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# ON THE CONVEXITY OF A SOLUTION OF LIOUVILLE'S EQUATION

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**1. Introduction.** Let $\Omega$ be a simply connected domain in the complex plane $\mathbb{C}$, $\Omega \neq \mathbb{C}$, and let

$$g(z, \zeta) = -\log|z - \zeta| - h(z, \zeta)$$

be its Green's function (for the Laplacian operator). The function

$$h(z) = h(z, z)$$

plays an important role in geometric function theory (see e.g. [9], [16]) and in a number of physical applications ([2], [4], [6], [7], [8], [12], [15]).

As has been noticed in [7, p. 548], [8] (and is directly verified using (3) below), $h(z)$ satisfies Liouville's equation

$$\Delta h = 4e^{2h} \qquad \text{in } \Omega. \tag{1}$$

The boundary behaviour is that $h(z) = -\log \delta(z) + O(1)$ as $z \to \partial\Omega$, where $\delta(z)$ denotes the distance to the boundary (see §3 below). Actually, $h(z)$ can be characterized as the unique solution of (1) with this boundary behaviour [3], [8], [15] and also as the maximum solution of (1) alone, [1, Lemma 1-1], [8], [15].

The main purpose of the present note is to give a simple, complex variable proof of the fact [3], [10], [11] that $h(z)$ is convex whenever $\Omega$ is convex. The major step in our proof can be formulated as a coefficient inequality for convex univalent functions. This part is not new [17], [5, Exercise 2, p. 70], but its connection with the convexity of $h(z)$ seems not to have been noticed in the literature. We also relate $h(z)$ to some other domain functions and expand a little on a physical interpretation of $h(z)$ in terms of vortex motion (thereby summarizing some parts of [8]).

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**2. The main result.** The following theorem was proved in [3]. Under suitable additional assumptions on $\partial\Omega$ it also becomes a special case of convexity results for solutions of more general PDEs (in arbitrary dimension) appearing in [10] and [11].

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Our proof is entirely different from those in [3], [10], [11] and moreover is quite simple and direct. On the other hand, it is certainly less flexible than e.g. the methods developed in [10].

THEOREM. *If $\Omega$ is convex then $h(z)$ is convex. Moreover, $h(z)$ is strictly convex unless $\Omega$ is a half-plane or an infinite strip.*

*Note.* Strict convexity is here to be taken in a strong sense, namely that both eigenvalues of the Hessian are (everywhere) strictly positive.

*Proof.* $h(z)$ is convex if and only if, at any point $z \in \Omega$ and for any $\lambda \in \mathbb{C}$,

$$\frac{\partial^2 h}{\partial z^2}\lambda^2 + 2\frac{\partial^2 h}{\partial z \partial \bar{z}^2}\lambda\bar{\lambda} + \frac{\partial^2 h}{\partial \bar{z}^2}\bar{\lambda}^2 \geqslant 0.$$

It is enough to consider $\lambda$ of modulus one, and varying $\lambda$ then gives that $h(z)$ is convex if and only if

$$\left|\frac{\partial^2 h}{\partial z^2}\right| \leqslant \frac{\partial^2 h}{\partial z \partial \bar{z}}.$$

Moreover strict convexity here corresponds to strict inequality holding everywhere. Since $\partial^2 h/\partial z \partial \bar{z} = \exp 2h$ by (1) we thus see that $h(z)$ is convex if and only if

$$\left|e^{-2h}\frac{\partial^2 h}{\partial z^2}\right| \leqslant 1, \tag{2}$$

with strict inequality corresponding to strict convexity.

Let $f\colon \mathbb{D} \to \Omega$ be any Riemann mapping function, where $\mathbb{D}$ is the unit disc. In terms of $f$, the Green's function for $\Omega$ is

$$g(z, w) = -\log\left|\frac{\zeta - \eta}{1 - \bar{\eta}\zeta}\right|$$

where $z = f(\zeta)$, $w = f(\eta)$. This gives

$$h(z, w) = -\log|1 - \bar{\eta}\zeta| - \log\left|\frac{z - w}{\zeta - \eta}\right|$$

and hence, letting $\eta \to \zeta$,

$$h(z) = -\log(1 - |\zeta|^2) - \log|f'(\zeta)| \tag{3}$$

$(z = f(\zeta))$. Let $z_0 \in \Omega$ be an arbitrary point for which we want to check (2). Then we can choose $f$ so that $f(0) = z_0$, by which (3) gives

$$h(z_0) = -\log|f'(0)|, \tag{4}$$

$$\frac{\partial h}{\partial z}(z_0) = -\frac{f''(0)}{2f'(0)^2},$$

$$\frac{\partial^2 h}{\partial z^2}(z_0) = \frac{f''(0)^2}{f'(0)^4} - \frac{f'''(0)}{2f'(0)^3}.$$

Thus (2) becomes

$$\left|\frac{f'''(0)}{2f'(0)} - \left(\frac{f''(0)}{f'(0)}\right)^2\right| \leqslant 1 \tag{5}$$

(to be proven).

Now (5) actually follows from a coefficient inequality for convex univalent functions given in [17], [5, Exercise 2, p. 70], but since we have to keep track of the equality cases and since the proof anyway is very short we provide also the proof of (5).

That $\Omega$ is convex means [14] for $f$ that

$$\operatorname{Re}\left(\zeta\frac{f''(\zeta)}{f'(\zeta)} + 1\right) \geqslant 0 \qquad \text{in } \mathbb{D},$$

or, mapping the right half-plane onto the unit disc by $u \mapsto (u-1)/(u+1)$, that

$$\left|\frac{\zeta f''(\zeta)}{\zeta f''(\zeta) + 2f'(\zeta)}\right| \leqslant 1 \qquad \text{in } \mathbb{D}.$$

In view of Schwarz' lemma this is also equivalent to that

$$|F(\zeta)| \leqslant 1 \qquad \text{in } \mathbb{D}, \tag{6}$$

where we have put

$$F(\zeta) = \frac{f''(\zeta)}{\zeta f''(\zeta) + 2f'(\zeta)}.$$

If equality holds in (6) at some point then $F(\zeta) \equiv A$ for some $|A| = 1$, and this readily gives that $f$ maps $\mathbb{D}$ onto a half-plane. When $\Omega$ is a half-plane $h(z)$ is convex but not strictly convex (the upper half-plane e.g. gives $h(z) = -\log 2y$ ($z = x + iy$)).

Excluding this case in the sequel we thus have strict inequality in (6), in particular $|F(0)| < 1$. From (6) we now obtain

$$\left|\frac{F(\zeta)-F(0)}{1-\overline{F(0)}F(\zeta)}\right| \leqslant 1 \qquad \text{in } \mathbb{D},$$

which implies

$$\left|\frac{d}{d\zeta}\right|_{\zeta=0}\left(\frac{F(\zeta)-F(0)}{1-\overline{F(0)}F(\zeta)}\right)\right| \leqslant 1, \tag{7}$$

i.e.,

$$|F'(0)| \leqslant 1 - |F(0)|^2. \tag{8}$$

In particular,

$$|F'(0) - F(0)^2| \leqslant 1. \tag{9}$$

But

$$F(0) = \frac{f''(0)}{2f'(0)}, \tag{10}$$

$$F'(0) = \frac{f'''(0)}{2f'(0)} - \frac{3}{4}\left(\frac{f''(0)}{f'(0)}\right)^2, \tag{11}$$

hence (9) is nothing else than (5), which thereby is proven.

*Note.* The aforementioned coefficient inequality in [17], [5] is (8) with (10, 11) inserted and $f'(0) = 1$.

To have equality in (5) we must first of all have equality in (7). This gives

$$\frac{F(\zeta)-F(0)}{1-\overline{F(0)}F(\zeta)} \equiv A\zeta \tag{12}$$

for some $|A| = 1$. By a rotation of the coordinate $\zeta$ we may assume that $A = 1$. Then $F'(0) > 0$. The fact that we moreover must have equality in the step from (8) to (9) (the triangle inequality) then gives that $F(0)^2 \leqslant 0$, i.e., that $F(0) = i\omega$ for some $\omega \in \mathbb{R}$ ($|\omega| < 1$). By all this, (12) becomes

$$\frac{f''(\zeta)}{f'(\zeta)} = \frac{2(\zeta + i\omega)}{1 - 2i\omega\zeta - \zeta^2},$$

which easily can be integrated to give

$$f(\zeta) = B \log \frac{\zeta + i\omega - \sqrt{1-\omega^2}}{\zeta + i\omega + \sqrt{1-\omega^2}} + C$$

($B$, $C$ constants of integration). Since all these functions map $\mathbb{D}$ onto infinite strips the proof is now complete.

**3. Connection with some other domain functions and some estimates.** The transformation behaviour of the Green's function under conformal mappings gives (cf. the derivation of (3)) that

$$ds = e^{h(z)}|dz| \tag{13}$$

is a conformally invariant metric. For the unit disc we have (by (3))

$$e^{h(z)} = \frac{1}{1-|z|^2}.$$

Therefore the metric (13) coincides with the *Poincaré metric* for an arbitrary simply connected $\Omega \neq \mathbb{C}$. Liouville's equation (1) simply is the statement that (13) has constant curvature $= -4$.

In terms of the *Bergman kernel* $K(z, \zeta)$ for $\Omega$ we have

$$K(z, z) = \frac{1}{\pi} e^{2h(z)},$$

since this is true for the unit disc and both members behave in the same way under conformal mapping (see [14]). By (4), the power series at the origin of the conformal map $f\colon \mathbb{D} \to \Omega$ satisfying $f(0) = z_0$, $f'(0) > 0$ starts

$$f(\zeta) = z_0 + e^{-h(z_0)}\zeta + \cdots. \tag{14}$$

Therefore $\exp(-h(z))$ is the so-called *mapping radius* for $\Omega$ at $z \in \Omega$ [9], [16].

The function $-h(z)$ is sometimes also called the *Robin constant*. See [16] where it is studied on general Riemann surfaces. If $\mathbb{C}\backslash\Omega$ is compact and $z = \infty$ is regarded as an interior point of $\Omega$ then there is a well-known relation between the Robin constant at $z = \infty$ and the *transfinite diameter* of $\mathbb{C}\backslash\Omega$ [1], [16]. By means of a Möbius transformation this relation can be carried over to the general case ($\Omega \subset \mathbb{C}$, $z \in \Omega$) and then gives the formula

$$h(z) = -\lim_{n\to\infty} \min \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \log \frac{|z - z_i|\,|z - z_j|}{|z_i - z_j|}$$

for $h(z)$. Here the minimum is taken over all collections of $n$ distinct points $z_1, \ldots, z_n$ in $\mathbb{C}\backslash\Omega$. See [8], [16] for more details.

The boundary data of $h(z)$ mentioned in the introduction can actually be stated in forms of explicit global estimates of $h(z)$, namely (when $\Omega \subset \mathbb{C}$ is simply connected),

$$-\log \delta(z) - \log 4 \leqslant h(z) \leqslant -\log \delta(z) \tag{15}$$

($z \in \Omega$). If $\Omega$ is convex, the lower bound in (15) can even be sharpened to $-\log \delta(z) - \log 2$. (For a better upper bound near $\partial\Omega$, see [3].)

The lower bound in (15) follows e.g. from the Koebe "one-quarter theorem" [1], [14] which states that $f(\mathbb{D}) \supset \mathbb{D}(z_0; (1/4)\exp(-h(z_0)))$ for a univalent function $f$ of the form (14). ($\mathbb{D}(a; r) = \{z \in \mathbb{C}: |z - a| < r\}$.) The other bounds follow from a simple comparison principle: if $\Omega_1 \subset \Omega_2$ then the corresponding Green's functions satisfy $g_1(z, \zeta) \leqslant g_2(z, \zeta)$ for $z, \zeta \in \Omega_1$ and this gives that $h_1(\zeta) \geqslant h_2(\zeta)$ for $\zeta \in \Omega_1$. Taking here $\Omega_1 = \mathbb{D}(z; \delta(z))$, $\Omega_2 = \Omega$ and $\zeta = z$ one gets the upper bound in (15). When $\Omega$ is convex $\Omega$ is (given $z \in \Omega$) contained in a half-plane whose boundary is at distance $\delta(z)$ from $z$. Thus one can take $\Omega_1 = \Omega$ and $\Omega_2$ that half-plane in this case and with $\zeta = z$ one then obtains the lower bound in the convex case.

*Note.* The function $h(z)$ makes sense also when $\Omega$ is multiply connected (or any Riemann surface admitting a Green's function) but then it does not satisfy Liouville's equation [8], [15]. Also the above connections with the Poincaré metric and the Bergman kernel break down in the multiply connected case. However, $h(z)$ is still related to the Bergman kernel by

$$\Delta h(z) = 4\pi K(z, z)$$

as follows from the formula [14] $K(z, \zeta) = -(2/\pi)\partial^2 g(z, \zeta)/\partial z \partial\bar{\zeta}$. As to the role of Liouville's equation when $\Omega$ is multiply connected we have e.g. that the maximum solution of (1) is the $h(z)$ defined in terms of the Green's function for the universal covering surface of $\Omega$. With this $h(z)$ (which turns out to be single-valued on $\Omega$ despite this is not true for the Green's function itself), (13) becomes the Poincaré metric of $\Omega$. See [1], [8].

**4. Interpretation of $h(z)$ in terms of vortex motion.** We shall consider in $\Omega$ (any simply connnected domain in $\mathbb{C}$, $\Omega \neq \mathbb{C}$) the two-dimensional flow of an ideal incompressible fluid. We assume that at time zero the flow is irrotational except for a point vortex of strength $\Gamma \in \mathbb{R}$ at some point $z_0 \in \Omega$. If $\Omega$ is unbounded we moreover assume that there is no flow at infinity. The above assumptions mean that the stream function [13] of the flow at time zero is (up to an additive constant)

$$\psi(z) = -\Gamma g(z, z_0).$$

In terms of $\psi(z)$ the velocity field of the flow is given by

$$w(z) = 2i\frac{\partial\psi}{\partial\bar{z}} = -\frac{\partial\psi}{\partial y} + i\frac{\partial\psi}{\partial x}.$$

Thus, in terms of $h(z, \zeta)$,

$$w(z) = \frac{i\Gamma}{\bar{z} - \bar{z}_0} + 2i\Gamma\frac{\partial h(z, z_0)}{\partial\bar{z}}.$$

Consider now the flow near $z_0$. Since the singular part $i\Gamma(\bar{z} - \bar{z}_0)^{-1}$ of $w(z)$ is rotationally symmetric around $z_0$ the only flow which can be experienced at the point $z_0$ itself is $2i\Gamma\partial h(z, z_0)/\partial\bar{z}|_{z=z_0}$, the "regular part" of $w(z)$ at $z_0$. From the symmetry $h(z, \zeta) = h(\zeta, z)$ it follows that $\partial h(z, z_0)/\partial\bar{z}|_{z=z_0} = (1/2)\partial h(z, z)/\partial\bar{z}|_{z=z_0}$. Thus in terms of $h(z) = h(z, z)$ the expression for the regular part of $w(z)$ at $z_0$ becomes $i\Gamma\partial h(z)/\partial\bar{z}|_{z=z_0}$.

According to general principles for vortex motion [12], [13] the above means that the vortex will have to move with velocity $i\Gamma\partial h(z)/\partial\bar{z}|_{z=z_0}$. At a later instant $t$ the flow will remain of the same type just that the vortex has moved to a new poisition $z(t) \in \Omega$. Since the point $z_0$ in the above discussion was arbitrary, it follows that the vortex moves along a path $z(t)$ satisfying the differential equation

$$\frac{dz(t)}{dt} = i\Gamma\frac{\partial h}{\partial\bar{z}}(z(t)). \tag{16}$$

Together with the initial datum $z(0) = z_0$ this determines $z(t)$ and (hence) the entire flow for all $t$.

Written out in components (16) becomes $dx/dt = -(\Gamma/2)\partial h/\partial y, dy/dt = (\Gamma/2)\partial h/\partial x$. Thus the motion of the vortex is a two-dimensional Hamiltonian system with phase space $\Omega$ (for the position of the vortex), symplectic form $dx\,dy$ and with $(\Gamma/2)h(z)$ as the Hamiltonian function (which in the context of vortex motion is sometimes called the Routh's stream function; see [12]).

A particular consequence of the above discussion (or just of (16)) is that the vortex always moves along a level curve of $h(z)$. Thus Theorem §2 shows that whenever $\Omega$ is convex the vortex moves along a convex curve (straight line if $\Omega$ is an infinite strip or a half-plane, otherwise a strictly convex curve).

For certain exceptional choices of $z_0 \in \Omega$ the vortex may actually remain at rest at $z_0$. Clearly these points are exactly the critical points of $h(z)$, i.e., those points at which $\partial h(z)/\partial\bar{z} = 0$. If $\Omega$ is unbounded there need not be any critical point, as the example (in §2) with a half-plane shows, but if $\Omega$ is bounded then it is obvious from the boundary behaviour of $h(z)$ that there must be at least one critical point (namely the global minimum point for $h(z)$). In general $h(z)$ can have any number of critical points, but Theorem §2 yields that if $\Omega$ is convex (bounded or not) and not an infinite strip then $h(z)$ has at most one critical point. (In the case of an infinite strip all points on the symmetry line are critical.) This result was conjectured by Bengt-Joel Andersson ([2] and private communication) and proved in [8]. However, stated in terms of the mapping radius, the result was proved already in [9].

The critical points of $h(z)$ are of interest in several physical applications, not only vortex motion. See e.g. [7], [4], [6]. It may therefore be of interest to notice that the uniqueness of critical points for convex domains does not extend to star-shaped domains in general. Indeed, consider the Riemann mapping function

$$f(\zeta) = \zeta \frac{1 - R^2}{\zeta^2 - R^2} = \frac{1 - R^2}{2}\left(\frac{1}{\zeta - R} + \frac{1}{\zeta + R}\right)$$

where $R > 1$. We have

$$\operatorname{Re}\frac{\zeta f'(\zeta)}{f(\zeta)} = \operatorname{Re}\frac{R^2 + \zeta^2}{R^2 - \zeta^2} \geqslant 0$$

for $|\zeta| < 1$, which shows [14, ch. V] that $f$ maps $\mathbb{D}$ onto a domain $\Omega$ which is star-shaped with respect to the origin. By (3)

$$h(z) = -\log(R^2 - 1) - \log\frac{(1 - |\zeta|^2)|R^2 + \zeta^2|}{|R^2 - \zeta^2|^2},$$

$$\frac{\partial h(z)}{\partial z} f'(\zeta) = \frac{\bar{\zeta}}{1 - |\zeta|^2} + \frac{\zeta(\zeta^2 + 3R^2)}{\zeta^4 - R^4}$$

$(z = f(\zeta))$ and this gives that for $1 < R < \sqrt{3}$ $h(z)$ has critical points at $z = 0$, $\pm(1/4)\sqrt{(3 - R^2)(3 - R^{-2})}$, corresponding to $\zeta = 0$, $\pm\sqrt{(3 - R^2)/(3 - R^{-2})}$.

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