Laplacian growth on branched Riemann surfaces

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February 10, 2018

Abstract

We study non-univalent solutions of the Polubarinova-Galin equation, describing the time evolution of the conformal map from the unit disk onto a Hele-Shaw blob of fluid subject to injection at one point. This moving boundary problem is also called Laplacian growth. In particular, we tackle the difficulties arising when the map is not even locally univalent, in which case one has to pass to weak solutions developing on a branched covering surface of the complex plane.

One major concern is the construction of this Riemann surface, which is not given in advance but has to be constantly up-dated along with the solution. Once the Riemann surface is constructed the weak solution is automatically global in time, but we have had to leave open the question whether the weak solution can be kept simply connected all the time (as is necessary to connect to the Polubarinova-Galin equation). A certain crucial statement, a kind of stability statement for free boundaries, has therefore been left as a conjecture only.

Another major part of the paper concerns the structure of rational solutions (as for the derivative of the mapping function). Here we have fairly complete results on the dynamics. Several examples are given.

Keywords: Hele-Shaw flow, weighted Hele-Shaw flow, Laplacian growth, Polubarinova-Galin equation, Löwner-Kufarev equation, Löwner chain, subordination, quadrature Riemann surface, Abelian domain, algebraic domain, contractive zero divisor, partial balayage.

MSC: 30C20, 31C12, 34M35, 35R37, 76D27.

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1 Introduction

1.1 General

This paper is a continuation of [10], in which the motion of zeros and poles associated to locally univalent solutions of the Polubarinova-Galin equation was studied. This differential equation, in one real variable (time) and one complex variable, describes the time evolution of a conformal map from the unit disk onto a growing blob of a viscous fluid squeezed between two parallel plates. The two-dimensional view of the fluid blob is modeled by a domain in the complex plane, and its growth is assumed to be caused by a source at the origin. The history of this problem goes back to an experiment and a subsequent paper [20] by Henry Selby Hele-Shaw in 1898, and the literature on it is by now quite considerable. A short selection is [6], [25], [42], [32], [28], [38], [8], [40], [27], [21], [3], [23], [24], [19], [44], [16], [22], [1], [30], [15]. The evolution process has many other physical interpretations, besides Hele-Shaw flows, and one common name for it is Laplacian growth. For the history of the subject we refer to [41].

In the present paper we try to extend previous results on univalent and locally univalent solutions, in particular those in [10], to the setting of conformal maps which are not even locally univalent. Such mappings are then considered as univalent maps onto subdomains of a suitable Riemann surface, a branched covering surface of the complex plane.

The Polubarinova-Galin equation for a time dependent normalized conformal map $f(\cdot,t):\mathbb{D}\to\Omega(t)$, where $\Omega(t)$ is the fluid domain at time t, reads

$$\operatorname{Re}\left[\dot f(\zeta,t)\overline{\zeta f'(\zeta,t)}\right]=q(t)\quad\text{for } \zeta\in\partial\mathbb{D}, \tag{1.1}$$

where $q(t)>0$ is the source strength and the normalization means that $f(0,t)=0$, $f'(0,t)>0$. The equation expresses more exactly that the speed of the boundary $\partial\Omega(t)$ in the outward normal direction equals $2\pi q(t)$ times the harmonic measure (or normal derivative of Green's function) with respect to the origin (interpret the left member as $|f'|$ times the inner product between $\dot{f}$ and the unit normal vector $\zeta f'/|f'|$).

The classical case is that f is univalent, but in this paper we shall allow arbitrary functions f, analytic in some neighborhood of the closed unit disk and subject to the above normalization. Then it turns out that it is appropriate to add to (1.1) the requirement that

$$\frac{d}{dt}f(\omega(t),t) = 0 \tag{1.2}$$

for every zero $\omega(t)$ of $f'(\cdot,t)$ inside $\mathbb{D}$. With this requirement, (1.1) and (1.2) taken together become equivalent to an equation of Löwner-Kufarev type (see (2.7), (2.8) below) and it follows that the family $f(\cdot,t)$ (for t in some interval) becomes a subordination chain. This entails that there exists a Riemann surface $\mathcal{M}$ such that the $f(\cdot,t)$ become univalent as mappings into $\mathcal{M}$. As such mappings we put a tilde on the names of functions and domains:

$$\tilde{f}(\cdot,t) : \mathbb{D} \to \mathcal{M}, \quad \tilde{\Omega}(t) = \tilde{f}(\mathbb{D},t).$$

Thus we obtain a Hele-Shaw evolution on a Riemann surface.

For (1.1), (1.2) we start with some given $f(\cdot,0)$, as initial condition. Even if this is taken to be univalent it may happen that zeros of $f'(\cdot,t)$ reach $\partial\mathbb{D}$ and try to enter $\mathbb{D}$. This causes problems for (1.1) when $q > 0$, and one has to pass to a weak solution in order to allow a zero to make the transition into $\mathbb{D}$. It turns out that the transition is indeed possible, but the solution will then not be smooth in time.

A major case under consideration will be when $f'(\zeta,0)$ is a rational function. Then $f'(\zeta,t)$ will remain a rational function for $t > 0$, but when a zero of f' passes through $\partial\mathbb{D}$ it turns out the structure of this rational function changes. In the simplest case it will acquire two new zeros and one pole of order two. The behavior is quite interesting, and it connects to the theory of contractive divisors on Bergman space [18].

So a considerable part of the paper deals with the structure of rational solutions. Another part of the paper, which however is incomplete at present, is the study of global in time solutions. The main assertion then is that, given any $f(\cdot,0)$, there exists a weak solution of (1.1), (1.2) defined for all $0 \le t < \infty$. The proof of this requires the construction of the appropriate

Riemann surface $\mathcal{M}$. This is a not completely trivial task because $\mathcal{M}$ has to be constructed along with the solution: every time a zero of $f'(\cdot,t)$ reaches $\partial\mathbb{D}$ the Riemann surface has to be updated with a new branch point (as a covering surface of the complex plane).

What we have at present is a complete proof, except for an isolated technical difficulty which still remains. More exactly, what we would need to prove is the conjecture stated below. The prerequisits for the conjecture are as follows.

Let g be a function analytic in a neighborhood of the closed unit disk. From the theory of quadrature domains [33], [34], or the related theory of partial balayage [13], [7], it follows that for every $t>0$ sufficiently small there exists a domain $D(t)\supset\mathbb{D}$, uniquely determined up to a null-set and compactly contained in the region of analyticity of g, such that (with $dm = dxdy$)

$$\int_{D(t)} h|g|^2 dm \geq \int_{\mathbb{D}} h|g|^2 dm + th(0)$$

for every function h which is subharmonic and integrable (with respect to $|g|^2 m$) in $D(t)$. What we will need is that this $D(t)$ is simply connected if $t>0$ is sufficiently small. In a slightly stronger form this is our conjecture:

Conjecture 1.1. *If $t>0$ is sufficiently small, then $D(t)$ is star-shaped with respect to the origin, in particular simply connected.*

If $g\neq 0$ on $\partial\mathbb{D}$, so that $|g|^2\geq c>0$ in a neighborhood of $\partial\mathbb{D}$, then Conjecture 1.1 holds, as a consequence of results on stability of free boundaries in [2], [5], for example. However, we need to use Conjecture 1.1 exactly when g has zeros on $\partial\mathbb{D}$. Even in that case there are strong intuitive and analytic support for Conjecture 1.1, but no rigorous proof that we are aware of. The difficulty of proving a statement like Conjecture 1.1 was recognized already by M. Sakai [35] (Section 5 there).

Despite the fact that we have not been able to settle the above conjecture we believe that the remaining parts of the paper contains enough interesting material to deserve publication, at least in preprint form. We consider the study of rational solutions (general structure, motion of zeros and poles) together with the general set-up for lifting non-univalent solutions to a Riemann surface, which is not *a priori* given, as our our main achievements. In addition, there are a number of enlightening examples.

1.2 Contents of paper

A more detailed description of the contents of the paper goes as follows. In Section 2 we review the set-up and terminology in the locally univalent case, following essentially [10]. Section 3 discusses Löwner chains and subordination, based on corresponding material in [26], and also clarifies the relationship between the Polubarinova-Galin equation and the corresponding equation of Löwner-Kufarev type in the non-locally univalent case.

Since the global solutions we are looking for will not be smooth in general we have to discuss weak solutions, of variational inequality type, which can be formulated in terms of quadrature domains for subharmonic functions, in the spirit of Sakai [33], or else in terms of partial balayage. These notions are explained in some detail in Section 4, in the planar case, and the corresponding Riemann surface versions are developed in Section 5.

Partial balayage can be considered as an orthogonal projection in a Hilbert space, and when performing this on a covering surface the question arises to what extent it commutes with the projection map which pushes, for example, measures on the covering space down to measures on the base space. An affirmative answer to this question is given in Section 6.

The main assertion concerning global in time weak solutions which stay simply connected all the time is stated in Section 7, and is proved with Conjecture 1.1 taken as an assumption. In Section 9 we elaborate in detail three different solutions of the Polubarinova-Galin equation starting out from an cardioid. Quite surprisingly, one of these solutions can be driven backward in time, as a solution representing suction out of a cardioid. However, this is at the price of allowing a pole inside the unit disk, so the interpretation of this solution remains somewhat unclear. In Section 10 we still make an attempt to explain this kind of matter in a more general setting. In Section 8, finally, we expose the general structure of rational solutions (the derivative of the mapping function being rational), and set up the motion of zeros and poles as a dynamical system. Particular emphasis is given to how the structure changes when a zero crosses the unit circle.

1.3 Dedication

We would like to dedicate this paper to the memory of *Makoto Sakai*, who passed away in December 2013, at the age of 70. Sakai made original and groundbreaking contributions in potential theory, highly relevant to the contents of the present paper. He was one of the creators of the theory of quadrature domains, and his books [33], [37] on the subject will have a long-standing impact on the development of the subject, and on potential theory and its applications in general. His papers and books are not always easy to read, but they are very sharp, and Sakai's work is now receiving increasing recognition in the general mathematical community.

The present paper is closely related to one of Sakai's least known papers, namely [35]. Sakai always wanted to obtain complete and sharp result, and he was usually successful in this respect. However, in [35] he did not reach that full perfection, he had to make a probably unnecessary assumption (that a certain domain has no cusps on its boundary) in order to prove his main result. Ironically, the authors of the present paper were stopped on essentially the same mathematical difficulty, which we now have left as a conjecture.

1.4 Acknowledgements

The authors are grateful to Michiaki Onodera for important information and discussions.

The authors also would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme 2014-FRB: Free Boundary Problems and Related Topics, where work on this paper was undertaken. This work was supported by EPSRC grant no. EP/K032208/1.

2 Preparatory material

2.1 List of notations

We here list some notations which will be used, but not always further explained, in the paper.

- $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, $\mathbb{D}(a, r) = \{\zeta \in \mathbb{C} : |\zeta - a| < r\}$.
- $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ = the Riemann sphere.
- $\Omega^e = \mathbb{C} \setminus \overline{\Omega}$, the exterior of a domain Ω in $\mathbb{C}$, P, or in an ambient space in general (depending on context).

- $dm = dm(z) = dx \wedge dy = \frac{1}{2\mathrm{i}} d\bar{z} \wedge dz$ ($z = x + \mathrm{i}y$), area measure in the z plane.
- $\omega^* = 1/\overline{\omega}$, for $\omega \in \mathbb{C}$.
- $h^*(\zeta) = \overline{h(1/\bar{\zeta})} = \sum_{j=1}^{m} \bar{b}_j \zeta^{-j}$, where $h(\zeta, t) = \sum_{j=1}^{m} b_j \zeta^j$.

  (There is a slight ambiguity in this notation: we have $\zeta^* = 1/\bar{\zeta}$ if ζ is considered as a point, whereas $f^*(\zeta) = 1/\zeta$ for the function $f(\zeta) = \zeta$.)
- $\dot{f}(\zeta, t) = \frac{\partial}{\partial t} f(\zeta, t)$, $f'(\zeta, t) = \frac{\partial}{\partial \zeta} f(\zeta, t)$.
- With $E \subset \mathbb{C}$ any set which contains the origin,

$$\begin{aligned}
\mathcal{O}(E) =& \{f : f \text{ is analytic in some neighborhood of } E\}, \\
\mathcal{O}_{\mathrm{norm}}(E) =& \{f \in \mathcal{O}(E) : f(0) = 0, f'(0) > 0\}, \\
\mathcal{O}_{\mathrm{locu}}(E) =& \{f \in \mathcal{O}_{\mathrm{norm}}(E) : f' \neq 0 \text{ on } E\}, \\
\mathcal{O}_{\mathrm{univ}}(E) =& \{f \in \mathcal{O}_{\mathrm{locu}}(E) : f \text{ is univalent (one-to-one) on } E\}.
\end{aligned}$$

- card = ‘number of elements in’.
- $SL^1(\Omega, \lambda)$ denotes the set of subharmonic functions in Ω which are integrable with respect to a measure λ.
- ν_f: counting function, see Definition 3.1.
- $\mathrm{Bal}\,(\mu, \lambda)$: partial balayage, see Definition 4.2.
- $\mathrm{supp}\,\nu$: the closed support of a measure, or distribution, ν.
- χ_E: the characteristic function of a set E.

2.2 Basic set up in the univalent case

The **Polubarinova-Galin equation** is the dynamical equation for the conformal map from the unit disk onto a domain in the complex plane representing the two-dimensional view of a blob of a viscous fluid, which grows or shrinks due to the presence of a source or sink at one point, chosen to be the origin. The type of flow in question, actually incompressible potential flow in the two dimensional picture, is traditionally called **Hele-Shaw flow** (see

[41], [15] for historical accounts), and in recent time also **Laplacian growth**, referring to the moving boundary problem.

A smooth map $t \mapsto f(\cdot, t) \in \mathcal{O}_{\rm univ}(\overline{\mathbb{D}})$ is a **(strong) solution** of the Polubarinova-Galin equation if it satisfies

$$\mathrm{Re}\left[\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}\right] = q(t) \quad \text{for } \zeta \in \partial\mathbb{D}. \tag{2.1}$$

Here $q(t)$ is a real-valued function, which is given in advance and which represents the strength of the source/sink. Typically $q = \pm 1$, which corresponds to injection (plus sign) or suction (minus sign) at a rate 2π. Since the transformation $t \mapsto -t$ changes q to $-q$ in (2.1) it is enough to discuss one of the cases $q > 0$ and $q < 0$. In general we shall take $q > 0$. To increase flexibility we allow q to depend on time, and occasionally also to vanish. The meaning of (2.1) in terms of the Green's function of the domain is shown in equation (5.8) below.

Equation (2.1) expresses that the image domains $\Omega(t) = f(\mathbb{D}, t)$ evolve in such a way that

$$\frac{d}{dt}\int_{\Omega(t)} h dm = 2\pi q(t) h(0) \tag{2.2}$$

for every function h which is harmonic in a neighborhood of $\overline{\Omega(t)}$. This means that the speed of the boundary $\partial\Omega(t)$ in the normal direction equals $q(t)$ times the normal derivative of the Green's function of $\Omega(t)$ with a pole at $z = 0$. The equivalence between (2.1) and (2.2) follows from the general formula

$$\frac{d}{dt}\int_{\Omega(t)} \varphi\, dm = \int_{\partial\mathbb{D}} \varphi(f(\zeta,t))\mathrm{Re}\left[\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}\right] d\theta \quad (\zeta = e^{\mathrm{i}\theta}), \tag{2.3}$$

valid for any smooth evolution $t \mapsto f(\cdot, t) \in \mathcal{O}_{\rm univ}(\overline{\mathbb{D}})$ and for any smooth test function φ in the complex plane. Cf. Lemma 4.1 below.

On choosing $h(z) = z^k$, $k = 0, 1, 2, \dots$ in (2.2) it follows that the harmonic moments

$$M_k(t) = \frac{1}{\pi}\int_{\Omega(t)} z^k dm(z) = \frac{1}{2\pi \mathrm{i}}\int_{\partial\mathbb{D}} f(\zeta,t)^k f^*(\zeta,t) f'(\zeta,t) d\zeta \tag{2.4}$$

are conserved quantities, except for the first one, which by (2.2) is related to $q(t)$ by $\frac{d}{dt}M_0(t) = 2q(t)$. Thus

$$M_0(t) = M_0(0) + 2Q(t), \tag{2.5}$$

where $Q(t)$ is the accumulated source up to time $t > 0$:

$$Q(t) = \int_0^t q(s)ds. \tag{2.6}$$

Occasionally we may use $Q(t)$ also for $t < 0$, in which case it is negative (if $q > 0$).

Within the class of smooth (or monotone) evolutions of simply connected domains, Laplacian growth is characterized by the preservation of the moments $M_1, M_2, \dots$. This is a consequence of Theorem 10.13 and Corollary 10.14 in [33]. See also Theorem 6.2 and Corollary 6.3 in [9].

One may consider the equation (2.1) on different levels of generality. It is natural to keep the normalization $f(0) = 0$, $f'(0) > 0$, in fact the coupling to (2.2) depends on this, but (2.1) makes sense for any $f \in \mathcal{O}_{\rm norm}(\overline{\mathbb{D}})$, at least as long one makes sure that $q(t) = 0$ whenever a zero of f' appears on $\partial\mathbb{D}$. In the locally univalent case, $f \in \mathcal{O}_{\rm locu}(\overline{\mathbb{D}})$, the mathematical treatment of (2.1) is exactly the same as in the 'physical' case $f \in \mathcal{O}_{\rm univ}(\overline{\mathbb{D}})$. We shall then speak of a **locally univalent solution** of the Polubarinova-Galin equation.

When $f \in \mathcal{O}_{\rm locu}(\overline{\mathbb{D}})$, then $\dot{f}/\zeta f' \in \mathcal{O}(\overline{\mathbb{D}})$ and equation (2.1) can be solved for $\dot{f}$ by dividing both sides with $|\zeta f'|^2$. The result is an equation which we shall refer to as the **Löwner-Kufarev equation**, namely

$$\dot{f}(\zeta, t) = \zeta f'(\zeta, t) P(\zeta, t) \quad (\zeta \in \mathbb{D}), \tag{2.7}$$

where $P(\zeta, t)$ is the analytic function in $\mathbb{D}$ whose real part has boundary value $q(t)|f'(\zeta, t)|^{-2}$ and which is normalized by $\operatorname{Im} P(0, t) = 0$. Explicitly $P(\zeta, t)$ is given by

$$P(\zeta, t) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{q(t)}{|f'(z, t)|^2} \frac{z + \zeta}{z - \zeta} \frac{dz}{z} \quad (\zeta \in \mathbb{D}). \tag{2.8}$$

When $f \in \mathcal{O}_{\rm locu}(\overline{\mathbb{D}})$ then $P \in \mathcal{O}(\overline{\mathbb{D}})$, in fact the right member of (2.7) extends analytically as far as f does (see [8]). We shall keep the notation $P = P(\zeta, t)$ also for the analytic extension of the Poisson integral beyond $\overline{\mathbb{D}}$.

As a general notation throughout the paper, we set

$$g(\zeta, t) = f'(\zeta, t). \tag{2.9}$$

The function g in fact turns out to be more fundamental than f itself. Of course, f can be recaptured from g by

$$f(z, t) = \int_0^z g(\zeta, t) d\zeta. \tag{2.10}$$

Part of the paper will deal with the case that g is a rational function, or perhaps better to say, $g\,d\zeta$ is a rational differential, in other words an Abelian differential on the Riemann sphere. If g has residues then f will have logarithmic poles, besides ordinary poles. The terminology **Abelian domain** for the image domain $\Omega = f(\mathbb{D})$ has been used [40] for this case. Alternatively one may speak of Ω being a **quadrature domain** (see [14] for the terminology and further references), which in the present case means that a finite quadrature identity of the kind

$$\int_\Omega h(z)dxdy = \sum_{j=1}^{r} c_j \int_{\gamma_j} h(z)dz + \sum_{j=0}^{\ell}\sum_{k=1}^{n_j-1} a_{jk}h^{(k-1)}(z_j) \tag{2.11}$$

holds for integrable analytic functions h in Ω. Here the z_j are fixed (i.e., independent of h) points in Ω, with specifically $z_0 = 0$, the c_j, a_{jk} are fixed coefficients, and the γ_j are arcs in Ω with end points among the z_j. This sort of structure is stable under Hele-Shaw flow because, as is seen from (2.2), what happens under the evolution is only that the right member is augmented by the term $2\pi Q(t)h(0)$, where $Q(t)$ is the accumulated source up to time t, see (2.6).

When g is rational we shall write it on the form

$$g(\zeta,t) = b(t)\frac{\prod_{k=1}^{m}(\zeta-\omega_k(t))}{\prod_{j=1}^{n}(\zeta-\zeta_j(t))} = b(t)\frac{\prod_{i=1}^{m}(\zeta-\omega_i(t))}{\prod_{j=1}^{\ell}(\zeta-\zeta_j(t))^{n_j}}. \tag{2.12}$$

Here $m \geq n = \sum_{j=1}^{\ell} n_j$, $|\zeta_j| > 1$ and repetitions are allowed among the ω_k, ζ_j to account for multiple zeros and poles. Then, with the argument of $b(t)$ chosen so that $g(0,t) > 0$, $f \in \mathcal{O}_{\mathrm{locu}}(\overline{\mathbb{D}})$ if and only if $|\omega_k| > 1$, $|\zeta_j| > 1$ for all k and j. The assumption $m \geq n$ means that $g\,d\zeta$, as a differential, has at least a double pole at infinity, which the Hele-Shaw evolution in any case will force it to have because the source/sink at the origin creates a pole of f at infinity.

The form (2.12) is stable in time, with the sole exception that when $m = n$ the pole of f at infinity may disappear at one moment of time (see [10], or Proposition 8.1 below). The rightmost member of (2.12) will be used when we need to be explicit about the orders of the poles. The convention then is that $\zeta_1, \dots, \zeta_\ell$ are distinct and $n_j \geq 1$. Thus $n = \sum_{j=1}^{\ell} n_j$, and in the full sequence $\zeta_1, \dots, \zeta_n$, the tail $\zeta_{\ell+1}, \dots, \zeta_n$ will be repetitions of (some of) the $\zeta_1, \dots, \zeta_\ell$ according to their orders. In equations (2.11) and (2.12), ℓ and the n_j are the same.

One can easily express the Löwner-Kufarev equation (2.7) directly in terms of g. In fact, on writing P_g for the Poisson integral in (2.8), equation (2.7) together with (2.10) is equivalent to

$$\frac{\partial}{\partial t}\log g(\zeta,t)=\zeta P_g(\zeta,t)\frac{\partial}{\partial\zeta}\log g(\zeta,t)+\frac{\partial}{\partial\zeta}(\zeta P_g(\zeta,t)). \tag{2.13}$$

When g is rational, as in (2.12), also $P_g(\zeta,t)$ will be a rational function (see more precisely (8.9) in Section 8 below), and so will the derivatives of $\log g$: we have

$$\log g(\zeta,t)=\log b(t)+\sum_{k=1}^{m}\log(\zeta-\omega_k(t))-\sum_{j=1}^{n}\log(\zeta-\zeta_j(t)), \tag{2.14}$$

$$\frac{\partial}{\partial t}\log g(\zeta,t)=\frac{\dot b(t)}{b(t)}-\sum_{k=1}^{m}\frac{\dot\omega_k(t)}{\zeta-\omega_k(t)}+\sum_{j=1}^{n}\frac{\dot\zeta_j(t)}{\zeta-\zeta_j(t)}, \tag{2.15}$$

$$\frac{\partial}{\partial\zeta}\log g(\zeta,t)=\sum_{k=1}^{m}\frac{1}{\zeta-\omega_k(t)}-\sum_{j=1}^{n}\frac{1}{\zeta-\zeta_j(t)}. \tag{2.16}$$

Thus (2.13) becomes an identity between rational functions.

3 Dynamics and subordination

3.1 Generalities

In the non locally univalent case the Polubarinova-Galin and Löwner-Kufarev equations are no longer equivalent. The Löwner-Kufarev equation is the stronger one, and solutions to it can still be viewed as univalent mapping functions, but then onto subdomains of a Riemann surface. The evolution of these subdomains is monotone, which amounts to saying that the function family is a subordination chain. Solutions to the more general Polubarinova-Galin equation are not unique, but still satisfy a weaker form of monotonicity, namely monotonicity of the counting function.

Definition 3.1. For any $f\in\mathcal{O}(\overline{\mathbb{D}})$, the **counting function**, or mapping degree, ν_f of f tells how many times a value $z\in\mathbb{C}$ is attained by f in $\mathbb{D}$. It is an integer valued function defined almost everywhere in $\mathbb{C}$ (namely outside $f(\partial\mathbb{D})$) by

$$\nu_f(z) = \operatorname{card}\{\zeta \in \mathbb{D} : f(\zeta) = z\} = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} d\log\left(f(\zeta) - z\right). \tag{3.1}$$

Clearly, f is univalent in $\mathbb{D}$ if and only if $0 \leq \nu_f \leq 1$.

Definition 3.2. Let $f, g \in \mathcal{O}_{\text{norm}}(\mathbb{D})$. We say that f is **subordinate** to g, and write $f \prec g$, if there exists a univalent function $\varphi : \mathbb{D} \to \mathbb{D}$ such that $f = g \circ \varphi$. Note that φ is automatically normalized, hence $\varphi \in \mathcal{O}_{\text{univ}}(\mathbb{D})$.

Let $I \subset [0, \infty)$ be any interval. A map $I \ni t \mapsto f(\cdot, t) \in \mathcal{O}_{\text{norm}}(\mathbb{D})$ is called a **subordination chain** on I if $f(\cdot, s) \prec f(\cdot, t)$ whenever $s \leq t$.

The following lemma shows that by increasing the level of abstraction (lifting the maps to a Riemann surface), subordination becomes nothing else than ordinary monotonicity. The result is not new (see [26] for the classical case of univalent f and g), but we give the proof because it will be a model for how our specific Riemann surfaces needed for the Hele-Shaw problem will be constructed.

Lemma 3.1. *Let $\{f(\cdot, t)\}_{t \in I} \subset \mathcal{O}_{\text{norm}}(\mathbb{D})$, where $I \subset [0, \infty)$ is any interval. Then the following are equivalent.*

(i) $\{f(\cdot, t)\}$ is a subordination chain on I.

(ii) There exists a Riemann surface $\mathcal{M}$, a nonconstant analytic function $p : \mathcal{M} \to \mathbb{C}$ ('covering map') and univalent analytic functions

$$\tilde{f}(\cdot, t) : \mathbb{D} \to \mathcal{M} \quad (t \in I)$$

('liftings' of the $f(\cdot, t)$) such that

(a) $f(\zeta, t) = p(\tilde{f}(\zeta, t))$;

(b) $\tilde{f}(\mathbb{D}, s) \subset \tilde{f}(\mathbb{D}, t)$ for $s \leq t$.

Proof. The proof that (ii) implies (i) is just a straight-forward verification, with the subordination functions defined by

$$\varphi(\zeta, s, t) = \tilde{f}^{-1}(\tilde{f}(\zeta, s), t) \tag{3.2}$$

for $s \le t$, and where $\tilde{f}^{-1}(\zeta, t)$ denotes the inverse of $\tilde{f}(\zeta, t)$ with respect to ζ.

To prove the opposite, assume (i). We have to construct the Riemann surface $\mathcal{M}$ and the covering map p. For each $t \in I$, let $\mathbb{D}_t$ be a copy of $\mathbb{D}$ and let $\mathcal{M}_t$ be $\mathbb{D}_t$ considered as an abstract Riemann surface (for which $\mathbb{D}_t$ serves as coordinate space). We define a covering map

$$p(\cdot, t) : \mathcal{M}_t \to \mathbb{C}$$

by declaring that, in the coordinate space $\mathbb{D}_t$, it shall be represented by

$$f(\cdot, t) : \mathbb{D}_t \to \mathbb{C}.$$

When $s \le t$ we have the embedding $\varphi(\cdot, s, t) : \mathbb{D}_s \to \mathbb{D}_t$ coming from the assumed subordination, which we on the level of the abstract Riemann surfaces consider as an inclusion map

$$\mathcal{M}_s \subset \mathcal{M}_t. \tag{3.3}$$

Note that these embeddings and inclusions commute with the covering maps because of the subordination relations

$$f(\varphi(\zeta, s, t), t) = f(\zeta, s) \quad (s \le t). \tag{3.4}$$

In view of the inclusions (3.3) we may define

$$\mathcal{M} = \cup_{t \in I} \mathcal{M}_t.$$

This is a Riemann surface because each point belongs to some $\mathcal{M}_t$, and there it has a neighborhood (e.g., all of $\mathcal{M}_t$) which can be identified with an open subset of the complex plane ($\mathcal{M}_t \cong \mathbb{D}_t \cong \mathbb{D}$), and the coordinates on $\mathcal{M}$ so obtained are related by invertible analytic functions (the $\varphi(\cdot, s, t)$). The covering map $p : \mathcal{M} \to \mathbb{C}$ is defined by declaring that on $\mathcal{M}_t$ it shall agree with $p(\cdot, t)$. Again, this is consistent.

Finally, the map $f(\cdot, t) : \mathbb{D} \to \mathbb{C}$ lifts to

$$\tilde{f}(\cdot, t) : \mathbb{D} \to \mathcal{M}_t \subset \mathcal{M}$$

by declaring that on identifying $\mathcal{M}_t$ with $\mathbb{D}_t$ it shall simply be the identity map: $\tilde{f}(\zeta, t) = \zeta \in \mathbb{D}_t$ for $\zeta \in \mathbb{D}$. Also this is consistent.

In the last picture, the evolution maps $\tilde{f}(\zeta, t)$ become trivial, while the covering maps are nontrivial ($p(\zeta, t) = f(\zeta, t)$):

$$\mathbb{D} \xrightarrow{\text{id}} \mathbb{D}_t \xrightarrow{f(\cdot,t)} \mathbb{C}.$$

For visualization it may however be better to have the view

$$\mathbb{D} \xrightarrow{\tilde{f}(\cdot,t)} \mathcal{M}_t \xrightarrow{\text{proj}} \mathbb{C}$$

in which the evolution maps $\tilde{f}(\zeta, t)$ really are liftings of the $f(\zeta, t)$, while the covering maps $p(\cdot, t)$ are trivial identifications (local identity maps, except at branch points).

Now, when $\mathcal{M}$ and p have been constructed the rest of the proof are easy verifications (omitted). □

Example 3.1. The functions

$$f(\zeta, t) = \frac{\zeta(t^3\zeta - 2t^2 + 1)}{\zeta - t}$$

can be shown to make up a non-univalent subordination family on the interval $1 < t < \infty$. The derivative $f'(\zeta, t)$ vanishes at $\zeta = t^{-1} \in \mathbb{D}$. The Riemann surface $\mathcal{M}$ appearing in Lemma 3.1 consists, when visualized as a covering surface over $\mathbb{C}$, of two copies of $\mathbb{C}$ joined by a branch point at $f(t^{-1}, t) = 1$. This example will be further discussed in Example 5.3, where also partial proofs of the above statements can be found.

Lemma 3.2. *For $f, g \in \mathcal{O}_{\text{norm}}(\mathbb{D})$, $f \prec g$ implies $\nu_f \leq \nu_g$ (almost everywhere).*

Proof. This is immediate from a change of variable in the integral appearing in ν_f: assuming $f = g \circ \varphi$ we have, using that φ is univalent,

$$\nu_f(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} d\log\left(f(\zeta) - z\right) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} d\log\left(g(\varphi(\zeta)) - z\right)$$

$$= \frac{1}{2\pi i} \int_{\varphi(\partial\mathbb{D})} d\log\left(g(\zeta) - z\right) \leq \frac{1}{2\pi i} \int_{\partial\mathbb{D}} d\log\left(g(\zeta) - z\right) = \nu_g(z).$$

□

3.2 The Polubarinova-Galin versus the Löwner-Kufarev equation

The relationship between the Polubarinova-Galin and the Löwner-Kufarev equations in the non-univalent case is the following.

Theorem 3.1. *Let $I \ni t \mapsto f(\cdot,t) \in \mathcal{O}_{\mathrm{norm}}(\overline{\mathbb{D}})$ be smooth on some time interval I and assume that $f' \neq 0$ on $\partial\mathbb{D}$ on this interval. Then for $q(t) \geq 0$ the following are equivalent.*

(i) $f(\zeta,t)$ solves the Löwner-Kufarev equation (2.7).

(ii) $f(\zeta,t)$ solves the Polubarinova-Galin equation (2.1) and $\dot{f}(\omega,t) = 0$ for every root $\omega \in \mathbb{D}$ of $f'(\omega,t) = 0$.

(iii) $f(\zeta,t)$ solves the Polubarinova-Galin equation (2.1) and $\{f(\cdot,t)\}$ is a subordination chain.

Remark 3.2. As a matter of terminology, if $\omega(t) \in \mathbb{D}$ is a zero of f', then $f(\omega(t),t)$ will be called a **branch point** of f, viewing f as a covering map. In a related terminology, it is branch point of the then multivalued lifting map f^{-1}. Since $\frac{d}{dt}f(\omega(t),t) = \dot{f}(\omega(t),t)$ when $f'(\omega(t),t) = 0$, the second condition in (ii) expresses that the branch points do not move.

Proof. The additional condition in (ii) means more precisely (taking multiplicities into account) that

$$\frac{\dot{f}(\zeta,t)}{\zeta f'(\zeta,t)} \in \mathcal{O}(\overline{\mathbb{D}}) \tag{3.5}$$

After dividing both members in (2.1) by $|\zeta f'(\zeta,t)|^2$ and using the defining properties (2.8), (2.9) of $P(\zeta,t)$ this condition is seen to be exactly what is needed to pass between (2.1) and (2.7). Thus (i) and (ii) are equivalent.

Assume next that (iii) holds. That $\{f(\cdot,t)\}$ is a subordination chain means that for $s \leq t$ there exist univalent functions $\varphi(\cdot,s,t) : \mathbb{D} \to \mathbb{D}$ such that (3.4) holds. By differentiating (3.4) with respect to t it immediately follows that (3.5) holds. Thus (iii) implies (ii).

We finally prove that (i) implies (iii). This is done exactly as in the corresponding proof for Löwner chains of univalent functions in Chapter 6 of [26]. To construct the subordination functions $\varphi(\cdot,s,t)$ one considers, for given $s\geq 0$ and $\zeta\in\mathbb{D}$, the initial value problem

$$\begin{cases}\frac{dw}{dt}=-wP(w,t), \quad t\geq s,\\ w(s)=\zeta.\end{cases} \tag{3.6}$$

It has a unique solution $w=w(t)$ defined on the time interval on which f, and hence P, is defined. In terms of $w(t)$ we then define, for $s\leq t$,

$$\varphi(\zeta,s,t)=w(t).$$

Since different trajectories for (3.6) never intersect $\varphi(\zeta,s,t)$ is a univalent function of ζ in the unit disk, and using the chain rule and (2.7) one sees that $\frac{d}{dt}f(\varphi(\zeta,s,t),t)=0$. Thus $f(\varphi(\zeta,s,t),t)$ is constantly equal to its initial value at $t=s$, which is $f(\zeta,s)$. This proves the subordination.

□

The Polubarinova-Galin equation itself is equivalent to a more general version of the Löwner-Kufarev equation, as follows.

Theorem 3.2. *Let $I\ni t\mapsto f(\cdot,t)\in\mathcal{O}_{\rm norm}(\overline{\mathbb{D}})$ be smooth on some time interval I and assume that $f'\neq 0$ on $\partial\mathbb{D}$ on this time interval. Then $f(\zeta,t)$ solves the Polubarinova-Galin equation (2.1) if and only if*

$$\dot f(\zeta,t)=\zeta f'(\zeta,t)\left(P(\zeta,t)+R(\zeta,t)\right), \tag{3.7}$$

where P is the Poisson integral (2.8) and where $R(\zeta,t)$ is any function of the form

$$R(\zeta,t)=-\mathrm{i}\,\mathrm{Im}\sum_{\omega_j\in\mathbb{D}}\sum_{k=1}^{r_j}\frac{2B_{jk}(t)}{(-\omega_j(t))^k}+\sum_{\omega_j\in\mathbb{D}}\sum_{k=1}^{r_j}\left(\frac{2B_{jk}(t)}{(\zeta-\omega_j(t))^k}-\frac{2\overline{B_{jk}(t)}\zeta^k}{(1-\overline{\omega_j(t)}\zeta)^k}\right). \tag{3.8}$$

Here $\{\omega_j\}$ are the zeros of f' in $\mathbb{D}$ (necessarily finitely many), r_j is the order of the zero at $\{\omega_j(t)\}$, and $B_{jk}(t)$ are arbitrary smooth functions of t.

Proof. The proof of Theorem 3.2 is immediate since the additional term $R(\zeta,t)$ satisfies

$$\mathrm{Re}\,R(\zeta,t)=0, \quad \zeta\in\partial\mathbb{D}, \tag{3.9}$$

$$\mathrm{Im}\,R(0,t)=0$$

and is allowed to contain exactly those kinds of singularities in $\mathbb{D}$ which will be killed by the factor f' in front of it in (3.7). The first term in (3.8) is just the normalization assuring that $\operatorname{Im} R(0,t) = 0$, and the other terms exchange polar parts between $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ without changing the real part on $\partial\mathbb{D}$.

$\square$

Remark 3.3. The term $R(\zeta, t)$ primarily regulates the motion of the branch points, but this means that it, indirectly, also affects the dynamics of the boundary $\partial\Omega(t)$. If $\omega_j \in \mathbb{D}$ is a simple zero, for example, then it follows from (3.7), (3.8) that B_{j1} is proportional to the speed of the corresponding branch point:

$$\frac{d}{dt} f(\omega_j(t), t) = 2\omega_j(t) f''(\omega_j(t), t) B_{j1}(t).$$

As for the dynamics of the boundary, we first remark that when $\zeta \in \partial\mathbb{D}$ is kept fixed the point $z = f(\zeta, t) \in \partial\Omega(t)$ generally does not move perpendicular to the boundary. On decomposing the speed into normal and tangential directions,

$$\dot{f} = \dot{f}_{\text{normal}} + \dot{f}_{\text{tangential}} = \operatorname{Re}(\dot{f} \cdot \frac{\overline{\zeta f'}}{|f'|}) \frac{\zeta f'}{|f'|} + \operatorname{Im}(\dot{f} \cdot \frac{\overline{\zeta f'}}{|f'|}) \frac{i\zeta f'}{|f'|},$$

one sees from (3.9) that, in the instantaneous picture, the term $R(\zeta, t)$ only affects the tangential component $\dot{f}_{\text{tangential}}$. Still the dynamics of $\partial\Omega(t)$ is influenced by $R(\zeta, t)$, as will be seen in examples in Section 9.

4 Weak solutions

4.1 Preliminaries and definition

Some of our main results will be formulated in terms of variational inequality weak solutions, just called weak solutions for short, which are expressed in terms of time independent test functions which are subharmonic in the domains $\Omega(t)$. We shall also need general smooth test functions, like Φ below. When such test functions are pulled back to the unit disk via the mapping functions f they become time dependent, and the time and space derivatives will be coupled. Indeed, if $\Phi(z)$ is any smooth function in $\mathbb{C}$ then, by the chain rule, the composed function $\Psi(\zeta, t) = \Phi(f(\zeta, t))$, defined for $\zeta \in \overline{\mathbb{D}}$, satisfies

$$|f'(\zeta,t)|^2\frac{\partial\Psi}{\partial t}=\dot f(\zeta,t)\overline{f'(\zeta,t)}\,\frac{\partial\Psi}{\partial\zeta}+\overline{\dot f(\zeta,t)}f'(\zeta,t)\frac{\partial\Psi}{\partial\bar\zeta}. \tag{4.1}$$

When working in $\mathbb{D}$ we shall need test functions Ψ which satisfy just (4.1) in itself, without necessarily being of the form $\Phi\circ f$ for some Φ.

Lemma 4.1. *For any smooth evolution $t\mapsto f\in\mathcal{O}_{\rm norm}(\overline{\mathbb{D}})$ and any smooth function $\Psi(\zeta,t)$ which satisfies (4.1) we have*

$$\frac{d}{dt}\int_{\mathbb{D}}\Psi(\zeta,t)|f'(\zeta,t)|^2\,dm(\zeta)=\int_{\partial\mathbb{D}}\Psi(\zeta,t)\mathrm{Re}\left[\dot f(\zeta,t)\overline{\zeta f'(\zeta,t)}\right]d\theta, \tag{4.2}$$

where $\zeta=e^{\mathrm{i}\theta}$ in the right member.

Proof. Differentiation under the integral sign gives, using (4.1),

$$\frac{d}{dt}\int_{\mathbb{D}}\Psi(\zeta,t)|f'(\zeta,t)|^2\,dm(\zeta)=\int_{\mathbb{D}}(\dot f\bar f'\frac{\partial\Psi}{\partial\zeta}+\dot{\bar f}f'\frac{\partial\Psi}{\partial\bar\zeta}+\Psi\dot f'\bar f'+\Psi f'\dot{\bar f}')\,dm$$

$$=\frac{1}{2\mathrm{i}}\int_{\mathbb{D}}(\bar f'\frac{\partial}{\partial\zeta}(\dot f\Psi)+f'\frac{\partial}{\partial\bar\zeta}(\dot{\bar f}\Psi))d\bar\zeta d\zeta=\frac{1}{2\mathrm{i}}\int_{\partial\mathbb{D}}\Psi(\dot{\bar f}f'd\zeta-\dot f\bar f'd\bar\zeta)$$

$$=\int_{\partial\mathbb{D}}\Psi(\zeta,t)\mathrm{Re}\left[\dot f(\zeta,t)\overline{\zeta f'(\zeta,t)}\right]d\theta.$$

□

Corollary 4.2. *For any smooth evolution $t\mapsto f(\cdot,t)\in\mathcal{O}_{\rm norm}(\overline{\mathbb{D}}))$ and any smooth function Φ in $\mathbb{C}$ we have*

$$\frac{d}{dt}\int_{\mathbb{C}}\Phi(z)\nu_{f(\cdot,t)}(z)dm(z)=\int_0^{2\pi}\Phi(f(\zeta,t))\mathrm{Re}\left[\dot f(\zeta,t)\overline{\zeta f'(\zeta,t)}\right]d\theta. \tag{4.3}$$

Proof. Pulling the left member back to the unit disk by means of f gives

$$\frac{d}{dt}\int_{\mathbb{C}}\Phi(z)\nu_{f(\cdot,t)}(z)\,dm(z)=\frac{d}{dt}\int_{\mathbb{D}}\Phi(f(\zeta,t))|f'(\zeta,t)|^2\,dm(\zeta).$$

Since the composed function $\Psi(\zeta,t)=\Phi(f(\zeta,t))$ satisfies (4.1) the corollary follows immediately from Lemma 4.1. □

Note that Corollary 4.2 is strictly weaker than Lemma 4.1. If for example $\nu_{f(\cdot,t)} = 2$ on some part of $\mathbb{C}$ then Lemma 4.1 allows the test function Ψ there to take different values on the two sheets of $f(\mathbb{D})$ lying above this part, which is not possible for the Φ in Corollary 4.2.

When $f(\zeta, t)$ solves the Polubarinova-Galin equation (2.1) we get

$$\frac{d}{dt} \int_{\mathbb{C}} \Phi(z)\nu_{f(\cdot,t)}(z)dm(z) = q(t) \int_0^{2\pi} \Phi(f(e^{\mathrm{i}\theta}, t))d\theta.$$

In particular, applying this to arbitrary $\Phi \geq 0$:

Corollary 4.3. *For any solution $t \mapsto f(\cdot, t) \in \mathcal{O}_{\mathrm{norm}}(\mathbb{D})$ of the Polubarinova-Galin equation (2.1) with $q(t) \geq 0$, $\nu_{f(\cdot,t)}$ is an increasing function of t.*

Specializing (4.3), on the other hand, to subharmonic and harmonic test functions (which we then denote h) we obtain, in view of the mean-value properties satisfied by such functions:

Corollary 4.4. *Let $t \mapsto f(\cdot, t) \in \mathcal{O}_{\mathrm{norm}}(\overline{\mathbb{D}})$ solve the Polubarinova-Galin equation (2.1) with $q(t) \geq 0$. Then*

$$\frac{d}{dt} \int_{\mathbb{C}} h\nu_{f(\cdot,t)}dm \geq 2\pi q(t)h(0)$$

for any h which is subharmonic in a neighborhood of $\operatorname{supp} \nu_f$. *If h is harmonic, equality holds.*

As a particular case we get the relevant version of moment conservation. Keeping the rightmost member of (2.4) as definition of the harmonic moments in the non-univalent case, so that

$$M_k(t) = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{D}} f(\zeta, t)^k |f'(\zeta, t)|^2 dm(\zeta) = \frac{1}{\pi} \int_{\mathbb{C}} z^k \nu_{f(\cdot,t)}(z)dm(z),$$

we have

$$\frac{d}{dt} M_k(t) = 0, \quad k = 1, 2, 3, \dots$$

under the assumptions of Corollary 4.4.

By integrating the inequality in Corollary 4.4 with respect to t we next obtain

Corollary 4.5. *Whenever $s \leq t$ and h is subharmonic in a neighborhood of* $\operatorname{supp} \nu_f(\cdot, t)$ *we have, when f solves the Polubarinova-Galin equation (2.1),*

$$\int_{\mathbb{C}} h\nu_{f(\cdot,t)} dm - \int_{\mathbb{C}} h\nu_{f(\cdot,s)} dm \geq 2\pi(Q(t) - Q(s))h(0), \tag{4.4}$$

where Q is the accumulated source (see (2.6)).

This corollary connects to a well-established notion of (variational inequality) weak solution for the Hele-Shaw problem with a source of strength $q(t) \geq 0$ at the origin. We formulate the definition first for domains (or open sets) in $\mathbb{C}$. It will later be extended to contexts of Riemann surfaces.

Definition 4.1. With $I \subset \mathbb{R}$ an interval (of any sort), a family of bounded open sets $\{\Omega(t) \subset \mathbb{C} : t \in I\}$ is a **weak solution** if for any $s, t \in I$ with $s \leq t$, $\Omega(s) \subset \Omega(t)$ and

$$\int_{\Omega(t)} h dm - \int_{\Omega(s)} h dm \geq 2\pi(Q(t) - Q(s))h(0) \tag{4.5}$$

holds for every h which is subharmonic and integrable in $\Omega(t)$. If the interval I is of the form $[0, T)$ (or $[0, T]$) then it is enough that (4.5) holds for $s = 0$ to have the full strength of (4.5).

Thus a solution of the Polubarinova-Galin equation is a weak solution as long as it is univalent, i.e., $0 \leq \nu_{f(\cdot,t)} \leq 1$. When $\nu_{f(\cdot,t)}$ takes values ≥ 2 it does not fit into the definition of a weak solution, but the exceeding parts of $\nu_{f(\cdot,t)}$ can still be swept out to produce a weak solution. This process, of partial balayage, will shortly be discussed in some detail.

Given any initial bounded open set $\Omega(0)$, a weak solution in the sense of Definition 4.1 always exists on the interval $I = [0, \infty)$, and it is unique up to nullsets. If $\Omega(0)$ is connected and $0 \in \Omega(0)$, then also $\Omega(t)$ is connected for all $t > 0$. However, the domains $\Omega(t)$ need not be simply connected all the time, hence may be out of reach for the Polubarinova-Galin equation, although they do become simply connected for large enough $Q(t)$. See [16], [15] and references therein.

4.2 Weak solutions in terms of balayage

The weak solution can be seen as an instance of a sweeping process called partial balayage, which under present circumstances results in quadrature domains for subharmonic functions. We formulate first this sweeping process in $\mathbb{C}$ and then adapt it to Riemannian manifolds according to our needs. Some general references are [13], [9], [39], [7], [29], [12].

The fixed data is a measure λ which (for the purpose of the present article) has a bounded density with respect to Lebesgue measure, i.e., satisfies $\lambda \leq Cm$ for some constant C, and on a sufficiently large set, say outside a compact set, moreover is bounded from below:

$$\lambda \geq cm \tag{4.6}$$

for some $c > 0$.

Definition 4.2. With λ as above, let μ be a positive Radon measure with compact support in $\mathbb{C}$. Then **partial balayage** of μ to λ is defined as

$$\operatorname{Bal}(\mu, \lambda) = \mu + \Delta u,$$

where u is the smallest non-negative locally integrable function satisfying

$$\mu + \Delta u \leq \lambda, \tag{4.7}$$

with Δu denoting the distributional Laplacian of u.

The assumption (4.6) guarantees that there exist functions $u \geq 0$ with compact support satisfying (4.7), and then it follows from general potential theory that a smallest such u exists, and also that it can be taken to be lower semicontinuous. See the above references. In particular, the result $\operatorname{Bal}(\mu, \lambda)$ of partial balayage will be a measure with compact support.

Assume for simplicity that also μ is absolutely continuous with respect to Lebesgue measure, say $d\mu = \rho\, dm$, and that $\lambda = m$. Then u is to be the smallest of all functions which satisfy

$$\begin{cases} u \geq 0, \\ \Delta u \leq 1 - \rho. \end{cases} \tag{4.8}$$

This statement constitutes an obstacle problem on ordinary form and it has a unique solution. This solution can also be characterized by the requirement that the two inequalities in (4.8) shall hold in the complementary sense

$$u(1 - \rho - \Delta u) = 0. \tag{4.9}$$

In general, $\mathrm{Bal}\,(\mu,\lambda)$ is squeezed between the two natural bounds,

$$\min\{\mu,\lambda\} \leq \mathrm{Bal}\,(\mu,\lambda) \leq \lambda,$$

and the more detailed structure is (under our assumptions) that

$$\mathrm{Bal}\,(\mu,\lambda) = \lambda\chi_\Omega + \mu\chi_{\mathbb{C}\setminus\Omega}. \tag{4.10}$$

Here Ω denotes the largest open set in which equality holds in (4.7), in other words, $\Omega = \mathbb{C} \setminus \mathrm{supp}\,(\lambda - \mathrm{Bal}\,(\mu,\lambda))$. It is called the **saturated set**, and it contains the noncoincidence set for the obstacle problem:

$$\{z \in \mathbb{C} : u(z) > 0\} \subset \Omega.$$

The inclusion may be strict, but under mild conditions the difference set is just a Lebesgue null-set.

In view of (4.10), the saturated set Ω contains all information of the result of partial balayage. Another characterization of this set, directly in terms of μ and λ, is as follows:

$$\mu < \lambda \text{ on } \mathbb{C} \setminus \Omega, \tag{4.11}$$

$$\int_\Omega h\,d\mu \leq \int_\Omega h\,d\lambda \text{ for all } h \in SL^1(\Omega,\lambda). \tag{4.12}$$

Here (4.11) shall be interpreted as saying that $\mathbb{C} \setminus \Omega \subset \mathrm{supp}\,((\lambda-\mu)_+)$, in other words that whenever $\mu \geq \lambda$ in some open set U it follows that $U \subset \Omega$.

Recall from Section 2.1 that $SL^1(\Omega,\lambda)$ denotes the set of subharmonic functions in Ω which are integrable with respect to λ. This class of test functions can, in (4.12), be replaced by just all logarithmic kernels $h(z) = \log|z-a|$ for $a \in \mathbb{C}$ together with all $h(z) = -\log|z-b|$ for $b \in \mathbb{C} \setminus \Omega$, see [33], [34]. With these test functions, (4.12) reduces to the statement that $u \geq 0$ in $\mathbb{C}$, $u = 0$ on $\mathbb{C} \setminus \Omega$, where u now denotes the logarithmic potential of $\mu\chi_\Omega - \lambda\chi_\Omega$ (so that $\Delta u = \lambda\chi_\Omega - \mu\chi_\Omega$). The proof of the equivalence between (4.10) and (4.11), (4.12) then becomes straight-forward, on noting in particular that the above u will be identical with the function u appearing in Definition 4.2.

We shall mostly consider $\mathrm{Bal}\,(\mu,\lambda)$ in cases when there exists an open set $D \subset \mathbb{C}$ such that $\mu \geq \lambda$ on D, $\mu = 0$ outside D. In such cases,

$$\mathrm{Bal}\,(\mu,\lambda) = \lambda\chi_\Omega. \tag{4.13}$$

When $\lambda = m$, (4.12) then expresses that Ω is a **quadrature domain for subharmonic functions** for μ. This means that $\mu = 0$ outside Ω and that

$$\int_\Omega h d\mu \leq \int_\Omega h dm \tag{4.14}$$

holds for all $SL^1(\Omega, m)$, see [33] for detailed information. In terms of partial balayage the weak solution $\Omega(t)$ at time t is obtained by

$$\mathrm{Bal}\,(2\pi Q(t)\delta_0 + \chi_{\Omega(0)}m, m) = \chi_{\Omega(t)}m \quad (t > 0).$$

Generally speaking, partial balayage destroys information: in for example (4.13), Ω is uniquely determined by μ, but many different measures μ give the same Ω. Therefore the balayage point of view, or the formulation with quadrature domains for subharmonic functions, embodies the fact that not only does Hele-Shaw flow preserve harmonic moments, so that the mass distributions $2\pi Q(t)\delta_0 + \chi_{\Omega(0)}m$ and $\chi_{\Omega(t)}m$ above are gravi-equivalent, but also that there is a time direction saying that the first mass distribution contains more information than the second. This reflects the fact that Laplacian growth is well-posed in one time direction (increasing t when $q > 0$) but ill-posed in the other, and also reminds of the role of entropy in statistical mechanics, which singles out one time direction.

Example 4.1. To illustrate the use of partial balayage, we note that the measure $\nu_{f(\cdot,t)}m$ in (4.4) may be swept to a measure of the form $\chi_{\Omega(t)}m$: $\mathrm{Bal}\,(\nu_{f(\cdot,t)}m, m) = \chi_{\Omega(t)}m$. This is the same as saying that $\int h\nu_{f(\cdot,t)}dm \leq \int_{\Omega(t)} hdm$ for $h \in SL^1(\Omega(t), m)$, as in (4.14) above. Taking $s = 0$ as initial time and assuming for simplicity that $f(\cdot, 0)$ is univalent, so that $\nu_{f(\cdot,0)} = \chi_{\Omega(0)}$ with $\Omega(0) = f(\mathbb{D}, 0)$, the inequality (4.4) gives

$$\int_{\Omega(t)} hdm - \int_{\Omega(0)} hdm \geq 2\pi Q(t)h(0)$$

for functions h subharmonic in $\Omega(t)$. In other words, $\{\Omega(t) : t \geq 0\}$ is the ordinary weak solution, possibly multiply connected, with initial domain $\Omega(0)$.

The evolution of $\nu_{f(\cdot,t)}$ can therefore be viewed as a refinement of the ordinary weak solution, a refinement in the sense that it contains more information. One can always pass from $\nu_{f(\cdot,t)}$ to $\chi_\Omega(t)$ by balayage, but there is in general no way to recover $\nu_{f(\cdot,t)}$ from $\chi_\Omega(t)$. An even more refined version of the evolution is obtained by lifting everything to a Riemann surface over $\mathbb{C}$, which we shall now discuss.

5 Lifting strong and weak solutions to a Riemann surface

5.1 Hele-Shaw flow on manifolds

Hele-Shaw flow makes sense on Riemannian manifolds (of any dimension). The only difference compared to the Euclidean case then is that the measure $dm = dx \wedge dy$ in, for example, (2.2) and (4.5) shall be replaced by the intrinsic volume form of the manifold. This also indicates how (4.5) changes under variable transformations ($dm = dx \wedge dy$ shall be treated as a 2-form). We shall need to make these things precise in the case that the Riemannian manifold is a branched covering Riemann surface over $\mathbb{C}$, with the metric inherited from the Euclidean metric on $\mathbb{C}$ via the covering map.

Let $\mathcal{M}$ be a Riemann surface and $p : \mathcal{M} \to \mathbb{C}$ a nonconstant analytic function, thought of as a, possibly branched, covering map. If $\tilde{z} = \tilde{x} + \mathrm{i}\tilde{y}$ is a local holomorphic coordinate on $\mathcal{M}$ and $z = x + \mathrm{i}y$ the usual coordinate on $\mathbb{C}$ then the Riemannian metric on $\mathcal{M}$ is taken to be the Euclidean metric $|dz|^2 = dx^2 + dy^2$, which is lifted to $\mathcal{M}$ by p, i.e.,

$$d\tilde{s}^2 = |dp|^2 = |p'(\tilde{z})|^2(|d\tilde{x}|^2 + |d\tilde{y}|^2). \tag{5.1}$$

The intrinsic area form on $\mathcal{M}$ is similarly the pull-back of $dm = dx \wedge dy$ to $\mathcal{M}$, namely

$$d\tilde{m} = \frac{1}{2\mathrm{i}} d\bar{p} \wedge dp = |p'(\tilde{z})|^2 d\tilde{x} \wedge d\tilde{y}. \tag{5.2}$$

In terms of the Hermitian bilinear form $d\bar{p} \otimes dp$ one can write $d\tilde{s}^2 = \operatorname{Re} d\bar{p} \otimes dp$, $d\tilde{m} = \operatorname{Im} d\bar{p} \otimes dp$.

Assume now that $0 \in p(\mathcal{M})$ and let $\tilde{0} \in \mathcal{M}$ be a point such that $p(\tilde{0}) = 0$. Then we may consider Hele-Shaw evolution on $\mathcal{M}$ with injection (or suction) at $\tilde{0}$. In case of a simply connected evolution $\tilde{\Omega}(t)$, let

$$\tilde{f}(\cdot, t) : \mathbb{D} \to \tilde{\Omega}(t) \subset \mathcal{M}$$

be conformal maps with $\tilde{f}(0, t) = \tilde{0}$ and $f'(0, t) > 0$, where $f = p \circ \tilde{f}$ is the projection of $\tilde{f}$ to $\mathbb{C}$,

$$f(\zeta, t) = p(\tilde{f}(\zeta, t)).$$

The latter relationship gives

$$\frac{\dot{f}(\zeta, t)}{\zeta f'(\zeta, t)} = \frac{\dot{\tilde{f}}(\zeta, t)}{\zeta \tilde{f}'(\zeta, t)}, \tag{5.3}$$

which expresses invariance of the Poisson integral (2.8) under changes of coordinates. In particular it follows that the evolution of $\tilde{f}$ is described by

$$\dot{\tilde{f}}(\zeta, t) = \zeta \tilde{f}'(\zeta, t) P_g(\zeta, t), \tag{5.4}$$

where $P_g(\zeta, t) = P(\zeta, t)$ is the Poisson integral (2.8) defined, not in terms of $\tilde{f}'$ but in terms of $g = f'$. The relationship between f' and $\tilde{f}'$ is

$$f'(\zeta, t) = p'(\tilde{f}(\zeta, t))\tilde{f}'(\zeta, t). \tag{5.5}$$

If p is thought of as just a local identity map (away from branch points) then f' and $\tilde{f}'$ are the same.

It should be noted that $\tilde{f}(\cdot, t)$ by definition always is univalent in $\mathbb{D}$, in particular $\tilde{f}' \neq 0$ in $\mathbb{D}$. If $f' = 0$ at some point in $\mathbb{D}$, then it is the factor $p'(\tilde{f}(\zeta, t))$ in (5.5) that vanishes there. When formulated as a Polubarinova-Galin equation the evolution of $\tilde{f}$ is given by

$$\operatorname{Re}\,[\dot{\tilde{f}}(\zeta, t)\overline{\zeta \tilde{f}'(\zeta, t)}] = \frac{q(t)}{|p'(\tilde{f}(\zeta, t))|^2} \quad (\zeta \in \partial\mathbb{D}). \tag{5.6}$$

This equation is an immediate consequence of (5.4), (5.5) and (2.8). It is actually the general form of the Polubarinova-Galin equation on a manifold with Riemannian metric given as in (5.1), even when the integral of p' is not interpreted as a covering map.

In general, $\dot{\tilde{f}}$ and $\zeta\tilde{f}'$ should be interpreted as vectors in the tangent space of $\mathcal{M}$ at $\tilde{z} = \tilde{f}(\zeta, t)$. More precisely, $\dot{\tilde{f}}$ is the speed of $\tilde{z} \in \partial\tilde{\Omega}$ when $\zeta \in \partial\mathbb{D}$ is kept fixed and $\zeta\tilde{f}'$ is a vector pointing in the outward normal direction of $\partial\tilde{\Omega}$. Equation (5.6) expresses that

$$< \dot{\tilde{f}}, \zeta\tilde{f}' >_{\mathcal{M}} = q \quad \text{on } \partial\mathbb{D}, \tag{5.7}$$

where $< \cdot, \cdot >_{\mathcal{M}}$ denotes the (real) inner product on the tangent space of $\mathcal{M}$. Alternatively, expressed in terms of the form $d\tilde{m} = \frac{1}{2\mathrm{i}} d\bar{p} \wedge dp$, (5.6) says that

$$d\tilde{m}(\dot{\tilde{f}}, \mathrm{i}\zeta\tilde{f}') = q \quad \text{on } \partial\mathbb{D},$$

which can be interpreted as a Poisson bracket relation. This has in some mathematical physics literature, see for example [1], [43], [15], been formalized under the name *string equation*.

Let $G_{\tilde{\Omega}}(\tilde{z},\tilde{0})$ be the Green's function of $\tilde{\Omega}$, vanishing on the boundary and with behavior

$$G_{\tilde{\Omega}}(\tilde{z},\tilde{0}) = -\frac{1}{2\pi}\log|\tilde{z}| + \text{harmonic}$$

at $\tilde{z}=\tilde{0}$. Then

$$G_{\tilde{\Omega}}(\tilde{z},\tilde{0}) = -\frac{1}{2\pi}\log|\zeta|,$$

where $\tilde{z}=\tilde{f}(\zeta,t)$. Since $2\frac{\partial}{\partial\tilde{z}}\log|\zeta| = \frac{\partial}{\partial\tilde{z}}\log\zeta = \frac{1}{\zeta\tilde{f}'(\zeta,t)}$, this shows that (5.7), when written on the form

$$< \dot{\tilde{f}}, \frac{\zeta\tilde{f}'}{|\zeta\tilde{f}'|} >_{\mathcal{M}} = \frac{q}{|\zeta\tilde{f}'|} \quad \text{on } \partial\mathbb{D},$$

expresses that

$$\dot{\tilde{f}}_{\text{normal}} = 2\pi q|\nabla G_{\tilde{\Omega}}(\tilde{z},\tilde{0})|, \tag{5.8}$$

i.e., that the boundary moves in the outward normal direction with speed proportional to the gradient of the Green's function. This is the classical description of Laplacian growth.

When $f(\cdot,t)$ solves the Löwner-Kufarev equation it is a subordination chain by Theorem 3.1 and hence it can be lifted to a Riemann surface $\mathcal{M}$ by Lemma 3.1. Most of the previous formulas have simple formulations on $\mathcal{M}$, for example (2.2) generalizes to

$$\frac{d}{dt}\int_{\tilde{\Omega}(t)} h\,d\tilde{m} = 2\pi q(t)h(\tilde{0}), \tag{5.9}$$

for h harmonic in a neighborhood of $\tilde{\Omega}(t)$, and where $\tilde{\Omega}(t)=\tilde{f}(\mathbb{D},t)$, $f = p\circ\tilde{f} : \mathbb{D}\to\mathcal{M}\to\mathbb{C}$. For subharmonic h we have inequality $\geq$. Thus on integrating (5.9) with respect to t we arrive at the natural notion of weak solution on the Riemann surface $\mathcal{M}$.

Definition 5.1. A family of open sets $\{\tilde{\Omega}(t)\subset\mathcal{M} : t\in I\}$ with compact closure in $\mathcal{M}$ is a **weak solution** on $\mathcal{M}$ if, for any $s,t\in I$ with $s<t$, $\tilde{\Omega}(s)\subset\tilde{\Omega}(t)$ and

$$\int_{\tilde{\Omega}(t)} hd\tilde{m} - \int_{\tilde{\Omega}(s)} hd\tilde{m} \geq 2\pi(Q(t)-Q(s))h(\tilde{0}) \tag{5.10}$$

holds for every $h\in SL^1(\tilde{\Omega}(t),\tilde{m})$.

In terms of partial balayage on $\mathcal{M}$ (which makes good sense) the property of being a weak solution, i.e., (5.10) together with $\tilde{\Omega}(s) \subset \tilde{\Omega}(t)$, translates into

$$\mathrm{Bal}\,(2\pi(Q(t)-Q(s))\delta_{\tilde{0}} + \chi_{\tilde{\Omega}(s)}\tilde{m}, \tilde{m}) = \chi_{\tilde{\Omega}(t)}\tilde{m} \quad (s < t). \tag{5.11}$$

5.2 Examples

Example 5.1. The purpose of this example is to prepare for the way the definition of a weak solution is going to be used in the proof of Theorem 7.1. For later clarity we spell out everything quite much in detail.

Choose $s = 0$ in Definition 5.1 and assume that the domain $\tilde{\Omega}(0) = \tilde{f}(\mathbb{D}, 0) = \tilde{f}(\mathbb{D})$ at time $s = 0$ is obtained by uniformization of some fixed $f \in \mathcal{O}_{\mathrm{norm}}(\overline{\mathbb{D}})$, as in Section 3. Thus $f = p \circ \tilde{f}$, where p is the covering map $p : \tilde{\Omega}(0) \to \mathbb{C}$ which, in terms of the trivial decomposition

$$\mathbb{D} \xrightarrow{\mathrm{id}} \mathbb{D} \xrightarrow{f} \mathbb{C},$$

is obtained by interpreting the second $\mathbb{D}$ as an abstract Riemann surface, identified as $\tilde{\Omega}(0)$. The names of the mappings are then shifted to

$$\mathbb{D} \xrightarrow{\tilde{f}} \tilde{\Omega}(0) \xrightarrow{p} \mathbb{C}.$$

By assumption, f is analytic in some larger disk, say in $\mathbb{D}(0, \rho)$, $\rho > 1$. Thus the two diagrams extend to

$$\mathbb{D}(0, \rho) \xrightarrow{\mathrm{id}} \mathbb{D}(0, \rho) \xrightarrow{f} \mathbb{C}, \tag{5.12}$$

$$\mathbb{D}(0, \rho) \xrightarrow{\tilde{f}} \mathcal{M} \xrightarrow{p} \mathbb{C}, \tag{5.13}$$

respectively, which defines the Riemann surface $\mathcal{M}$ as being the conformal image of $\mathbb{D}(0, \rho)$ under $\tilde{f}$. In particular, $\tilde{\Omega}(0) \subset \mathcal{M}$, and for small enough $t > 0$ the weak solution with initial domain $\tilde{\Omega}(0)$ will stay compactly in $\mathcal{M}$. The defining property of the solution domain $\tilde{\Omega}(t) \supset \tilde{\Omega}(0)$ at time $t > 0$ is, when formulated in terms of the abstract Riemann surface notations of (5.13),

$$\int_{\tilde{\Omega}(t)} \tilde{h}\,d\tilde{m} - \int_{\tilde{\Omega}(0)} \tilde{h}\,d\tilde{m} \geq 2\pi Q(t)\tilde{h}(\tilde{0}). \tag{5.14}$$

This is to hold for all integrable (with respect to $\tilde{m}$) subharmonic functions $\tilde{h}$ in $\tilde{\Omega}(t)$. When the same property is formulated by identifying $\mathcal{M}$ with $\mathbb{D}(0,\rho)$ as in (5.12) it becomes

$$\int_{D(t)} h|g|^2 dm - \int_{\mathbb{D}} h|g|^2 dm \geq 2\pi Q(t)h(0), \tag{5.15}$$

to hold for all subharmonic h in $D(t)$, integrable with respect to the measure $|g|^2 dm$. Here $D(t) = \tilde{f}^{-1}(\tilde{\Omega}(t)) \subset \mathbb{D}(0,\rho)$, $g = f'$, and $h = \tilde{h} \circ \tilde{f}$, which is subharmonic if and only if $\tilde{h}$ is. Note that (by definition, (5.1)) $d\tilde{m} = p^*(dm)$ in the picture (5.13), which becomes $|g|^2 dm$ in the picture (5.12).

The domains $\tilde{\Omega}(t)$ and $D(t)$ are not necessarily simply connected when $t > 0$, as they are defined only in terms of a weak solution. Eventually, however, we want to assert that they are simply connected if $t > 0$ is small enough (Conjecture 7.3).

The only thing which can make a weak solution break down is that it runs out of the manifold, $\mathcal{M}$. Then the natural thing to do is to try to extend $\mathcal{M}$ to a larger manifold. Weak solutions are unique (up to null-sets), but they of course depend on the choice of $\mathcal{M}$ and p. If we take $\mathcal{M}$ to be, for example, a disk $\mathbb{D}(0,a) \subset \mathbb{C}$ then, assuming $Q(t) \to \infty$ as $t \to \infty$, any Hele-Shaw evolution will eventually run out of $\mathcal{M}$. The following example shows that there are always many different ways of enlarging $\mathcal{M}$, which then give rise to different Hele-Shaw evolutions.

Example 5.2. Choose a point $a > 0$ on the positive real axis, to be used as a stopping point and also as a branch point. Let $\mathcal{M} = \mathbb{D}(0,a)$ be the disk reaching out to a, and consider it as a Riemann surface with trivial projection map $p(z) = z$ to $\mathbb{C}$. Then starting from empty space a Hele-Shaw flow evolution on $\mathcal{M}$ with injection at the origin gives a family of growing disks, say $\Omega(t) = \mathbb{D}(0,at)$ on the time interval $0 < t < 1$, as a weak (and strong) solution for the source strength $q(t) = a^2 t$ (so that $Q(t) = \frac{1}{2\pi} m(\mathbb{D}(0,at))$). At time $t = 1$ it runs out of $\mathcal{M}$, but it can be continued without any changes on the trivially extended Riemann surface $\mathcal{M}_1 = \mathbb{C}$, for $0 < t < \infty$.

However, it can also be continued in many other ways. Let for example $\mathbb{C}_1$, $\mathbb{C}_2$ be two copies of $\mathbb{C}$ and consider

$$\mathcal{M}_2 = (\mathbb{C}_1 \setminus \{a\}) \cup (\mathbb{C}_2 \setminus \{a\}) \cup \{a\}$$

as a covering surface of $\mathbb{C}$ with a branch point at $z = a$. The covering map $p : \mathcal{M}_2 \to \mathbb{C}$ identifies any point on $\mathbb{C}_j$ ($j = 1, 2$) with the corresponding point on $\mathbb{C}$. This is also true at $z = a$, but a more accurate description there has to be given in terms of a local coordinate. We may for example choose a local coordinate $\tilde{z}$ on $\mathcal{M}_2$ so that $\tilde{z} = 0$ corresponds to $z = a$ and, more precisely, so that

$$p_2(\tilde{z}) = \tilde{z}^2 + a.$$

Thus, with $z = p_2(\tilde{z})$, $\tilde{z} = \sqrt{z-a}$. This coordinate $\tilde{z}$ is actually a global coordinate on $\mathcal{M}_2$ and it makes $\mathcal{M}_2$ appear as the classical Riemann surface of the multivalued function $\sqrt{z-a}$.

In terms of the above coordinate, the area form of $\mathcal{M}_2$ is

$$d\tilde{m}_2 = \frac{1}{2\mathrm{i}} d\bar{p}_2 \wedge dp_2 = 4|\tilde{z}|^2 d\tilde{x} d\tilde{y}.$$

The source point is to be one of the two points $\pm\sqrt{-a}$ on $\mathcal{M}_2$ above $0 \in \mathbb{C}$, let it be $\tilde{0} = \mathrm{i}\sqrt{a}$, $\sqrt{a}$ denoting the positive root. Now the definition of a weak solution on $\mathcal{M}_2$ becomes, explicitly,

$$4\int_{\tilde{\Omega}(t)} h(\tilde{z})|\tilde{z}|^2 d\tilde{x} d\tilde{y} - 4\int_{\tilde{\Omega}(s)} h(\tilde{z})|\tilde{z}|^2 d\tilde{x} d\tilde{y} \geq 2\pi(t-s)h(\mathrm{i}\sqrt{a}),$$

to hold for all integrable (with respect to $\tilde{m}_2$) subharmonic functions h in $\tilde{\Omega}(t)$. Expressed in the coordinate $\tilde{z}$ it is thus a weighted Hele-Shaw flow, as discussed in for example [19]. It exists for all $0 < t < \infty$, but it is certainly different from the solution $\Omega(t)$ on $\mathcal{M}_1 = \mathbb{C}$. For $t > 1$, and when viewed on $\mathcal{M}_2$, part of $\tilde{\Omega}(t)$ continues on the original ('lower') sheet, say $\mathbb{C}_1$, while part goes to the 'upper' sheet $\mathbb{C}_2$. Hedenmalm and Shimorin [19] use the terminology 'wrapped Hele-shaw flow' when the solution goes up on a Riemann covering surface, at least in the case when there are no branch points.

Example 5.3. This example can be viewed as a continuation of Example 5.2 (and also of Example 3.1), but from a different point of view. It is based on an example of Sakai [35], and it appears also, from a different point of view, in [17]. Let

$$f(\zeta, t) = b(t)\, \frac{\zeta(\zeta - 2t^{-1} + t^{-3})}{\zeta - t},$$

where $1 < t < \infty$ and $b(t) \in \mathbb{R}$ are parameters. The derivative is

$$g(\zeta, t) = b(t)\, \frac{(\zeta - t^{-1})(\zeta - 2t + t^{-1})}{(\zeta - t)^2}. \tag{5.16}$$

We see that g has two zeros, $\omega_1(t) = t^{-1} \in \mathbb{D}$ and $\omega_2(t) = 2t - t^{-1} \in \mathbb{C}\setminus\overline{\mathbb{D}}$, and that $g(\zeta, t)d\zeta$, as a differential, has double poles at $\zeta_1(t) = t = \frac{1}{2}(\omega_1(t)+\omega_2(t))$ and at infinity. The data of g are special in two ways: first of all ω_1 and ζ_1 are reflections of each other with respect to the unit circle, and secondly ω_2 is chosen so that $gd\zeta$ has no residues (which is immediately clear since f has no logarithmic poles).

Since $\omega_1(t) \in \mathbb{D}$, $f(\cdot, t)$ is not locally univalent in $\mathbb{D}$, but it generates the same moments as a disk: all moments (defined by the rightmost member in (2.4)) vanish, except the first one which is

$$M_0(t) = b(t)^2\,\frac{2t^2 - 1}{t^4}.$$

More generally, the corresponding quadrature identity is

$$\frac{1}{\pi}\int_{\mathbb{D}} h(\zeta)|g(\zeta,t)|^2 dm(\zeta) = b(t)^2\,\frac{2t^2-1}{t^4}h(0), \tag{5.17}$$

holding for h analytic and integrable in $\mathbb{D}$. This formula also shows that, for the special choice $b(t) = \frac{t^2}{\sqrt{2t^2-1}}$, $g(\zeta, t)$ is a contractive (inner) zero divisor in the sense of Hedenmalm [17], [18].

Despite (5.17), $f(\zeta, t)$ in general does not solve the Polubarinova-Galin equation (2.1). Only for one particular choice of $b(t)$ it does. This choice is determined by the requirement that $f(\omega_1(t), t)$ shall be time independent. Since

$$f(\omega_1(t), t) = f(t^{-1}, t) = \frac{b(t)}{t^3}$$

this condition gives

$$b(t) = at^3, \tag{5.18}$$

where a is a constant. A calculation shows that for this particular choice of $b(t)$, the Polubarinova-Galin equation indeed holds with $q(t) = a^2t(4t^2 - 1)$:

$$\operatorname{Re}\left[\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}\right] = a^2t(4t^2-1) \quad \text{for } \zeta \in \partial\mathbb{D}.$$

Note that $q(t) > 0$. Also the Löwner-Kufarev equation holds, because $f(\omega_1(t), t) = a$ is fixed (cf. Theorem 3.1).

Now we shall see that, taking $a > 0$, the above solution, namely

$$f(\zeta, t) = \frac{a\zeta(t^3\zeta - 2t^2 + 1)}{\zeta - t},$$

is exactly the projection under p_2 of the evolution on $\mathcal{M}_2$ in Example 5.2. In fact, since $f(\cdot,t)$ maps the zero $\omega_1(t) \in \mathbb{D}$ of $g(\zeta,t)$ onto the fixed point a, $f(\cdot,t)$ lifts to a map $\tilde{f}(\cdot,t)$ into the surface $\mathcal{M}_2$. Inverting $p_2(\tilde{z}) = \tilde{z}^2 + a$ gives the explicit expression

$$\tilde{f}(\zeta,t) = \sqrt{f(\zeta,t) - a} = \sqrt{\frac{at(t\zeta-1)^2}{\zeta - t}}.$$

This function, for any fixed $t > 1$, is univalent, $\tilde{f}(\cdot,t) : \mathbb{D} \to \tilde{\Omega}(t) \subset \mathcal{M}_2$, and as a function of t it represents, in the coordinate $\tilde{z}$, the Hele-Shaw evolution on $\mathcal{M}_2$. Indeed, it satisfies the Polubarinova-Galin equation on $\mathcal{M}_2$:

$$\mathrm{Re}\left[\dot{\tilde{f}}(\zeta,t)\overline{\zeta\tilde{f}'(\zeta,t)}\right] = \frac{q(t)}{4|\tilde{f}(\zeta,t)|^2} \quad \text{for } \zeta \in \partial\mathbb{D}.$$

This is an instance of (5.6), as $p_2'(\tilde{z}) = 2\tilde{z}$.

As a summary, we write up in coordinates, z and $\tilde{z}$, and $0 < t < \infty$, the complete evolution in Example 5.2, namely the growing disk which at the point a climbs up to the Riemann surface $\mathcal{M}_2$:

(i) In terms of z, solving the ordinary Löwner-Kufarev equation (2.7), (2.8),

$$f(\zeta,t) = \begin{cases} at\zeta & (0 < t < 1), \\ \frac{a\zeta(t^3\zeta - 2t^2+1)}{\zeta - t} & (1 < t < \infty). \end{cases} \tag{5.19}$$

Notice that both of the expressions above are (real) analytic in t, even across the junction value $t = 0$. Thus the combined function $f(\zeta,t)$ is piecewise real analytic with respect to t.

(ii) In terms of $\tilde{z}$, for which we have (5.4) and (5.6) when $t \neq 1$, and for which the entire solution (across $t = 1$) is a weak solution on $\mathcal{M}_2$,

$$\tilde{f}(\zeta,t) = \begin{cases} \sqrt{a(t\zeta-1)} & (0 < t < 1), \\ \sqrt{\frac{at(t\zeta-t)^2}{\zeta-t}} & (1 < t < \infty). \end{cases}$$

The source strength is

$$q(t) = \begin{cases} a^2 t & (0 < t < 1), \\ a^2 t(4t^2 - 1) & (1 < t < \infty). \end{cases} \tag{5.20}$$

Here we can see a discontinuity of $q(t)$ at $t = 1$. However this is harmless, and can be avoided by using another time parametrization. For example one could define $f(\zeta, t) = at^3\zeta$, $\tilde{f}(\zeta, t) = \sqrt{a(t^3\zeta - 1)}$ for $0 < t < 1$, which gives the same family of domains, just traversed with a different speed. This would give $q(t) = 3a^2t^5$ for $0 < t < 1$, making $q(t)$ continuous across $t = 1$.

5.3 The Riemann surface solution pulled back to the unit disk

For a function $h(\zeta, t)$ which is holomorphic in ζ, the requirement (4.1), with $\Psi = h$, reduces to the simpler statement

$$\dot{h}(\zeta, t)f'(\zeta, t) = \dot{f}(\zeta, t)h'(\zeta, t). \tag{5.21}$$

This can be viewed as the vanishing of a functional determinant and can alternatively be written as

$$\frac{\dot{h}(\zeta, t)}{\zeta h'(\zeta, t)} = \frac{\dot{f}(\zeta, t)}{\zeta f'(\zeta, t)}, \tag{5.22}$$

where (on dividing by ζ) we also have used that $f(0, t) = 0$ for all t. When f solves the Löwner-Kufarev equation (2.7) the right member is holomorphic in $\mathbb{D}$ and equals $P(\zeta, t)$. Then (5.22) means that h solves the same Löwner-Kufarev equation as f. This can be interpreted as saying that 'h flows with f', and it also follows that the $h(\zeta, t)$ are subordinated by the same functions as $f(\zeta, t)$:

$$h(\varphi(\zeta, s, t), t) = h(\zeta, s) \quad (s \leq t). \tag{5.23}$$

Here $\varphi(\zeta, s, t)$ are the subordination functions in (3.4). Note that (5.23), or (5.21), implies that $h(0, t) = h(0, s)$ (or $\dot{h}(0, t) = 0$).

We can now assert

Proposition 5.1. *Let $t \mapsto f(\cdot, t) \in \mathcal{O}_{\rm norm}(\mathbb{D})$ be a smooth evolution on some time interval and assume that $f' \neq 0$ on $\partial\mathbb{D}$ on this time interval. Then $f(\cdot, t)$ solves the Polubarinova-Galin equation (2.1) if and only if*

$$\frac{d}{dt}\int_{\mathbb{D}} h(\zeta, t)|f'(\zeta, t)|^2\, dm(\zeta) = 2\pi q(t)h(0, t). \tag{5.24}$$

for every function $h(\cdot, t) \in \mathcal{O}(\overline{\mathbb{D}})$ which satisfies (5.21) (equivalently, (4.1) or (5.23)), and it solves the Löwner-Kufarev equation (2.7) if and only if moreover (3.5) holds (equivalently, $f(\cdot, t)$ is a subordination chain).

Proof. The proposition follows immediately from Lemma 4.1 and Theorem 3.1 since $\operatorname{Re} h|_{\partial\mathbb{D}}$ and $\operatorname{Im} h|_{\partial\mathbb{D}}$ range over a dense set of functions in (5.24). $\square$

Also the Riemann surface weak formulation (5.10) can, in case relevant domains are simply connected, be pulled back to the unit disk in various ways. In Example 5.1 this was done by pulling the initial domain back to $\mathbb{D}$, which works well for discussing solutions on a short time interval $0 \leq t < \varepsilon$. However, to discuss global solutions it is better to fix a final time $t = T$ under consideration, and then pull back the domain $\tilde{\Omega}(T) \subset \mathcal{M}$ at that time to $\mathbb{D}$, assuming that $\tilde{\Omega}(T)$ is simply connected. Then all previous domains become subdomains of $\mathbb{D}$.

Thus fixing T and identifying $\tilde{\Omega}(T)$ with $\mathbb{D}$ via $\tilde{f}(\cdot, T)$, equation (5.10) becomes, for $s < t \leq T$ and on setting $g = f'$ as usual,

$$\int_{D(t,T)} h(z)|g(z,T)|^2\,dm(z) - \int_{D(s,T)} h(z)|g(z,T)|^2\,dm(z) \tag{5.25}$$
$$\geq 2\pi(Q(t) - Q(s))h(0),$$

to hold for $h \in SL^1(D(t,T), m)$. Here the domains $D(s,T) = \tilde{f}^{-1}(\tilde{\Omega}(s), T)$, $D(t,T) = \tilde{f}^{-1}(\tilde{\Omega}(t), T)$, satisfying $D(s,T) \subset D(t,T) \subset \mathbb{D}$, need not be simply connected. Choosing $t = T = 0$ with $s < 0$ gives

$$\int_{\mathbb{D}} h(z)|g(z,0)|^2\,dm(z) - \int_{D(s)} h(z)|g(z,0)|^2\,dm(z) \geq -2\pi Q(s)h(0), \tag{5.26}$$

where $D(s) = D(s,0) \subset \mathbb{D}$ and $Q(s) < 0$. This is a counterpart of (5.15) for negative times. It also connects to the theory of finite contractive zero divisors: starting, as in Example 5.2, a Hele-Shaw evolution on a Riemann surface $\mathcal{M}$ from empty space, we have $D(s) = \emptyset$ at the initial time $s < 0$, and then (5.26) can be identified with the definition of an inner divisor (namely $g(z,0)$ in the above equation), as in [17], [18].

The weak solution can be coupled to the Löwner-Kufarev equation only if the domains $\tilde{\Omega}(t)$, or $D(t,T)$, are simply connected. When this is the case we have $D(t,T) = \varphi(\mathbb{D}, t, T)$, where $\varphi(\zeta, s, t)$ are the subordination functions associated to the conformal maps $f(\cdot, t) : \mathbb{D} \to \tilde{\Omega}(t)$. In such a case, and returning to (5.25), choosing $T = t$ there and making the variable transformation $z = \varphi(\zeta, s, t)$ in the last integral, one gets

$$\int_{\mathbb{D}} h(z)|g(z,t)|^2\,dm(z) - \int_{\mathbb{D}} h(\varphi(\zeta,s,t))|g(\zeta,s)|^2\,dm(\zeta) \tag{5.27}$$

$$\geq 2\pi(Q(t)-Q(s))h(0),$$

to hold for h subharmonic and integrable in $\mathbb{D}$. For harmonic h we have equalities in the above inequalities because both of $\pm h$ are then subharmonic. The relation (5.27) can also be obtained directly by integrating (5.24) and using (5.23). Note that for time dependent test functions, $h(z,t)$, which satisfy (5.23), the relation (5.27) takes the simpler form

$$\int_{\mathbb{D}} h(z,t)|g(z,t)|^2\,dm(z) - \int_{\mathbb{D}} h(z,s)|g(z,s)|^2\,dm(z) \geq \tag{5.28}$$

$$\geq 2\pi(Q(t)-Q(s))h(0,t).$$

As for the right member, $h(0,t)$ is actually independent of t by (5.23).

We summarize:

Proposition 5.2. *A family* $\{f(\cdot,t)\in\mathcal{O}_{\rm norm}(\overline{\mathbb{D}}) : 0\leq t\leq T\}$ *represents a weak solution as in Definition 5.1 (with* $I=[0,T]$*) if an only if it is a subordination chain as in Definition 3.2 and (5.27) holds for* $0\leq s<t\leq T$.

6 Compatibility between balayage and covering maps

The family of branched covering surfaces over $\mathbb{C}$ form a partially ordered set in a natural way. Within in each of the surfaces one can perform partial balayage, sweeping to the area form lifted from $\mathbb{C}$. Thus we have two kinds of projection maps, reducing refined objects to cruder objects containing less information:

(i) The first is the balayage operator taking, for example, an initial domain $\Omega(0)$ to the domain at a later time $\Omega(t)$ by sweeping out the accumulated source:

$$\mathrm{Bal}\,(2\pi Q(t)\delta_{\tilde 0}+\chi_{\Omega(s)}\tilde m,\tilde m)=\chi_{\Omega(t)}\tilde m.$$

This map is really an orthogonal projection in a Hilbert space (e.g., the Sobolev space $H^1_0(\mathcal{M})=W^{1,2}_0(\mathcal{M})$ if the Dirac measures are suitably smoothed out). It is a 'horizontal' projection, within each covering surface.

(ii) The second is the branched covering map between two Riemann surfaces, by which a measure on the higher surface can be pushed down to a measure on the lower surface. One may think of this as a 'vertical' projection.

The aim of the present section is to show that these two projections commute in an appropriate sense. Let $p : \mathcal{M} \to \mathbb{C}$ be a branched covering map, i.e., p is a nonconstant analytic function. By p_* we denoted the push-forward map, which can be applied to measures on $\mathcal{M}$, to (parametrized) chains for integration (simply by composition), etc. Similarly, p^* denotes the pull-back map, which can be applied to functions and differential forms on $\mathbb{C}$. If for example $\tilde{\Omega}$ is a domain in $\mathcal{M}$, thought of as the oriented 2-chain parametrized by some $\tilde{f} : \mathbb{D} \to \mathcal{M}$ ($\tilde{\Omega} = \tilde{f}(\mathbb{D})$), then $p_*\tilde{\Omega}$ is the 2-chain parametrized by $f = p \circ \tilde{f} : \mathbb{D} \to \mathbb{C}$, which can be thought of as $\Omega = f(\mathbb{D})$ with appropriate multiplicities. In other words, p_* takes the measure $\chi_{\tilde{\Omega}}\tilde{m}$, on $\mathcal{M}$, where $d\tilde{m} = p^*dm = d(p \circ x) \wedge d(p \circ y)$, to $\nu_f m$ on $\mathbb{C}$, ν_f being the counting function, Definition 3.1.

Note that p_* and p^* are linear maps on suitable vector spaces and that they, in some formal sense, are each others adjoints. For example, for measures μ with compact support on $\mathcal{M}$ and continuous functions φ on $\mathbb{C}$ we have

$$\int_{\mathbb{C}} \varphi \, d(p_*\mu) = \int_{\mathcal{M}} (\varphi \circ p) \, d\mu = \int_{\mathcal{M}} (p^*\varphi) \, d\mu.$$

The first identity here can be used as a definition of p_* when it acts on measures, and $p^*(\varphi)$ is simply defined as $\varphi \circ p$. See further Section 4.1.7 in [4].

In order to be able to use systematic notations we now denote the complex plane by $\mathcal{M}$, and we call the covering surface $\tilde{\mathcal{M}}$. This makes the proposition below look like a quite general result (which it in fact is, but we shall only prove it under the stated assumptions).

Proposition 6.1. *With $p : \tilde{\mathcal{M}} \to \mathcal{M}$ a nonconstant proper analytic map between two Riemann surfaces, where $\mathcal{M} = \mathbb{C}$, let $\tilde{\mu}$ be a measure with compact support in $\tilde{\mathcal{M}}$, λ a measure on $\mathcal{M}$, absolutely continuous with respect to m and satisfying (4.6), and let $\tilde{\lambda}$ be a measure on $\tilde{\mathcal{M}}$ satisfying $\tilde{\lambda} \geq p^*\lambda$. Then*

$$\mathrm{Bal}\,(p_*\mathrm{Bal}\,(\tilde{\mu}, \tilde{\lambda}), \lambda) = \mathrm{Bal}\,(p_*\tilde{\mu}, \lambda).$$

Proof. Since $\mathcal{M} = \mathbb{C}$ and p is proper, $\tilde{\mathcal{M}}$ will be large enough for $\mathrm{Bal}\,(\tilde{\mu}, \tilde{\lambda})$ to exist and have compact support in $\tilde{\mathcal{M}}$. Set then

$$\tilde{\nu} = \mathrm{Bal}\,(\tilde{\mu}, \tilde{\lambda}),$$

$$\nu' = \mathrm{Bal}\,(p_*\tilde{\nu}, \lambda),$$

$$\mu = p_*\tilde{\mu},$$
$$\nu = \mathrm{Bal}\,(\mu, \lambda)$$

and we shall show that $\nu' = \nu$.

By the general structure of partial balayage (4.10) we have

$$\tilde{\nu} = \tilde{\lambda}\chi_{\tilde{\Omega}} + \tilde{\mu}\chi_{\tilde{\mathcal{M}}\setminus\tilde{\Omega}}, \tag{6.1}$$

where $\tilde{\Omega} \subset \tilde{\mathcal{M}}$ is the maximal open set in which $\tilde{\nu} = \tilde{\lambda}$. Recall (4.11), (4.12) that this $\tilde{\Omega}$ can also be characterized by

$$\begin{cases} \tilde{\mu} < \tilde{\lambda} \text{ on } \tilde{\mathcal{M}} \setminus \tilde{\Omega}, \\ \int_{\tilde{\Omega}} \psi\, d\tilde{\mu} \leq \int_{\tilde{\Omega}} \psi\, d\tilde{\lambda} \text{ for all } \psi \in SL^1(\tilde{\Omega}, \tilde{\lambda}). \end{cases} \tag{6.2}$$

Here $SL^1(\tilde{\Omega}, \tilde{\lambda})$ may be replaced by a smaller test class, as discussed after (4.11), (4.12), to avoid some possible integrability problems below.

Similarly to the above we have

$$\nu' = \lambda\chi_{\Omega'} + (p_*\tilde{\nu})\,\chi_{\mathcal{M}\setminus\Omega'} \tag{6.3}$$

where $\Omega' \subset \mathcal{M}$ is characterized by

$$\begin{cases} p_*\tilde{\nu} < \lambda \text{ on } \mathcal{M} \setminus \Omega', \\ \int_{\Omega'} \varphi\, d(p_*\tilde{\nu}) \leq \int_{\Omega'} \varphi\, d\lambda \quad (\varphi \in SL^1(\Omega', \lambda)), \end{cases} \tag{6.4}$$

and

$$\nu = \lambda\chi_{\Omega} + \mu\,\chi_{\mathcal{M}\setminus\Omega}, \tag{6.5}$$

with $\Omega \subset \mathcal{M}$ characterized by

$$\begin{cases} \mu < \lambda \text{ on } \mathcal{M} \setminus \Omega, \\ \int_{\Omega} \varphi\, d\mu \leq \int_{\Omega} \varphi\, d\lambda \quad (\varphi \in SL^1(\Omega, \lambda)). \end{cases}$$

Since p_* is a linear operator (6.1) gives

$$p_*\tilde{\nu} = p_*(\tilde{\lambda}\chi_{\tilde{\Omega}}) + p_*(\tilde{\mu}\chi_{\tilde{\mathcal{M}}\setminus\tilde{\Omega}}). \tag{6.6}$$

By the assumption $\tilde{\lambda} \geq p^*(\lambda)$ we have $p_*(\chi_{\tilde{\Omega}}\tilde{\lambda}) \geq \lambda\chi_{p(\tilde{\Omega})}$. Thus (6.6) shows that $p_*\tilde{\nu} \geq \lambda$ in $p(\tilde{\Omega})$. It follows that $\nu' \geq \lambda$ in $p(\tilde{\Omega})$, hence

$$p(\tilde{\Omega}) \subset \Omega'. \tag{6.7}$$

By definition of $p_*\tilde\nu$, the second part of (6.4) spells out to

$$\int_{p^{-1}(\Omega')} (\varphi\circ p)\, d\tilde\nu \le \int_{\Omega'} \varphi\, d\lambda \quad (\varphi\in SL^1(\Omega',\lambda)),$$

which in view of (6.1) gives that

$$\int_{p^{-1}(\Omega')\cap\tilde\Omega} (\varphi\circ p)\, d\tilde\lambda + \int_{p^{-1}(\Omega')\setminus\tilde\Omega} (\varphi\circ p)\, d\tilde\mu \le \int_{\Omega'} \varphi\, d\lambda \quad (\varphi\in SL^1(\Omega',\lambda)).$$

Next we take $\psi = p^*\varphi = \varphi\circ p$ in (6.2). This gives

$$\int_{\tilde\Omega} (\varphi\circ p)\, d\tilde\mu \le \int_{\tilde\Omega} (\varphi\circ p)\, d\tilde\lambda \quad (\varphi\in SL^1(\Omega',\lambda)).$$

Combining with the previous inequality, and using that $p^{-1}(\Omega')\supset\tilde\Omega$ by (6.7), gives, for $\varphi\in SL^1(\Omega',\lambda)$,

$$\int_{\Omega'}\varphi\, d\mu = \int_{p^{-1}(\Omega')} (\varphi\circ p)\, d\tilde\mu = \int_{\tilde\Omega} (\varphi\circ p)\, d\tilde\mu + \int_{p^{-1}(\Omega')\setminus\tilde\Omega} (\varphi\circ p)\, d\tilde\mu$$

$$\le \int_{\tilde\Omega} (\varphi\circ p)\, d\tilde\lambda + \int_{p^{-1}(\Omega')\setminus\tilde\Omega} (\varphi\circ p)\, d\tilde\mu \le \int_{\Omega'} \varphi\, d\lambda \quad (\varphi\in SL^1(\Omega',\lambda)).$$

In summary,

$$\int_{\Omega'}\varphi\, d\mu \le \int_{\Omega'}\varphi\, d\lambda \quad (\varphi\in SL^1(\Omega',\lambda)). \tag{6.8}$$

We also have, by (6.1), (6.4) and, respectively, (6.7),

$$p_*(\tilde\mu\chi_{\tilde{\mathcal M}\setminus\tilde\Omega}) \le p_*\tilde\nu < \lambda \quad \text{on } \mathcal M\setminus\Omega',$$

$$p_*(\tilde\mu\chi_{\tilde\Omega}) = 0 \quad \text{in } \mathcal M\setminus\Omega'.$$

Therefore $\mu = p_*\tilde\mu < \lambda$ on $\mathcal M\setminus\Omega'$. In combination with (6.8) this gives

$$\nu = \lambda\chi_{\Omega'} + \mu\chi_{\mathcal M\setminus\Omega'}.$$

Now (6.3), (6.6), (6.7) finally give

$$\nu' = \lambda\chi_{\Omega'} + (p_*\tilde\nu)\,\chi_{\mathcal M\setminus\Omega'} = \lambda\chi_{\Omega'} + (p_*(\tilde\lambda\chi_{\tilde\Omega}) + p_*(\tilde\mu\chi_{\tilde{\mathcal M}\setminus\tilde\Omega}))\chi_{\mathcal M\setminus\Omega'}$$

$$= \lambda\chi_{\Omega'} + (p_*\tilde\mu)\,\chi_{\mathcal M\setminus\Omega'} = \lambda\chi_{\Omega'} + \mu\,\chi_{\mathcal M\setminus\Omega'} = \nu,$$

as desired.

$\square$

7 Global simply connected weak solutions

As already mentioned, given $p : \mathcal{M} \to \mathbb{C}$ as in Section 5 and any initial domain $\tilde{\Omega}(0) \subset \mathcal{M}$ with $\tilde{0} \in \Omega(0)$, a unique global weak solution $\{\tilde{\Omega}(t) : 0 \leq t < \infty\}$, in the sense of Definition 5.1, exists if just $\mathcal{M}$ is large enough. And if $\mathcal{M}$ is not large enough from outset it may always be extended, in many ways (cf. Example 5.2), to allow for such a global weak solution. However, even if the initial domain $\tilde{\Omega}(0)$ is simply connected the weak solution will in general not remain simply connected all the time.

Now, our main statement, Theorem 7.1, asserts that if $\tilde{\Omega}(0)$ is simply connected and has analytic boundary, then it is indeed always possible to choose $\mathcal{M} \supset \tilde{\Omega}(0)$ so that the solution $\tilde{\Omega}(t)$ in $\mathcal{M}$ remains simply connected all the time. Without referring to any Riemann surface the assertion may be formulated simply as saying that there exists a global weak solution of the Löwner-Kufarev equation, for any given $f(\cdot, 0) \in \mathcal{O}_{\text{norm}}(\overline{\mathbb{D}})$. The solution cannot not always be smooth in t, because if zeros of $g = f'$ reach the unit circle then it is in most cases necessary to change the structure of g in order to make the solution go on. The Riemann surfaces involved are needed mainly to make the appropriate notion of a weak solution precise (Definition 5.1).

The difficulty in constructing $\mathcal{M}$ lies in the fact that it cannot be constructed right away, but has to be created along with the solution. It has to be updated every time a zero of g for the corresponding Löwner-Kufarev equation reaches the unit circle. Unfortunately, as we have not been able to settle Conjecture 1.1 stated in the introduction, we have to include the validity of this conjecture among the assumptions in the theorem below. The precise formulation is as follows.

Theorem 7.1. *Let $f(\cdot, 0) \in \mathcal{O}_{\text{norm}}(\overline{\mathbb{D}})$ be given, together with $q(t) \geq 0$ ($0 \leq t < \infty$) such that $Q(t) \to \infty$ as $t \to \infty$. Then, under the assumption that Conjecture 1.1 (or Conjecture 7.3 below) is true, there exists a Riemann surface $\mathcal{M}$, a nonconstant holomorphic function $p : \mathcal{M} \to \mathbb{C}$ and a point $\tilde{0} \in \mathcal{M}$ with $p(\tilde{0}) = 0$ such that the following assertions hold.*

- *(i) $f(\cdot, 0)$ factorizes over $\mathcal{M}$, i.e., there exists a univalent function $\tilde{f}(\cdot, 0) : \mathbb{D} \to \mathcal{M}$ with $\tilde{f}(0, 0) = \tilde{0}$ such that $f(\cdot, 0) = p(\tilde{f}(\cdot, 0))$.*
- *(ii) On setting $\tilde{\Omega}(0) = \tilde{f}(\mathbb{D}, 0)$, the weak Hele-Shaw evolution $\{\tilde{\Omega}(t)\}$ on $\mathcal{M}$ with initial domain $\tilde{\Omega}(0)$ exists for all $0 \leq t < \infty$ and $\tilde{\Omega}(t)$ is simply connected for each t.*

(iii) *Let* $\nu_{f(\cdot,t)}$ *denote the counting function of* $f(\cdot,t) = p(\tilde{f}(\cdot,t))$ *and let* $\Omega(t)$ *denote the domain obtained by partial balayage of* $\nu_{f(\cdot,t)}m$ *onto Lebesgue measure* m*:*

$$\mathrm{Bal}\,(\nu_{f(\cdot,t)}m, m) = \chi_{\Omega(t)}m.$$

Then the family $\{\Omega(t)\}$ *is a weak solution in the ordinary sense on* $\mathbb{C}$*, with the domains* $\Omega(t)$ *possibly multiply connected.*

For the proof of Theorem 7.1 we shall need a few auxiliary results, stated below.

Lemma 7.1. *Let* $f \in \mathcal{O}_{\rm norm}(\overline{\mathbb{D}})$ *and let* $0 < r < 1$*. Then the following are equivalent.*

(i) f *extends to be meromorphic in* $\mathbb{D}(0, \frac{1}{r})$ *with poles only at the reflected (in* $\partial\mathbb{D}$*) zeros of* g*, more precisely so that* $fg^* \in \mathcal{O}(\mathbb{D}(0,\frac{1}{r}) \setminus \overline{\mathbb{D}})$*.*

(ii) *For every number* ρ *with* $r < \rho < 1$ *there exists a constant* C_ρ *such that*

$$|\int_{\mathbb{D}} h|g|^2 dm| \leq C_\rho \sup_{\mathbb{D}(0,\rho)} |h| \quad (h \in \mathcal{O}(\overline{\mathbb{D}})). \tag{7.1}$$

(iii) *For every number* ρ *with* $r < \rho < 1$ *there exists a (signed) measure* σ *with* $\operatorname{supp}\sigma \subset \overline{\mathbb{D}(0,\rho)}$ *such that*

$$\int_{\mathbb{D}} h|g|^2 dm = \int h d\sigma \quad (h \in \mathcal{O}(\overline{\mathbb{D}})). \tag{7.2}$$

Proof. Assume (i). Then for every $r < \rho < 1$ we have

$$\int_{\mathbb{D}} h|g|^2 dm = \frac{1}{2\mathrm{i}} \int_{\partial\mathbb{D}} h\bar{f} df = \frac{1}{2\mathrm{i}} \int_{\partial\mathbb{D}} hf^* g d\zeta = \frac{1}{2\mathrm{i}} \int_{\partial\mathbb{D}(0,\rho)} hf^* g d\zeta,$$

where we used that $f^*g = (fg^*)^* \in \mathcal{O}(\mathbb{D} \setminus \overline{\mathbb{D}(0,r)})$, by assumption. Now (ii) follows with $C_\rho = \frac{1}{2}\int_{\partial\mathbb{D}(0,\rho)} |f^*g||d\zeta|$.

That (ii) implies (iii) follows from general functional analysis (the Hahn-Banach theorem and the Riesz representation theorem for functionals on $C(\overline{\mathbb{D}(0,\rho)})$, see [31]).

Assume now (iii) and we shall prove (i). Consider the Cauchy transforms of σ and, $|g|^2\chi_{\mathbb{D}}$, defined by

$$\hat{\sigma}(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{d\sigma(\zeta)}{z-\zeta},$$

$$G(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{|g(\zeta)|^2 dm(\zeta)}{z-\zeta}, \tag{7.3}$$

respectively. Here G is defined and continuous in all $\mathbb{C}$ and satisfies, in the sense of distributions,

$$\frac{\partial G}{\partial \bar{z}} = \bar{g}g\chi_{\mathbb{D}}.$$

Thus, in $\mathbb{D}$,

$$G = \bar{f}g + H$$

for some $H \in \mathcal{O}(\mathbb{D})$. This equality also defines H on $\partial\mathbb{D}$, by which it becomes continuous on $\overline{\mathbb{D}}$.

On the other hand, (7.2) shows that $\hat{\sigma} = G$ outside $\overline{\mathbb{D}}$, and by continuity this also holds on $\partial\mathbb{D}$. Hence

$$f^* g = \bar{f}g = G - H = \hat{\sigma} - H$$

on $\partial\mathbb{D}$, and since the right member is holomorphic in $\mathbb{D}\setminus\overline{\mathbb{D}(0,\rho)}$ the desired meromorphic extension of f follows.

$\square$

If $f(\cdot,t) \in \mathcal{O}_{\mathrm{univ}}(\overline{\mathbb{D}})$ is a univalent weak solution then it is known [11], [16] that the radius of analyticity of f is an increasing function of time. In the non-univalent case this is no longer true, but there is a related radius (essentially $1/r$ in the previous lemma) which is stable in time (actually increases), and this will be a good enough statement for our needs.

Lemma 7.2. *Let $\tilde{\Omega}(\cdot,t) = \tilde{f}(\mathbb{D},t)$ be a simply connected weak solution on a Riemann surface $\mathcal{M}$ with projection $p : \mathcal{M} \to \mathbb{C}$ and let $f(\zeta,t) = p(\tilde{f}(\zeta,t))$. Assume $q(t) \geq 0$ and that for a certain $0 < r < 1$ the equivalent conditions in Lemma 7.1 hold for $f = f(\cdot,0)$. Then they hold with the same r for all $f(\cdot,t)$, $t > 0$.*

Proof. If $f(\cdot,t) \in \mathcal{O}_{\mathrm{norm}}(\overline{\mathbb{D}})$ is a weak solution on $\mathcal{M}$ starting at $t = 0$ then, by (5.27),

$$\int_{\mathbb{D}} h(z)|g(z,t)|^2\, dm(z) = \int_{\mathbb{D}} h(\varphi(\zeta,0,t))|g(\zeta,0)|^2\, dm(\zeta) + 2\pi Q(t)h(0)$$

for all $h \in \mathcal{O}(\overline{\mathbb{D}})$. Assume now that condition (ii) of Lemma 7.1 holds at $t = 0$ for some $0 < r < 1$. Since $|\varphi(\zeta,0,t)| \leq |\zeta|$ by Schwarz' lemma we then get, for an arbitrary ρ with $r < \rho < 1$,

$$|\int_{\mathbb{D}} h(\varphi(\zeta,0,t))|g(\zeta,0)|^2 dm(\zeta)| \leq C_\rho \sup_{\zeta\in\mathbb{D}(0,\rho)} |h(\varphi(\zeta,0,t))| \leq C_\rho \sup_{z\in\mathbb{D}(0,\rho)} |h(z)|,$$

hence

$$|\int_{\mathbb{D}} h(z)|g(z,t)|^2\, dm(z)| \leq (C_\rho + 2\pi Q(t)) \sup_{z\in\mathbb{D}(0,\rho)} |h(z)|.$$

This shows that (ii) of Lemma 7.1 holds also at any time $t > 0$, with the same r as for $t = 0$, which is what we needed to prove.

□

The final auxiliary result is a conjecture, which is very likely to be true but for which we have no complete proof at present. It was therefore was listed among the assumptions in Theorem 7.1. It concerns the issue of keeping $\tilde{\Omega}(t)$ simply connected all the time. With the weak solution pulled back to $\mathbb{D}$, as in Example 5.1, the crucial statement becomes the following, formulated in terms of (5.15).

Conjecture 7.3. *Let $g \in \mathcal{O}(\overline{\mathbb{D}})$ be fixed (independent of t) and denote by $\{D(t) : 0 \leq t < \varepsilon\}$ the weak solution for the weight $|g|^2$ and initial domain $D(0) = \mathbb{D}$, with $\varepsilon > 0$ is so small that all domains $D(t)$ are compactly contained in the region of analyticity of g. In other words,*

$$\int_{D(t)} h|g|^2 dm \geq \int_{\mathbb{D}} h|g|^2 dm + 2\pi Q(t)h(0)$$

for every $h \in SL^1(D(t), |g|^2 m)$, and $0 \leq t \leq \varepsilon$. Then, if $\varepsilon > 0$ is sufficiently small, the domains $D(t)$ are star-shaped with respect to the origin, in particular simply connected.

In terms of partial balayage, $D(t) \supset \mathbb{D}$ is given by

$$\mathrm{Bal}\,(2\pi Q(t)\delta_0, |g|^2\chi_{\mathcal{G}\setminus\mathbb{D}}m) = |g|^2\chi_{D(t)\setminus\mathbb{D}}m,$$

where $\mathcal{G} \supset \overline{\mathbb{D}}$ is the domain of analyticity of g. If g has no zeros on $\partial\mathbb{D}$, then Conjecture 7.3 indeed holds, by virtue of stability results for free boundaries [2], [5], or else by existence of classical solutions [3]. But we need Conjecture 7.3 exactly in the case when g has zeros on $\partial\mathbb{D}$.

Some steps towards a proof of Conjecture 7.3. In terms of the function $u = u(z,t)$ appearing in Definition 4.2 for the choice $\mu = 2\pi Q(t)\delta_0$, $\lambda = |g|^2\chi_{\mathcal{G}\setminus\mathbb{D}}m$, the weak solution $\{D(t) : 0 \leq t < \varepsilon\}$ is given by

$$D(t) = \{z \in \mathbb{C} : u(z,t) > 0\}$$

with u satisfying (and determined by)

$$\begin{cases} u \geq 0 \quad \text{in} \quad \mathbb{C}, \\ \Delta u = |g|^2\chi_{D(t)\setminus\mathbb{D}} - 2\pi Q(t)\delta_0 \quad \text{in} \quad \mathbb{C}, \\ u = |\nabla u| = 0 \quad \text{outside} \quad D(t). \end{cases}$$

These properties follow in a standard manner from the complementarity system (of type (4.8), (4.9)) satisfied by u.

Now write, in terms of polar coordinates $z = re^{\mathrm{i}\theta}$,

$$v = r\frac{\partial u}{\partial r}.$$

This function is continuous in $\mathbb{C} \setminus \{0\}$ because the elliptic partial differential equation which u satisfies (in the sense of distributions) shows that u is continuously differentiable, even across $\partial D(t)$. In order to show that $D(t)$ is star-shaped it is enough to show that u decreases in each radial direction, i.e., that $v \leq 0$. This is what one hopes to show, for $t > 0$ small enough.

In the region $D(t) \setminus \mathbb{D}$ we have $\Delta u = |g|^2$, and easy computations show that this translates into the equation

$$\Delta v = 2|g|^2\,\mathrm{Re}(1 + \frac{zg'}{g}), \quad z \in D(t) \setminus \mathbb{D}, \tag{7.4}$$

for v. As to boundary conditions we have

$$v = 0 \quad \text{on} \quad \partial D(t),$$

since u vanishes together with its first derivative there. Inside $\mathbb{D}$, $u(z) = -Q(t)\log|z| + \text{harmonic}$, and except for the logarithmic singularity at the origin, u is continuously differentiable in all $D(t)$. It follows that v is harmonic in $\mathbb{D}$ with $v(0) = -Q(t)$ and that v is continuous in all $D(t)$. (On $\partial\mathbb{D}$ there is a jump in the first derivatives.)

Unfortunately we cannot be sure of the sign of the right member of (7.4). If we knew that it was nonnegative, then the desired conclusion $v \leq 0$ would follow immediately from the maximum principle. To clarify the situation we make a local analysis around a point on $\partial\mathbb{D}$ at which g vanishes. We may assume that this point is $z = 1$, and then we can write

$$g(z) = (z-1)^d h(z),$$

where d is the order of the zero and h is analytic with $h(1) \neq 0$. We compute the right member of (7.4) as

$$2|g(z)|^2 \operatorname{Re}(1 + \frac{zg'(z)}{g(z)}) = 2|z-1|^{2d}|h(z)|^2 \operatorname{Re}(1 + \frac{dz}{z-1} + \frac{zh'(z)}{h(z)})$$

$$= 2|z-1|^{2d}\operatorname{Re}(zh'(z)\overline{h(z)}) + 2|z-1|^{2(d-1)}|h(z)|^2 \operatorname{Re}((z-1)(\bar{z}-1) + d\cdot z(\bar{z}-1))$$

$$= 2|z-1|^{2(d-1)}|h(z)|^2 \left(|z-1|^2 \operatorname{Re}\frac{zh'(z)}{h(z)} + (d+1)(|z - \frac{d+2}{2d+2}|^2 - (\frac{d}{2d+2})^2\right).$$

Here the second term inside the bracket is positive outside the circle with center $\frac{d+2}{2d+2}$ and radius $\frac{d}{2d+2}$, in particular outside $\mathbb{D}$, while the first term may have any sign.

Thus, the right member in (7.4) is positive in major parts of neighborhoods (outside $\mathbb{D}$) of points on $\partial\mathbb{D}$ where g vanishes. In some examples, like if h is constant, which will be the case in the example in Section 9.3 below, it follows that the right member in (7.4) is positive in all $D(t) \setminus \mathbb{D}$, and the star-shapedness can be inferred. Close to other points on $\partial\mathbb{D}$ one can perform an analysis based on known stability behavior of free boundaries [2], [5]. This gives at least that, outside any fixed neighborhood of $z = 1$, $D(t)$ does not have any holes if $t > 0$ is small enough.

Proof. (of theorem)

To get started, observe that we can find $\mathcal{M}$ so that (i) holds. It is just to take $\mathcal{M} = \mathbb{D}$, $\tilde{f}(\zeta, 0) = \zeta$ and $p = f(\cdot, 0)$. Compare the proof of Lemma 3.1. The remaining part of the proof consists of extending the Riemann surface $\mathcal{M}$ so that (ii) remains valid; (i) will automatically remain valid.

So assume that we have constructed $\mathcal{M}$ so that (ii) holds on a time interval $[0, T]$, where $T \geq 0$ ($T = 0$ not excluded). We shall show how to extend $\mathcal{M}$ (if necessary) and the solution, to some interval $[0, T + \varepsilon]$, $\varepsilon > 0$.

We have $\tilde{\Omega}(t) = \tilde{f}(\mathbb{D}, t)$, where $\tilde{f}(\cdot, t) : \mathbb{D} \to \mathcal{M}$, $f(\cdot, t) = p \circ \tilde{f}(\cdot, t) \in \mathcal{O}_{\text{norm}}(\overline{\mathbb{D}})$, $f(\cdot, t)$ is a subordination chain and $f(\cdot, t)$ is meromorphic in a disk $\mathbb{D}(0, \frac{1}{r})$, with $0 < r < 1$ independent of t by Lemma 7.2.

Set $\mathcal{M}_T = \tilde{\Omega}(T) \subset \mathcal{M}$. This is the only part of $\mathcal{M}$ which is needed up to time T, and it can be identified with $\mathbb{D}_T = \mathbb{D}$ via $\tilde{f}(\cdot, T)$. Now choose $1 < \rho < \frac{1}{r}$ (with r as in Lemma 7.2) so that $f'(\zeta, T)$ has no zeros for $1 < |\zeta| < \rho$ (but may have it for $|\zeta| = 1$). Viewing $\mathbb{D}(0, \rho) \supset \mathbb{D}$ as a Riemann surface over $\mathbb{C}$ with covering map $f(\cdot, T)$ we get, on the level of abstract Riemann surfaces, an extension $\mathcal{M}' \supset \mathcal{M}$ of $\mathcal{M}$. On $\mathcal{M}'$ we can continue the weak solution to some time interval $[0, T + \varepsilon]$, $\varepsilon > 0$. Compare the discussion in Example 5.1. For $\varepsilon > 0$ small enough this solution $\tilde{\Omega}(t)$ remains simply connected, assuming Conjecture 7.3. Then set $\mathcal{M}_{T+\varepsilon} = \tilde{\Omega}(T + \varepsilon)$.

Thus we can always extend a weak solution defined on a closed time interval to a larger interval. We also have to show that whenever we have a solution on a half-open interval $[0, T)$ (with $T > 0$) it can be extended to the closure $[0, T]$. However, this is fairly immediate because we can simply define $\tilde{\Omega}(T) = \mathcal{M}_T = \cup_{0 \leq t < T} \mathcal{M}_t$ (cf. proof of Lemma 3.1). This surface is easily seen to be simply connected (because any closed curve in $\mathcal{M}_T$ will lie entirely in $\mathcal{M}_t$ for some $t < T$). Moreover, the defining property (5.10) of a weak solution will hold on all $[0, T]$, and since the radius of analyticity of $f(\cdot, T) : \mathbb{D} \to \tilde{\Omega}(T)$ is larger than one (Lemma 7.2), $\tilde{\Omega}(T) \cong \mathbb{D}$ will have compact closure in a larger Riemann surface $\mathcal{M} \supset \mathcal{M}_T$, on which the evolution may continue.

The above arguments show that there is no finite stopping time for the construction of $\mathcal{M}$ and a simply connected weak solution in $\mathcal{M}$. Therefore part (ii) of the theorem follows.

Assertion (iii) of the theorem is an easy consequence of Proposition 6.1. $\square$

8 General structure of rational solutions

In this section we shall prove that the property of $g = f'$ being a rational function is preserved in time for weak solutions as long as they remain simply connected. In other words, it is preserved by the Löwner-Kufarev equation, even under transition of zeros of g through $\partial\mathbb{D}$. However, g acquires additional zeros and poles under such an event, and the transition will not be smooth. We shall also show that for certain other solutions of the Polubarinova-Galin equation rationality is also preserved.

We shall give two avenues to the question of rationality: first a direct approach, just making an 'Ansatz' of a rational g in a suitable version of the Polubarinova-Galin equation, and secondly via quadrature identities, which are related to the concept of a weak solution and which can incorporate transitions of zeros through $\partial\mathbb{D}$.

8.1 Direct approach

We assume that g rational of the form (2.12), and we address the question to which extent this form is preserved in time for solutions of the Polubarinova-Galin or Löwner-Kufarev equations when g is allowed to have zeros in $\mathbb{D}$.

Recall (Theorem 3.2) that the Polubarinova-Galin equation is equivalent to a relaxed version of the Löwner-Kufarev equation,

$$\dot{f}(\zeta,t)=\zeta f'(\zeta,t)\left(P(\zeta,t)+R(\zeta,t)\right), \tag{8.1}$$

where $P(\zeta,t)$ is the Poisson integral (2.8) and where $R(\zeta,t)$ is any function of the form (3.8). We shall here assume, for simplicity, that the zeros ω_j of g are simple, and then (3.8) becomes

$$R(\zeta,t)=\mathrm{i}\,\mathrm{Im}\sum_{\omega_j\in\mathbb{D}}\frac{2B_j(t)}{\omega_j(t)}+\sum_{\omega_j\in\mathbb{D}}\left(\frac{2B_j(t)}{\zeta-\omega_j(t)}-\frac{2\overline{B_j(t)}\zeta}{1-\overline{\omega_j(t)}\zeta}\right). \tag{8.2}$$

The interpretation of the free constants $B_j(t)$ here is that they determine the speed of the branch points $f(\omega_j(t),t)$. Using (8.2) we have

$$\frac{d}{dt}f(\omega_j(t),t)=2\omega_j(t)f''(\omega_j(t)),t)B_j(t).$$

The equation (8.1) for f is equivalent to the equation, generalizing (2.13),

$$\frac{\partial}{\partial t}(\log g)=\zeta(P+R)\frac{\partial}{\partial\zeta}(\log g)+\frac{\partial}{\partial\zeta}(\zeta(P+R)) \tag{8.3}$$

for g. Here the derivatives of $\log g$ are obtained from (2.15), (2.16), and it only remains to evaluate the Poisson integral $P(\zeta,t)$. This can be done by a simple residue calculus in (2.8), using that $|g(\zeta,t)|^2 = g(\zeta,t)g^*(\zeta,t)$ when $\zeta \in \partial\mathbb{D}$, where the right member is a rational function in ζ. However, the calculation becomes more transparent if everything is done at an algebraic level, by which it essentially reduces to an expansion in partial fractions.

Recall that, by definitions of P and R,

$$P(\zeta,t) + P^*(\zeta,t) = \frac{2q(t)}{g(\zeta,t)g^*(\zeta,t)}, \tag{8.4}$$

$$R(\zeta,t) + R^*(\zeta,t) = 0.$$

The rational function $q(t)/g(\zeta,t)g^*(\zeta,t)$ has poles at the zeros of g and g^*, i.e., at $\omega_1,\dots,\omega_m,\omega_1^*,\dots,\omega_m^*$. At infinity it has the behavior (by (2.12))

$$\lim_{\zeta\to\infty} \frac{q(t)}{g(\zeta,t)g^*(\zeta,t)} = A_\infty = \begin{cases} \frac{q\prod_{j=1}^{n}\bar{\zeta}_j}{|b|^2\prod_{j=1}^{m}\bar{\omega}_j} & \text{if } m=n, \\ 0 & \text{if } m>n. \end{cases} \tag{8.5}$$

We shall assume, in addition to the zeros ω_k being simple, that no two zeros are reflections of each other with respect to the unit circle, i.e., we assume that $\omega_k \neq \omega_j^*$ for all k,j and, in particular ($k=j$), that there are no zeros on the unit circle. These assumptions are necessary in order to expect the existence of a smooth solution of the Polubarinova-Galin equation (2.1), and even more so for the Löwner-Kufarev equation. Indeed, spelling out (2.1) as

$$\dot{f}(\zeta,t)\cdot\zeta^{-1}g^*(\zeta,t) + \dot{f}^*(\zeta,t)\cdot\zeta g(\zeta,t) = 2q(t) \tag{8.6}$$

we see that if, for some particular value of t, g and g^* have a common zero, with $\dot{f}$ and $\dot{f}^*$ finite, then $q(t)$ must be zero.

It will be seen shortly (equation (8.9)) that P is a rational function whenever g is rational, hence, by (8.1), also $\dot{f}$ is rational. So (8.6) is an identity between rational functions, and so is valid throughout the Riemann sphere.

With the above assumptions in force we can write

$$\frac{q(t)}{g(\zeta,t)g^*(\zeta,t)} = A_\infty + \sum_{k=1}^{m}\frac{\overline{A}_k}{\overline{\omega}_k} + \sum_{k=1}^{m}\left[\frac{A_k}{\zeta-\omega_k} + \frac{\overline{A}_k\zeta}{1-\overline{\omega}_k\zeta}\right], \tag{8.7}$$

where the coefficients $A_k = A_k(t,b,\omega_1,\dots,\omega_m,\zeta_1,\dots,\zeta_n)$ are given by

$$A_k = \frac{q(t)}{g'(\omega_k,t)g^*(\omega_k,t)} = \frac{q}{|b|^2}\cdot\frac{\prod_j(\omega_k-\zeta_j)\prod_j\overline{(\omega_k^*-\zeta_j)}}{\prod_{j\neq k}(\omega_k-\omega_j)\prod_j\overline{(\omega_k^*-\omega_j)}} \tag{8.8}$$

for $1 \le k \le m$. Notice that some of the A_k may vanish: if $\omega_k \in \mathbb{D}$ and ω_k^* coincides with one of the poles ζ_j, then $A_k = 0$.

Now, $P(\zeta, t)$ is to be that holomorphic function in $\mathbb{D}$ whose real part has boundary values $q(t)/g(\zeta, t)g^*(\zeta, t)$ and whose imaginary part vanishes at the origin. The function (8.7) itself certainly has the right boundary behaviour on $\partial\mathbb{D}$, but it is not holomorphic in $\mathbb{D}$. On the other hand, the two types of polar parts occurring in (8.7) have the same real parts on the boundary:

$$\operatorname{Re} \frac{A_k}{\zeta - \omega_k} = \operatorname{Re} \frac{\overline{A}_k \zeta}{1 - \overline{\omega}_k \zeta} \quad \text{on } \partial\mathbb{D}.$$

Therefore, without changing the real part on the boundary we can make the function (8.7) holomorphic in $\mathbb{D}$ by a simple exchange of polar parts. In addition, one can add a purely imaginary constant to account for the normalization of P at the origin. This gives

$$P(\zeta, t) = A_0 + \sum_{\omega_j \in \mathbb{C}\setminus\overline{\mathbb{D}}} \frac{2A_j}{\zeta - \omega_j} + \sum_{\omega_j \in \mathbb{D}} \frac{2\overline{A}_j \zeta}{1 - \overline{\omega}_j \zeta}, \tag{8.9}$$

with the $A_j = A_j(t)$ given by (8.8) for $1 \le j \le m$. For A_0 we have

$$\operatorname{Re} A_0 = \operatorname{Re} A_\infty + \operatorname{Re} \sum_{k=1}^{m} \frac{A_k}{\omega_k},$$

$$\operatorname{Im} A_0 = \operatorname{Im} \sum_{\omega_j \in \mathbb{C}\setminus\overline{\mathbb{D}}} \frac{2A_j}{\omega_j},$$

so that the real part of (8.7) remains unaffected in the passage to (8.9), and so that the normalization $\operatorname{Im} P(0, t) = 0$ is achieved. Note that $\operatorname{Re}(P(\zeta, t) + R(\zeta, t)) \ge 0$ in $\mathbb{D}$ if and only if $R = 0$ (because R has poles in $\mathbb{D}$ if $R \ne 0$).

Thus

$$P(\zeta, t) + R(\zeta, t) =$$

$$= A_0 + \mathrm{i} \operatorname{Im} \sum_{\omega_j \in \mathbb{D}} \frac{2B_j}{\omega_j} + \sum_{\omega_j \in \mathbb{C}\setminus\overline{\mathbb{D}}} \frac{2A_j}{\zeta - \omega_j} + \sum_{\omega_j \in \mathbb{D}} \frac{2B_j}{\zeta - \omega_j} + \sum_{\omega_j \in \mathbb{D}} \frac{2(\overline{A}_j - \overline{B}_j)\zeta}{1 - \overline{\omega}_j \zeta},$$

$$= C + \sum_{\omega_j \in \mathbb{C}\setminus\overline{\mathbb{D}}} \frac{2A_j}{\zeta - \omega_j} + \sum_{\omega_j \in \mathbb{D}} \frac{2B_j}{\zeta - \omega_j} - \sum_{\omega_j \in \mathbb{D}} \frac{2(\overline{A}_j - \overline{B}_j)(\omega_j^*)^2}{\zeta - \omega_j^*},$$

where

$$C = A_0 + \mathrm{i}\,\mathrm{Im} \sum_{\omega_j \in \mathbb{D}} \frac{2B_j}{\omega_j} - \sum_{\omega_j \in \mathbb{D}} 2(\overline{A}_j - \overline{B}_j)\omega_j^*.$$

Also,

$$\zeta(P(\zeta,t) + R(\zeta,t)) =$$

$$= C\zeta + D + \sum_{\omega_j \in \mathbb{C}\setminus\overline{\mathbb{D}}} \frac{2A_j\omega_j}{\zeta - \omega_j} + \sum_{\omega_j \in \mathbb{D}} \frac{2B_j\omega_j}{\zeta - \omega_j} - \sum_{\omega_j \in \mathbb{D}} \frac{2(\overline{A}_j - \overline{B}_j)(\omega_j^*)^3}{\zeta - \omega_j^*},$$

with

$$D = \sum_{\omega_j \in \mathbb{C}\setminus\overline{\mathbb{D}}} 2A_j + \sum_{\omega_j \in \mathbb{D}} 2B_j - \sum_{\omega_j \in \mathbb{D}} 2(\overline{A}_j - \overline{B}_j)(\omega_j^*)^2.$$

In view of (2.15), (2.16) the dynamical law (8.3) becomes

$$\frac{\dot{b}}{b} - \sum_{k=1}^{m} \frac{\dot{\omega}_k}{\zeta - \omega_k} + \sum_{j=1}^{n} \frac{\dot{\zeta}_j}{\zeta - \zeta_j} = \tag{8.10}$$

$$= \big(C\zeta + D + \sum_{\omega_j \in \mathbb{C}\setminus\overline{\mathbb{D}}} \frac{2A_j\omega_j}{\zeta - \omega_j} + \sum_{\omega_j \in \mathbb{D}} \frac{2B_j\omega_j}{\zeta - \omega_j} - \sum_{\omega_j \in \mathbb{D}} \frac{2(\overline{A}_j - \overline{B}_j)(\omega_j^*)^3}{\zeta - \omega_j^*}\big)$$

$$\cdot\big(\sum_{k=1}^{m} \frac{1}{\zeta - \omega_k} - \sum_{j=1}^{n} \frac{1}{\zeta - \zeta_j}\big) +$$

$$+C - \sum_{\omega_j \in \mathbb{C}\setminus\overline{\mathbb{D}}} \frac{2A_j\omega_j}{(\zeta - \omega_j)^2} - \sum_{\omega_j \in \mathbb{D}} \frac{2B_j\omega_j}{(\zeta - \omega_j)^2} + \sum_{\omega_j \in \mathbb{D}} \frac{2(\overline{A}_j - \overline{B}_j)(\omega_j^*)^3}{(\zeta - \omega_j^*)^2}.$$

The derivatives $\dot{b}$, $\dot{\omega}_k$, $\dot{\zeta}_j$ to be determined appear as coefficients in the constant term and poles of order one. Therefore (8.10) can be satisfied only if all terms with poles of higher order cancel out. This automatically occurs for the terms of the form $\frac{2A_j\omega_j}{(\zeta-\omega_j)^2}$ $(\omega_j \in \mathbb{C}\setminus\overline{\mathbb{D}})$ and $\frac{2B_j\omega_j}{(\zeta-\omega_j)^2}$ $(\omega_j \in \mathbb{D})$.

In order that the remaining terms $\sum_{\omega_j \in \mathbb{D}} \frac{2(\overline{A}_j - \overline{B}_j)(\omega_j^*)^3}{(\zeta - \omega_j^*)^2}$ with poles of the second order shall disappear we must have, for each j with $\omega_j \in \mathbb{D}$, that

$$A_j = B_j. \tag{8.11}$$

These second order poles cannot cancel in any other way. In order to allow the rational form (2.12) to be stable in time we therefore assume from now on that (8.11) holds.

Note that R under this assumption becomes uniquely determined. When (8.11) holds,

$$C = A_0 + \mathrm{i}\,\mathrm{Im} \sum_{\omega_j \in \mathbb{D}} \frac{2A_j}{\omega_j} = \mathrm{Re}\, A_\infty + \mathrm{Re} \sum_{k=1}^{m} \frac{A_k}{\omega_k} + \mathrm{i}\,\mathrm{Im} \sum_{j=1}^{m} \frac{2A_j}{\omega_j}, \tag{8.12}$$

$$D = \sum_{j=1}^{m} 2A_j \tag{8.13}$$

and $P + R$ takes the simpler form

$$P(\zeta, t) + R(\zeta, t) = C + \sum_{j=1}^{m} \frac{2A_j}{\zeta - \omega_j}.$$

The dynamical law (8.10) now becomes

$$\frac{\dot{b}}{b} - \sum_{k=1}^{m} \frac{\dot{\omega}_k}{\zeta - \omega_k} + \sum_{j=1}^{n} \frac{\dot{\zeta}_j}{\zeta - \zeta_j} = \tag{8.14}$$

$$= \big(C\zeta + D + \sum_{j=1}^{m} \frac{2A_j \omega_j}{\zeta - \omega_j}\big) \cdot \big(\sum_{k=1}^{m} \frac{1}{\zeta - \omega_k} - \sum_{j=1}^{n} \frac{1}{\zeta - \zeta_j}\big) + C - \sum_{j=1}^{m} \frac{2A_j \omega_j}{(\zeta - \omega_j)^2}.$$

Therefore (8.10) results in the following system of ordinary differential equations for ω_k, ζ_j, b.

Theorem 8.1. *Under the assumption that g has only simple zeros, that g and g^* have no common zeros (in particular g has no zero on $\partial\mathbb{D}$), and that in addition (8.11) holds, the Polubarinova-Galin equation (2.1), or (8.3), gives the following rational dynamics for g:*

$$\frac{d}{dt} \log \omega_k = -C - \frac{2A_k}{\omega_k} - \sum_{j=1,\, j \neq k}^{m} \frac{2(A_k + A_j)}{\omega_k - \omega_j} + \sum_{j=1}^{n} \frac{2A_k}{\omega_k - \zeta_j}$$

$$= P^*(\omega_k) + R^*(\omega_k) - \frac{2A_k}{\omega_k}\Big(1 + \sum_{j=1}^{m} \frac{1}{1 - \overline{\omega}_j \omega_k} - \sum_{j=1}^{n} \frac{1}{1 - \overline{\zeta}_j \omega_k}\Big), \tag{8.15}$$

$$\frac{d}{dt} \log \zeta_j = -C - \sum_{k=1}^{m} \frac{2A_k}{\zeta_j - \omega_k} = P^*(\zeta_j) + R^*(\zeta_j), \tag{8.16}$$

$$\frac{d}{dt}\log b = (m-n+1)C. \tag{8.17}$$

Here the coefficients A_j, C are given by (8.5), (8.8), (8.12).

Since the $B_j(t)$ are completely free functions we can simply define them by (8.11). Then (8.15)–(8.17) is a regular system of ordinary differential equations, having a unique solution as long as the stated conditions on the zeros of g and g^* hold.

The above unique rational solution of the Polubariova-Galin equation solves the Löwner-Kufarev equation if and only if $R = 0$. By (8.11) this requires that $A_k = 0$ for each k with $\omega_k \in \mathbb{D}$. Looking at (8.8) we see that, if $q \neq 0$, the only way that A_k can vanish at a given instant is that $\omega_k^* = \zeta_j$ for some j. However, it will be seen in Example 8.3 below that A_k may vanish for some particular value of t without vanishing for all t. Therefore we also need that

$$\dot{\omega}_k^* = \dot{\zeta}_j,$$

so that the condition $A_k = 0$ persists in time. We shall show that this is the case if and only if ω_k^* is a pole of g of order at least two (more generally, of strictly higher order than that of the zero).

Assume that at one particular moment, say $t = 0$, $\omega_k^* = \zeta_j$ for some pair k, j. Then $A_k = 0$ at that moment so that (8.15), (8.16) (with $R = 0$) become

$$\begin{cases} \frac{d}{dt}\log\omega_k = P^*(\omega_k), \\ \frac{d}{dt}\log\zeta_j = P^*(\zeta_j). \end{cases} \tag{8.18}$$

This can also be written

$$\begin{cases} \frac{d}{dt}\log\omega_k^* = -P(\omega_k^*), \\ \frac{d}{dt}\log\zeta_j = P^*(\omega_k^*). \end{cases}$$

Thus we see that $\frac{d}{dt}\log\omega_k^* = \frac{d}{dt}\log\zeta_j$ holds if and only if $P(\omega_k^*) + P^*(\omega_k^*) = 0$, which, recalling (8.4), happens if and only if gg^* has a pole at $\omega_k^* = \zeta_j$. Looking at the expression (2.12) for g one sees that this occurs if and only if the pole of g at ζ_j is of higher order than the zero of g at ω_k.

Finally, an informal remark about multiple zeros (a situation which is not covered by the above analysis). If a zero $\omega_k \in \mathbb{C} \setminus \overline{\mathbb{D}}$ is of order ≥ 2 then P will have a pole at ω_k of the same order, and it is easy to see that this pole will never cancel out in the dynamical equation (8.10). For this reason multiple zeros outside $\overline{\mathbb{D}}$ can never survive, even though collisions may occur. The solution will remain smooth over a collision because if two roots, ω_1 and ω_2, collide the equations still will be regular when reformulated in terms of the combinations $\omega_1 + \omega_2$ and $\omega_1\omega_2$.

On the other hand, if g has a multiple zero ω_k in $\mathbb{D}$ this will not cause any higher order pole of P if g has a pole of at least the same order at ω_k^*, and if the order of the pole is of strictly higher order then the same situation will persist in time. Therefore a solution will be obtained as before.

By now we have proved the following theorem on local behavior of solutions of (8.1), or (8.3).

Theorem 8.2. *Given $g(\zeta, 0)$ of the form (2.12) such that no two zeros of $g(\zeta, 0)$ are related by $\omega_k = \omega_j^*$, then for exactly one choice of $R(\zeta, t)$, namely that defined by (8.11), there exists a solution $g(\zeta, t)$ of (8.3) which remains on the original rational form (2.12).*

Necessary and sufficient condition for this rational solution to also solve the Löwner-Kufarev equation (2.7) is that $R(\zeta, t) = 0$. This occurs precisely under the condition that whenever $g(\zeta, t)$ has a zero ω_k in $\mathbb{D}$, the reflected point ω_k^* is a pole of $g(\zeta, t)$ of order strictly greater than that of the zero. This property is conserved in time.

Every pole ζ_j of g moves out from the origin, and every zero ω_k inside the unit disk moves towards the origin, as time increases.

The last statement follows from (8.18) together with the fact that P is positive in $\mathbb{D}$, negative outside, and the opposite for P^*.

Remark 8.1. A particular consequence of Theorem 8.2 is that the there are no polynomial solutions of the Löwner-Kufarev equation with zeros of g in the unit disk.

Example 8.2. Consider g being of the form

$$g(\zeta, t) = b(t)(\zeta - \omega_1(t)),$$

with $\omega_1(t) > 0$ and $b(t)$ real. In the notations of (2.12) and Theorem 8.1 we have $m = 1$, $n = 0$, $A_\infty = 0$, $A_1 = \frac{q\omega_1}{b^2(1-|\omega_1|^2)}$, $C = \frac{A_1}{\omega_1}$. This gives the dynamical system

$$\begin{cases} \dot{\omega}_1 = -\frac{3q\omega_1}{b^2(1-\omega_1^2)}, \\ \dot{b} = \frac{2q}{b(1-\omega_1^2)}, \end{cases} \tag{8.19}$$

expressing the necessary and sufficient conditions for (8.1) or (8.3), i.e., the Polubarinova-Galin equation (2.1), to hold.

Starting with $b(0) = -1$, $\omega_1(0) = 1$, and choosing $q(t) = e^{2t} - e^{-4t}$, which is positive for $t > 0$, one verifies that $\omega_1(t) = e^{3t}$, $b(t) = -e^{-2t}$ is a solution of (8.19). The corresponding mapping function $f(\zeta, t)$ is then univalent, and starts out from $f(\zeta, 0) = \zeta - \frac{1}{2}\zeta^2$, which maps $\mathbb{D}$ onto a cardioid with a cusp on the boundary. To be more precise, the system (8.19) seems to be singular for the initial location $\omega_1 = 1$ of the zero of g. However, the data above are chosen so that q vanishes at the same time. Indeed, $\frac{q(t)}{1-\omega_1^2(t)} = -e^{-4t}$, which is smooth for all $t \in \mathbb{R}$. It follows that the given solution actually satisfies (8.19) for all $t \in \mathbb{R}$. This solution will be further discussed in Section 9.2.

For $t < 0$ we have $q(t) < 0$ and $0 < \omega_1(t) < 1$. It follows that if one lets t run backwards, from $t = 0$ to $t = -\infty$, then one will still be in the injection (source) case, but with the mapping function $f(\zeta, t)$ non-univalent. See Section 9.2 for some more discussion of this solution (then with t replaced by $-t$).

When $0 < \omega_1 < 1$ the above polynomial solution of the Polubarinova-Galin equation does not satisfy the Löwner-Kufarev equation because the 'branch point' $f(\omega_1(t), t)$ moves: $f(e^{3t}, t) = \frac{1}{2}e^{4t}$. However, the Löwner-Kufarev equation must also have a solution (at least a weak one). This will have a different structure, namely

$$g(\zeta, t) = b(t)\,\frac{(\zeta - \omega_1(t))(\zeta - \omega_2(t))(\zeta - \omega_3(t))}{(\zeta - \zeta_1(t))^2},$$

with $\zeta_1 = 1/\omega_1$. Here $0 < \omega_1 < 1$, while $\omega_{2,3} \in \mathbb{D}^e$ may be nonreal. In the notation of (2.12) and Theorem 8.1 we now have $m = 3$, $n = 2$ and

$$\begin{cases} A_\infty = 0, \\ A_1 = 0, \\ A_2 = \frac{q\omega_2(1-\omega_1\omega_2)(\omega_2-\omega_1)}{b^2\omega_1^4(\omega_2-\omega_3)(1-|\omega_2|^2)(1-\omega_2\bar{\omega}_3)}, \\ A_3 = \frac{q\omega_3(1-\omega_1\omega_3)(\omega_3-\omega_1)}{b^2\omega_1^4(\omega_3-\omega_2)(1-|\omega_3|^2)(1-\omega_3\bar{\omega}_2)}, \\ A_0 = C = \frac{A_2}{\omega_2} + \frac{A_3}{\omega_3} + \mathrm{i}\,\mathrm{Im}(\frac{A_2}{\omega_2} + \frac{A_3}{\omega_3}). \end{cases}$$

By insertion into (8.15), (8.16), (8.17) one gets a dynamical system for the data ω_1, ω_2, ω_3, ζ_1, b (with $\zeta_1 = 1/\omega_1$ given from outset). Despite this system looking quite complicated, the solution can, for a suitable choice of $q(t)$, be spelled out in full detail, using different tools. This will be done in Section 9.3.

Example 8.3. Let g initially be given by

$$g(\zeta,0)=b(0)\frac{\zeta-\omega_1(0)}{\zeta-\zeta_1(0)}$$

for some $0<\omega_1(0)<1$, $\zeta_1(0)>1$, $b(0)>0$. Then, first of all, there exists a unique solution of the Polubarinova-Galin equation of the same form

$$g(\zeta,t)=b(t)\frac{\zeta-\omega_1(t)}{\zeta-\zeta_1(t)} \tag{8.20}$$

with $0<\omega_1(t)<1$, $\zeta_1(t)>1$, $b(t)>0$. The system of ordinary differential equations in Theorem 8.1 for $\omega_1(t)$, $\zeta_1(t)$, $b(t)$ explicitly becomes

$$\begin{cases}\dot{\omega}_1=-\frac{q\zeta_1}{b^2}-3A_1+\frac{2A_1\omega_1}{\omega_1-\zeta_1},\\ \dot{\zeta}_1=-\frac{q\zeta_1^2}{b^2\omega_1}-\frac{A_1\zeta_1}{\omega_1}+\frac{2A_1\zeta_1}{\omega_1-\zeta_1},\\ \dot{b}=\frac{q\zeta_1}{b\omega_1}+\frac{A_1b}{\omega_1},\end{cases}$$

where

$$A_1=\frac{q(\omega_1-\zeta_1)(1-\omega_1\zeta_1)}{b^2(1-|\omega_1|^2)},$$

and we have taken into account that all quantities are real. It is seen immediately that the solution will go on as long as $\omega_1(t)$, $\zeta_1(t)$, $b(t)$ stay in the above specified intervals. However, the so obtained solution cannot solve the Löwner-Kufarev equation because, by Theorem 8.2, that equation requires that g has a pole of order at least two at the reflected point of $\omega_1(t)$.

The Löwner-Kufarev equation also has a solution. This represents an evolution on a Riemann surface $\mathcal{M}$ above $\mathbb{C}$ with a branch point over $f(\omega_1(t),t)$, which has to be a fixed (time-independent) point. If $\zeta_1(0)\neq\omega_1^*(0)$ this solution (it will be unique after the Riemann surface $\mathcal{M}$ has been fixed) will be of the form

$$g(\zeta,t)=b(t)\frac{(\zeta-\omega_1(t))(\zeta-\omega_2(t))(\zeta-\omega_3(t))}{(\zeta-\zeta_1(t))(\zeta-\zeta_2(t))^2},$$

where $\zeta_2(t)=\omega_1(t)^*$ and $\omega_2(0)=\omega_3(0)=\zeta_2(0)$. If $\zeta_1(0)=\omega_1^*(0)$ it will be of the slightly simpler form

$$g(\zeta,t)=b(t)\frac{(\zeta-\omega_1(t))(\zeta-\omega_2(t))}{(\zeta-\zeta_1(t))^2},$$

with $\zeta_1(t)=\omega_1(t)^*$ and $\omega_2(0)=\zeta_1(0)$. One then obtains the evolution in Example 5.3, where ζ_1 was used as time parameter. Thus, by (5.16), (5.18), $\omega_2(t)=2t\zeta_1(t)-\zeta_1(t)^{-1}$, $b(t)=b(0)\zeta_1(0)^{-3}\zeta_1(t)^3$.

8.2 Approach via quadrature identities

This approach to rational solutions has the advantage that it can incorporate transitions of zeros through $\partial\mathbb{D}$, even when $q \neq 0$.

In the previous subsection we saw that structural properties such as having an identity of the kind (7.2), or an estimate (7.1), holding are preserved in time when $f = f(\cdot, t)$ represents a weak solution. The same type of argument also shows that the property of g being a rational function is preserved, because such property is equivalent to an identity (7.2) holding with σ of a particularly simple form. We start by elaborating a lemma making this statement precise.

When g is rational the computation in the beginning of the proof of Lemma 7.1 can be made more explicit and ends up with a quadrature formula for $h \in \mathcal{O}(\overline{\mathbb{D}})$. Specifically we get

$$\frac{1}{\pi}\int_{\mathbb{D}} h|g|^2 dm = \frac{1}{2\pi \mathrm{i}}\int_{\partial\mathbb{D}} hf^* df = \sum \operatorname*{Res}_{\mathbb{D}}(hf^* g d\zeta) + \sum_j c_j \int_{\gamma_j} hg d\zeta. \quad (8.21)$$

Here the γ_j are arcs in $\mathbb{D}$ connecting the points where f^* has logarithmic poles. The above computation actually does not require h to be holomorphic in $\mathbb{D}$, it is enough that hg is holomorphic. Thus one can allow h to have a pole at any zero of g in $\mathbb{D}$.

Equation (8.21) is a Riemann surface version, pulled back to $\mathbb{D}$, of (2.11). In the terminology of [35] the corresponding domain $\tilde{\Omega} = \tilde{f}(\mathbb{D})$ ($\tilde{f}$ being the lift of f, as in Section 3)), then is a **quadrature Riemann surface**. The formula (8.21) may alternatively be presented without line integrals by expressing the right member in terms of an integral of h, namely

$$H(z) = \int_0^z h(\zeta)g(\zeta)d\zeta.$$

The quadrature identity then becomes

$$\frac{1}{\pi}\int_{\mathbb{D}} h|g|^2 dm = -\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{D}} dH \wedge d\bar{f} = -\sum \operatorname*{Res}_{\mathbb{D}}(Hdf^*) = \sum \operatorname*{Res}_{\mathbb{D}} \frac{H(\zeta)g^*(\zeta)d\zeta}{\zeta^2}.$$

By spelling out the results of the residue calculations, and taking into account the other direction we have the following.

Proposition 8.1. *Let $f \in \mathcal{O}_{\rm norm}(\overline{\mathbb{D}})$. Then $g = f'$ is a rational function if and only if there exist α_j, γ_j, a_{kj}, c_j, r, ℓ, n_j so that the quadrature identity*

$$\frac{1}{\pi}\int_{\mathbb{D}} h|g|^2 dm = \sum_{j=1}^{r} c_j \int_{\gamma_j} hg d\zeta + \sum_{j=0}^{\ell}\sum_{k=1}^{n_j-1} a_{jk}h^{(k-1)}(\alpha_j) \tag{8.22}$$

holds for all $h \in \mathcal{O}(\overline{\mathbb{D}})$. Here we have used the same numbering as in (2.11). In particular, $\alpha_0 = 0$. The end points of the γ_j are the logarithmic poles of f^* and the α_j are the ordinary poles of f^*g in $\mathbb{D}$. In other words, with g of the form (2.12), these points are from the set $\{\zeta_0^*, \zeta_1^*, \dots, \zeta_\ell^*\}$, $\zeta_0 = \infty$.

In addition, if $I \ni t \mapsto f(\cdot, t) \in \mathcal{O}_{\rm norm}(\overline{\mathbb{D}})$ represents a weak solution of the evolution problem in Definition 5.1 and Proposition 5.2, then the form (8.22) is stable over time, assuming that it holds initially. However, the data depend on time: $a_{jk} = a_{jk}(t)$, $\alpha_j = \alpha_j(t)$, $\gamma_j = \gamma_j(t)$, with the qualification that $\alpha_0 = 0$ is fixed and that for a_{01} we have the precise behavior $a_{01}(t) = a_{01}(0) + 2Q(t)$. (Thus $a_{01}(t)$ may become zero at one moment of time.)

Remark 8.4. The time dependence of the data, for the evolution problem, is caused by the chain rule, and disappears on using test functions of the form $h(f(\zeta, t))$, or more generally time dependent test functions $h(\zeta, t)$ satisfying (5.21) or (5.23).

The weak solution concept is based on the Löwner-Kufarev equation, but (8.22) is actually stable in time also for solutions of the Polubarinova-Galin equation. This follows from Proposition 5.1.

Proof. For the first statement in the proposition, the 'only if' part follows by evaluating the residues in the previous formulas, the ζ_j being the poles of f^*g in $\mathbb{D}$. Note that a zero ω of g in $\mathbb{D}$ will allow f^* to have a pole at the same point, and of the same order, hence g to have a pole of one order higher at the reflected point ω^*, without causing a contribution in the right member of (8.22). Alternatively, one may allow the test function h to have a pole at ω.

To prove the 'if' part we use in (8.22) the test functions

$$h(\zeta) = \frac{1}{z - \zeta} \quad (\zeta \in \mathbb{D}),$$

with $z \notin \overline{\mathbb{D}}$. Then the left member of (8.22) becomes the previously used (see (7.3)) Cauchy transform G of $|g|^2\chi_{\mathbb{D}}$ while the right hand side takes the form

$R(z) + Q(z)$, where $R(z)$ is a rational function and $Q(z)$ is the contribution from the line integrals:

$$Q(z) = \sum_j c_j \int_{\gamma_j} \frac{g(\zeta)d\zeta}{z-\zeta}.$$

Reasoning as in the proof of Lemma 7.1 we first get $G = \bar{f}g + H$ on $\overline{\mathbb{D}}$ for some $H \in \mathcal{O}(\mathbb{D})$ which is continuous up to $\partial\mathbb{D}$, and then the identity

$$\bar{f}g + H = R + Q$$

on $\partial\mathbb{D}$. The latter relation can also be written as

$$f^*(z) = \frac{R(z)}{g(z)} - \frac{H(z)}{g(z)} + \frac{Q(z)}{g(z)} \quad (z \in \partial\mathbb{D}). \tag{8.23}$$

The integrals appearing in the definition of $Q(z)$ make jumps of magnitude $\pm 2\pi \mathrm{i} g(z)$ as z crosses γ_j from one side to the other. It follows that the first two terms in the right member of (8.23) are meromorphic functions in $\mathbb{D}$ while the last term is holomorphic except for constant ($= 2\pi \mathrm{i} c_j$) jumps across the arcs γ_j. These jumps disappear when differentiating (8.23). The conclusion is that $df(z) = g(z)dz$ is a rational (Abelian) differential (or f an Abelian integral), because the right member gives the appropriate extension of f to the Riemann sphere. Thus g is a rational function, as claimed. Note that (8.23) then holds identically in $\mathbb{C}$.

The second statement in the proposition, about weak solutions, is an easy consequence of (5.27).

□

Example 8.5. With

$$g(\zeta) = b\,\frac{(\zeta-\omega_1)(\zeta-\omega_2)}{(\zeta-\zeta_1)^2}$$

the quadrature identity is in general of the form

$$\frac{1}{\pi}\int_{\mathbb{D}} h|g|^2 dm = a_0 h(0) + a_1 h(\zeta_1^*) + c\int_0^{\zeta_1^*} hg\,d\zeta.$$

However, if $\zeta_1^* = \omega_1$ (or $\zeta_1^* = \omega_2$) then $a_1 = 0$ and if $\zeta_1 = \frac{1}{2}(\omega_1+\omega_2)$ (implying that g has no residues) then $c = 0$. Both of this occurred in Example 5.3.

Taking the full Hele-Shaw evolution, as in Examples 5.2 and 5.3, into account we therefore see that one can achieve a quadrature identity description of the evolution on the unified form

$$\frac{1}{\pi}\int_{\mathbb{D}} h(\zeta)|g(\zeta,t)|^2 dm(\zeta) = 2Q(t)h(0)$$

for $0 < t < \infty$, despite the fact that $f(\zeta,t)$ changes behavior as in (5.19) when the zero of g passes through the unit circle.

Example 8.6. Similarly, with

$$g(\zeta) = b\,\frac{(\zeta-\omega_1)(\zeta-\omega_2)(\zeta-\omega_3)}{(\zeta-\zeta_1)^2}$$

one gets in general a quadrature identity of the form

$$\frac{1}{\pi}\int_{\mathbb{D}} h|g|^2 dm = a_{01}h(0) + a_{02}h'(0) + a_{11}h(\zeta_1^*) + c\int_0^{\zeta_1^*} hg d\zeta.$$

If g has no residues and $\zeta_1^* = \omega_1$ then the constants a_{11} and c vanish and we get just

$$\frac{1}{\pi}\int_{\mathbb{D}} h|g|^2 dm = a_{01}h(0) + a_{02}h'(0).$$

This case will be discussed in Section 9.3.

9 Examples: several evolutions of a cardioid

In order to illustrate Theorem 7.1, as well as the structure theory in Section 8, we shall consider several different Hele-Shaw evolutions which all start out from the cardioid $\Omega(0) = f(\mathbb{D},0)$, where

$$f(\zeta,0) = \zeta - \frac{1}{2}\zeta^2. \tag{9.1}$$

Thus $g(\zeta,0) = 1-\zeta$, having the root $\omega_1(0) = 1$, which maps onto a cusp on $\partial\Omega(0)$ at $f(1,0) = \frac{1}{2}$. It is a major open problem to find some natural way to make Hele-Shaw suction ($q < 0$) starting from the above cardioid, and we shall briefly discuss this problem in Sections 9.4 and 10. We shall first construct three solutions which correspond to injection ($q > 0$), one of them (in Section 9.3) imitating the proof of Theorem 7.1.

9.1 The univalent solution

This is the ordinary univalent Hele-Shaw evolution $f(\cdot,t) \in \mathcal{O}_{\text{univ}}(\overline{\mathbb{D}})$, which by conservation of $M_1 = a_1^2 \bar{a}_2 = -\frac{1}{2}$ is given by

$$f(\zeta,t) = a_1(t)\zeta + a_2(t)\zeta^2 = a_1(t)\zeta - \frac{1}{2a_1(t)^2}\zeta^2.$$

Adapting $q(t)$ so that $a_1(t) = e^t$ $(0 \le t < \infty)$, for example, gives

$$f(\zeta,t) = e^t\zeta - \frac{1}{2}e^{-2t}\zeta^2, \quad q(t) = e^{2t} - e^{-4t}.$$

Note that $\omega_1(t) = e^{3t}$ starts out with finite speed, despite the cusp. This is possible because $q(0) = 0$. For $t > 0$, $q(t) > 0$.

The solution domains $\Omega(t) = f(\mathbb{D},t)$ $(0 \le t < \infty)$ enjoy the quadrature identities

$$\frac{1}{\pi}\int_{\Omega(t)} h dm = (2Q(t) + \frac{3}{2})h(0) - \frac{1}{2}h'(0), \tag{9.2}$$

holding for any $h \in \mathcal{O}(\overline{\Omega(t)})$.

9.2 A non-univalent solution of the Polubarinova-Galin equation

In the univalent solution, the coefficient a_1 ranges over the interval $1 \le a_1 < \infty$, and the moment $M_0 = a_1^2 + 2|a_2|^2 = a_1^2 + \frac{1}{2}a_1^{-4}$ is an increasing function of a_1. But, as a function of a_1, M_0 is strictly convex on the entire interval $0 < a_1 < \infty$, and it has a minimum for $a_1 = 1$. Thus M_0 increases also as a_1 decreases from 1 to 0. Choosing then $a_1 = e^{-t}$, $0 \le t < \infty$, gives our second solution

$$f(\zeta,t) = e^{-t}\zeta - \frac{1}{2}e^{2t}\zeta^2, \quad q(t) = e^{4t} - e^{-2t}.$$

This is not even locally univalent, but it does solve the Polubarinova-Galin equation. The zero of $g(\zeta,t)$, $\omega_1(t) = e^{-3t}$, moves from the unit circle towards the origin, and its image point, $f(e^{-3t},t) = \frac{1}{2}e^{-4t}$ also moves. Therefore the solution cannot be lifted to a fixed Riemann surface, and $f(\zeta,t)$ does not solve the Löwner-Kufarev equation (see Theorem 3.1).

A quadrature identity similar to (9.2) still holds. Considering first h to be defined in the image domain, it can be pulled back to $\mathbb{D}$ or be expressed in terms of the counting function (3.1). Exhibiting both we have

$$\frac{1}{\pi}\int_{\mathbb{D}} h(f(\zeta,t))|g(\zeta,t)|^2 dm(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} h\nu_{f(\cdot,t)}dm =$$

$$= (2Q(t)+\frac{3}{2})h(0)-\frac{1}{2}h'(0).$$

A slightly stronger form is obtained by using a time dependent test function, $h(\zeta,t)$, defined in $\mathbb{D}$ and such that (5.21) or (5.23) holds. Using (5.28) with $s=0$ then gives

$$\frac{1}{\pi}\int_{\mathbb{D}} h(\zeta,t)|g(\zeta,t)|^2 dm(\zeta) = (2Q(t)+\frac{3}{2})h(0,0)-\frac{1}{2}h'(0,0). \qquad (9.3)$$

Choosing instead h to be independent of time, the coefficients become time dependent:

$$\frac{1}{\pi}\int_{\mathbb{D}} h(\zeta)|g(\zeta,t)|^2 dm(\zeta) = (2Q(t)+\frac{3}{2})h(0)-\frac{e^t}{2}h'(0). \qquad (9.4)$$

9.3 A non-univalent solution of the Löwner-Kufarev equation

Even though the univalent solution in Section 9.1 is perfectly good in all respects, the solution which is constructed in the proof of Theorem 7.1 is a different one, namely one which goes up on a Riemann surface with two sheets. This is because the solution in the proof is constructed in such a way that at any time, say $t=t_0$, at which a zero of $g(\zeta,t)$ reaches $\partial\mathbb{D}$, the continued solution propagates on the Riemann surface which uniformizes $f^{-1}(\zeta,t_0)$ in a neighborhood of $\overline{\mathbb{D}}$. This is a necessary step in most cases, but occasionally (as in the present example, with $t_0=0$) it turns out that the original Riemann surface itself was actually good enough.

Below we calculate that solution which the proof of Theorem 7.1 would have given us. This has the additional advantage of giving a reference solution which can be used as comparison in order to obtain estimates for other solutions. The idea (cf. Example 8.2) is that the initial g, which we write as

$$g(\zeta,0) = -(\zeta-1)\cdot\frac{(\zeta-1)(\zeta-1)}{(\zeta-1)^2}, \qquad (9.5)$$

continues as

$$g(\zeta,t)=b(t)(\zeta-\omega_1(t))\cdot\frac{(\zeta-\omega_2(t))(\zeta-\omega_3(t))}{(\zeta-\zeta_1(t))^2},$$

where one of the zeros, say $\omega_1(t)$, moves into $\mathbb{D}$, $\zeta_1(t)=\omega_1^*(t)$ and where $\omega_2(t)$, $\omega_3(t)$ in addition are chosen so that $g(\zeta,t)$ has no residues. This means that $f(\zeta,t)$ will be of the form

$$f(\zeta,t)=-\frac{b_1\zeta+b_2\zeta^2+b_3\zeta^3}{\zeta-\zeta_1} \tag{9.6}$$

with $b_1=b_1(t)$, $b_2=b_2(t)$, $b_3=b_3(t)$ and $\zeta_1=\zeta(t)$ all real. The parameters b_1 and ζ_1 will turn out to be positive and strictly increasing in time. At time $t=0$ we have

$$\begin{cases} b(0)=-1,\\ b_1(0)=1,\\ b_2(0)=-\frac{3}{2},\\ b_3(0)=\frac{1}{2},\\ \omega_1(0)=\omega_2(0)=\omega_3(0)=\zeta_1(0)=1. \end{cases} \tag{9.7}$$

From (9.6) we obtain

$$g(\zeta,t)=\frac{b_1\zeta_1+2b_2\zeta_1\zeta+(3b_3\zeta_1-b_2)\zeta^2-2b_3\zeta^3}{(\zeta-\zeta_1)^2}, \tag{9.8}$$

$$f^*(\zeta,t)=\frac{b_1\zeta^2+b_2\zeta+b_3}{\zeta^2(\zeta_1\zeta-1)}.$$

The coefficients $b_j=b_j(t)$ and the pole $\zeta_1=\zeta_1(t)$ are to be determined according to the following principles:

- The reflected point of $\zeta_1(t)$ is to be a zero of g:

$$g(1/\zeta_1(t),t)=0.$$

- $f(\cdot,t)$ shall map the above point $1/\zeta_1(t)$ to a point which does not move:

$$f(1/\zeta_1(t),t)=\text{constant}=f(1,0)=\frac{1}{2}.$$

- The moment $M_1(t)$ is conserved in time:

$$M_1(t) = \operatorname*{Res}_{\zeta=0}(ff^*gd\zeta) = M_1(0) = -\frac{1}{2}.$$

- $M_0(t)$ evolves according to

$$M_0(t) = \operatorname*{Res}_{\zeta=0}(f^*gd\zeta) = M_0(0) + 2Q(t) = \frac{3}{2} + 2Q(t).$$

The constant values $\pm\frac{1}{2}$ and $\frac{3}{2}$ above are obtained from the initial data (9.7). Spelling out, the above equations become

$$\begin{cases} b_1\zeta_1^4 + 2b_2\zeta_1^3 + (3b_3\zeta_1 - b_2)\zeta_1 - 2b_3 = 0, \\ b_1\zeta_1^2 + b_2\zeta_1 + b_3 + \frac{1}{2}\zeta_1^2(1-\zeta_1^2) = 0, \\ b_1^2 b_3 - \frac{1}{2}\zeta_1^2 = 0, \\ b_1b_2\zeta_1 + 2b_2b_3\zeta_1 + b_1b_3(\zeta_1^2+2) + (\frac{3}{2}+2Q)\zeta_1^2 = 0. \end{cases} \tag{9.9}$$

Here we have four equations for the five time dependent parameters b_1, b_2, b_3, ζ_1 and Q. It turns out that it is possible to solve this system by expressing all paramenters in terms of b_1:

$$\begin{cases} \zeta_1 = +\sqrt{\frac{1}{2}(1+2b_1-\frac{1}{b_1^2})}, \\ b_2 = -\frac{\zeta_1}{4}(1+2b_1+\frac{3}{b_1^2}), \\ b_3 = \frac{2b_1^3+b_1^2-1}{4b_1^4}, \\ Q = \frac{1}{16b_1^6}(4b_1^8+2b_1^7-12b_1^6+b_1^4+6b_1^3+2b_1^2-3). \end{cases}$$

The range for $\zeta_1 = \zeta(t)$ is $1 \leq \zeta_1 < \infty$. At time $t=0$ we shall have $\zeta_1 = 1$. Then also $b_1 = 1$, and since one easily checks that $\frac{d\zeta_1}{db_1} > 0$ it is appropriate to fix the time scale by setting

$$b_1(t) = e^t.$$

By this all parameters b_1, b_2, b_3, ζ_1, Q become explicit functions of t. Including expansions for small $t > 0$ we have

$$\begin{cases} \zeta_1(t) = \sqrt{\frac{1}{2}(1+2e^t-e^{-2t})} = 1+t-\frac{3}{4}t^2+O(t^3), \\ b_1(t) = e^t = 1+t+\frac{1}{2}t^2+O(t^2), \\ b_2(t) = -\frac{1}{4\sqrt{2}}(1+2e^t+3e^{-2t})\sqrt{1+2e^t-e^{-2t}} = -\frac{3}{2}-\frac{1}{2}t+\frac{3}{8}t^2+O(t^3), \\ b_3(t) = \frac{1}{4}(2e^{-t}+e^{-2t}-e^{-4t}) = \frac{1}{2}-\frac{5}{4}t^2+O(t^3), \\ Q(t) = \frac{1}{16}(4e^{2t}+2e^t-12+e^{-2t}+6e^{-3t}+2e^{-4t}-3e^{-6t}) = 4t^3+O(t^4). \end{cases} \tag{9.10}$$

This gives

$$q(t) = \frac{1}{8}(4e^{2t}+e^t-e^{-2t}-9e^{-3t}-4e^{-4t}+9e^{-6t}) = 12t^2+O(t^3), \tag{9.11}$$

$$f(\zeta,t) = -\frac{2(1+t)\zeta-(3+t)\zeta^2+\zeta^3+O(t^2)}{2(\zeta-1-t+O(t^2))}.$$

Thus $q(0)=\dot{q}(0)=0$, while for $t>0$, $q(t)>0$, so the evolution is very slow in the beginning, in fact so slow that it is not at all singular at $t=0$.

As for $g(\zeta,t)$, we already know (by construction) that one of its zeros is $\omega_1(t)=1/\zeta_1(t)$. By dividing out this zero in (9.8) one gets g on the form

$$g(\zeta,t) = -2b_3(t)\,\frac{(\zeta-1/\zeta_1(t))(\zeta^2-\frac{1}{2}(b_1(t)^2+3)\zeta_1(t)\,\zeta+b_1(t)^3)}{(\zeta-\zeta_1(t))^2},$$

and the remaining two zeros $\omega_2(t)$, $\omega_3(t)$ are the zeros of the second degree polynomial in the numerator. One easily checks that the discriminant of that polynomial is negative on some interval $0<t<\varepsilon$, hence $\omega_2(t)$, $\omega_3(t)$ are non-real (a complex conjugate pair) for those values of t. For large t they are however real (the discriminant is positive). From

$$\omega_2(t)+\omega_3(t) = \frac{1}{2}(b_1(t)^2+3)\zeta_1(t), \quad \omega_2(t)\omega_3(t) = b_1(t)^3$$

one also realizes that the real parts of the two roots are increasing functions of t, for all $0<t<\infty$. For small t, (9.10) gives the expansions

$$\begin{cases} \omega_1(t) &= 1-t+O(t^2), \\ \omega_{2,3}(t) &= 1+(\frac{3}{2}\pm i\frac{\sqrt{7}}{2})t+O(t^2). \end{cases} \tag{9.12}$$

The solution $f(\zeta,t)$ represents an evolution of the cardioid which is non-univalent regarded as a map into $\mathbb{C}$ but which can be viewed as a univalent map $\tilde{f}(\zeta, t)$ into a two-sheeted Riemann surface over $\mathbb{C}$ (actually over the whole Riemann sphere $\mathbb{P}$). It is that solution which comes out of the construction in the proof of Theorem 7.1. This means that, with $f(\zeta, 0) = \zeta - \frac{1}{2}\zeta^2$, one initially considers $\mathcal{M}_0 = f(\mathbb{D}, 0) = \tilde{f}(\mathbb{D}, 0)$ as a Riemann surface over $\mathbb{C}$ and then gradually extends it, first for $\varepsilon > 0$ small, to $\mathcal{M}_\varepsilon = \tilde{f}(\mathbb{D}(0, 1+\varepsilon), 0)$, which simply is a copy of $\mathbb{D}(0, 1+\varepsilon)$. In the present case one can go on with this same procedure for arbitrary $\varepsilon > 0$, which eventually gives the two-sheeted covering surface $\mathcal{M} = \tilde{f}(\mathbb{P}, 0)$ of the Riemann sphere. It has branch points over $z = 1/2$ (by construction) and over $z = \infty$.

Conjecture 7.3 now concerns the solution pulled back to the unit disk by $f(\zeta, 0) = \zeta - \frac{1}{2}\zeta^2$. Thus the function g in that conjecture is $g(\zeta) = f'(\zeta, 0) = 1 - \zeta$. The inverse of $f(\zeta, 0) = \tilde{f}(\zeta, 0)$ is $\tilde{f}^{-1}(z, 0) = 1 - \sqrt{1 - 2z}$, hence one gets the function

$$\Phi(\zeta, t, 0) = f^{-1}(f(\zeta, t), 0) = 1 - \sqrt{1 + \frac{2}{\zeta - \zeta_1(t)}(b_1(t)\zeta + b_2(t)\zeta^2 + b_3(t)\zeta^3)},$$

which, for $0 < t < \varepsilon$ say, maps $\mathbb{D}$ conformally onto the slightly larger domain $D(t)$ (in the notation of Conjecture 7.3). Because of the square root it is not entirely trivial that $\Phi(\zeta, t, 0)$ is single-valued in $\mathbb{D}$. The pole at $\zeta = \zeta_1(t)$ causes no problem in this respect since $\zeta_1(t) \notin \mathbb{D}$, but since $1 \in D(t) = \Phi(\mathbb{D}, t, 0)$ there must be a point in $\mathbb{D}$ for which the expression under the square root vanishes. However, despite this the square root does in fact resolve into a single-valued function in $\mathbb{D}$. In terms of the notations in Lemma 3.1 and Section 5 we have $\Phi(\zeta, t, 0) = \tilde{f}(\zeta, t)$, $f(\zeta, 0) = p(\zeta)$, $g(\zeta) = p'(\zeta)$.

The function $\Phi(\zeta, t, 0)$ is similar to the subordination functions $\varphi(\zeta, s, t)$ (for $s < t$) in (3.2), but it goes the other way. It 'superordinates' a function $f(\zeta, t)$ at a time $t > 0$ in terms of an earlier function $f(\zeta, 0)$:

$$f(\zeta, t) = f(\Phi(\zeta, t, 0), 0). \tag{9.13}$$

The inverses of the superordination functions are defined in domains $D(t) \supset \mathbb{D}$, and their restrictions to the unit disk are simply the subordination functions:

$$\varphi(\zeta, 0, t) = \Phi^{-1}(\zeta, t, 0) \quad \text{for } \zeta \in \mathbb{D},\ t > 0.$$

The construction of $f(\zeta, t)$ is made in such a way that quadrature identities like (9.2), (9.4) remain valid. With a time dependent test function $h(\zeta, t)$, defined in $\mathbb{D}$ and satisfying (5.21) or (5.23), we have exactly the same identity (9.3) as in Section 9.2 (which is also valid for the example in Section 9.1). With h independent of time we get, as in (9.4), coefficients which depend on time, however now in a different way:

$$\frac{1}{\pi}\int_{\mathbb{D}} h(\zeta)|g(\zeta,t)|^2 dm = (2Q(t)+\frac{3}{2})\, h(0) - \frac{1}{2}\frac{\sqrt{2}e^t}{\sqrt{1+2e^t-e^{-2t}}}\, h'(0).$$

This formula follows by a straightforward calculation using (9.9). The additional time dependent factor in the last term is simply $1/g(0,t)$ (so also in (9.4)).

We may also write the quadrature identity in a form which connects to Conjecture 7.3 and in the proof of Theorem 7.1: using the fixed transition function $z = f(\zeta, 0)$ between the parameter space and the image space one finds, in terms of $D(t) = \Phi(\mathbb{D}, t, 0)$,

$$\frac{1}{\pi}\int_{D(t)} h(\zeta)|g(\zeta,0)|^2 dm = (2Q(t)+\frac{3}{2})h(0) - \frac{1}{2}h'(0). \tag{9.14}$$

This is in agreement with (5.14), and it thereby essentially confirms that the somewhat *ad hoc* attempt, starting with (9.5), for construction of a solution which uniformizes the cusp in fact gives exactly that solution which is produced in the proof of Theorem 7.1.

As for the geometry, the domains $D(t)$ are star-shaped with respect to the origin, hence the solution exists for all $0 < t < \infty$, cf. [11]. Indeed, the star-shapedness follows from equation (7.4) in the proof of Theorem 7.1, which in the present context becomes

$$\Delta v = 4(|z - \frac{3}{4}|^2 - \frac{1}{16}), \quad z \in D(t) \setminus \mathbb{D}.$$

Here, the right member is non-negative, and as $\Delta v = 0$ in $\mathbb{D}$, v is continuous in $D(t)$ and $v = 0$ on ∂D, the required inequality $v \leq 0$ in $D(t)$ follows from the maximum principle.

9.4 A solution for the suction case

An interesting aspect is that the now fully explicit solution $f(\zeta, t)$, defined for $0 < t < \infty$ by (9.6), (9.10), is not only smooth at $t = 0$, it even has a real analytic continuation across $t = 0$. This extended solution, defined for $-\varepsilon < t < \infty$ (say), has the drawback that it has a pole inside $\mathbb{D}$, but $q(t)$ remains positive for $t < 0$, as can be seen from (9.11). This means that the solution represents suction out of the cardioid as t decreases to negative values.

A closer look at $f(\zeta, t)$ for $t < 0$ shows that (keeping the notation from Section 9.3) the zero $\omega_1(t)$ is now outside the unit disk, while the two complex conjugate zeros $\omega_2(t)$, $\omega_3(t)$ are inside, as well as the pole $\zeta_1(t)$. Since $f(\zeta, t)$ is no longer holomorphic in $\mathbb{D}$ we are strictly speaking outside the scope of the previously developed theory, but it is easy to see that the Polubarinova-Galin equation (2.1) still makes sense. Because of the real analyticity of all data, the fact that (2.1) holds on the interval $0 < t < \infty$ implies that it automatically holds on $-\varepsilon < t < \infty$. Of course, this can also be verified by a direct (but quite tedious) calculation. The boundary curve $f(\partial\mathbb{D}, t)$ is for each $t < 0$ a smooth closed loop contained in $\Omega(0)$, and it recedes as t decreases.

To get a clear picture of the situation, note first that, for $t \neq 0$, $f(\zeta, t)$ is a rational function of order three, hence it maps the Riemann sphere $\mathbb{P}$ onto the sphere covered thrice. Viewed as a covering map, f has branch points over $\beta_j = f(\omega_j(t), t)$, $j = 1, 2, 3$, and $\beta_4 = \infty$. Obviously, β_4 does not depend on t, and the same is actually true also for β_1, by the construction of f in Section 9.3. In fact, $\beta = 1/2 =$ the cusp point of $\partial\Omega(0)$. As in previous situations we shall write $\tilde{f}$ when we think of f as a univalent map into a Riemann surface.

To make it perfectly clear, we first have a trivial decomposition of f as

$$\mathbb{P} \xrightarrow{\text{id}} \mathbb{P} \xrightarrow{f(\cdot,t)} \mathbb{P}.$$

Then the idea is to consider the middle $\mathbb{P}$ as an abstract Riemann surface, denoted $\mathcal{F}(t)$, and think of the last map as a covering projection. The first map will then be called $\tilde{f}(\cdot, t)$:

$$\mathbb{P} \xrightarrow{\tilde{f}(\cdot,t)} \mathcal{F}(t) \xrightarrow{\text{proj}} \mathbb{P}.$$

Because of the covering projection, $\mathcal{F}(t)$ is more than an abstract Riemann surface, it inherits a Riemannian metric from $\mathbb{P}$. And as a Riemannian manifold it really depends on t. Using the variable ζ in $\tilde{z} = \tilde{f}(\zeta, t)$ (where $\tilde{z} \in \mathcal{F}(t)$) as a coordinate, the metric on $\mathcal{F}(t)$ is given by

$$ds^2 = |f'(\zeta, t)|^2 |d\zeta|^2. \tag{9.15}$$

The branch points, together with appropriate 'cuts' between them, define an exact division of $\mathcal{F}(t)$ into sheets $\mathbb{P}^{(j)}$ (copies of $\mathbb{P}$) such that the pre-image $\tilde{f}^{-1}(a, t)$ of any point $a \in \mathbb{P}$ has one point on each sheet. If a is a branch point, two or more of these pre-images are common to some sheets. We may write the above as

$$\mathcal{F}(t) = (\mathbb{P}^{(1)} \cup \mathbb{P}^{(2)} \cup \mathbb{P}^{(3)})/\{\frac{1}{2}, \beta_2(t), \beta_3(t), \infty\}. \tag{9.16}$$

We have assumed that $t \neq 0$, and we choose the numbering so that $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$ are connected by the branch points at $\beta_1 = 1/2$ and ∞, and $\mathbb{P}^{(1)}$, $\mathbb{P}^{(3)}$ are connected at $\beta_2(t)$ and $\beta_3(t)$. To be precise about the cuts one may, for example, have one cut along the positive real axis, from $1/2$ to ∞, which serves as a passage from $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$, and another cut, disjoint from the previous, between $\beta_2(t)$ and $\beta_3(t)$, serving as a passage between $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(3)}$. The pre-images of these cuts under $f^{-1}(\cdot, t)$ are closed curves which divide $\mathbb{P}$ into three pieces, thereby they also divide $\mathcal{F}(t)$ into three pieces, which then become $\mathbb{P}^{(1)}$, $\mathbb{P}^{(2)}$, $\mathbb{P}^{(3)}$.

When $t = 0$ there are only two copies of $\mathbb{P}$, say

$$\mathcal{M} = \tilde{f}(\mathbb{P}, 0) = (\mathbb{P}^{(1)} \cup \mathbb{P}^{(2)})/\{\frac{1}{2}, \infty\},$$

since $f(\zeta, 0) = \zeta - \frac{1}{2}\zeta^2$ has order two. The notation $\mathcal{M}$ for this surface was introduced in Section 9.3. Thus the geometric meaning of changing $g(\zeta, 0) = 1 - \zeta$ to the form (9.5) is that one adds a new Riemann sphere $\mathbb{P}^{(3)}$, which for $t = 0$ is disconnected from $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$, but which gets attached via the branch points $\beta_2(t)$, $\beta_3(t)$ when t moves away from $t = 0$. For $t = 0$ it is convenient to define $\mathcal{F}(0)$ to be the disconnected Riemann surface corresponding to (9.5):

$$\mathcal{F}(0) = \mathcal{M} \cup \mathbb{P}^{(3)}.$$

For $0 < t < \varepsilon$, $\tilde{\Omega}(t) = \tilde{f}(\mathbb{D}, t)$ lies mainly in $\mathbb{P}^{(1)}$ (choosing the numbering this way), but also go up with a small piece, around $\beta_1 = 1/2$, to $\mathbb{P}^{(2)}$.

For example, for the counting function (3.1) we have $\nu_{f(\cdot,t)} = 2$ in a certain neighborhood of $z = 1/2$, while $\nu_{f(\cdot,t)} = 1$ in the remaining part of $\Omega(t) = f(\mathbb{D}, t)$.

Even though the Riemannian manifold $\mathcal{F}(t)$ as a whole depends on t, there is, when $0 < t < \varepsilon$, a certain neighborhood, say $\mathcal{M}_\varepsilon$, of the closure of $\tilde{\Omega}(t) = \tilde{f}(\mathbb{D}, t)$ which does not. This is because the branch points $\beta_2(t)$, $\beta_3(t)$, which are responsible for the time dependence, are outside $\mathcal{M}_\varepsilon$. A more precise argument for the time independence can be given in terms of superordination: instead of using, as in (9.15, ζ as a local variable we can in $\mathcal{M}_\varepsilon$ use $z = \Phi(\zeta, t, 0)$. This gives, by (9.13),

$$f'(\zeta, t)d\zeta = f'(\Phi(\zeta, t, 0), 0)\Phi'(\zeta, t, 0)d\zeta = f'(z, 0)dz,$$

which brings the metric in $\mathcal{M}_\varepsilon$ on the time independent form

$$ds^2 = |f'(z, 0)|^2|dz|^2.$$

When $-\varepsilon < t < 0$, on the other hand, $\omega_2(t), \omega_3(t) \in \mathbb{D}$ while $\omega_1(t) \in \mathbb{D}^e$, and it can be seen (by an argument to be given in Section 10) that

$$\tilde{\Omega}(t) = (\Omega^{(1)}(t) \cup \mathbb{P}^{(3)})/\{\beta_2(t), \beta_3(t)\}, \tag{9.17}$$

where $\Omega^{(1)}(t) \subset \mathbb{P}^{(1)}$ is a simply connected subdomain of $\Omega(0)$ which connects to $\mathbb{P}^{(3)}$ via $\beta_2(t), \beta_3(t) \in \Omega^{(1)}(t)$. Thus

$$\tilde{\Omega}^e(t) = ((\mathbb{P}^{(1)} \setminus \overline{\Omega^{(1)}(t)}) \cup \mathbb{P}^{(2)})/\{\frac{1}{2}, \infty\},$$

and since the branch points here are fixed there is a neighborhood, say $\mathcal{N}_\varepsilon$, of $\tilde{\Omega}^e(t)$ in $\mathcal{F}(t)$ on which the short time evolution takes place and can be generated by partial balayage.

A consequence of the above is that, when $-\varepsilon < t < 0$, $\partial\tilde{\Omega}(t) = \partial\Omega^{(1)}(t)$ is a simple closed curve which lies entirely in $\Omega(0)$. Forgetting about $\mathbb{P}^{(3)}$ in (9.17), it is tempting to view $\Omega^{(1)}(t)$ as a result of suction from $\Omega(0)$. However, since the surrounding Riemann surface is time dependent, as one sees from (9.17), this will be only a rather relaxed form of suction, satisfying the Polubarinova-Galin, but not the Löwner-Kufarev, equation. From a classical perspective it is always impossible to suck from a domain having that type of cusp $\Omega(0)$ has (cf. [36]), but allowing extra sheets and movable branch points makes the situation more flexible. We shall discuss these matters in some generality in the next section.

10 Injection versus suction in a Riemann surface setting

Recall the meaning of the Polubarinova-Galin equation (2.1), or (5.7), in terms of the Green's function for the Riemann surface domain $\tilde\Omega(t) = \tilde f(\mathbb{D}, t)$, as spelled out in (5.8):

$$\dot{\tilde f}_{\rm normal}(\tilde z, t) = 2\pi q(t)|\nabla G_{\tilde\Omega(t)}(\tilde z, \tilde 0)|, \quad \tilde z \in \partial\tilde\Omega(t). \tag{10.1}$$

The Green's function is simply obtained from that of the unit disk, by conformal invariance:

$$G_{\tilde\Omega(t)}(\tilde z, \tilde 0) = -\frac{1}{2\pi}\log|\zeta|, \tag{10.2}$$

where $\tilde z = \tilde f(\zeta, t)$.

Now assume that f is a rational function, of order r say. So we assume that $g = f'$ is a rational function free of residues. In terms of the structure (2.12) this means that the number m there lies in the interval $r - 1 \le m \le 2(r-1)$, where the extreme cases correspond to f having a single pole of order r (then f must be polynomial of degree r), respectively f having r distinct simple poles. When we consider f as a conformal map into a Riemann surface, and denote it by $\tilde f$ instead, then it maps $\mathbb{P}$ onto an r-fold covering surface $\mathcal{F}(t)$ of $\mathbb{P}$. As explained in the previous section, after a choice of cuts we can think of $\mathcal{F}(t)$ as consisting of r copies of $\mathbb{P}$ connected by branch points:

$$\tilde f(\cdot, t) : \mathbb{P} \to \mathcal{F}(t) = (\mathbb{P}^{(1)} \cup \cdots \cup \mathbb{P}^{(r)})/\{\text{branch points}\}.$$

Now $\tilde\Omega(t) = \tilde f(\mathbb{D}, t) \subset \mathcal{F}(t)$. Let $\tilde\Omega^e(t) = \tilde f(\mathbb{D}^e, t)$ be the complementary domain in $\mathcal{F}(t)$. Since the Green's function for $\mathbb{D}^e$ with pole at infinity is $\frac{1}{2\pi}\log|\zeta|$ it follows that

$$G_{\tilde\Omega^e(t)}(\tilde z, \tilde\infty) = \frac{1}{2\pi}\log|\zeta|,$$

with $\tilde z = \tilde f(\zeta, t)$ as before, and $\tilde\infty = \tilde f(\infty, t) \in \mathcal{F}(t)$. Thus, comparing with (10.2), $G_{\tilde\Omega^e}(\tilde z, \tilde\infty)$ is after a sign change simply the harmonic continuation of $G_{\tilde\Omega}(\tilde z, \tilde 0)$. In particular it follows that, on the common boundary $\partial\tilde\Omega(t) = \partial\tilde\Omega^e(t)$,

$$|\nabla G_{\tilde\Omega^e}(\tilde z, \tilde\infty)| = |\nabla G_{\tilde\Omega}(\tilde z, \tilde 0)|.$$

Returning to (10.1) this means that we also have

$$\dot{\tilde{f}}_{\rm normal}(\tilde{z},t)=2\pi q(t)|\nabla G_{\tilde{\Omega}^e(t)}(\tilde{z},\tilde{\infty})|,\quad \tilde{z}\in\partial\tilde{\Omega}^e(t). \tag{10.3}$$

The left member here is the same as in (10.1), but in relation to $\tilde{\Omega}^e(t)$ it is an inward pointing normal vector.

Thus the equation which describes injection at $\tilde{0}$ in $\tilde{\Omega}(t)$ at the same time describes suction at $\tilde{\infty}$ from $\tilde{\Omega}^e(t)$, and conversely. So the two problems are in the present setting equivalent, which might seem remarkable since the suction problem is known in general to be highly unstable and ill-posed, while injection always is stable and well-posed. The explanation is that we have lifted everything to a Riemann covering surface $\mathcal{F}(t)$, which has branch points allowed to move, and which is conformally equivalent to the Riemann sphere in such a way that $\tilde{\Omega}(t)$ and $\tilde{\Omega}^e(t)$ correspond to $\mathbb{D}$ and $\mathbb{D}^e$, respectively. With movable branch points in $\tilde{\Omega}(t)$ the solution of (10.1) is not unique, as follows from Theorem 3.2. Similarly for $\tilde{\Omega}^e(t)$ and (10.3). And some branch points must be allowed to move because $\mathcal{F}(t)$ as a whole is time dependent.

Thus lifting to a Riemann surface with movable branch points can be viewed as a kind of relaxation, opening up for more suction solutions. Indeed, if we can arrange that the branch points in $\tilde{\Omega}^e(t)$ are fixed, then we may perform injection at $\tilde{\infty}=\tilde{f}(\infty,t)\in\tilde{\Omega}^e(t)$ by partial balayage, and this will correspond, in some sense, to suction at $\tilde{0}$ from $\tilde{\Omega}(t)$.

As an example, we can explain the suction from $\Omega(0)$ obtained at the end of Section 9.4. We identify this initial domain, which has a cusp at $z=1/2$, with $\tilde{\Omega}(0)\subset\mathcal{F}(0)$, which lies entirely in $\mathbb{P}^{(1)}$, in the notation of Section 9.4. The complementary domain in $\mathcal{F}(0)$ then is $\tilde{\Omega}^e(0)=((\mathbb{P}^{(1)}\setminus\overline{\tilde{\Omega}(0)})\cup\mathbb{P}^{(2)})/\{1/2,\infty\}\cup\mathbb{P}^{(3)}$.

Now, each of $\mathbb{P}^{(1)}$, $\mathbb{P}^{(2)}$ and $\mathbb{P}^{(3)}$ has its own point of infinity, denote them by $\tilde{\infty}^{(1)}$, $\tilde{\infty}^{(2)}$, $\tilde{\infty}^{(3)}$, respectively, and we much choose from which of these to inject. But there is actually no choice, it is impossible to inject at $\tilde{\infty}^{(3)}$ because $\mathbb{P}^{(3)}$ is isolated (at this initial stage $t=0$), and the other two points are actually the same point in $\mathcal{F}(0)$ since they represent the branch point β_4 for $f(\zeta,0)$. In fact, $\tilde{\infty}^{(1)}=\tilde{\infty}^{(2)}=\tilde{f}(\infty,0)$, which is the correct source point.

In terms of partial balayage on $\mathcal{M}=\mathcal{F}(0)$, the desired evolution $\tilde{\Omega}^e(t)$, with initial domain $\tilde{\Omega}^e(0)$ and source $\tilde{\infty}^{(1)}=\tilde{\infty}^{(2)}=\tilde{f}(\infty,0)$, in principle becomes

$$\mathrm{Bal}(2\pi Q(-t)\delta_{\tilde{\infty}^{(1)}}+\chi_{\tilde{\Omega}^e(0)}\tilde{m},\tilde{m})=\chi_{\tilde{\Omega}^e(t)}\tilde{m}. \tag{10.4}$$

Here we are using the same time variable as in Section 9.4, which means that $-\varepsilon < t < 0$ and $Q(-t) > 0$.

Unfortunately, (10.4) does not really make sense because the measure we are sweeping has infinite mass. But it is easy to remedy the situation by subtracting $\chi_{\tilde{\Omega}^e(0)}\tilde{m}$ from both sides. Using that $\partial\tilde{\Omega}(0)$ is a nullset with respect to $\tilde{m}$ this gives

$$\mathrm{Bal}(2\pi Q(-t)\delta_{\tilde{\infty}^{(1)}}, \chi_{\tilde{\Omega}(0)}\tilde{m}) = \chi_{\tilde{\Omega}(0)\setminus\tilde{\Omega}(t)}\tilde{m}, \tag{10.5}$$

where then $\tilde{\Omega}(t)$ is the result of the suction out of $\tilde{\Omega}(0)$. Equation (10.5) makes perfectly good sense for $-\varepsilon < t < 0$, where $\varepsilon > 0$ is chosen so that $2\pi Q(\varepsilon) = \tilde{m}(\tilde{\Omega}(0))$, and it is the correct formula for the describing the evolution of $\tilde{\Omega}(t)$ (or $\tilde{\Omega}^e(t)$). Of course, everything can be pulled back from $\mathcal{M}$ to $\mathbb{P}$ by using ζ in $\tilde{z} = \tilde{f}(\zeta, 0)$ as global variable on $\mathcal{M}$. Then one gets

$$\mathrm{Bal}(2\pi Q(-t)\delta_\infty, |g|^2\chi_{\mathbb{D}}) = |g|^2\chi_{\mathbb{D}\setminus\hat{D}(t)} \quad (-\varepsilon < t < 0),$$

where $g(\zeta) = g(\zeta, 0) = 1 - \zeta$ and $\hat{D}(t) = \tilde{f}^{-1}(\tilde{\Omega}(t), 0)$.

References

[1] A. Abanov, M. Mineev-Weinstein, and A. Zabrodin, *Multi-cut solutions of Laplacian growth*, Phys. D, 238 (2009), pp. 1787–1796.

[2] L. A. Caffarelli, *A remark on the Hausdorff measure of a free boundary, and the convergence of coincidence sets*, Boll. Un. Mat. Ital. A (5), 18 (1981), pp. 109–113.

[3] J. Escher and G. Simonett, *Classical solutions of multidimensional Hele-Shaw models*, SIAM J. Math. Anal., 28 (1997), pp. 1028–1047.

[4] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[5] A. Friedman, *Variational principles and free-boundary problems*, Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1982. A Wiley-Interscience Publication.

[6] L. A. GALIN, *Unsteady filtration with a free surface*, C. R. (Doklady) Acad. Sci. URSS (N.S.), 47 (1945), pp. 246–249.

[7] S. J. GARDINER AND T. SJÖDIN, *Partial balayage and the exterior inverse problem of potential theory*, in Potential theory and stochastics in Albac, vol. 11 of Theta Ser. Adv. Math., Theta, Bucharest, 2009, pp. 111–123.

[8] B. GUSTAFSSON, *On a differential equation arising in a Hele-Shaw flow moving boundary problem*, Ark. Mat., 22 (1984), pp. 251–268.

[9] ——, *Lectures on balayage*, in Clifford algebras and potential theory, vol. 7 of Univ. Joensuu Dept. Math. Rep. Ser., Univ. Joensuu, Joensuu, 2004, pp. 17–63.

[10] B. GUSTAFSSON AND Y.-L. LIN, *On the dynamics of roots and poles for solutions of the Polubarinova-Galin equation*, Ann. Acad. Sci. Fenn. Math., 38 (2013), pp. 259–286.

[11] B. GUSTAFSSON, D. PROKHOROV, AND A. VASIL'EV, *Infinite lifetime for the starlike dynamics in Hele-Shaw cells*, Proc. Amer. Math. Soc., 132 (2004), pp. 2661–2669 (electronic).

[12] B. GUSTAFSSON AND J. ROOS, *Partial balayage on Riemannian manifolds*, arXiv.1605.03102, (2016).

[13] B. GUSTAFSSON AND M. SAKAI, *Properties of some balayage operators, with applications to quadrature domains and moving boundary problems*, Nonlinear Anal., 22 (1994), pp. 1221–1245.

[14] B. GUSTAFSSON AND H. S. SHAPIRO, *What is a quadrature domain?*, in Quadrature domains and their applications, vol. 156 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 2005, pp. 1–25.

[15] B. GUSTAFSSON, R. TEODERSCU, AND A. VASIL′EV, *Classical and stochastic Laplacian growth*, Advances in Mathematical Fluid Mechanics, Birkhäuser Verlag, Basel, 2014.

[16] B. GUSTAFSSON AND A. VASIL′EV, *Conformal and potential analysis in Hele-Shaw cells*, Advances in Mathematical Fluid Mechanics, Birkhäuser Verlag, Basel, 2006.

[17] H. Hedenmalm, *A factorization theorem for square area-integrable analytic functions*, J. Reine Angew. Math., 422 (1991), pp. 45–68.

[18] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*, vol. 199 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

[19] H. Hedenmalm and S. Shimorin, *Hele-Shaw flow on hyperbolic surfaces*, J. Math. Pures Appl. (9), 81 (2002), pp. 187–222.

[20] H. S. Hele-Shaw, *The flow of water*, Nature, 58 (1898), pp. 33–36.

[21] Y. E. Hohlov, S. D. Howison, C. Huntingford, J. R. Ockendon, and A. A. Lacey, *A model for nonsmooth free boundaries in Hele-Shaw flows*, Quart. J. Mech. Appl. Math., 47 (1994), pp. 107–128.

[22] D. Khavinson, M. Mineev-Weinstein, and M. Putinar, *Planar elliptic growth*, Complex Anal. Oper. Theory, 3 (2009), pp. 425–451.

[23] Y.-L. Lin, *Large-time rescaling behaviours of Stokes and Hele-Shaw flows driven by injection*, European J. Appl. Math., 22 (2011), pp. 7–19.

[24] ——, *Perturbation theorems for Hele-Shaw flows and their applications*, Ark. Mat., 49 (2011), pp. 357–382.

[25] P. Y. Polubarinova-Kochina, *On a problem of the motion of the contour of a petroleum shell*, Dokl. Akad. Nauk USSR, 47 (1945), pp. 254–257.

[26] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975. With a chapter on quadratic differentials by Gerd Jensen, Studia Mathematica/Mathematische Lehrbücher, Band XXV.

[27] M. Reissig and L. von Wolfersdorf, *A simplified proof for a moving boundary problem for Hele-Shaw flows in the plane*, Ark. Mat., 31 (1993), pp. 101–116.

[28] S. Richardson, *Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel*, J. Fluid Mech., 56 (1972), pp. 609–618.

[29] J. Roos, *Equilibrium measures and partial balayage*, Complex Anal. Oper. Theory, 9 (2015), pp. 65–85.

[30] J. Ross and D. W. Nyström, *The Hele-Shaw flow and moduli of holomorphic discs*, Compos. Math., 151 (2015), pp. 2301–2328.

[31] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York, third ed., 1987.

[32] P. G. Saffman and G. Taylor, *The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid*, Proc. Roy. Soc. London. Ser. A, 245 (1958), pp. 312–329. (2 plates).

[33] M. Sakai, *Quadrature domains*, vol. 934 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1982.

[34] ——, *Applications of variational inequalities to the existence theorem on quadrature domains*, Trans. Amer. Math. Soc., 276 (1983), pp. 267–279.

[35] ——, *Finiteness of the family of simply connected quadrature domains*, in Potential theory (Prague, 1987), Plenum, New York, 1988, pp. 295–305.

[36] ——, *Regularity of free boundaries in two dimensions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 20 (1993), pp. 323–339.

[37] ——, *Small modifications of quadrature domains*, Mem. Amer. Math. Soc., 206 (2010), pp. vi+269.

[38] B. Shraiman and D. Bensimon, *Singularities in nonlocal interface dynamics*, Phys. Rev. A (3), 30 (1984), pp. 2840–2842.

[39] T. Sjödin, *On the structure of partial balayage*, Nonlinear Anal., 67 (2007), pp. 94–102.

[40] A. N. Varchenko and P. I. Etingof, *Why the Boundary of a Round Drop Becomes a Curve of Order Four*, American Mathematical Society AMS University Lecture Series, Providence, Rhode Island, third ed., 1992.

[41] A. Vasil′ev, *From the Hele-Shaw experiment to integrable systems: a historical overview*, Complex Anal. Oper. Theory, 3 (2009), pp. 551–585.

[42] Y. P. Vinogradov and P. P. Kufarev, *On a problem of filtration*, Akad. Nauk SSSR. Prikl. Mat. Meh., 12 (1948), pp. 181–198.

[43] P. B. Wiegmann and A. Zabrodin, *Conformal maps and integrable hierarchies*, Comm. Math. Phys., 213 (2000), pp. 523–538.

[44] X. Xie and S. Tanveer, *Rigorous results in steady finger selection in viscous fingering*, Arch. Ration. Mech. Anal., 166 (2003), pp. 219–286.

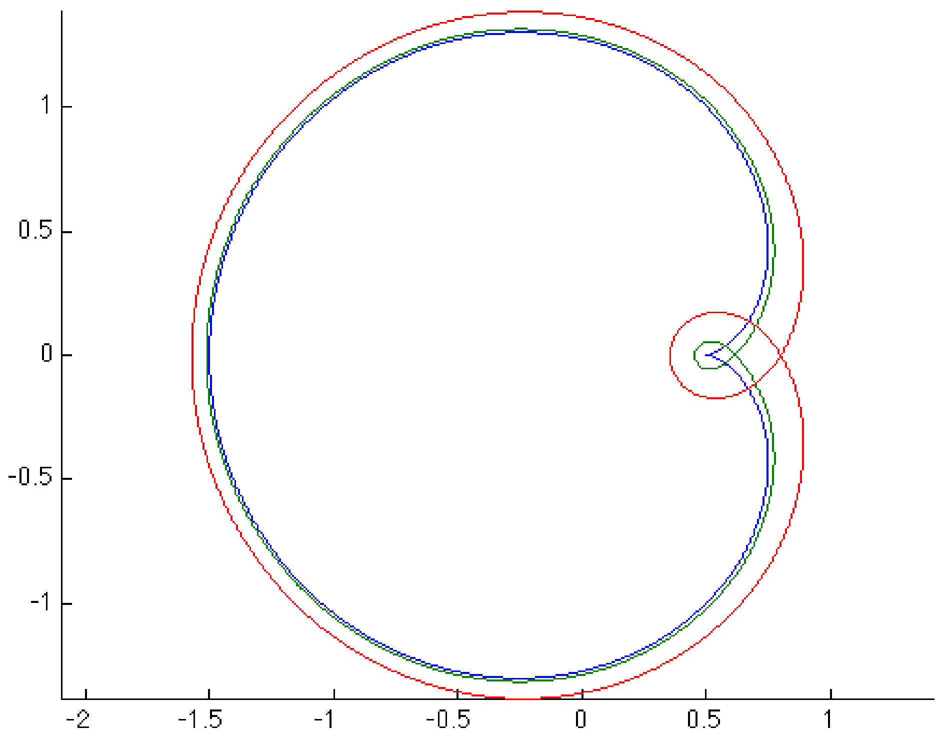

Figure 9.1: Solution described in Section 9.3

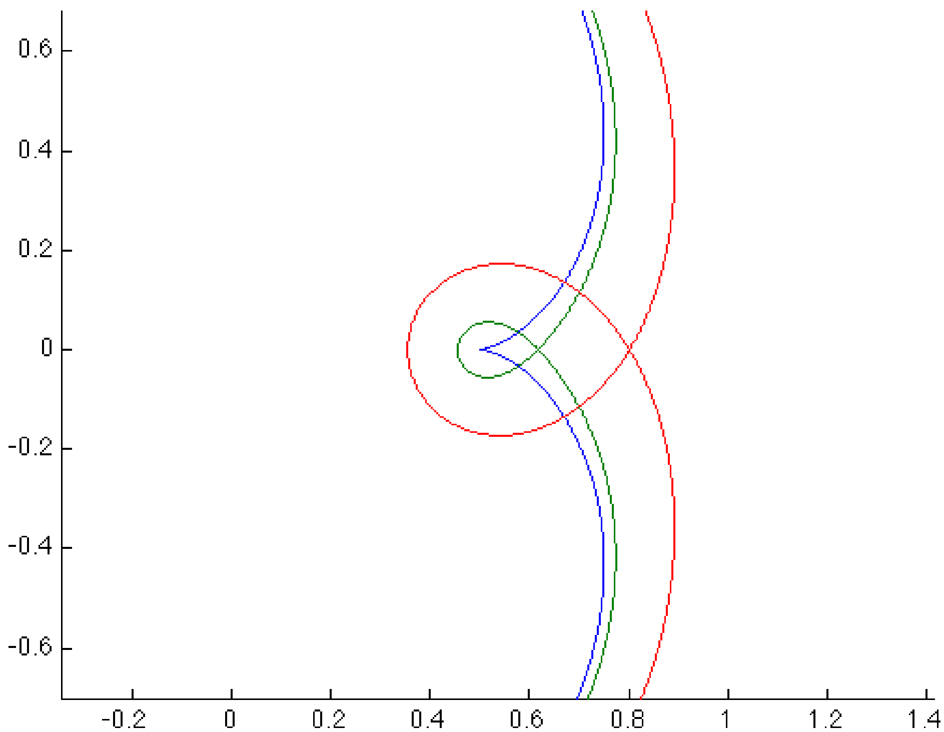

Figure 9.2: The same solution enlarged

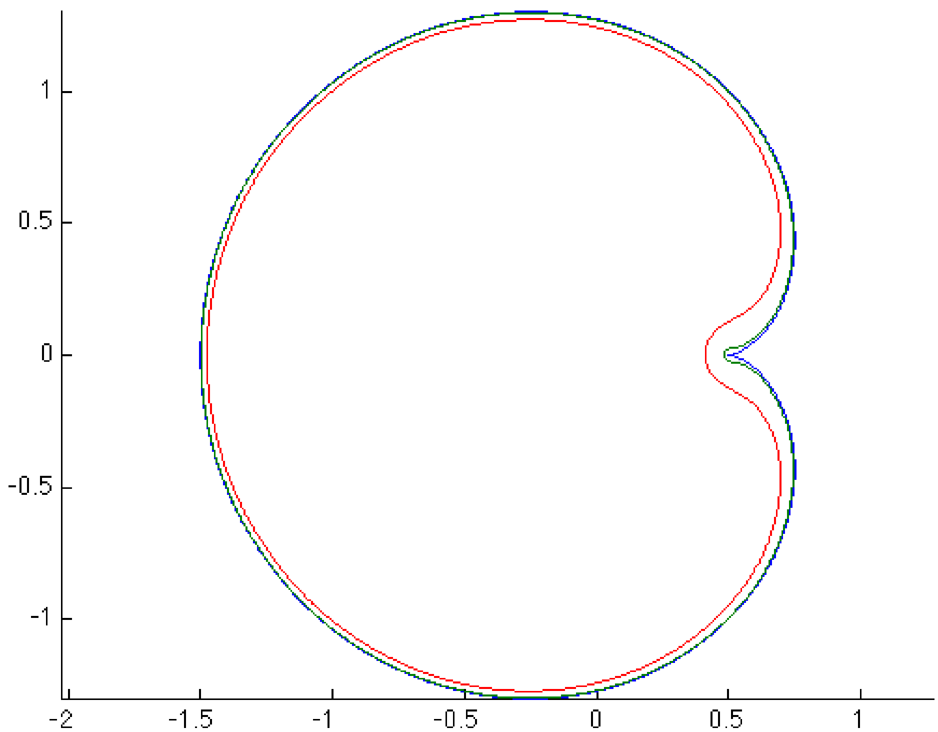

Figure 9.3: Solution described in Section 9.4

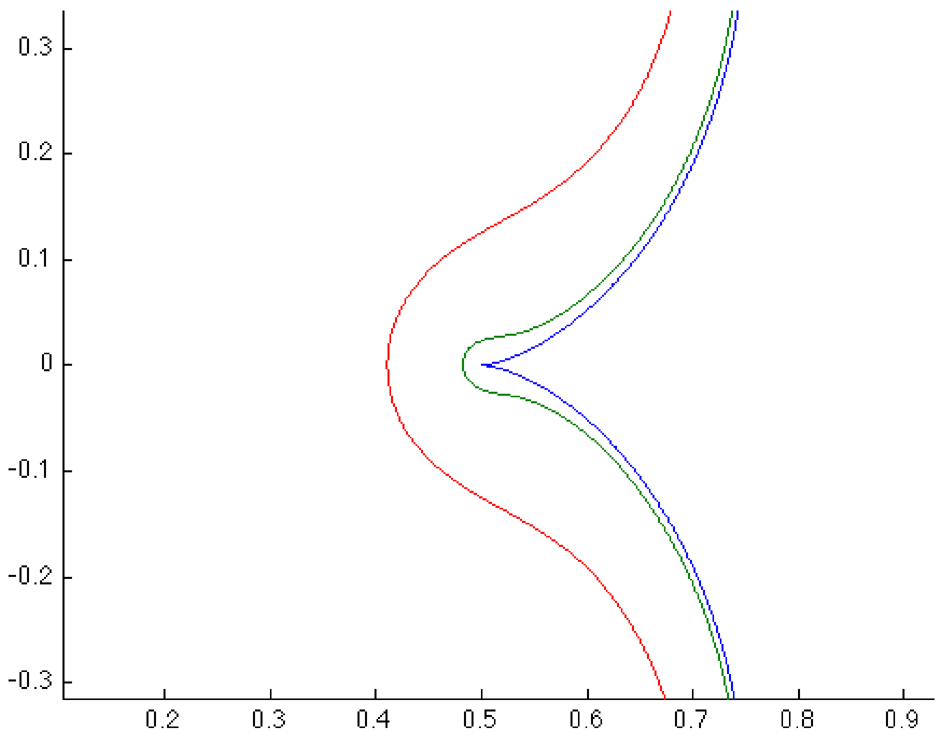

Figure 9.4: The same solution enlarged