

ON EXACT QUADRATURE FORMULAS FOR HARMONIC FUNCTIONS ON POLYHEDRA

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ABSTRACT. A classical quadrature result for analytic functions of a complex variable due to Motzkin and Schoenberg is extended to higher dimensions. A general scheme for integrating on polyhedra solutions of partial differential equations is discussed.

1. INTRODUCTION

A formula due to Motzkin-Schoenberg and later refined by Davis [1] asserts that the area integral on a triangle of the second derivative of an analytic function is a linear combination of its values at the vertices of the triangle. This fact was recently used in inverting the shape of a planar polygon from a finite sequence of its moments (see [2] and [3]). This note collects a few simple remarks which represent possible generalizations of the Motzkin-Schoenberg-Davis formula in higher dimensions.

The basis of all these formulas is of course Stokes' theorem. Our main result states that the integral of any three derivatives of a harmonic function on a polyhedron in \mathbf{R}^n can be evaluated, with the loss of two derivatives, on the codimension-2 skeleton. This phenomenon is actually more general, and it persists for solutions of homogeneous partial differential operators of order two, for instance as long as the faces of the polyhedron are not characteristic. A better situation, when iterated integration is possible and the result can be pushed down to the 0-th skeleton, is in the case of pluriharmonic functions in \mathbf{C}^n integrated on cartesian products of planar polyhedra.

A variety of other remarkable quadrature formulas for harmonic functions integrated on polyhedra are known. However, in general the support of the representing measure is not contained in any level of the skeleton of the polyhedron (see [4] and [7]).

Notation. Below we denote by P_n a bounded open polyhedron in \mathbf{R}^n , $n \geq 2$, by P_k its k -dimensional skeleton, and by dx^k the k -dimensional linear measure on P_k induced from the (oriented) volume n -measure in \mathbf{R}^n , $0 \leq k \leq n$. The partial derivatives will be denoted as $\partial_j = \frac{\partial}{\partial x_j}$, $1 \leq j \leq n$. The dual L^* of a differential operator L is always considered with respect to the bilinear pairing between smooth functions of compact support and distributions.

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2. MAIN RESULTS

To start with, we present a general formula for harmonic functions of n variables.

Theorem 1. *Let P_n be a bounded polyhedron in \mathbf{R}^n and let L_3 be a third order homogeneous partial differential operator with constant coefficients.*

Then for each codimension-2 face P_{n-2}^i of P_n , $i \in I$, there exists a linear, first order, homogeneous differential operator with constant coefficients L_1^i , such that:

$$(1) \quad \int_{P_n} L_3(u) dx^n = \sum_{i \in I} \int_{P_{n-2}^i} L_1^i(u) dx^{n-2}$$

for every harmonic function u defined in a neighbourhood of $\overline{P_n}$.

Note that, since in case $n = 2$ an analytic function is the gradient of a harmonic function, the above formula is equivalent (at least for simply connected polygons) to the Motzkin-Schoenberg-Davis formula; see [1] and [2].

A simple example shows that Theorem 1 is not valid for the skeleton of codimension 3. Indeed, take the standard simplex in \mathbf{R}^3 :

$$P_3 = \{(x, y, z); x + y + z \leq 1, x \geq 0, y \geq 0, z \geq 0\}$$

and the harmonic function $u(x, y, z) = xyz$. Then the gradient of u vanishes on all vertices of P_3 , but

$$\int_{P_3} \frac{\partial^3 u(x, y, z)}{\partial x \partial y \partial z} dx dy dz \neq 0.$$

Formula (1) can be improved at least in two ways: by adapting the differential equation satisfied by u to the polyhedron P_n , or by using direct products of polygons and a more restrictive class of functions.

In the first case, we state the following result.

Theorem 2. *Let P_n be a bounded polyhedron in \mathbf{R}^n , let L_m be an m -th order homogeneous partial differential operator with constant coefficients and let u be a smooth function defined in a neighbourhood of $\overline{P_n}$.*

Assume that for each face F of P_n at least one of the following two conditions is satisfied:

(i) F is characteristic for L_m ;

(ii) u satisfies in a neighbourhood of F an equation $Q_{m-1}(u) = 0$, and F is non-characteristic for Q_{m-1} , a homogeneous linear partial differential operator with constant coefficients of order $m - 1$, which may depend on the face F .

Then for each codimension-2 face P_{n-2}^i of P_n , $i \in I$, there exists a homogeneous partial differential operator L_{m-2}^i , of order $m - 2$, with constant coefficients, such that:

$$\int_{P_n} L_m(u) dx^n = \sum_{i \in I} \int_{P_{n-2}^i} L_{m-2}^i(u) dx^{n-2}.$$

The L_{m-2}^i depend only on P_n , L_m and the Q_{m-1} .

For instance the Laplacian in Theorem 1 can be replaced by the wave operator, as long as P_n is in general position with respect to the characteristic cone. A related general observation, well known for the wave equation, is contained in the following direct consequence of Stokes' theorem.

Let u be a smooth function defined in a neighbourhood of the unit cube $C_n = \{x \in \mathbf{R}^n; 0 \leq x_j \leq 1, 1 \leq j \leq n\}$ and let $C_0 = \{v_k; 1 \leq k \leq 2^n\}$ be its vertices. Then:

$$(2) \quad \int_{C_n} \partial_1 \partial_2 \dots \partial_n (u) = \sum_{k=1}^{2^n} \epsilon_k u(v_k),$$

where $\epsilon_k = \pm 1, 1 \leq k \leq 2^n$.

Denoting by χ_F the characteristic function of a set F , the above result can equivalently be written in the sense of distributions:

$$(-1)^n \partial_1 \partial_2 \dots \partial_n \chi_{C_n} = \sum_k \epsilon_k \delta_{v_k}.$$

More generally, returning to an arbitrary polyhedron, we can restate Theorem 1 (with Δ replaced by a general second order operator Q_2) as follows.

Proposition 3. *Let L_3, Q_2 be linear partial differential operators with constant coefficients of order 3, respectively 2. Then, for solutions u of the equation $Q_2(u) = 0$ in a neighbourhood of $\overline{P_n}$, the integral $\int_{P_n} L_3(u) dx^n$ can be pushed down to the $(n - 2)$ -skeleton P_{n-2} of P_n if and only if there are distributions $\sigma, \tau \in \mathcal{E}'(\mathbf{R}^n)$ satisfying:*

$$(3) \quad L_3^* \chi_{P_n} = \sigma + Q_2^* \tau,$$

with

$$\text{supp}(\sigma) \subset \overline{P_{n-2}}, \quad \text{supp}(\tau) \subset \overline{P_n}.$$

Moreover, the structure of the distribution τ above turns out to be simple. More specifically, if $F_i, i \in I$, are the codimension-1 faces of P_n , then, up to functions in the kernel of Q_2^* , $\tau = \sum_{i \in I} a_i \delta_{F_i}$, with some constants a_i .

Indeed, assuming that the face F_i is given by the equation $x_n = 0$ we infer that in a neighbourhood of F_i :

$$L_3^* \chi_{P_n}(x) = C \partial_n^3 H(x_n) = C \delta''(x_n)$$

for some constant C , and where H is the Heaviside function and δ the Dirac measure. Thus, locally on the face F_i , we have $\tau(x) = c \delta(x_n)$, with c a constant.

The next result is somewhat analogous to example (2), but with the ordinary cube in \mathbf{R}^n replaced by a ‘‘polycube’’ (or rather ‘‘poly-polygon’’) in \mathbf{C}^n . We denote $z = (z_1, \dots, z_n) \in \mathbf{C}^n$, and $z_j = x_j + \sqrt{-1}y_j, 1 \leq j \leq n$.

Theorem 4. *Let $P_{2n} \subset \mathbf{C}^n$ be a product of polygons in \mathbf{C} , and let $L_n = \partial/\partial z_1 \dots \partial/\partial z_n$. Then there exist constants $\gamma_v \in \mathbf{C}$, where $v \in P_0$, so that for every analytic function f defined in a neighbourhood of $\overline{P_{2n}}$:*

$$(4) \quad \int_{P_{2n}} L_n^2(f) dx^{2n} = \sum_{v \in P_0} \gamma_v f(v).$$

The real valued version of Theorem 4 reads then as follows.

Corollary 5. *In the conditions of Theorem 4, let L_3^j be homogeneous, constant coefficients, partial differential operators of order three in $x_j, y_j, 1 \leq j \leq n$. Then there are vectors γ_v with 2^n components such that for every pluriharmonic function defined in a neighbourhood of $\overline{P_{2n}}$:*

$$\int_{P_{2n}} L_3^1 L_3^2 \dots L_3^n (u) dx^{2n} = \sum_{v \in P_0} \gamma_v \cdot \nabla_{(x_1, y_1)} \nabla_{(x_2, y_2)} \dots \nabla_{(x_n, y_n)} (u)|_v.$$

Pluriharmonic functions in \mathbf{C}^n are functions which are harmonic when restricted to any complex linear subspace. They arise as real parts of complex analytic functions (see [5] and the references cited there for details).

3. PROOFS

Proof of Theorem 1. Although Theorem 1 is a consequence of Theorem 2, we include below a separate simple proof for better illustrating the main idea.

We shall first push the integral in the left member of (1) out to ∂P_n and then further push the contribution from any given $(n - 1)$ -face $F \subset \partial P_n$ to ∂F . We may assume that the face F is contained in the hyperplane $x_n = 0$. The operator L_3 is a sum of monomials, so we can assume that $L_3 = \partial_i \partial_j \partial_k$, with $i, j, k \in \{1, 2, \dots, n\}$, not necessarily distinct.

By using Stokes' theorem we are led to evaluate, up to a sign:

$$\int_{\partial P_n} \partial_i \partial_j (u) dx_1 \wedge \dots \wedge \hat{dx}_k \wedge \dots \wedge dx_n.$$

The contribution from the face F vanishes if $k \neq n$ (because $dx_n = 0$ along F) and equals

$$\int_F \partial_i \partial_j (u) dx_1 \wedge \dots \wedge dx_{n-1}$$

if $k = n$.

In the case $i = j = n$ we can use Laplace's equation and write:

$$\partial_n^2 (u) = - \sum_{k=1}^{n-1} \partial_k^2 (u).$$

Therefore, we can assume that at least one of the indices i, j , say j , is different from n . Consequently, by Stokes' theorem again, the integral over F reduces to (up to a sign):

$$\int_{\partial F} \partial_i (u) dx_1 \wedge \dots \wedge \hat{dx}_j \wedge \dots \wedge dx_{n-1},$$

and this proves Theorem 1.

Proof of Theorem 2. We have, for arbitrary smooth functions u ,

$$L_m(u) dx_1 \wedge \dots \wedge dx_n = d\omega,$$

where

$$\omega = \sum_{k=1}^n L_{m-1}^k(u) dx_1 \wedge \dots \wedge \hat{dx}_k \wedge \dots \wedge dx_n,$$

for suitable $(m - 1)$ -th order operators L_{m-1}^k satisfying:

$$(5) \quad \sum_{k=1}^n (-1)^{k-1} L_{m-1}^k(\partial_k u) = L_m(u).$$

Hence

$$\int_{P_n} L_m(u) dx^n = \int_{\partial P_n} \omega.$$

Consider a face $F \subset \partial P_n$. We may assume that it is given by the equation $x_n = 0$. Then

$$(6) \quad \int_F \omega = \int_F L_{m-1}^n(u) dx_1 \wedge \dots \wedge dx_{n-1}.$$

Case 1. Assume that F is characteristic for L_m . This means that $L_m(u)$ does not contain the derivative $\partial_n^m u$. By relation (5), $L_{m-1}^n(u)$ then does not contain $\partial_n^{m-1} u$. Thus every term of L_m^n contains a derivative ∂_j with $j \neq n$. Hence the integrand in (6) is exact and the integral can be pushed to ∂F (with no assumption on u).

Case 2. Assume that $Q_{m-1}(u) = 0$ and F is non-characteristic for Q_{m-1} . The latter means that $Q_{m-1}(u)$ contains $\partial_n^{m-1} u$. Thus, if $\partial_n^{m-1} u$ occurs in $L_{m-1}^n(u)$, then it can be removed, and we can continue as in case 1.

This finishes the proof of Theorem 2.

Proof of Proposition 3. If formula (3) holds, then for u satisfying $Q_2(u) = 0$ in a neighbourhood of $\overline{P_n}$ we have:

$$\begin{aligned} \langle \chi_{P_n}, L_3 u \rangle &= \langle L_3^* \chi_{P_n}, u \rangle = \langle \sigma, u \rangle + \langle Q_2^* \tau, u \rangle \\ &= \langle \sigma, u \rangle + \langle \tau, Q_2 u \rangle = \langle \sigma, u \rangle, \end{aligned}$$

i.e., the integral of $L_3 u$ over P_n is pushed to the $(n - 2)$ -skeleton.

Conversely, suppose that

$$\langle \chi_{P_n}, L_3 u \rangle = \langle \sigma, u \rangle,$$

for all u satisfying $Q_2(u) = 0$ in a neighbourhood of $\overline{P_n}$, with $\text{supp}(\sigma) \subset \overline{P_{n-2}}$. Then the linear map

$$Q_2(u) \longrightarrow \langle \chi_{P_n}, L_3 u \rangle - \langle \sigma, u \rangle$$

is well defined and continuous in the topology of $\mathcal{E}(\mathbf{R}^n)$. So, by Hahn-Banach Theorem, there exists a distribution τ with $\text{supp}(\tau) \subset \overline{P_n}$, satisfying

$$\langle \tau, Q_2(u) \rangle = \langle \chi_{P_n}, L_3(u) \rangle - \langle \sigma, u \rangle,$$

or

$$\langle Q_2^* \tau, u \rangle = \langle L_3^* \chi_{P_n}, u \rangle - \langle \sigma, u \rangle,$$

for all smooth functions u defined in a neighbourhood of $\overline{P_n}$.

Proof of Theorem 4. Let $P_{2n} = \Delta_1 \times \dots \times \Delta_n$ be a product of planar polygons, and let $f(z_1, z_2, \dots, z_n)$ be an analytic function in a neighbourhood of $\overline{P_{2n}}$. Fix the variables $z_j = a_j \in \Delta_j$, $2 \leq j \leq n$. Then the function $g(z_1) = f(z_1, a_2, \dots, a_n)$ is analytic in a neighbourhood of $\overline{\Delta_1}$. By the Motzkin-Schoenberg-Davis theorem there are constants $c_v \in \mathbf{C}$, depending only on Δ_1 and labelled by the vertices v of Δ_1 , so that:

$$\int_{\Delta_1} \frac{\partial^2(g)}{\partial z_1^2} d\bar{z}_1 \wedge dz_1 = \sum_{v \in (\Delta_1)_0} c_v g(v).$$

An iterated integration then finishes the proof. The proof of Corollary 5 is entirely similar.

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