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# Some Isoperimetric Inequalities in Electrochemistry and Hele Shaw Flows

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Isoperimetric inequalities are applied to a moving-boundary problem for doubly-connected domains. This problem occurs for example in electrochemistry, in which case the domains in question are the electrolyte of an electrolytic cell. The two electrodes surrounding the electrolyte are assumed to grow or dissolve, at different rates in general, by electrochemical reaction. We obtain optimal estimates showing, for example, that the least change in volume of each electrode always occurs in spherical symmetry.

## 1. Introduction

### (a) *The Mathematical Problem*

WE CONSIDER domains $\omega$ in $\mathbb{R}^N$ $(N \geqslant 2)$ of the kind $\omega = \omega_0 \backslash \bar{\omega}_1$, where $\omega_j$ $(j = 0, 1)$ are bounded open sets with $\gamma_j = \partial \omega_j$ smooth (say of class $C^2$) and satisfying $\bar{\omega}_1 \subset \omega_0$. For such domains $\omega$ the problem

$$\Delta u = 0 \quad \text{in } \omega, \qquad u = \begin{cases} 0 & \text{on } \gamma_0, \\ 1 & \text{on } \gamma_1, \end{cases} \tag{1.1}$$

has a unique solution $u = u_\omega \in C^1(\bar{\omega})$.

Let $(\alpha_0, \alpha_1) \in \mathbb{R}^2$, $(\alpha_0, \alpha_1) \neq (0, 0)$. This paper is concerned with the following moving-boundary problem for domains $\omega(t)$ of the above kind. Given $\omega(0)$ find $\omega(t)$ for $t$ in some time-interval $[0, t_0)$ such that

$$\gamma_j(t) \text{ moves with the velocity } \alpha_j \nabla u(t) \quad (j = 0, 1) \tag{1.2}$$

(we write $u(t)$ instead of $u_{\omega(t)}$ for simplicity).

This problem has been considered by, for example, C. M. Elliott (1980) for $\alpha_0 = -1$, $\alpha_1 = 0$ and B. Gustafsson (1987) for $\alpha_0 = \alpha_1 = -1$. In (Elliott, 1980) existence and uniqueness of weak (variational inequality) solutions, global in time, are proved. The method actually works whenever $\alpha_0 < 0$, $\alpha_1 = 0$ or $\alpha_0 = 0$, $\alpha_1 > 0$. In (Gustafsson, 1987) existence for small time of weak solutions is proved under the assumption that $\partial\omega(0)$ is analytic (this assumption is necessary because the problem is ill-posed if $\alpha_1 < 0$ (or $\alpha_0 > 0$)). The method of proof works whenever $\alpha_0$ and $\alpha_1$ have the same sign.

Here our aim is complementary: we use isoperimetric inequalities to compare a general solution $\omega(t)$ with a corresponding 'symmetrized' solution $\Omega(t)$. We thereby obtain optimal estimates for the 'stopping time' (if any) for $\omega(t)$ and we show that the variation of the measures $m_j(t) = |\omega_j(t)|$ are slowest for the symmetrized solution (for certain choices of $\alpha_j$), etc.

This paper develops and generalizes a previous note (Mossino, 1985). Some more results of the same kind are found in (Mossino, 1986). (Sections 1 to 3 of the present paper comprise essentially a revised and shortened version of (Mossino, 1986).)

## (b) *Physical Applications*

There are several conceivable physical interpretations of our problem. The first comes from electrochemistry ('electrochemical machining') (see, for example, (de Barr & Oliver, 1967; Elliott, 1980; FitzGerald & McGeough, 1969, 1970; FitzGerald *et al.*, 1969)). Then $N = 2$ or 3 and the configuration $\omega_0$, $\omega_1$ represents the cross-section of an electrolytic machine: the electrolyte occupies the domain $\omega(t)$ while $\gamma_j(t)$ are the boundaries of the electrodes. The electric potential is equal to some real constant times $u(t)$ and the electric current is proportional to $\nabla u(t)$.

The physically most interesting case is that of a melting anode and a growing or unchanging cathode. If for example $\gamma_0(t)$ represents the anode surface and $\gamma_1(t)$ the cathode surface then we have $\alpha_0 < 0$, $\alpha_1 \leq 0$, the values of $\alpha_j$ depending on the particular kinds of materials involved and on the constant of proportionality relating $u(t)$ to the electric potential. The case in which $\alpha_0 = \alpha_1 < 0$ corresponds to a simple mass transfer between the electrodes, which occurs for example if the electrodes are made of copper and the electrolyte is a solution of copper sulphate. The case in which $\alpha_0 < 0$, $\alpha_1 = 0$ occurs in industrial processes called anodic smoothing, anodic shaping etc. See (de Barr & Oliver, 1967; Elliott, 1980; FitzGerald & McGeough, 1969, 1970; FitzGerald *et al.* 1969).

A second application (pointed out by J. Ockendon) concerns the flow of a viscous incompressible fluid in the narrow region between two slightly separated parallel surfaces, the so-called Hele Shaw flow (see, for example, (Elliott & Ockendon, 1982; Richardson, 1972)). Then $N = 2$ and $\omega(t)$ represents the region occupied at time $t$ by the viscous fluid in question while $\omega_1(t)$ is a bubble of some fluid of negligible viscosity, for example, air, and $\mathbb{R}^2 \backslash \omega_0(t)$ is a vacuum or possibly air at a different pressure. In this application $\alpha_0 = \alpha_1$.

There is a related application, valid for $N = 2$ or 3 (with $\alpha_0 = \alpha_1$) in which $\mathbb{R}^N$ is a porous medium with $\omega(t)$ occupied (and saturated) by some incompressible viscous fluid. As in the Hele Shaw model $\omega_1(t)$ is a bubble of, for example, air and $\mathbb{R}^N \backslash \omega_0(t)$ is a vacuum or possibly air at a different pressure. See (Di Benedetto & Friedman, 1986; Howison, 1986) in which the limiting case with $\omega_0(t) = \mathbb{R}^N$ ('exterior problem') is treated.

There may also be other interpretations of our problem, for example (as pointed out by M. E. Gurtin) as a degenerate Stefan problem with three phases.

## 2. The starting (in)-equations; preliminaries

We shall consider 'classical' solutions $\omega(t)$ on some interval $[0, t_0)$ of the problem described in Section 1. Thus $\omega(t)=\omega_0(t)\backslash\overline{\omega_1(t)}$, where $\omega_j(t)$ are bounded open sets in $\mathbb{R}^N$ satisfying $\overline{\omega_1(t)}\subset\omega_0(t)$ and with $\gamma_j(t)=\partial\omega_j(t)$ smooth. Setting $m_j(t)=|\omega_j(t)|=\text{meas}\,\omega_j(t)$ we have

$$0<m_1(t)<m_0(t) \tag{2.1}$$

for all $t\in[0, t_0)$. As to the concept of classical solution we only need that $m_j\in C^1[0, t_0)$ and (2.3), (2.4) below hold. Thus one can simply take the equations (2.3), (2.4) (with $c$ defined by (2.2)) as the starting point and there is no need to give here a more detailed description of the solution concept.

With $\nu$ the outward normal of $\omega$ on $\partial\omega$ we define

$$c(t)=\int_{\omega(t)}|\nabla u|^2\,\mathrm{d}x=-\int_{\gamma_0(t)}\frac{\partial u}{\partial\nu}(t)\,\mathrm{d}\gamma=\int_{\gamma_1(t)}\frac{\partial u}{\partial\nu}(t)\,\mathrm{d}\gamma, \tag{2.2}$$

to be the *capacity* of $\omega(t)$. Then the moving-boundary condition (1.2) yields

$$\frac{\mathrm{d}m_0}{\mathrm{d}t}=\alpha_0\int_{\gamma_0(t)}\frac{\partial u}{\partial\nu}(t)\,\mathrm{d}\gamma=-\alpha_0 c(t), \tag{2.3}$$

$$\frac{\mathrm{d}m_1}{\mathrm{d}t}=-\alpha_1\int_{\gamma_1(t)}\frac{\partial u}{\partial\nu}(t)\,\mathrm{d}\gamma=-\alpha_1 c(t). \tag{2.4}$$

In particular $\mathrm{d}m_j/\mathrm{d}t$ always has opposite sign to $\alpha_j$ since $c(t)>0$.

If $\alpha_0$, $\alpha_1\neq 0$, then (2.3), (2.4) give

$$\frac{1}{\alpha_0}\frac{\mathrm{d}m_0}{\mathrm{d}t}=\frac{1}{\alpha_1}\frac{\mathrm{d}m_1}{\mathrm{d}t}=-c(t). \tag{2.5}$$

Set

$$\beta=\frac{m_0(0)}{\alpha_0}-\frac{m_1(0)}{\alpha_1}. \tag{2.6}$$

Then, integrating (2.5),

$$\beta=\frac{m_0(t)}{\alpha_0}-\frac{m_1(t)}{\alpha_1} \tag{2.7}$$

for all $t\in[0, t_0)$. In the particular case in which $\alpha_0=\alpha_1$,

$$m(t)=|\omega(t)|=m_0(t)-m_1(t)$$

remains constant.

Now using rearrangement techniques as in (Bandle, 1980; Mossino, 1984; Talenti, 1976) we get

$$N^2\rho_N^{2/N} \leqslant c(t)[\Phi(m_0(t)) - \Phi(m_1(t))], \tag{2.8}$$

where $\rho_N$ is the measure of the unit ball in $\mathbb{R}^N$ and

$$\Phi'(x) = x^{(2/N)-2}. \tag{2.9}$$

Equation (2.8) is our fundamental inequality and most results in this paper are (in principle) elementary consequences of (2.3), (2.4), and (2.8).

When $\alpha_0$, $\alpha_1 \neq 0$ equations (2.3), (2.4), (2.8) give

$$N^2\rho_N^{2/N} \leqslant \frac{1}{\alpha_1}\,\Phi(m_1(t))\,\frac{\mathrm{d}m_1}{\mathrm{d}t} - \frac{1}{\alpha_0}\,\Phi(m_0(t))\,\frac{\mathrm{d}m_0}{\mathrm{d}t} \tag{2.10}$$

and, by integration,

$$\begin{aligned} N^2\rho_N^{2/N}t &\leqslant \left[\frac{1}{\alpha_1}\,\psi(m_1(\tau)) - \frac{1}{\alpha_0}\,\psi(m_0(\tau))\right]_{\tau=0}^{\tau=t} \\ &= \frac{1}{\alpha_1}\,\psi(m_1(t)) - \frac{1}{\alpha_0}\,\psi(m_0(t)) - \left(\frac{1}{\alpha_1}\,\psi(m_1(0)) - \frac{1}{\alpha_0}\,\psi(m_0(0))\right), \end{aligned} \tag{2.11}$$

where

$$\psi' = \Phi. \tag{2.12}$$

If one of $\alpha_j$ is zero, say $\alpha_1 = 0$, $\alpha_0 \neq 0$, then

$$m_1(t) = \text{constant} = m_1(0) \tag{2.13}$$

(replacing (2.7)). In place of (2.10) we have (using (2.3))

$$N^2\rho_N^{2/N} \leqslant \frac{1}{\alpha_0}\,[\Phi(m_1(0)) - \Phi(m_0(t))]\,\frac{\mathrm{d}m_0}{\mathrm{d}t} \tag{2.14}$$

and integration now gives

$$\begin{aligned} N^2\rho_N^{2/N}t &\leqslant \frac{1}{\alpha_0}\,[\Phi(m_1(0))m_0(\tau) - \psi(m_0(\tau))]_{\tau=0}^{\tau=t} \\ &= \frac{1}{\alpha_0}\,(\Phi(m_1(0))m_0(t) - \psi(m_0(t))) - \frac{1}{\alpha_0}\,(\Phi(m_1(0))m_0(0) - \psi(m_0(0))). \end{aligned} \tag{2.15}$$

Similarly, if $\alpha_0 = 0$, $\alpha_1 \neq 0$ then $m_0(t) = \text{constant} = m_0(0)$,

$$N^2\rho_N^{2/N} \leqslant \frac{1}{\alpha_1}\,[\Phi(m_1(t)) - \Phi(m_0(0))]\,\frac{\mathrm{d}m_1}{\mathrm{d}t},$$

$$\begin{aligned} N^2\rho_N^{2/N}t &\leqslant \frac{1}{\alpha_1}\,[\psi(m_1(\tau)) - \Phi(m_0(0))m_1(\tau)]_{\tau=0}^{\tau=t} \\ &= \frac{1}{\alpha_1}\,(\psi(m_1(t)) - \Phi(m_0(0))m_1(t)) - \frac{1}{\alpha_1}\,(\psi(m_1(0)) - \Phi(m_0(0))m_1(0)). \end{aligned}$$

In order to interpret our inequalities we shall consider a 'symmetrized' version of our problem, in which $\omega(t)$ is replaced by $\Omega(t)=\Omega_0(t)\backslash\overline{\Omega_1(t)}$, where $\Omega_j(t)$ are the balls of appropriate radius in $\mathbb{R}^N$, centred at the origin and determined by the initial condition

$$M_j(0)=m_j(0) \qquad (j=0,\,1) \tag{2.16}$$

(where $M_j(t)=|\Omega_j(t)|$) together with the condition that the boundaries $\Gamma_j(t)=\partial\Omega_j(t)$ move with velocities $\alpha_j\nabla U(t)$ $(j=0,\,1)$, where

$$\Delta U(t)=0 \quad \text{in } \Omega(t),$$
$$U(t)=\begin{cases}0 & \text{on } \Gamma_0(t),\\ 1 & \text{on } \Gamma_1(t).\end{cases}$$

It is obvious that *local* (small time) solutions of this form exist and that they are uniquely determined by the previous conditions. In fact, (2.7) together with (2.11)′ (if $\alpha_j\neq 0$) below are implicit formulae determining $M_0(t)$, $M_1(t)$ and hence $\Omega(t)$. The solution $\Omega(t)$ exists in some time interval $[0\,,\,T)$, where either $T=+\infty$ or $T$ is finite and determined by the breakdown of one of the inequalities in $0<M_1(t)<M_0(t)<\infty$.

One should notice that $\Omega(t)$ is not the 'symmetrized' domain of $\omega(t)$, at any time; that is, $M_j(t)=m_j(t)$ is false for $t\neq 0$. However, for simplicity we shall call $\Omega(t)$ the 'symmetrized solution' instead of 'the solution of the symmetrized problem'. We shall use capital letters for quantities related to the symmetrized problem while the corresponding quantities for the original problem are denoted by small letters. Our aim is to compare these two kinds of quantities ($m_j(t)$ with $M_j(t)$, $c(t)$ with $C(t)$, etc).

For the symmetrized problem equality is achieved in (2.8) (see (Mossino, 1984)). Thus

$$N^2\rho_N^{2/N}=C(t)[\Phi(M_0(t))-\Phi(M_1(t))] \tag{2.8$'$}$$

and, if $\alpha_0$, $\alpha_1\neq 0$,

$$N^2\rho_N^{2/N}=\frac{1}{\alpha_1}\,\Phi(M_1(t))\,\frac{\mathrm{d}M_1}{\mathrm{d}t}-\frac{1}{\alpha_0}\,\Phi(M_0(t))\,\frac{\mathrm{d}M_0}{\mathrm{d}t}, \tag{2.10$'$}$$

$$N^2\rho_N^{2/N}t=\left[\frac{1}{\alpha_1}\,\psi(M_1(\tau))-\frac{1}{\alpha_0}\,\psi(M_0(\tau))\right]_{\tau=0}^{\tau=t} \tag{2.11$'$}$$

If $\alpha_1=0$, $\alpha_0\neq 0$ then

$$N^2\rho_N^{2/N}=\frac{1}{\alpha_0}[\Phi(M_1(0))-\Phi(M_0(t))]\,\frac{\mathrm{d}M_0}{\mathrm{d}t}, \tag{2.14$'$}$$

$$N^2\rho_N^{2/N}t=\frac{1}{\alpha_0}[\Phi(M_1(0))M_0(\tau)-\psi(M_0(\tau))]_{\tau=0}^{\tau=t} \tag{2.15$'$}$$

and, if $\alpha_0 = 0$, $\alpha_1 \neq 0$,

$$N^2 \rho_N^{2/N} = \frac{1}{\alpha_1} [\Phi(M_1(t)) - \Phi(M_0(0))] \frac{dM_1}{dt},$$

$$N^2 \rho_N^{2/N} t = \frac{1}{\alpha_1} [\psi(M_1(\tau)) - \Phi(M_0(0)) M_1(\tau)]_{\tau=0}^{\tau=t}.$$

Explicitly the functions $\Phi$ and $\psi$ can be chosen to be

$$\Phi(x) = \begin{cases} \text{Log } x & \text{if } N = 2, \\ -\dfrac{N}{N-2} x^{(2/N)-1} & \text{if } N > 2, \end{cases} \tag{2.17}$$

$$\psi(x) = \begin{cases} x(\text{Log } x - 1) & \text{if } N = 2, \\ -\dfrac{N^2}{2(N-2)} x^{2/N} & \text{if } N > 2. \end{cases} \tag{2.18}$$

For later use we set

$$\alpha = \alpha_0 / \alpha_1 \quad (\text{if } \alpha_1 \neq 0). \tag{2.19}$$

## 3. Main results

Let $\omega(t)$ be an arbitrary (classical) solution defined on $[0, t_0)$ and let $\Omega(t)$ be the corresponding symmetrized solution satisfying (2.16). We shall estimate stopping times for $\omega(t)$ (that is, give upper bounds for $t_0$), compare $m_j(t)$ with $M_j(t)$, $c(t)$ with $C(t)$ and $dm_j/dt$ with $dM_j/dt$.

### (a) *Stopping Times, Comparison Between $m_j(t)$ and $M_j(t)$, etc.*

By (2.1), $(m_1(t), m_0(t))$ and $(M_1(t), M_0(t))$ always move in the set

$$D = \{(m_1, m_0) \in \mathbb{R}^2 : 0 < m_1 < m_0\}. \tag{3.1}$$

We first consider the case where $\alpha_j \neq 0$ $(j = 0, 1)$. Define $F: D \to \mathbb{R}$ by

$$F(m_1, m_0) = N^{-2} \rho_N^{-2/N} \left( \frac{1}{\alpha_1} \psi(m_1) - \frac{1}{\alpha_0} \psi(m_0) - A \right), \tag{3.2}$$

where

$$A = \frac{1}{\alpha_1} \psi(m_1(0)) - \frac{1}{\alpha_0} \psi(m_0(0)) \tag{3.3}$$

and $\psi$ is given by (2.18). Now (2.18) shows that $F$ extends continuously to

$$\bar{D} = \{(m_1, m_0) \in \mathbb{R}^2 : 0 \leq m_1 \leq m_0\}.$$

Observe that $F$ also depends on $(m_1(0), m_0(0))$ but that changing $(m_1(0), m_0(0))$

just affects $F$ by an additive constant. (We always have $F(m_1(0), m_0(0))=0$.) Also recall (2.16).

In terms of $F$, equations (2.11) and (2.11)′ say that

$$t \leqslant F(m_1(t), m_0(t)), \tag{3.4}$$

$$t = F(M_1(t), M_0(t)), \tag{3.5}$$

for all $t \geqslant 0$ for which $\omega(t)$ and $\Omega(t)$ (respectively) exist.

From (2.7) we see that $(m_1(t), m_0(t))$ actually moves along the straight line

$$I=\left\{(m_1, m_0)\in D : \frac{m_0}{\alpha_0}-\frac{m_1}{\alpha_1}=\beta\right\}. \tag{3.6}$$

The same is true for $(M_1(t), M_0(t))$ since (by (2.6)) the value of $\beta$ only depends on $(m_1(0), m_0(0))=(M_1(0), M_0(0))$. Moreover $(m_1(t), m_0(t))$ and $(M_1(t), M_0(t))$ move in the same direction on $I$ (this direction being determined by the signs of $\alpha_j$) and by (3.5) this is the direction in which $F$ increases (strictly). It also follows that $F$ is monotone on $I$. We shall call the direction on $I$ in which $F$ increases (and $(m_1(t), m_0(t))$, $(M_1(t), M_0(t))$ move) the *positive direction* of $I$. Notice the interpretation that (3.5) gives to $F$: for $(m_1, m_0)$ located on $I$, after $(m_1(0), m_0(0))$, $F(m_1, m_0)$ simply tells at what time the symmetrized solution reaches $(m_1, m_0)$.

An immediate consequence of $F(M_1(t), M_0(t)) \leqslant F(m_1(t), m_0(t))$ and the above discussion is that for each $t \geqslant 0$ $(m_1(t), m_0(t))$ has always progressed further on $I$ than $(M_1(t), M_0(t))$ has. Therefore, for each $j=(0, 1)$ and for $t \geqslant 0$ such that both $\omega(t)$ and $\Omega(t)$ exist,

$$m_j(t)\begin{cases}\leqslant M_j(t) & \text{if } \alpha_j>0, \quad (3.7)\\ \geqslant M_j(t) & \text{if } \alpha_j<0. \quad (3.8)\end{cases}$$

It also follows that if $(M_1(t), M_0(t))$ reaches $\partial D$ or goes to infinity in a finite time then, if the solution $\omega(t)$ does not break down earlier for other reasons, $(m_1(t), m_0(t))$ reaches $\partial D$ or goes to infinity in a shorter time. Actually, the latter time is strictly shorter (if $\omega(0)$ is not spherically symmetric) as the following argument shows.

With $\tau = F(m_1(t), m_0(t)) \geqslant t$ we have $(M_1(\tau), M_0(\tau))=(m_1(t), m_0(t))$ by (3.5) and hence $C(\tau) \leqslant c(t)$ by classical isoperimetric inequalities (Polya & Szegö, 1951). It follows from (2.3), (2.4) that

$$\left|\frac{\mathrm{d}M_j(\tau)}{\mathrm{d}\tau}\right| \leqslant \left|\frac{\mathrm{d}m_j(t)}{\mathrm{d}t}\right|. \tag{3.9}$$

Hence at any given point on $I$, $(M_1(\tau), M_0(\tau))$ always moves more slowly than $(m_1(t), m_0(t))$. (Later we shall compare $\mathrm{d}M_j(t)/\mathrm{d}t$ and $\mathrm{d}m_j(t)/\mathrm{d}t$.) Therefore the time separation between the arrivals of $(m_1(t), m_0(t))$ and $M_1(\tau), M_0(\tau))$ at points on $I$ never decreases in the positive direction of $I$. From this the assertion above follows (since also we have strict inequality in, for example, (3.4) for $t>0$ unless $\omega(0)$ is spherically symmetric).

By (3.5) the time needed for $(M_1(t), M_0(t))$ to reach $\partial D$ or go to infinity is

$$T = \sup_I F. \tag{3.10}$$

When $T$ is finite it is the *stopping time* for the symmetrized solution (since this solution exists as long as $(M_1(t), M_0(t)) \in D$). By the above discussion, if $T<\infty$ and $\omega(0)$ is not spherically symmetric the stopping time for the original solution $\omega(t)$ is strictly less than $T$. (Generally $\omega(t)$ breaks down before $(m_1(t), m_0(t))$ leaves $D$ because of the development of singularities on $\partial\omega(t)$; $\gamma_0(t)$ (respectively $\gamma_1(t)$) will be unstable and tend to develop singularities if $\alpha_0>0$ (respectively $\alpha_1<0$). See also (Gustafsson, 1987).)

We now investigate the finiteness of $T$. The equation for $I$ can be written

$$m_0 = \alpha m_1 + \alpha_0\beta \tag{3.11}$$

$(\alpha = \alpha_0/\alpha_1)$. From this we see that

(i) $I$ is bounded if $\alpha<1$,
(ii) $I$ is unbounded (in one direction) if $\alpha \geqslant 1$.

Thus, if $\alpha<1$ we immediately obtain $T<\infty$ since $F$ is continuous on $\bar{I} \subset \bar{D}$. Moreover, it follows from (3.5) and the fact that $\mathrm{d}M_1/\mathrm{d}t$ has opposite sign to $\alpha_1$ that the supremum is attained on $m_1=0$ if $\alpha_1>0$ and on the line $m_1=m_0$ if $\alpha_1<0$. This gives

$$T = \begin{cases} F(0, \alpha_0\beta) & \text{if } \alpha_1>0, \\ F\left(\dfrac{\alpha_0\beta}{1-\alpha}, \dfrac{\alpha_0\beta}{1-\alpha}\right) & \text{if } \alpha_1<0 \end{cases}$$

(when $\alpha<1$).

If $\alpha\geqslant 1$ we have to distinguish between the cases in which $\alpha_j>0$ and $\alpha_j<0$. When $\alpha_j>0$ $(j=0, 1)$, $M_j(t)$ decreases with increasing $t$ and so it follows from (3.5) that $F$ attains its supremum on $I$ on $\partial D$. Thus $T<\infty$ in this case. More exactly we get

$$T = \begin{cases} F(0, \alpha_0\beta) & \text{if } \beta>0, \\ F\left(\dfrac{\alpha_0\beta}{1-\alpha}, \dfrac{\alpha_0\beta}{1-\alpha}\right) & \text{if } \beta<0 \end{cases}$$

(when $\alpha\geqslant 1$, $\alpha_j>0$).

When $\alpha\geqslant 1$ and $\alpha_j<0$ $(j=0, 1)$, $M_j(t)$ increases with $t$ and (3.5) shows that $F$ increases in the unbounded direction of $I$. Using (3.10), (3.11) we see that

$$T = N^{-2}\rho_N^{-2/N}\left(\lim_{x\to\infty}\left(\frac{1}{\alpha_1}\psi(x) - \frac{1}{\alpha_0}\psi(\alpha x + \alpha_0\beta)\right) - A\right).$$

The explicit expressions (2.18) for $\psi(x)$ give

$$\frac{1}{\alpha_1}\psi(x)-\frac{1}{\alpha_0}\psi(\alpha x+\alpha_0\beta)$$

$$=\begin{cases}-\dfrac{1}{\alpha_1}x\operatorname{Log}\left(\alpha+\dfrac{\alpha_0\beta}{x}\right)-\beta\operatorname{Log}x-\beta\operatorname{Log}\left(\alpha+\dfrac{\alpha_0\beta}{x}\right)+\beta & \text{if } N=2,\\ -\dfrac{N^2x^{2/N}}{2(N-2)\alpha_1}\left(1-\dfrac{1}{\alpha}\left(\alpha+\dfrac{\alpha_0\beta}{x}\right)^{2/N}\right) & \text{if } N>2.\end{cases}$$

When $\alpha>1$ (and $\alpha_j<0$) we see that

$$\lim_{x\to\infty}\left(\frac{1}{\alpha_1}\psi(x)-\frac{1}{\alpha_0}\psi(\alpha x+\alpha_0\beta)\right)=+\infty$$

for all $N\geqslant 2$. When $\alpha=1$ (and $\alpha_j<0$), $\alpha_0\beta>0$ (otherwise $I$ would be empty) and hence $\beta<0$. One then finds that

$$\lim_{x\to\infty}\left(\frac{1}{\alpha_1}\psi(x)-\frac{1}{\alpha_0}\psi(\alpha x+\alpha_0\beta)\right)=\begin{cases}+\infty & \text{if } N=2,\\ 0 & \text{if } N>2.\end{cases}$$

Thus, when $\alpha\geqslant 1$, $\alpha_j<0$,

$$T=\begin{cases}+\infty & \text{if } \alpha>1 \text{ or } N=2,\\ -AN^{-2}\rho_N^{-2/N} & \text{if } \alpha=1 \text{ and } N>2\end{cases}$$

(where $A$ is given by (3.3)).

Suppose now that one of $\alpha_j$ is zero, say $\alpha_1=0$, $\alpha_0\neq 0$. In view of (2.15) $F$ should then be defined by

$$F(m_1,m_0)=F(m_0)=\frac{N^{-2}\rho_N^{-2/N}}{\alpha_0}[\Phi(m_1(0))m_0-\psi(m_0)-A], \tag{3.12}$$

where

$$A=\Phi(m_1(0))m_0(0)-\psi(m_0(0)).$$

Then (3.4), (3.5) remain true, by (2.15), (2.15)$'$.

The line $I$ is now vertical and defined by (2.13), that is,

$$I=\{(m_1,m_0)\in D: m_1=m_1(0)\}.$$

The discussion after (3.6) remains the same, hence (3.7), (3.8) hold ($j=0$). As to the stopping time $T=\sup_I F$, we have

$$T=F(m_1(0),m_1(0))<\infty$$

if $\alpha_0>0$ ($m_0(t)$ decreasing) while

$$T=\frac{N^{-2}\rho_N^{-2/N}}{\alpha_0}\left[\lim_{x\to\infty}\Phi(m_1(0))x-\psi(x)-A\right]=+\infty$$

if $\alpha_0<0$, as is seen from (2.17), (2.18).

The case in which $\alpha_0 = 0,\ \alpha_1 \neq 0$ is treated similarly. With $F$ defined by

$$F(m_1, m_0) = F(m_1) = \frac{N^{-2}\rho_N^{-2/N}}{\alpha_1}[\psi(m_1) - \Phi(m_0(0))m_1 - A] \qquad (3.13)$$

for $A = \psi(m_1(0)) - \Phi(m_0(0))m_1(0)$, and

$$I = \{(m_1, m_0) \in D : m_0 = m_0(0)\},$$

the stopping time $T = \sup_I F$ becomes

$$T = \begin{cases} F(0, m_0(0)) & \text{if } \alpha_1 > 0, \\ F(m_0(0), m_0(0)) & \text{if } \alpha_1 < 0; \end{cases}$$

this is finite in both cases.

We summarize in the following.

THEOREM 3.1. *The solution $\Omega(t)$ of the symmetrized problem exists for all time if $\alpha_0 < \alpha_1 \leq 0$ or if $\alpha_0 = \alpha_1 < 0$ and $N = 2$. In all other cases it has a finite stopping time $T$, which is*

$$T = F\left(0,\ m_0(0) - \frac{\alpha_0}{\alpha_1} m_1(0)\right)$$

*if $\alpha_1 > 0,\ \alpha_0 \leq \alpha_1\, m_0(0)/m_1(0)$ (in this case $T$ corresponds to $M_1(T) = 0$);*

$$T = F\left(\frac{\alpha_1 m_0(0) - \alpha_0 m_1(0)}{\alpha_1 - \alpha_0}, \frac{\alpha_1 m_0(0) - \alpha_0 m_1(0)}{\alpha_1 - \alpha_0}\right)$$

*if $\alpha_0 > \max\{\alpha_1,\ \alpha_1\, m_0(0)/m_1(0)\}$ (in this case $T$ corresponds to $M_1(T) = M_0(T)$);*

$$T = -AN^{-2}\rho_N^{-2/N}$$

*if $\alpha_0 = \alpha_1 < 0,\ N > 2$ (in this case $T$ corresponds to $M_1(T) = \infty,\ M_0(T) = \infty$).*

*Here, $F$ is given by (3.2), (3.12), (3.13) and $A$ by (3.3). When $T < \infty$, also the original problem has a finite stopping time, which is strictly less than $T$ unless $\omega(0)$ is spherically symmetric. In all cases and for all $t \geq 0$ for which the original solution exists we have*

$$m_j(t) \leq M_j(t) \quad \text{if } \alpha_j \geq 0, \qquad m_j(t) \geq M_j(t) \quad \text{if } \alpha_j \leq 0$$

$(j = 0, 1)$.

REMARK From the comparison results for $m_j(t)$ one immediately obtains comparison results for quantities such as $q(t) = m_0(t)/m_1(t)$ and $m(t) = m_0(t) - m_1(t)$. In fact, the expressions $m_0/m_1$ and $m_0 - m_1$ are monotone functions on every $I$ so every order relation between $(m_1(t), m_0(t))$ and $(M_1(t), M_0(t))$ on $I$ gives rise to an order relation between $q(t)$ and $Q(t)$ and between $m(t)$ and $M(t)$.

Consider for example the case when $\alpha_0 \leq \alpha_1 < 0$, so that $\alpha \geq 1$ and $I$ is unbounded in its positive direction. Then $\alpha < q(t) \leq Q(t)$ and $Q(t) \downarrow \alpha$ if $\beta < 0$, $\alpha > q(t) \geq Q(t)$ and $Q(t) \uparrow \alpha$ if $\beta > 0$ while $q(t) \equiv Q(t) \equiv \alpha$ if $\beta = 0$. In all these cases $m(t)$ and $M(t)$ are increasing and $M(t) \leq m(t)$.

(b) *Comparison Between $c(t)$ and $C(t)$, etc.*

Next we want to compare $c(t)$ with $C(t)$ and $\mathrm{d}m_j/\mathrm{d}t$ with $\mathrm{d}M_j/\mathrm{d}t$. From Polya & Szegö (1951) we know that $C(0) \leqslant c(0)$ with equality only if $\omega(0)$ is spherically symmetric. Therefore it is natural to ask whether

$$C(t) \leqslant c(t) \tag{3.14}$$

for all $t > 0$. Of course, (3.14) remains true in some short time-interval just by continuity ($c(t)$ is continuous by (2.3), (2.4) and the smoothness assumption on $m_j(t)$). We shall prove that (3.14) holds in certain cases for all $t \geqslant 0$ (or for all sufficiently large $t$) for which the original solution exists.

Whenever (3.14) holds we also have

$$\frac{\mathrm{d}M_j(t)}{\mathrm{d}t} \geqslant \frac{\mathrm{d}m_j(t)}{\mathrm{d}t} \quad \text{if } \alpha_j \geqslant 0, \tag{3.15}$$

$$\frac{\mathrm{d}M_j(t)}{\mathrm{d}t} \leqslant \frac{\mathrm{d}m_j(t)}{\mathrm{d}t} \quad \text{if } \alpha_j \leqslant 0 \tag{3.16}$$

($j = 0, 1$) by (2.3), (2.4). The inequalities (3.15), (3.16) can also be written as

$$\left|\frac{\mathrm{d}M_j(t)}{\mathrm{d}t}\right| \leqslant \left|\frac{\mathrm{d}m_j(t)}{\mathrm{d}t}\right|.$$

Define

$$G(m_1, m_0) = \Phi(m_0) - \Phi(m_1)$$

on $D$. Then $G$ is positive since $\Phi$ is increasing by (2.9). From (2.8), (2.8)′ we have

$$C(t)G(M_1(t), M_0(t)) \leqslant c(t)G(m_1(t), m_0(t)) \tag{3.17}$$

for $t \geqslant 0$. Suppose that we can prove that

$$G(M_1(t), M_0(t)) \geqslant G(m_1(t), m_0(t)) \tag{3.18}$$

for some $t \geqslant 0$. Then (3.14) follows from (3.17) for that particular $t$.

Now (3.18) holds if $G$ decreases (in the non-strict sense) in the positive direction of $I$ between $(M_1(t), M_0(t))$ and $(m_1(t), m_0(t))$, that is, if

$$\frac{\mathrm{d}}{\mathrm{d}\tau} G(M_1(\tau), M_0(\tau)) \leqslant 0 \tag{3.19}$$

for $t \leqslant \tau \leqslant F(m_1(t), m_0(t))$. We shall prove that (3.19) holds for all $\tau \geqslant t$ provided it holds for $\tau = t$.

We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau} G(M_1(\tau), M_0(\tau)) &= \Phi'(M_0(\tau))\frac{\mathrm{d}M_0}{\mathrm{d}\tau} - \Phi'(M_1(\tau))\frac{\mathrm{d}M_1}{\mathrm{d}\tau} \\ &= C(\tau)(\alpha_1 M_1(\tau)^{(2/N)-2} - \alpha_0 M_0(\tau)^{(2/N)-2}) \\ &= C(\tau)\alpha_0\alpha_1(M_0(\tau)M_1(\tau))^{(2/N)-2}\left(\frac{M_0(\tau)^{2-(2/N)}}{\alpha_0} - \frac{M_1(\tau)^{2-(2/N)}}{\alpha_1}\right).\end{aligned} \tag{3.20}$$

From (3.20) we see that (3.19) is always true if $\alpha_0 \geqslant 0$ and $\alpha_1 \leqslant 0$, and never true if $\alpha_0 \leqslant 0$ and $\alpha_1 \geqslant 0$.

In the remaining cases $\alpha_0\alpha_1 > 0$ and (3.19) holds if and only if

$$\frac{M_0(\tau)^{2-(2/N)}}{\alpha_0} - \frac{M_1(\tau)^{2-(2/N)}}{\alpha_1} \leqslant 0. \tag{3.21}$$

By (2.5) the derivative of this expression is

$$-(2-(2/N))C(\tau)(M_0(\tau)^{1-(2/N)} - M_1(\tau)^{1-(2/N)}),$$

which is always zero or less (and exactly zero if $N=2$). It follows that if (3.21), or (3.19), holds for some $\tau = t$ it holds for all $\tau \geqslant t$ as claimed. To summarize, we have the following result.

THEOREM 3.2 *If $\alpha_0 \geqslant 0$, $\alpha_1 \leqslant 0$ then* (3.14) (*and hence* (3.15), (3.16)) *holds for all $t \geqslant 0$ for which the original solution exists. If $\alpha_0\alpha_1 > 0$ and if*

$$\alpha_1\left(\frac{M_0}{M_1}\right)^{2-2/N} \leqslant \alpha_0 \tag{3.22}$$

*for some $(M_1, M_0) \in I$ then* (3.14) *holds for all $t \geqslant \max\{0, \tau\}$ for which the original solution exists, where $\tau$ is determined by $(M_1(\tau), M_0(\tau)) = (M_1, M_0)$, that is $\tau = F(M_1, M_0)$. In particular, if $\alpha_1(M_0(0)/M_1(0))^{2-2/N} \leqslant \alpha_0$ then* (3.14) *holds for all $t \geqslant 0$ for which the original solution exists.*

EXAMPLE When $\alpha_0 = \alpha_1 < 0$ (3.22) always holds. Hence (3.14) to (3.16) hold in this case whenever the original solution exists.

## 4. Generalizations

(a) *Other Boundary Conditions for u*

Some of the results in Section 3 remain valid, with minor changes in the proofs, if the boundary conditions in (1.1) are replaced by the more general

$$u = \begin{cases} 0 & \text{on } \gamma_0(t), \\ f & \text{on } \gamma_1(t), \end{cases} \tag{4.1}$$

where $f$ is constant on $\gamma_1(t)$ but depends, directly or indirectly, on $t$. We shall briefly indicate the changes needed, assuming for simplicity that $f > 0$. (The case in which $f < 0$ may be handled by changing the signs of $\alpha_0$, $\alpha_1$.)

The case when $f = f(t)$ is an explicit function of $t$ essentially corresponds to just a (monotone) change of time-scale. Of the results in Section 3 only the formulae for $T$ will be affected, the new $T$ being obtained by replacing (3.10) by

$$\int_0^T f(t)\,\mathrm{d}t = \sup_I F.$$

The case when $f$ is a function of $m_0$ and $m_1$, $f = f(m_1, m_0)$, is a little more interesting and may also be of some physical significance. In the Hele Shaw model or in the porous medium case one may think, for example, of having the fluid in $\omega_1(t)$ trapped there. Assuming that this fluid is a gas, the simplest model (Boyles law) yields the pressure on $\gamma_1(t)$ proportional to $1/m_1$, hence

$$f(m_1, m_0) = \frac{a}{m_1} - b \tag{4.2}$$

($a > 0$, $b \geqslant 0$ constants).

Now, for $f = f(m_1, m_0)$ smooth and positive (the latter at least in a neighbourhood of $(m_1(0), m_0(0)) \in D$) the results of comparison between $m_j(t)$ and $M_j(t)$ in Theorem 3.1 are easily seen to remain true. In fact, (3.4) and (3.5) remain valid with the definition (3.2) of $F$ changed to

$$F(m_1, m_0) = N^{-2}\rho_N^{-2/N} \int_{(m_1(0), m_0(0))}^{(m_1, m_0)} \frac{1}{\alpha_1} \frac{\Phi(x)}{f(x, y)}\, \mathrm{d}x - \frac{1}{\alpha_0} \frac{\Phi(y)}{f(x, y)}\, \mathrm{d}y$$

(integration along $I$). With this $F$ the stopping time $T$ is still given by (3.10). Also, the comparison result (Theorem 3.2) between

$$c(t) = \frac{1}{f(m_1(t), m_0(t))} \int_{\gamma_1(t)} \frac{\partial u}{\partial \nu}(t)\, \mathrm{d}\gamma$$

and $C(t)$ remains true, the only change in the proof being that the positive factor $f(M_1(\tau), M_0(\tau))$ will occur in (3.20).

As to the quantities $\mathrm{d}m_j/\mathrm{d}t$ and $\mathrm{d}M_j/\mathrm{d}t$, their couplings with $c(t)$ and $C(t)$ are now given by

$$\frac{\mathrm{d}m_j}{\mathrm{d}t} = -\alpha_j f(m_1(t), m_0(t))c(t)$$

(with a similar result for $\mathrm{d}M_j/\mathrm{d}t$) instead of (2.3), (2.4); methods of Section 3 now give that

$$\left|\frac{\mathrm{d}M_j(t)}{\mathrm{d}t}\right| \leqslant \left|\frac{\mathrm{d}m_j(t)}{\mathrm{d}t}\right|, \tag{4.3}$$

provided the function $(\Phi(m_0) - \Phi(m_1))/f(m_1, m_0)$ is decreasing (in the non-strict sense) in the positive direction of $I$ from $(M_1(t), M_0(t))$ onwards. Unfortunately, the latter condition is not satisfied in some of the more interesting cases with non-constant $f$, for example, when $\alpha_0 = \alpha_1 = -1$ and $f$ is given by (4.2) with $b = 0$. (If we modify equation (4.2) to $f(m_1, m_0) = a/(m_1 + d)$, $d > 0$, we do get (4.3) for all $t \geqslant 0$ for which the original solution exists if $d/m \geqslant 1$ $(N = 2)$, $d/m \geqslant (N-1)/N$ $(N \geqslant 3)$; here $m = |\omega(0)| = m_0(0) - m_1(0)$. Moreover, (4.3) holds for all sufficiently large $t$ if $N \geqslant 3$ or if $N = 2$ and $d/m \geqslant \frac{1}{2}$.)

(b) *Exterior Domains*

We finally consider the limiting case when $\omega_0(t) = \mathbb{R}^N$ for all $t$, that is, when $\omega(t) = \mathbb{R}^N \backslash \overline{\omega_1(t)}$ is the complement of a compact set. Thus $\partial\omega(t)$ just consists of

$\gamma_1(t)$ (assumed smooth). When $N \geq 3$ we then define $u$ harmonic in $\omega(t)$ by

$$u = 1 \quad \text{on } \gamma_1(t) \tag{4.4}$$

$$u(x) = O(|x|^{2-N}) \quad \text{as } |x| \to \infty. \tag{4.5}$$

This problem has a unique solution. In fact, if say $0 \notin \overline{\omega(t)}$ it reduces to an ordinary Dirichlet problem in a bounded domain for $v(x) = |x|^{2-N} u(x/|x|^2)$, the 'Kelvin transform' of $u$, since the condition (4.5) means precisely that the singularity of $v$ at the origin is removable. Also, it is easy to check that the solution of (4.4), (4.5) is identical to the limit or supremum of the solutions $u = u_R$ of (1.1) with $\omega_0(t) = \{x \in \mathbb{R}^N : |x| < R\}$ $(R \to \infty)$.

What has been said above also applies when $N = 2$, but then the solution of (4.4), (4.5) will be identically one, which does not give rise to any interesting moving-boundary problem. Thus, when $N = 2$ we must allow $u$ to have a singularity at infinity. Since $\log |x|$ is the simplest one, the most natural candidate for $u$ when $N = 2$ seems to be the harmonic function in $\omega(t)$ determined by

$$u = 0 \quad \text{on } \gamma_1(t), \tag{4.6}$$

$$u(x) = -\operatorname{Log} |x| + O(1) \quad \text{as } |x| \to \infty. \tag{4.7}$$

(Observe that the choice of constant in (4.6) is immaterial, so we select zero.)

With $u$ as above $(N \geq 2)$ we consider the moving-boundary problem in which $\gamma_1(t)$ moves with the velocity $\alpha_1 \nabla u$, $\alpha_1 \in \mathbb{R}$. Of course, it is enough to consider just $\alpha_1 = \pm 1$. The physical applications of this problem are essentially the same as in the doubly-connected case (with appropriate modifications). See also (DiBenedetto & Friedman, 1986; Howison, 1986), where however the condition (4.5) is replaced by one which keeps

$$\int_{\gamma_1(t)} \frac{\partial u}{\partial \nu} \, \mathrm{d}\gamma$$

fixed (as is the case when $N = 2$ and $u$ is given by (4.6), (4.7)).

Consider first the case in which $N \geq 3$ (the case when $N = 2$ turns out to be trivial). As in (2.2) the capacity of $\omega(t)$ (or rather $\overline{\omega_1(t)}$) is given by

$$c(t) = \int_{\omega(t)} |\nabla u|^2 \, \mathrm{d}x = \int_{\gamma_1(t)} \frac{\partial u}{\partial \nu} \, \mathrm{d}\gamma,$$

and the moving-boundary condition gives

$$\frac{\mathrm{d}m_1}{\mathrm{d}t} = -\alpha_1 \int_{\gamma_1(t)} \frac{\partial u}{\partial \nu} \, \mathrm{d}\gamma = -\alpha_1 c(t). \tag{4.8}$$

In place of (2.8) one now has

$$N^2 \rho_N^{2/N} \leq -c(t) \Phi(m_1(t)) \tag{4.9}$$

with $\Phi$ given by (2.17). Thus by (4.8)

$$N^2 \rho_N^{2/N} \leq \frac{1}{\alpha_1} \Phi(m_1(t)) \frac{\mathrm{d}m_1}{\mathrm{d}t} \tag{4.10}$$

and, by integration,

$$N^2\rho_N^{2/N}t \leqslant \frac{1}{\alpha_1}(\psi(m_1(t)) - \psi(m_1(0))) \tag{4.11}$$

(with $\psi$ given by (2.18)).

We also consider the symmetrized problem determined by

$$M_1(0) = m_1(0),$$

and for this we have equalities in (4.9) to (4.11). Thus we can write (4.11) as

$$t = F(M_1(t)) \leqslant F(m_1(t)), \tag{4.12}$$

where

$$F(m_1) = \frac{N^{-2}\rho_N^{-2/N}}{\alpha_1}(\psi(m_1) - \psi(m_1(0))).$$

As usual this gives

$$\begin{aligned} M_1(t) &\geqslant m_1(t) \quad \text{if } \alpha_1 > 0,\\ M_1(t) &\leqslant m_1(t) \quad \text{if } \alpha_1 < 0, \end{aligned} \tag{4.13}$$

for all $t \geqslant 0$ for which the original solution exists. Observe by (4.12) that $F$ is now simply the inverse function of $M_1(t)$.

From (4.12), (2.18) we see that

$$\sup F = \begin{cases} \lim_{m_1 \to 0} F(m_1) = \dfrac{\rho_N^{-2/N}}{2(N-2)\alpha_1} m_1(0)^{2/N} < \infty & \text{if } \alpha_1 > 0, \\ \lim_{m_1 \to \infty} F(m_1) = +\infty. & \text{if } \alpha_1 < 0. \end{cases} \tag{4.14}$$

Hence the solution of the symmetrized problem will have a stopping time $T = \sup F$ given by (4.14) if $\alpha_1 > 0$ while it exists for all time if $\alpha_1 < 0$.

Going back to (4.9) and its counterpart for the symmetrized solution (and using (2.17)) we get

$$C(t)M_1(t)^{(2/N)-1} \leqslant c(t)m_1(t)^{(2/N)-1}. \tag{4.15}$$

If $\alpha_1 < 0$ this gives, by (4.13),

$$C(t) \leqslant c(t)$$

and hence, by (4.8),

$$\frac{\mathrm{d}M_1}{\mathrm{d}t} \leqslant \frac{\mathrm{d}m_1}{\mathrm{d}t}$$

for all $t \geqslant 0$ for which the original solution exists (for $\alpha_1 > 0$ we do not get anything).

Let us briefly consider the case in which $N = 2$ with $u$ given by (4.6), (4.7). We then still have the first part of (4.8), namely

$$\frac{\mathrm{d}m_1}{\mathrm{d}t} = -\alpha_1 \int_{\gamma_1(t)} \frac{\partial u}{\partial \nu}\,\mathrm{d}\gamma,$$

but now the right-hand member is constant (and equal to $-2\pi\alpha_1$) so we simply obtain

$$M_1(t) = m_1(t) = m_1(0) - 2\pi\alpha_1 t$$

for all $t \geq 0$. (The same is true (with other constants) for $N \geq 3$ when $u$ is defined as in (DiBenedetto & Friedman, 1986; Howison, 1986).) Equation (4.7) can be written

$$u(x) = -\mathrm{Log}\frac{|x|}{c(t)} + O(|x|^{-1}) \quad \text{as } |x| \to \infty,$$

where $c(t)$ is uniquely determined and called the capacity of $\omega(t)$ (or rather of $\omega_1(t)$). From $M_1(t) = m_1(t)$ one easily derives

$$C(t) \leq c(t) \tag{4.16}$$

for all $t \geq 0$ (whatever $\alpha_1$ is). In fact, there is a conformal map of $\{z \in \mathbb{C} : |z| < 1\}$ onto $\omega(t) \cup \{\infty\}$ of the form

$$f(z) = c(t)\left(\frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \ldots\right)$$

and using Green's formula one obtains

$$m_1(t) = \frac{1}{2\mathrm{i}}\int_{\partial\omega_1} \bar{w}\,\mathrm{d}w = -\frac{1}{2\mathrm{i}}\int_{|z|=1} \overline{f(z)} f'(z)\,\mathrm{d}z$$
$$= \pi c(t)^2\left(1 - \sum_{n=1}^{\infty} n\,|b_n|^2\right),$$

from which (4.16) follows (with equality only if $b_1 = b_2 = \ldots = 0$, that is, only if $\omega(t)$ is the exterior of a disc). Equation (4.16) can also be proved using rearrangement techniques as in (Mossino, 1984).

REMARK It is proved in (DiBenedetto & Friedman, 1986; Howison, 1986) that, when $\alpha_1 < 0$, $\mathbb{R}^N$ is emptied (that is, $\bigcap_{t>0} \omega(t) = \varnothing$) by a solution $\omega(t)$ of the exterior problem if and only if $\omega(0)$ is the complement of an ellipsoid ($N \geq 2$) (then also $\omega(t)$ are complements of ellipsoids for all $t > 0$). It would be interesting to have a result of this kind also in the doubly-connected case (when $\alpha_0$, $\alpha_1$ are both negative). One conjecture is that whenever $\omega(0)$ is not a spherical-shell domain the solution breaks down strictly before $(m_1(t), m_0(t))$ leaves $D$ (3.1). In general it seems hard to construct explicit solutions (besides the symmetrical ones) in the doubly-connected case. One explicit solution, which breaks down in finite time due to development of cusps on $\partial\omega(t)$, is constructed in (Gustafsson, 1986). For the exterior problem several examples are given in (Howison, 1986).

## 5. Conclusion

We have applied isoperimetric inequalities to a moving-boundary problem arising in electrochemistry, Hele Shaw flows, etc. Concerning the electrochemical model, the two electrodes of an electrolytic cell are assumed to grow or dissolve (at different rates in general) by electrochemical reaction, and we have found for example that the least change of volume of the electrodes and the electrolyte always occurs in spherical symmetry. From this, also, optimal upper bounds for the lifetime of a solution were obtained in certain cases, namely in those cases in which the corresponding symmetrized solution breaks down in finite time.

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