$ yields

$$(9) \qquad \int_{\partial\Omega'} f dz + 2\int_{\partial\Omega'} (\partial_{\bar z}\psi)^2 d\bar z = 0$$

by Stokes' formula. This is a weak form of Euler's equation that permits to pass to the point vortex limit provided none of the vortices is on the boundary of Ω'. To see this let

$$\psi(z,\tau) := \sum_p \omega_p G_{z_p(\tau)}(z) = \sum_p \left(-\frac{\omega_p}{2\pi}\log|z - z_p(\tau)| - \omega_p H_{z_p(\tau)}(z)\right).$$

LEMMA 6 *For any $\Omega' \subset \Omega$ with no vortices on $\partial\Omega'$ we have*

$$\int_{\partial\Omega'} f dz = \sum_{z_p \in \Omega'} \omega_p \dot z_p,$$

$$\int_{\partial\Omega'} (\partial_z\psi)^2 dz = -i\sum_{z_p\in\Omega'} \partial_{z_p} E.$$

Proof. We have

$$\partial_\tau\psi = \sum_p \frac{\omega_p}{4\pi}\left(\frac{\dot z_p}{z - z_p} + \frac{\dot{\bar z}_p}{\bar z - \bar z_p} + \text{regular terms}\right)$$
$$= Re\left(\sum_p \frac{\omega_p}{2\pi}\frac{\dot z_p}{z - z_p} + \text{regular analytic function}\right).$$

By definition of f the function $\partial_\tau\psi$ is the real part of if. Therefore the expression in the last bracket above is if up to an additive imaginary constant. The first claim follows from the residue theorem. Moreover

$$\partial_z\psi = \sum_p \omega_p \partial_z G_{z_p} = \sum_p \left(-\frac{\omega_p}{4\pi}\frac{1}{z - z_p} - \omega_p\partial_z H_{z_p}\right)$$

so that

$$(\partial_z\psi)^2$$
$$= \sum_p \left(\frac{\omega_p^2}{16\pi^2}\frac{1}{(z - z_p)^2} + \frac{\omega_p^2}{2\pi}\frac{1}{z - z_p}\partial_z H_{z_p} + \omega_p^2 (\partial_z H_{z_p})^2 + \sum_{q\neq p}\omega_p\omega_q \partial_z G_{z_p}\partial_z G_{z_q}\right).$$

Here only the second and the fourth term have non-zero residues. The residue at z_p is

$$\frac{\omega^2}{2\pi}\partial_z H_{z_p}(z_p) + 2\sum_{q\neq p}\omega_p\omega_q \left(Res_{z_p}\partial_z G_{z_p}\right)\partial_z G_{z_q}(z_p)$$
$$= \frac{\omega_p^2}{4\pi}\partial_z t(z_p) - \frac{1}{2\pi}\sum_{q\neq p}\omega_p\omega_q\partial_z G_{z_q}(z_p) = -\frac{1}{2\pi}\partial_{z_p}E.$$

Here we used that $Res_{z_p}\partial_z G_{z_p} = \frac{1}{2\pi}$. This proves the second claim. □

Lemma 6 inserted in (9) gives the equation of motion (6). Using the definition of f equation (9) can be written as

$$2\int_{\partial\Omega'} \partial_{\bar z}\psi\partial_z\psi dz + 2\int_{\partial\Omega'} (\partial_{\bar z}\psi)^2 d\bar z + \int_{\partial\Omega'}\left(\frac{p}{\rho} - i\partial_\tau\psi\right) dz = 0.$$

If at any particular instant $\partial\Omega'$ consists of streamlines, so that $d\psi = 0$ along $\partial\Omega'$, then

$$(10) \qquad \int_{\partial\Omega'} \partial_{\bar z}\psi\partial_z\psi dz + \int_{\partial\Omega'} (\partial_{\bar z}\psi)^2 d\bar z = \int_{\partial\Omega'} \partial_{\bar z}\psi d\psi = 0.$$

Hence

$$\int_{\partial\Omega'} p \cdot idz = -\rho \int_{\partial\Omega'} \partial_\tau \psi dz$$

in this case. Here the left side represents the total force on Ω' exerted by the surrounding fluid. As a special case, take $\Omega' = \Omega$. Then

$$\int_{\partial\Omega} p \cdot i\, dz = -\rho \frac{d}{d\tau} \int_{\partial\Omega} \psi dz = 0$$

since ψ is constant (as a function of z) on each component of $\partial\Omega$. Taking the complex conjugate of (10) and combining it with Lemma 6 gives

$$\int_{\partial\Omega'} |\partial_z \psi|^2 \, d\bar{z} = i \sum_{z_p \in \Omega'} \partial_{z_p} E$$

provided $\partial\Omega'$ consists of streamlines. If $\Omega' = \Omega$ then

$$\int_{\partial\Omega} |\partial_z \psi|^2 \, d\bar{z} = i \sum_p \partial_{z_p} E. \tag{11}$$

In the case of a single vortex this is the formula for ∇t in Proposition 9.

8 Properties of hydrodynamic Green's and Robin function

The capacity potential or harmonic measure of Γ_k with respect to the rest of the boundary is defined by

$$\begin{aligned} \Delta u_k &= 0 \text{ in } \Omega, \\ u_k &= \delta_{kl} \text{ on } \Gamma_l. \end{aligned}$$

These functions form a basis of the space of harmonic functions which are locally constant on the boundary. They satisfy $\sum u_k \equiv 1$. By the general assumptions on the domain each u_k has bounded gradient. We show that difference between the hydrodynamic Green's function and the Dirichlet Green's function G_z^0 can be written in terms of these capacity potentials.

PROPOSITION 7 *Under the general assumptions of Section 2 on Ω we have:*

1. *The hydrodynamic Green's function exists and is well defined if and only if the periods satisfy the consistency relation*

$$\sum_k \gamma_k = -1. \tag{12}$$

2. *The hydrodynamic Green's function is unique, symmetric and*

$$G_z(\zeta) = G_z^0(\zeta) + \sum_{k,l} g^{kl} u_k(z) u_l(\zeta), \tag{13}$$

where g^{kl} is a symmetric positive semi definite matrix with one-dimensional kernel spanned by the vector $(\gamma_1, \ldots, \gamma_K)$.

Proof. Symmetry and uniqueness was already proved in Section 3. The necessity of (12) follows from

$$0 = \int_{\Omega \setminus B_z^\rho} \Delta G_z = \sum_k \int_{\Gamma_k} \partial_\nu G_z - \int_{\partial B_z^\rho} \partial_\nu G_z = 1 + \sum_k \gamma_k$$

for $\rho > 0$ small enough. The difference between the two Green's function

$$G_z - G_z^0 \quad = \quad H_z^0 - H_z$$

is harmonic and locally constant on the boundary. The values of the hydrodynamic Green's function on the respective boundary components are g_z^k. Then

$$G_z - G_z^0 \quad = \quad \sum_k g_z^k u_k.$$

By symmetry of both Green's functions

$$g_z^k \quad = \quad \sum_l g^{kl} u_l(z)$$

with a symmetric matrix g^{kl}. Actually the symmetry is a consequence of the construction below. To construct G_z we take (13) as an ansatz and try to determine the matrix g^{kl}. This will prove existence. The first two properties of G_z in Definition 1 are automatically satisfied. For the remaining properties we need the following facts. The capacity coefficients

$$(14) \qquad p_{kl} \quad = \quad \int_{\Gamma_k} \partial_\nu u_l \quad = \quad \int_{\partial\Omega} u_k \partial_\nu u_l \quad = \quad \int_\Omega \nabla u_k \cdot \nabla u_l$$

form a symmetric, positive semi definite matrix with one-dimensional kernel

$$(15) \qquad \ker(p) \quad = \quad \operatorname{span}(1, \ldots, 1)$$

because $\sum_k u_k = 1$. Moreover

$$(16) \qquad u_k(z) \quad = \quad -\int_{\partial\Omega} u_k \partial_\nu G_z^0 \quad = \quad -\int_{\Gamma_k} \partial_\nu G_z^0.$$

Using (14) and (16) the circulation property of G_z reduces to

$$\begin{aligned} \gamma_k \quad &= \quad \int_{\Gamma_k} \partial_\nu G_z \quad = \quad \int_{\Gamma_k} \partial_\nu G_z^0 + \sum_{i,j} g^{ij} u_i(z) \int_{\Gamma_k} \partial_\nu u_j \\ &= \quad -u_k(z) + \sum_{i,j} p_{kj} g^{ij} u_i(z) \quad = \quad \sum_i \left(\sum_j p_{kj} g^{ij} - \delta_{ki} \right) u_i(z). \end{aligned}$$

By linear independence of the u_i's and $\sum_i u_i = 1$ the term in bracket must be γ_k. Therefore

$$(17) \qquad \sum_j p_{kj} g^{ij} \quad = \quad \gamma_k + \delta_{ki}$$

for every i and k. For fixed i this is a linear equation for the vector $\left(g^{ij}\right)_{j=1\ldots K}$. It is uniquely solvable if and only if the vector on the right is orthogonal to the kernel of the capacity matrix (p_{kj})

$$\sum_k 1 \cdot (\gamma_k + \delta_{ki}) \quad = \quad 0,$$

i.e. if (12) holds. Moreover every solution is of the form $\left(g^{ij} + c_i\right)_{j=1\ldots K}$. We can adjust the constant such that

$$\sum_j \gamma_j g^{ij} \quad = \quad 0$$

for each i. This determines $\left(g^{ij}\right)$ uniquely. The above equation is the normalization condition. The symmetry of $\left(g^{ij}\right)$ follows from (17) and symmetry of (p_{kl}):

$$g^{ji} \quad = \quad \sum_k (\gamma_k + \delta_{ki})\, g^{jk} \quad = \quad \sum_{k,l} p_{kl} g^{jk} g^{il}.$$

Concerning the definiteness consider the quadratic form

$$Q(\xi) \; := \; \sum_{i,j} g^{ij}\xi_i\xi_j \; = \; \sum_{k,l} p_{kl}\left(\sum_j g^{jk}\xi_j\right)\left(\sum_i g^{il}\xi_i\right).$$

By the properties of (p_{kl}) this form is positive and $Q(\xi) = 0$ if and only if $\sum_j g^{jk}\xi_j$ is a constant independent of k. When this occurs (17) gives

$$0 \; = \; \sum_{i,l} p_{kl}g^{il}\xi_i \; = \; \sum_i (\gamma_k + \delta_{ki})\,\xi_i \; = \; \xi_k + \gamma_k \sum_i \xi_i,$$

i.e. $\xi_k = c\gamma_k$. This means that the kernel of (g^{il}) is spanned by $(\gamma_1, \ldots, \gamma_K)$. □

In particular $\left|G_z - G_z^0\right| \le ||g||$ where

$$||g|| \; := \; \sup\left\{\sum_{k,l} g^{kl}\xi_k\xi_l \, : \, \sum_k |\xi_k| \le 1\right\}$$

which is finite by finite connectivity of Ω. The Robin function associated to the Dirichlet Green's function is denoted by t^0. This type of Robin function together with its applications in complex analysis, geometry, and calculus of variations is extensively studied in [3]. The notation differs by a factor 2π. Denote by

$$d(z) \; := \; \operatorname{dist}(z, \partial\Omega)$$

the distance of a point from the boundary.

PROPOSITION 8 *If Ω satisfies the general assumptions of Section 2 and the exterior ball condition with radius ρ then the hydrodynamic Robin function is bounded by*

$$t^0 - ||g|| \; \le \; t \; \le \; t^0$$

where

$$-\frac{1}{2\pi}\log\left(d(z)\left(2 + \frac{d(z)}{\rho}\right)\right) \; \le \; t^0(z) \; \le \; -\frac{1}{2\pi}\log(d(z)).$$

Therefore

$$t(z) \; = \; -\frac{1}{2\pi}\log(d(z)) + O(1)$$

uniformly as z tends to $\partial\Omega$.

Proof. By (13)

$$t(z) - t^0(z) \; = \; -\sum_{k,l} g^{kl}u_k(z)u_l(z).$$

The first claim follows from positivity of the matrix g^{kl} and the definition of $||g||$. By the exterior ball condition we can compare t^0 with the corresponding Robin function for the exterior of a ball $(B_\zeta^\rho)^c \supset \Omega$. By monotonicity of t^0 with respect to the domain and [3, Table 2] we have

$$\begin{aligned} t^0(z) \; &\ge \; -\frac{1}{2\pi}\log\frac{|z-\zeta|^2 - \rho^2}{\rho} \\ &= \; \frac{1}{2\pi}\log\left((|z-\zeta| - \rho)\frac{|z-\zeta| + \rho}{\rho}\right) \\ &\ge \; -\frac{1}{2\pi}\log\left(d(z)\left(2 + \frac{d(z)}{\rho}\right)\right) \end{aligned}$$

if we choose ζ such that $|z - \zeta| = d(z) + \rho$. The upper bound for $t^0(z)$ follows similarly by comparison with the Robin function of the ball $B_z^{d(z)} \subset \Omega$. □

Next we derive a variational formula for the hydrodynamic Green's function subject to a regular variation of the domain and the circulations. Hadamard's variational formula for the Dirichlet Green's function is a special case. A necessary condition for the existence of the perturbed Green's function is that the variation of the total circulation vanishes (Proposition 7). Recall that g_z^k denotes the value of G_z on Γ_k.

PROPOSITION 9 *Under a regular variation $\delta\nu$ of $\partial\Omega$ in exterior normal direction and variations $\delta\gamma_k$ of the circulations, satisfying $\sum\delta\gamma_k = 0$, the hydrodynamic Green's and Robin functions change according to*

$$\begin{aligned} \delta G_z(\zeta) &= \int_{\partial\Omega} \partial_\nu G_z \partial_\nu G_\zeta \delta\nu + \sum_k \left(g_z^k + g_\zeta^k\right)\delta\gamma_k, \\ \delta t(z) &= -\int_{\partial\Omega} |\partial_\nu G_z|^2\, \delta\nu - 2\sum_k g_z^k \delta\gamma_k. \end{aligned}$$

In particular

$$\begin{aligned} \nabla G_z(\zeta) + \nabla G_\zeta(z) &= -\int_{\partial\Omega} \partial_\nu G_z \partial_\nu G_\zeta \nu, \\ \nabla t(z) &= \int_{\partial\Omega} |\partial_\nu G_z|^2\, \nu. \end{aligned}$$

Proof. We apply Green's identity to the harmonic function δG_z to obtain

$$\delta G_z(\zeta) = \int_{\partial\Omega} \left(G_\zeta \partial_\nu \delta G_z - \delta G_z \partial_\nu G_\zeta\right).$$

In the Dirichlet case the result follows from $\delta G_z = -\partial_\nu G_z \delta\nu$ on $\partial\Omega$. For the hydrodynamic Green's function the normalization condition gives

$$\delta G_z(\zeta) = \sum_k g_\zeta \delta\gamma_k - \int_{\partial\Omega} G_z^\delta \partial_\nu G_\zeta$$

where $G_z^\delta = G_z + \delta G_z$ denotes the perturbed Green's function defined on the perturbed domain Ω^δ. Using Green's identity and the normalization condition for the perturbed Green's function the second integral can be written as

$$\begin{aligned} \int_{\partial\Omega} G_z^\delta \partial_\nu G_\zeta &= \int_{\partial\Omega^\delta} G_z^\delta \partial_\nu G_\zeta^\delta \int_{\partial\Omega} G_z^\delta \partial_\nu G_\zeta^\delta + \int_{\partial\Omega} G_z^\delta \partial_\nu \delta G_\zeta \\ &= \int_{\Omega^\delta\setminus\Omega} \nabla G_z^\delta \cdot \nabla G_\zeta^\delta + \sum_k g_z^k \delta\gamma_k + O\left(|\delta\nu|^2\right). \end{aligned}$$

Noting that G_z^δ is locally constant on $\partial\Omega^\delta$ and that the width of the strip $\Omega^\delta \setminus \Omega$ is $\delta\nu$ the claim for the Green's function follows. As to the Robin function note that the singular part of the hydrodynamic Green's function does not change. Hence $\delta H_z(\zeta) = -\delta G_z(\zeta)$ and $\delta t(z) = \delta H_z(z)$. □

9 Motion of isolated point vortices

In this section we discuss qualitative properties of the point vortex model that can be derived from conservation of the renormalized kinetic energy. We adopt the general assumptions on the domain of Section 2.

COROLLARY 10 (Motion of a single vortex) *The center z of a single vortex of strength ω behaves as follows.*

1. *It moves along the level lines of the hydrodynamic Robin function at speed*

$$\dot{z} \;=\; -\frac{\omega}{2} J\nabla t(z).$$

In particular almost all orbits are periodic.

2. *Every critical point of the hydrodynamic Robin function is a rest point. There is at least one.*

3. *A vortex center close to a local minimum z_0 of the hydrodynamic Robin function circulates along the boundary of a small "ellipse". As the ellipse shrinks to a point the time of revolution tends to*

$$T \;=\; \frac{4\pi}{\omega\sqrt{\det D^2 t(z_0)}}.$$

Proof. The first and second claim are immediate consequences of Theorem 3 and Proposition 8. For the third one we normalize: $z_0 = 0$, $t(z_0) = 0$ and $t(z) = \frac{1}{2}\left(t_{11}z_1^2 + t_{22}z_2^2\right) + O(|z|^3)$ as $|z| \to 0$. For small $h > 0$ the area of the "ellipse" is

$$|\{t < h\}| \;=\; \frac{2\pi h}{\sqrt{t_{11}t_{22}}} + O\left(h^2\right).$$

By the co-area formula the corresponding time of revolution is

$$\int_{\partial\{t<h\}} \frac{ds}{|\dot{z}|} \;=\; \frac{2}{\omega}\int_{\partial\{t<h\}} \frac{ds}{|\nabla t|} \;=\; \frac{2}{\omega}\partial_h \left|\{t < h\}\right|.$$

□

Figure 3 in Section 11.1 shows some trajectories of a single vortex. The statement on rest points compares to a result for a vector valued problem obtained by Bethuel, Brézis, Hélein [4]. They show that stationary Ginzburg-Landau vortices tend to concentrate at the critical points of a certain Robin function.

COROLLARY 11 (Motion of multiple vortices) *The following types of collisions are excluded as long as no other collisions occur simultaneously.*

1. *Collision of two vortices in the interior of the domain.*

2. *Collision of a single vortex with the boundary.*

3. *Collision of multiple vortices in the interior of the domain unless $\sum_{q\neq p}\omega_p\omega_q \log|z_p - z_q|$ has a finite limit.*

4. *The motion of two vortices of different orientation exists for all time and admits a stationary constellation at $\max E(z_1, z_2)$.*

5. *If the total vorticity $\omega := \sum \omega_p$ does not vanish then the speed of the center of vorticity*

$$z \;:=\; \frac{1}{\omega}\sum \omega_p z_p$$

is given by

$$\dot{z} \;=\; -\frac{\omega}{2}\int_{\partial\Omega} |v|^2 \, J\nu.$$

Proof. All claims follow from conservation of the renormalized kinetic energy and the boundary behaviour of the hydrodynamic Robin function. The last one follows from (11); in the case of a single vortex also from Proposition 9. □

Remarks

1. Several examples with two stationary point vortices in an unbounded flow region can be found in the article of Elcrat and Miller [7].
2. At a collision the vortex model breaks down. The vector field is no longer Lipschitz. In particular there is no unique continuation.
3. An example of a triple collision in $\mathbb{R}^2$ can be found in Kimura [10] and in the book of Marchioro and Pulvirenti [15]. In this example the relative distances of the vortices are preserved and $\omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1 = 0$. Thus the singular contributions to the renormalized energy cancel at the collision. The collision happens in finite time and the velocities of the vortex centers are unbounded.

Much more can be said in the case of a single vortex moving in a simply connected domain. In this case the normalization yields $G_z = 0$ on $\partial\Omega$, i.e. $G_z = G_z^0$ and $t = t^0$. We invoke some results from [3].

COROLLARY 12 (Motion of a single vortex in a simply connected domain) *Let Ω be a simply connected domain. Then:*

1. *The Robin function is the maximal solution of Liouville's equation*

$$\Delta t = \frac{2}{\pi} e^{4\pi t} \quad in \ \Omega.$$

2. *Near a smooth boundary point z*

$$t(z - s\nu) = -\frac{1}{2\pi} \log\left(2s - hs^2 + o\left(s^2\right)\right) \quad as \ s \to 0,$$

where h denotes the curvature of the boundary at z with respect to the exterior normal. In particular a vortex center at a small distance s from the boundary stays within a distance $s + O\left(s^2\right)$. Its speed is

$$\dot{z} = -\frac{\omega}{4\pi s} J\nu + O(1).$$

3. *The Robin function of a convex domain is convex.*
4. *The Robin function of a convex bounded domain has a unique minimum point z_0. Its distance from the boundary is at least*

$$d(z_0) \geq \frac{\pi}{4} e^{-2\pi t(z_0)}.$$

10 Motion of vortex clusters

In this section we investigate the dynamics of vortex clusters. First we show that on a macroscopic scale the motion of the cluster converges to that of a point vortex as the cluster shrinks to a point. In Section 10.1 we will apply this result to the motion of vortex pairs. Then we study the dynamics within a cluster.

Consider a cluster $\{z_p : p \in C\}$ of non-vanishing total vorticity $\omega := \sum_{p\in C} \omega_p$ with center of vorticity

$$z := \frac{1}{\omega} \sum_{p\in C} \omega_p z_p.$$

Let ρ denote the diameter of the cluster. Obviously $z_p \in B_z^{\theta\rho}$ for a suitable factor θ. We show that the center of vorticity essentially moves like a single vortex of vorticity ω. The dependence of the error term on the distance from the boundary will be essential for our stability result for vortex pairs. Theorem 13 does not apply to Example 5 where $\omega = 0$. In this case the motion is governed by higher order derivatives of the Robin function.

THEOREM 13 (Macroscopic cluster dynamics) *Assume the general hypotheses of Section 2 on Ω and that the vortex cluster $\{z_p : p \in C\}$ of non-vanishing total vorticity has small diameter and is well separated from the other vortices, i.e. $|z_p - z_{p'}| < \rho$ for $p, p' \in C$ and $|z_q - z_q| > R$ for $p \in C$, $q \notin C$ where $\rho \ll R$ and R is kept fixed. Then its center of vorticity moves according to*

$$\dot{z} = \sum_{p \notin C} \omega_p J \nabla G_{z_p}(z) - \frac{\omega}{2} J \nabla t(z) + O\left(\frac{\rho^2}{d(z)^3}\right) \quad as \quad \rho \to 0.$$

Proof. By the equation of motion (6) the center of vorticity moves according to

$$\begin{aligned}
\omega \dot{z} &= \sum_{p \in C} \omega_p \dot{z}_p \\
&= \sum_{p \in C} \sum_{q \neq p} \omega_p \omega_q J \nabla G_{z_q}(z_p) - \frac{1}{2} \sum_{p \in C} \omega_p^2 J \nabla t(z_p) \\
&= \sum_{p \in C} \sum_{q \notin C} \omega_p \omega_q J \nabla G_{z_q}(z_p) - \sum_{p,p' \in C} \omega_p \omega_{p'} J \nabla H_{z_{p'}}(z_p) \qquad (18)
\end{aligned}$$

because $\nabla F_{z_q}(z_p) + \nabla F_{z_p}(z_q) = 0$ and $\nabla t(z) = 2\nabla H_z(z)$. On the other hand the right side in Theorem 13 multiplied with ω is

$$\sum_{p \in C} \sum_{q \notin C} \omega_p \omega_q J \nabla G_{z_q}(z) - \sum_{p,p' \in C} \omega_p \omega_{p'} J \nabla H_z(z) \qquad (19)$$

The difference between (18) and (19) is

$$\sum_{p \in C} \sum_{q \notin C} \omega_p \omega_q J \left(\nabla G_{z_q}(z_p) - \nabla G_{z_q}(z)\right) - \sum_{p,p' \in C} \omega_p \omega_{p'} J \left(\nabla H_{z_{p'}}(z_p) - \nabla H_z(z)\right).$$

The first term is

$$\sum_{q \notin C} \omega_q \sum_{p \in C} \omega_p J D^2 G_{z_q}(z)(z_p - z) + O\left(\rho^2\right) = O\left(\rho^2\right)$$

because $\sum_{p \in C} \omega_p (z_p - z) = 0$. The second term satisfies the same estimate. More precisely the total error is of order

$$\rho^2 \sup_{\zeta \in B_z^{\varrho \rho}} \left(\sup_{q \notin C} \|D^3 G_{z_q}(\zeta)\| + \sup_{p \in C} \|D^3 H_z(z_p)\| + \|D^3 H_z(\zeta)\| \right).$$

By Lemma 14 below this is of order $O\left(\frac{\rho^2}{d(z)^3}\right)$. □

For $\Omega = \mathbb{R}^2$ Marchioro and Pulvirenti [15] derived the asymptotic behaviour of the center of vorticity of equally oriented vortices by estimating the growth of the moment of inertia

$$\sum_{p \in C} \omega_p \left|z_p - z\right|^2.$$

We are left to prove the following estimates for the derivatives of the regular part of the hydrodynamic Green's function.

LEMMA 14 *If Ω satisfies the assumptions of Section 2 then*

$$\left|\nabla_z^m \nabla_\zeta^n H_z(\zeta)\right| \leq \frac{c_\Omega(m,n)}{d(z)^m d(\zeta)^n}$$

for every $m + n \geq 1$.

Proof. Recall that the regular part is symmetric and harmonic in each variable. At the boundary

$$H_z = F_z - \sum_j g^{kj} u_j(z) \quad \text{on } \Gamma_k.$$

Thus

$$H_z(\zeta) = -\sum_k \int_{\Gamma_k} \left(F_z - \sum_j g^{kj} u_j(z) \right) \partial_\nu G^0_\zeta. \tag{20}$$

Using

$$|\nabla_z F_z(w)| = \frac{1}{2\pi |w-z|} \le \frac{1}{2\pi d(z)} \quad \text{for } w \in \partial\Omega$$

together with $|\nabla u_j| \le c$, $-\partial_\nu G^0_\zeta \ge 0$ and $-\int_{\partial\Omega} \partial_\nu G^0_\zeta = 1$ we get

$$|\nabla_z H_\zeta(z)| = |\nabla_z H_z(\zeta)| \le \frac{c}{d(z)} \quad \text{for every } \zeta \in \partial\Omega.$$

By the maximum principle this yields the desired estimate for $m+n=1$. If h is harmonic in B^ρ_z then

$$|\nabla h(z)| \le \frac{1}{\rho} \sup_{\partial B^\rho_z} |h|.$$

Application of this estimate to $h = \nabla H_\zeta$ on $B^\rho_z \subset B^{2\rho}_z \subset \Omega$ proves the lemma for $m+n=2$. Similarly we estimate higher order derivatives. □

Next we analyze the dynamics within a vortex cluster normalized with respect to the center of vorticity.

THEOREM 15 (Microscopic vortex dynamics) *Under the assumptions of Theorem 13 and* $d(z) \ge \rho > 0$ *we have*

$$(z_p - z)^{\cdot} = \sum_{p' \in C,\ p' \neq p} \omega_{p'} J \nabla F_{z_{p'}}(z_p) + O\left(\frac{\rho}{d(z)^2}\right) \quad \text{as } \rho \to 0$$

for every $p \in C$.

Proof. By (6), Theorem 13 and Lemma 14 we have

$$\begin{aligned}
(z_p - z)^{\cdot} &= \sum_{q \notin C} \omega_q J \left(\nabla G_{z_q}(z_p) - \nabla G_{z_q}(z) \right) + \sum_{p' \in C,\, p' \neq p} \omega_{p'} J \nabla G_{z_{p'}}(z_p) \\
&\quad - \frac{\omega_p}{2} J \nabla t(z_p) + \frac{\omega}{2} J \nabla t(z) + O\left(\frac{\rho^2}{d(z)^3}\right) \\
&= \sum_{p' \in C,\, p' \neq p} \omega_{p'} J \nabla F_{z_{p'}}(z_p) \\
&\quad - \sum_{p' \in C} \omega_{p'} J \left(\nabla H_{z_{p'}}(z_p) - \nabla H_z(z) \right) + O\left(\frac{\rho}{d(z)^2}\right) \\
&= \sum_{p' \in C,\ p' \neq p} \omega_{p'} J \nabla F_{z_{p'}}(z_p) + O\left(\frac{\rho}{d(z)^2}\right).
\end{aligned}$$

□

Also this formula is second order accurate because the leading term is of order $O(1/\rho)$.

10.1 Stability of vortex pairs

In this section we prove long time existence for vortex pairs. They do not collide with each other nor with the boundary and they stay close together if they are close initially. In particular there is no recombination of vortex pairs of opposite strength. If further vortices are present the same holds as long as no other collisions occur (see also Corollary 20).

Example 16 As an illustration consider the motion of two equally oriented vortices in the half-space $\Omega = \{z \in \mathbb{R}^2 \, : \, y > 0\}$. The Hamiltonian is translation invariant in the x-direction, i.e.

$$0 \;=\; \frac{d}{dx} E\left(z_1 + \begin{pmatrix} x \\ 0 \end{pmatrix}, z_2 + \begin{pmatrix} x \\ 0 \end{pmatrix}\right)\Big|_{x=0} \;=\; (\partial_{x_1} + \partial_{x_2})\, E.$$

Thus the linear momentum $\omega_1 y_1 + \omega_2 y_2$ is conserved:

$$(\omega_1 y_1 + \omega_2 y_2)^{\cdot} \;=\; -\partial_{x_1} E - \partial_{x_2} E \;=\; 0.$$

This is Noether's theorem. Since $\omega_1\omega_2 > 0$ the coordinates y_1 and y_2 cannot go to zero simultaneously.

THEOREM 17 *The motion of two vortices in a domain satisfying the uniform exterior ball condition exists for all time. In particular they do not collide in finite time. In all cases* $|z_1(\tau) - z_2(\tau)| \le D\,|z_1(0) - z_2(0)|$ *with a constant* D *depending only on* Ω.

1. *If* $\omega_1\omega_2 < 0$ *then the motion stays away from the boundary of the phase space*

$$\{(z_1, z_2) \in \Omega \times \Omega \, : \, z_1 \neq z_2\}$$

by a uniform distance.

2. *If* $\omega_1\omega_2 > 0$ *and one of the vortices tends to the boundary then so does the other and*

$$|z_1 - z_2| \;\le\; C d(z_p)^{2+\alpha},$$

$$d\,(z_p(\tau)) \;\ge\; \begin{cases} Ae^{-B\tau} & (always), \\ (A + B\tau)^{-\frac{1}{2\alpha}} & (\omega_1 \neq \omega_2) \end{cases}$$

where

$$\alpha \;:=\; \frac{(\omega_1 - \omega_2)^2}{2\omega_1\omega_2} \ge 0.$$

for $p = 1,\, 2$ *and* $\tau \ge 0$ *with positive constants* $A,\, B,\, C$.

3. *Also in the presence of further vortices two vortices do not collide with each other nor with the boundary as long as no other collisions occur simultaneously.*

Proof. If $\omega_1\omega_2 < 0$ then each term in

$$E(z_1, z_2) \;=\; \omega_1\omega_2 G_{z_1}(z_2) - \frac{1}{2}\omega_1^2 t\,(z_1) - \frac{1}{2}\omega_2^2 t(z_2)$$

is bounded above. By conservation of E they are also bounded below. From the asymptotics for the Robin function at the boundary (Proposition 8) we get a lower bound $d(z_p) \ge \delta > 0$. On the remaining set boundedness of $G_{z_1}(z_2)$ leads to a uniform lower bound for $|z_1 - z_2|$. Energy conservation also implies that

$$-\log|z_1(\tau) - z_2(\tau)| \;=\; -\log|z_1(0) - z_2(0)| + f(\tau)$$

with a bounded function f whose bounds only depend on Ω.

For equally oriented vortices energy conservation does not lead to a uniform distance from the boundary of the phase space. However, we already know that the vortices do not collide in the interior and that none of them tends to the boundary alone. Thus we can restrict our attention to the case that they tend to the boundary simultaneously. We invoke Theorem 13 and adopt the notation introduced thereby. Note that the total vorticity is nonzero. By energy conservation and the Propositions 7 and 8 we have

$$G^0_{z_1}(z_2) - \frac{\omega_1}{2\omega_2} t^0(z_1) - \frac{\omega_2}{2\omega_1} t^0(z_2) \geq \frac{E}{\omega_1 \omega_2} - c. \tag{21}$$

Comparison with the Dirichlet Green's function of the complement of a disk $(B^\rho_\zeta)^c \supset \Omega$ yields

$$-\frac{1}{2\pi} \log \left| \frac{\rho(z_1 - z_2)}{\rho^2 - (z_2 - \zeta)\overline{(z_1 - \zeta)}} \right| \geq G^0_{z_1}(z_2).$$

Choosing ζ such that $|z_2 - \zeta| = d(z_2) + \rho$ we can estimate

$$\begin{aligned} \left|\rho^2 - (z_2 - \zeta)\overline{(z_1 - \zeta)}\right| &\leq |z_2 - \zeta|^2 - \rho^2 + |(z_2 - \zeta)(\overline{z_1} - \overline{z_2})| \\ &\leq c\left(d(z_2) + |z_1 - z_2|\right). \end{aligned}$$

By Proposition 8 exponentiation of the energy inequality (21) yields

$$\frac{|z_1 - z_2|}{d(z_p) + |z_1 - z_2|} \leq Ce^{-\frac{2\pi}{\omega_1\omega_2}E} d(z_1)^{\frac{\omega_1}{2\omega_2}} d(z_2)^{\frac{\omega_2}{2\omega_1}}$$

for $p = 2$. By symmetry also for $p = 1$. In particular

$$\frac{|z_1 - z_2|}{d(z_p)} \to 0 \quad \text{as} \quad z_1 \quad \text{or} \quad z_2 \to \partial\Omega$$

and so

$$\frac{|z_1 - z_2|}{d(z_p)} \leq Ce^{-\frac{2\pi}{\omega_1\omega_2}E} d(z_1)^{\frac{\omega_1}{2\omega_2}} d(z_2)^{\frac{\omega_2}{2\omega_2}}.$$

By the exterior ball condition also $z \in \Omega$, $d(z) \to 0$ and we can replace $d(z_p)$ by $d(z)$ up to lower order terms. This yields

$$|z_1 - z_2| \leq Cd(z)^{2+\alpha} \quad \text{as} \quad z_1 \quad \text{or} \quad z_2 \to \partial\Omega \tag{22}$$

because $\frac{\omega_1}{2\omega_2} + \frac{\omega_2}{2\omega_1} + 1 = 2 + \alpha$. For convex domains the same estimate extends to all of $\Omega \times \Omega$. According to Theorem 13 the center of vorticity moves with speed

$$\dot{z} = -\frac{\omega}{2} J\nabla t(z) + O\left(\frac{|z_1 - z_2|^2}{d(z)^3}\right)$$

which is orthogonal to $\nabla t(z)$ up to an error which by (22) is of order $O(d(z)^{1+2\alpha})$. Therefore

$$\partial_\tau t(z(\tau)) = \nabla t(z(\tau)) \cdot \dot{z}(\tau) \leq Cd(z)^{2\alpha}$$

by Lemma 14. Proposition 8 together with elementary comparison arguments for ordinary differential equations leads to the lower bound for $d(z)$. □

Thus a pair of equally oriented vortices approaching the boundary rotates rapidly around their common center of vorticity and slides along one of the boundary components at exponentially increasing speed. Simple examples or inspection of the proof show that these estimates are sharp. The constants C and B only depend on Ω, ω_1, ω_2, and the energy level. The constant A also depends on the initial position of the vortices. The third statement will be sharpened in Section 10.2.

10.2 Local stability of vortex clusters

The dynamics in vortex cluster is essentially independent of the "rest of the world". In fact we show that the energy exchange rate between the cluster and the other vortices is bounded whatever happens outside the cluster. We introduce the cluster energy

$$E_C(z_1, \ldots, z_P) \quad := \quad \sum_{p \in C} \sum_{q > p} \omega_p \omega_q G_{z_q}(z_p) - \frac{1}{2} \sum_{p \in C} \omega_p^2 t(z_p).$$

Regarding E as a quadratic form in $(\omega_1, \ldots, \omega_P)$ the cluster energy consists of those terms for which at least one of the vortices is in the cluster. In particular

$$(23) \qquad \nabla_{z_p} E_C \quad = \quad \nabla_{z_p} E \quad \text{for every} \quad p \in C.$$

THEOREM 18 *If $\partial\Omega$ is smooth and satisfies the assumptions of Section 2 and $R > 0$ then*

$$\left| \frac{d}{d\tau} E_C(z_1, \ldots, z_P) \right| \quad \leq \quad C \quad < \quad \infty$$

as long as $|z_p - z_q| \geq R$ for every $p \in C$, $q \notin C$.

Proof. By the equations of motion and (23) we have

$$\begin{aligned}
\frac{d}{d\tau} E_C \quad &= \quad \sum_{p \in C} \nabla_{z_p} E_C \cdot \dot{z}_p + \sum_{q \notin C} \nabla_{z_q} E_C \cdot \dot{z}_q \\
&= \quad \sum_{p \in C} \nabla_{z_p} E \cdot \frac{1}{\omega_p} J \nabla_{z_p} E \\
&\quad + \sum_{q \notin C} \left(\sum_{p \in C} \omega_p \omega_q \nabla G_{z_p}(z_q) \right) \cdot \left(\sum_{q' \neq q} \omega_{p'} J \nabla G_{z_{q'}}(z_q) - \frac{1}{2} \omega_q J \nabla t(z_q) \right) \\
&= \quad \sum_{p \in C} \sum_{q \notin C} \sum_{q' \neq q} \omega_p \omega_q \omega_{q'} \nabla G_{z_p}(z_q) \cdot J \nabla G_{z_{q'}}(z_q) \\
&\quad - \frac{1}{2} \sum_{p \in C} \sum_{q \notin C} \omega_p \omega_q^2 \nabla G_{z_p}(z_q) J \nabla t(z_q).
\end{aligned}$$

For the first term one immediately gets a uniform bound if $q' \in C$. The remaining terms can be rearranged as

$$\sum_{p \in C} \sum_{q, q' \notin C,\ q < q'} \omega_q \omega_{q'} \left(\nabla G_{z_p}(z_q) \cdot J \nabla G_{z_{q'}}(z_q) + \nabla G_{z_p}(z_{q'}) \cdot J \nabla G_{z_q}(z_{q'}) \right)$$

which is bounded by the first inequality in Lemma 19 below. Similarly, the boundedness of the last term follows from the second estimate in Lemma 19. □

LEMMA 19 *Suppose $\partial\Omega$ is smooth. Given $\rho > 0$ there is a constant $C < \infty$ such that*

$$\begin{aligned}
|\nabla G_a(z) \cdot J \nabla G_\zeta(z) + \nabla G_a(\zeta) \cdot J \nabla G_z(\zeta)| \quad &\leq \quad C, \\
|\nabla G_a(z) \cdot J \nabla H_\zeta(z) + \nabla G_a(\zeta) \cdot J \nabla H_z(\zeta)| \quad &\leq \quad C
\end{aligned}$$

for every $a \in \Omega, z, \zeta \in \Omega \setminus B_a^\rho$.

Proof. The function ∇G_a is Lipschitz outside B_a^ρ with a Lipschitz constant independent of a. Since $\nabla F_\zeta(z) + \nabla F_z(\zeta) = 0$ the difference between the two quantities to be estimated is

$$(\nabla G_a(z) - \nabla G_a(\zeta)) \cdot J \nabla F_\zeta(z)$$

which is bounded by uniform Lipschitz continuity of ∇G_a outside B_a^ρ. Thus we only need to prove one of the estimates, say the first one, which we write as

$$|(J\nabla G_a(z)\cdot\nabla_z + J\nabla G_a(\zeta)\cdot\nabla_\zeta)\, G_z(\zeta)| \;\le\; C.$$

For simplicity we only consider simply connected domains. In this case the vector field

$$\xi(z) \;:=\; -\frac{1}{2\pi}\frac{J\nabla G_a(z)}{|\nabla G_a(z)|^2} \tag{24}$$

generates a one-parameter group of conformal automorphisms $(\phi^s)_{s\in\mathbb{R}}$ of Ω having a as a common fixed point. If $\Omega = B_0^1$ and $a=0$ it consists of the rotations

$$\phi^s(z) \;=\; e^{-is}z \quad\text{with}\quad \xi(z) \;=\; \partial_s\phi^s(z)|_{s=0} \;=\; Jz.$$

These special automorphisms can be transplanted to an arbitrary simply connected domain by means of a conformal change of variables. The relation (24) is conformally equivariant. Now we decompose

$$\begin{aligned}
&(J\nabla G_a(z)\cdot\nabla_z + J\nabla G_a(\zeta)\cdot\nabla_\zeta)\, G_z(\zeta)\\
=\;& -\frac{1}{4\pi}\left(|\nabla G_a(z)|^2 + |\nabla G_a(\zeta)|^2\right)(\xi(z)\cdot\nabla_z + \xi(\zeta)\cdot\nabla_\zeta)\, G_z(\zeta)\\
&-\frac{1}{4\pi}\left(|\nabla G_a(z)|^2 - |\nabla G_a(\zeta)|^2\right)(\xi(z)\cdot\nabla_z - \xi(\zeta)\cdot\nabla_\zeta)\, G_z(\zeta).
\end{aligned}$$

The first term vanishes because $G_z(\zeta)$ is conformally invariant and

$$\partial_s G_{\phi^s(z)}(\phi^s(\zeta))\big|_{s=0} \;=\; (\xi(z)\cdot\nabla_z + \xi(\zeta)\cdot\nabla_\zeta)\, G_z(\zeta).$$

We now show that the second term is bounded independently of a. By Lipschitz continuity of ∇G_a,

$$\left||\nabla G_a(z)|^2 - |\nabla G_a(\zeta)|^2\right| \;\le\; C\,|z-\zeta|.$$

The generating vector field ξ is bounded since Ω is bounded. Finally

$$|\nabla G_z(\zeta)| \;\le\; \frac{C}{|z-\zeta|}.$$

This is obvious except when the arguments are close to each other and close to the boundary. Since $\partial\Omega$ is smooth this case can be reduced to the following standard situation via a conformal change of variables with locally bounded first and second derivatives. We assume that Ω is the upper half plane and $\zeta = 0$. Then

$$2\pi\,|\nabla G_z(\zeta)| \;\le\; 2\pi\,|\nabla F_z(\zeta) - \nabla F_{\overline{z}}(\zeta)| \;=\; \left|\frac{z-\zeta}{|z-\zeta|^2} - \frac{\overline{z}-\zeta}{|\overline{z}-\ z|^2}\right| \;\le\; \frac{2}{|z-\zeta|}.$$

□

COROLLARY 20 (Collisions within a cluster) *Under the assumptions of Theorem 13 the term*

$$\sum_{p,p'\in C,\; p\neq p'} \omega_p\omega_{p'}\log|z_p - z_{p'}|$$

is locally bounded in time as long as $d(z_p)\ge d_0>0$ for every $p\in C$. In particular a cluster of equally oriented vortices never collapses.

Proof. The above expression is the only part of E_C which may become infinite. □

COROLLARY 21 *Two vortices z_1, z_2 can not collide as long as $|z_p - z_q| > R$ for $p = 1, 2$ and $q \notin \{1, 2\}$.*

Proof. The proof of Theorem 17 was based on conservation of energy for the two vortex system. In the presence of additional vortices E is replaced by E_C. The claim follows from Theorem 18. □

The evolution of a continuous vorticity distribution can be approximated by the evolution of vortex clusters. This fact has been used by Christiansen [6] for numerical calculations in $\Omega = \mathbb{R}^2$.

11 Numerical approximation of point vortex dynamics

11.1 Solution of Liouville's equation

On a simply connected domain the orbits of a single vortex center are easily obtained by solving Liouville's equation (Corollary 10 and 12). In order to avoid infinite boundary data we replace Ω by

$$\Omega_h \ := \ \{x \in \Omega \ : \ d(z) > h\} .$$

On this domain we solve Liouville's equation with boundary values according to Corollary 12

$$\begin{aligned} \Delta t &= \frac{2}{\pi} e^{4\pi t} \ \text{ in } \ \Omega_h, \\ t &= -\frac{1}{2\pi} \log(2h) \ \text{ on } \ \partial\Omega_h. \end{aligned}$$

This can be done by Newton's method. The approximation $t^{(k+1)}$ is computed from the preceding one by solving the boundary value problem

$$\begin{aligned} \left(-\Delta + 8e^{4\pi t^{(k)}}\right)\left(t^{(k+1)} - t^{(k)}\right) &= \Delta t^{(k)} - \frac{2}{\pi} e^{4\pi t^{(k)}} \ \text{ in } \ \Omega_h, \\ t^{(k+1)} &= -\frac{1}{2\pi} \log(2h) \ \text{ on } \ \partial\Omega_h. \end{aligned}$$

By convexity of the nonlinearity an initial supersolution leads to a pointwise decreasing sequence of supersolutions. Details are given in [3, Section 3.3]. The level sets of t are the desired trajectories (Figure 3).

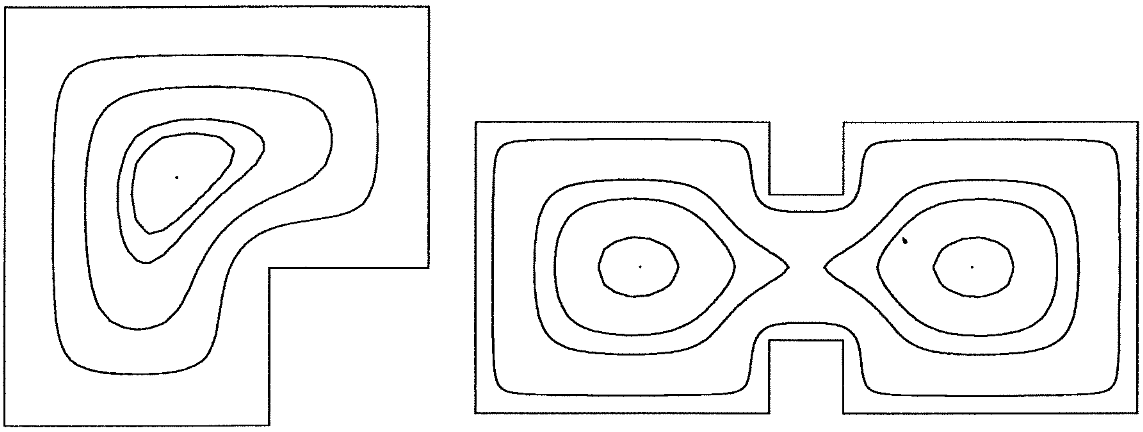

Figure 3: Orbits of a single vortex in simply connected domains computed by Newton's method [8].

11.2 Boundary element method

For multiply connected domains and multiple vortices we use a variant of the boundary element method. This approach is based on harmonicity of the regular part H_z of the hydrodynamic Green's function whose value and gradient is needed at P^2 points. We use the single layer representation [3]:

$$\int_{\partial\Omega} q_z F_\zeta \ = \ \begin{cases} H_z(\zeta), & \zeta \in \overline{\Omega}, \\ H_z^c(\zeta), & \zeta \in \overline{\Omega^c} \end{cases}$$

with the harmonic function H_z^c defined by this relation. The density q_z represents the jump of the normal derivative

$$q_z \ = \ \partial_\nu H_z - \partial_\nu H_z^c$$

while $H_z = H_z^c$ on $\partial\Omega$. If $\int_{\partial\Omega} q_z = 0$ then H_z^c is regular at infinity. This can be seen as follows. Let h be the regular harmonic extension of $H_z|_{\partial\Omega}$ to $\Omega^c \cup \{\infty\}$. Then $g := H_z^c - h$ is harmonic in Ω^c, vanishes at the boundary and $\lim_{\zeta\to\infty} g(\zeta)$ exists in $I\!R$. Thus $g = \lambda G_\infty$ must be a constant multiple of the Dirichlet Green's function on Ω^c with singularity $-F_0$ at infinity. This implies $\lambda = 0$ because

$$0 = \int_{\partial\Omega} q_z = \int_{\partial\Omega} \partial_\nu H_z - \int_{\Gamma_0} \partial_\nu (\lambda G_\infty + h) = -\lambda$$

where Γ_0 denotes the exterior boundary component. In particular

$$\int_{\Gamma_k} q_z = \int_{\Gamma_k} (\partial_\nu H_z - \partial_\nu H_z^c) = \int_{\Gamma_k} \partial_\nu H_z$$

for every k. Using the decomposition $G_z = F_z - H_z$ and $\nabla_\zeta F_\zeta(z) = -\nabla_z F_\zeta(z)$ the definition of the hydrodynamic Green's function (Definition 1) translates into the subsequent scheme.

1. Discretize the boundary $\partial\Omega$. On every boundary component choose an arbitrary point $\zeta_k \in \Gamma_k$. Fix a time step $d\tau$.

2. Suppose we know the position of the vortex centers at time τ. For $z = z_1, \ldots, z_P$ solve

$$\begin{aligned} \int_{\partial\Omega} q_z \partial_{J\nu(\zeta)} F_\zeta &= \partial_{J\nu(\zeta)} F_z(\zeta) \qquad (\zeta \in \partial\Omega)\,, \\ \int_{\Gamma_k} q_z &= \int_{\Gamma_k} \partial_\nu F_z - \gamma_k \qquad (k = 1, \ldots, K)\,, \\ \int_{\partial\Omega} q_z \sum_k \gamma_k F_{\zeta_k} &= \sum_k \gamma_k F_{\zeta_k}(z) \end{aligned} \tag{25}$$

for $q_z : \partial\Omega \to I\!R$. By Proposition 7 this system has a unique solution. Then evaluate

$$\begin{aligned} t(z) &= \int_{\partial\Omega} q_z F_z\,, \\ \nabla t(z) &= -2 \int_{\partial\Omega} q_z \nabla F_z\,. \end{aligned}$$

3. For $\zeta = z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_P$ evaluate

$$\begin{aligned} G_z(\zeta) &= F_z(\zeta) - \int_{\partial\Omega} q_z F_\zeta\,, \\ \nabla G_z(\zeta) &= \nabla F_z(\zeta) + \int_{\partial\Omega} q_z \nabla F_\zeta\,. \end{aligned}$$

4. Compute $\dot{z}_p$ according to the equation of motion (6) and use any standard method for ordinary differential equations to update the vortex positions. In the case of a single vortex we only need to follow a level line of the hydrodynamic Robin function. Since we know its gradient this can be done at a uniform accuracy of order $O\left(d\tau^2\right)$.

Note that (25) has some similarities with the Fredholm integral equation of the second kind used for the classical double layer representation. In particular the kernel changes sign at the singularity. Thus it would be interesting to know whether the condition number of (25) also remains bounded as the mesh size tends to 0.

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