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A Distortion Theorem for Quadrature Domains for Harmonic Functions

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We prove that any finitely connected domain in the plane can be distorted so that it becomes ‘‘graviequivalent’’ to a signed measure with arbitrarily small support. Precisely: if $D \subset \mathbb{C}$ is a bounded, finitely connected domain with analytic boundary then for any $a \in D$ and $r > 0$, $\varepsilon > 0$ with $B(a, r + \varepsilon) \subset D$ there exists a univalent function g in D with $|g(z) - z| < \varepsilon$ $(z \in D)$ and a signed measure μ with support in $B(a, r)$ such that for every integrable harmonic function h in $\Omega := g(D)$ we have $\int_\Omega h\, dx\, dy = \int h\, d\mu$. © 1996 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to prove the following theorem, stated, but not proved, in [8].

(*Notation.* $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.)

THEOREM 1. *Let D be a bounded domain in the complex plane $\mathbb{C}$ with ∂D consisting of finitely many disjoint analytic Jordan curves, let $a \in D$, and let $r > 0$, $\varepsilon > 0$ be any numbers satisfying $B(a, r + \varepsilon) \subset D$. Then there exists a univalent function g in D with $|g(z) - z| < \varepsilon$ $(z \in D)$ such that the image domain $\Omega = g(D)$ has the following property: there exists a signed measure μ*

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with support in $B(a,r)$ *such that*

$$\int_\Omega h\,dx\,dy = \int h\,d\mu \tag{1.1}$$

for every integrable harmonic function h *in* Ω.

That an equality of type (1.1) holds for (e.g.) all harmonic integrable h is often referred to by saying that Ω is a "quadrature domain" for the measure μ (see, e.g., [11, 17] and references therein). By taking $h(z) = \log|z-\zeta|$ with $\zeta \notin \Omega$ one sees that (1.1) means (or at least implies) that Ω regarded as a body with density one produces the same gravitational (or logarithmic) potential outside Ω as does the signed mass distribution μ.

Thus the theorem roughly says that the set of bodies graviequivalent to signed mass distributions supported in any fixed ball $B(a,r)$ are dense in the set of all bodies containing $B(a,r)$. One main point with this result is that if one instead asks for *positive* mass distribution doing the same job, i.e., if one in the theorem requires also $\mu \geq 0$, then it becomes tremendously false. Indeed, one may show that if (1.1) holds with $\mu \geq 0$ and with area $|\Omega| \geq 4r^2\pi$ then Ω is essentially ball shaped (e.g., Ω is simply connected, $\partial\Omega$ is analytic, and for any $z \in \partial\Omega$ the inward normal of $\partial\Omega$ at z intersects $B(a,r)$). This follows from results in [11, 6, 7, 16, 10].

The corresponding theorem with (1.1) required to hold for all integrable *analytic* functions h was (essentially) proved in [5]. From this follows the theorem for harmonic h in case Ω is simply connected. A short direct proof for this case is also given in [8]. In the present paper we shall extend the results of [5] to prove Theorem 1.

Notation.

$$C(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is continuous}\},$$

$$A(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is analytic}\},$$

$$M(\Omega) = \{f : \Omega \to \mathbb{C} \cup \{\infty\} : f \text{ is meromorphic}\},$$

$$H(\Omega) = \{h : \Omega \to \mathbb{R} : h \text{ is harmonic}\},$$

$$AL^1(\Omega) = \left\{f \in A(\Omega) : \int_\Omega |f|\,dx\,dy < \infty\right\},$$

$$HL^1(\Omega) = \left\{h \in H(\Omega) : \int_\Omega |h|\,dx\,dy < \infty\right\},$$

$$\delta_a = \text{Dirac measure at the point } a \in \mathbb{C},$$

$$\delta_{kj} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases}$$

$$|\Omega| = \text{area of } \Omega.$$

2. QUADRATURE DOMAINS FOR ANALYTIC AND HARMONIC FUNCTIONS

Let $D \subset \mathbb{C}$ be a finitely connected bounded domain with ∂D analytic, as in Section 1. Then D can be completed to a compact Riemann surface $\hat{D}$, called the "Schottky double" of D. It consists of D, a copy $\tilde{D}$ (the "back-side") of D provided with the opposite conformal structure, and their common boundary ∂D through which they are welded together. See [9, 5]. If $z \in D$ we denote by $\tilde{z}$ the corresponding point on $\tilde{D}$.

There is a natural anticonformal involution $\varnothing$ on $\hat{D}$ given by $\varnothing(z) = \tilde{z}$, $\varnothing(\tilde{z}) = z$ if $z \in D$ ($\tilde{z} \in \tilde{D}$) and by $\varnothing(z) = z$ if $z \in \partial D$. Together, the pair $(\hat{D}, \varnothing)$ constitutes a so-called symmetric Riemann surface. If f is any meromorphic function on $\hat{D}$ then

$$f^* = \overline{f \circ \varnothing}$$

is also meromorphic on $\hat{D}$. On ∂D, $f^* = \bar{f}$. Setting $f_1 = f|_{\overline{D}}, f_2 = f^*|_{\overline{D}}$ it follows that f_1 and f_2 are meromorphic functions in D continuous (in the extended sense) up to ∂D and satisfying

$$f_1 = \bar{f}_2 \qquad \text{on } \partial D. \tag{2.1}$$

Conversely, any pair of meromorphic functions f_1, f_2 on $\overline{D}$ satisfying (2.1) represents a meromorphic function on $\hat{D}$. Note that if f is represented as above by (f_1, f_2), then f^* is represented by (f_2, f_1).

We shall first recall a result for quadrature domains for analytic functions which has been proved in various parts in [3, 1, 5]. See also, e.g., [17, 13, 14].

PROPOSITION 1. *With D and $\hat{D}$ as above, let $\Omega \subset \mathbb{C}$ be a bounded domain conformally equivalent to D and let $g : D \to \Omega$ be a conformal map. Then the following assertions are equivalent.*

(i) *There exists a complex-valued distribution μ with support in a finite number of points in Ω such that*

$$\int_\Omega f \, dx \, dy = \langle \mu, f \rangle \qquad \textit{for all } f \in AL^1(\Omega). \tag{2.2}$$

(ii) *There exists a meromorphic function $S(z)$ in Ω, continuously extendible up to $\partial\Omega$ with*

$$S(z) = \bar{z} \qquad \textit{on } \partial\Omega. \tag{2.3}$$

(iii) *g extends to a meromorphic function on $\hat{D}$.*

When (i)–(iii) *hold,* μ, S, *and* g *are related by*

$$\begin{aligned} \langle \mu, f \rangle &= \pi \sum_{z\in\Omega} \operatorname{Res}(S(z)f(z)) \qquad (f \in AL^1(\Omega)) \\ S(g(\zeta)) &= g^*(\zeta) \qquad (\zeta \in D) \end{aligned} \tag{2.4}$$

and $S(z)$ *is called the Schwarz function of* $\partial\Omega$ [3, 17].

As to the proof we just mention that (ii) $\Rightarrow$ (i) follows from partial integration and the residue theorem,

$$\begin{aligned} \int_\Omega f\,dx\,dy &= \frac{1}{2i}\int_\Omega f\,d\bar z\,dz \\ &= \frac{1}{2i}\int_{\partial\Omega} f(z)\bar z\,dz = \frac{1}{2i}\int_{\partial\Omega} f(z)S(z)\,dz \\ &= \pi \sum_{z\in\Omega} \operatorname{Res}(f(z)S(z)), \end{aligned}$$

and that (iii) simply is the statement (ii) pulled back to D by means of g. Note that, in terms of the representation (2.1), Eq. (2.3) says that the pair $(z, S(z))$ represents a meromorphic function on $\hat\Omega$.

Any distribution $\mu \in \mathscr{D}'(\Omega)$ with support in a finite number of points is of the form

$$\langle \mu, \varphi \rangle = \sum_{k=1}^{m} \sum_{i,j=0}^{n} a_{kij} \frac{\partial^{i+j}\varphi}{\partial x^i \partial y^j}(z_k) \qquad (\varphi \in C^\infty(\Omega)) \tag{2.5}$$

for suitable m, n and $z_1, \ldots, z_m \in \Omega$, $a_{kij} \in \mathbb{C}$. Since $\partial f/\partial x = -i(\partial f/\partial y)$ when f is analytic, many different distributions μ of the form (2.5) have the same action on $AL^1(\Omega)$. It is always possible to take a_{kij} to be real whenever $i + j \geq 1$, but this still does not make μ uniquely determined in general, since, e.g., $\partial^2 f/\partial x^2 = -\partial^2 f/\partial y^2$.

One way to write the action of μ on $AL^1(\Omega)$ in a canonical form is to write (2.5) as

$$\langle \mu, f \rangle = \sum_{k=1}^{m} \sum_{j=0}^{2n} c_{kj} f^{(j)}(z_k)$$

for $f \in AL^1(\Omega)$.

When (2.2) holds we see, by taking $f = 1$, that $\sum_{k=1}^m a_{k00} = \sum_{k=1}^m c_{k0}$ is real and positive, but the individual $a_{k00} = c_{k0}$ may very well be nonreal. To be precise, we have the following. Suppose that (2.2) holds and choose

a_{kij} for $i + j \geq 1$ to be real. Then $a_{k00} \in \mathbb{R}$ for all k if and only if the identity

$$\int_\Omega u\,dx\,dy = \langle \mu, u \rangle \tag{2.6}$$

holds for every harmonic function u which is the real part of some $f \in AL^1(\Omega)$.

Indeed, if $a_{k00} \in \mathbb{R}$ then μ is real-valued (i.e., takes real values on real-valued test functions) and (2.6) follows from (2.2) by taking real parts. Conversely, (2.6) alone implies (2.2) and then for every $f \in AL^1(\Omega)$, $\langle \mu, \operatorname{Re} f \rangle = \int_\Omega (\operatorname{Re} f)\,dx\,dy = \operatorname{Re} \int_\Omega f\,dx\,dy = \operatorname{Re}\langle \mu, f \rangle$, which implies that $a_{k00} \in \mathbb{R}$ when $a_{kij} \in \mathbb{R}$, $i + j \geq 1$. (If, e.g., $a_{100} \notin \mathbb{R}$ then we get a contradiction by taking f to be a polynomial with $f(z_1) = 1$, $f(z_k) = 0$ for $k \neq 1$.)

Example 1. It is possible to construct Ω such that for a suitably large $a > 0$

$$\int_\Omega f\,dx\,dy = af(0) + i(f(1) - f(-1)) \tag{2.7}$$

for all $f \in AL^1(\Omega)$. (See Proposition 4 below.) Here the right member is $\langle \mu, f \rangle$ with $\mu = a\delta_0 + i\delta_1 - i\delta_{-1}$. This μ is nonreal and its action on $AL^1(\Omega)$ does not coincide with that of any real-valued distribution with support in a finite number of points. Therefore, the identity (2.6) does not hold for all $u = \operatorname{Re} f$, $f \in AL^1(\Omega)$.

In order to get a quadrature identity for this Ω and holding for all $u = \operatorname{Re} f$, $f \in AL^1(\Omega)$, write $f = u + iv$ and let γ be the straight line segment from -1 to $+1$. Then

$$\begin{aligned} i(f(1) - f(-1)) &= i\int_\gamma \frac{\partial f}{\partial x}\,dx \\ &= i\int_\gamma \frac{\partial u}{\partial x}\,dx - \int_\gamma \frac{\partial v}{\partial x}\,dx \\ &= i\int_\gamma \frac{\partial u}{\partial x}\,dx + \int_\gamma \frac{\partial u}{\partial y}\,dx \end{aligned}$$

so that, taking real parts in (2.7),

$$\int_\Omega u\,dx\,dy = au(0) + \int_\gamma \frac{\partial u}{\partial y}\,dx \tag{2.8}$$

for all $u = \operatorname{Re} f$, $f \in AL^1(\Omega)$. Thus (2.6) holds, but with μ a distribution of a more general form.

Next we wish to discuss quadrature identities

$$\int_\Omega h\,dx\,dy = \langle \mu, h \rangle \tag{2.9}$$

with μ of the form (2.5) holding for all $h \in HL^1(\Omega)$. Clearly, if (2.9) holds with μ complex-valued it also holds for $\operatorname{Re}\mu$ (and $\operatorname{Im}\mu$ annihilates $HL^1(\Omega)$), so we may as well assume from the beginning that μ is real-valued. Recall from the previous discussion that (2.9) then holds for all $h \in HL^1(\Omega)$ of the form $h = \operatorname{Re} f$, $f \in AL^1(\Omega)$ if and only if the equivalent conditions in Proposition 1 hold.

PROPOSITION 2. *With D and $\hat{D}$ as in the beginning of this section, let $\Omega \subset \mathbb{C}$ be a bounded domain conformally equivalent to D and let $g : D \to \Omega$ be a conformal map. Then the following assertions are equivalent.*

(i) *There exists a real-valued distribution μ with support in a finite number of points in Ω such that (2.9) holds for all $h \in HL^1(\Omega)$.*

(ii) *There exists a meromorphic function $S(z)$ in Ω, continuously extendible up to $\partial\Omega$ with*

$$S(z) = \bar{z} \qquad \text{on } \partial\Omega \tag{2.10}$$

and satisfying in addition

$$\operatorname{Re} \int_\gamma (\bar{z} - S(z))\,dz = 0 \tag{2.11}$$

for every smooth curve $\gamma \subset \overline{\Omega}$ avoiding the poles of $S(z)$ and satisfying $\partial\gamma \subset \partial\Omega$ (i.e., γ either is closed or joins two components of $\partial\Omega$).

(iii) *g extends to a meromorphic function on $\hat{D}$ satisfying the additional condition that*

$$\operatorname{Re} \int_\gamma g^*\,dg = 0 \tag{2.12}$$

for every closed curve γ in $\hat{D}$ avoiding the poles of g^*dg.

Proof. We first prove that (i) and (ii) are equivalent. To large parts this is actually well-known (see [12, 14], e.g.), but since there are some subtle points we prefer to outline the full proof.

Let $E(z) = -(1/2\pi)\log|z|$ be the standard fundamental solution of $-\Delta$ (so that $-\Delta E = \delta_0$). If (i) holds, set

$$u = (\mu - \chi_\Omega) * E.$$

Then u is a real-valued distribution, continuously differentiable outside supp μ and satisfying

$$\Delta u = \chi_\Omega - \mu \qquad \text{in } \mathbb{C}, \tag{2.13}$$

$$u = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \qquad \text{on } \Omega^c. \tag{2.14}$$

The latter equation follows by taking $h(z) = E(z - \zeta)$, $(\partial E/\partial x)(z - \zeta)$, $(\partial E/\partial y)(z - \zeta)$ for $\zeta \in \Omega^c$ in (2.9).

Define $S(z)$ by

$$S(z) = \bar{z} - 4\frac{\partial u}{\partial z}, \qquad z \in \overline{\Omega}.$$

Then $\partial S/\partial \bar{z} = 1 - \Delta u = \mu$ in Ω so that $S(z)$ is meromorphic in Ω. By (2.14), $S(z)$ is continuous up to $\partial\Omega$ with (2.10) holding there. Finally, (2.11) holds since

$$\begin{aligned} \operatorname{Re}\int_\gamma (\bar{z} - S(z))\, dz &= 4\operatorname{Re}\int_\gamma \frac{\partial u}{\partial z}\, dz \\ &= 2\int_\gamma du, \end{aligned}$$

which is zero since $u = 0$ on $\partial\gamma$ (or $\partial\gamma$ is empty). Thus (i) implies (ii).

Conversely, suppose (ii) holds. Let $z_1, \ldots, z_m \in \Omega$ be the poles of $S(z)$, pick a point $z_0 \in \partial\Omega$, and define $u(z)$ on $\overline{\Omega} \setminus \{z_1, \ldots, z_m\}$ by

$$u(z) = \frac{1}{2}\operatorname{Re}\int_{z_0}^{z} (\bar{\zeta} - S(\zeta))\, d\zeta$$

and extend it by zero outside $\overline{\Omega}$. Then, due to (2.10), (2.11), u is a well-defined continuously differentiable real-valued function in $\mathbb{C} \setminus \{z_1, \ldots, z_m\}$ satisfying (2.14). Moreover,

$$\Delta u = \chi_\Omega \qquad \text{in } \mathbb{C} \setminus \{z_1, \ldots, z_m\}. \tag{2.15}$$

This follows from the definition of u together with the known fact [1, 13] that $\partial\Omega$ is piecewise smooth (algebraic) when (2.10) holds.

At $z_1, \ldots, z_m$, u has polar singularities. Unfortunately, u need not be integrable over these, so u need not, in a canonical way, be a distribution on $\mathbb{C}$. Therefore one cannot define μ right away by (2.13), as one is tempted to do, with Δu the distributional Laplacian of u on $\mathbb{C}$. Nevertheless, with $B_\varepsilon = B(z_1, \varepsilon) \cup \cdots \cup B(z_m, \varepsilon)$ for $\varepsilon > 0$ small it follows, using (2.14), (2.15), and the cutting-off technique in [11, p. 60], that for $h \in HL^1(\Omega)$

$$\begin{aligned}\int_\Omega h\,dx\,dy &= \lim_{\varepsilon\to 0}\int_{\Omega\setminus B_\varepsilon} h\Delta u\,dx\,dy \\ &= \lim_{\varepsilon\to 0}\left(\int_{\partial B_\varepsilon} u\frac{\partial h}{\partial n}\,ds - \int_{\partial B_\varepsilon} h\frac{\partial u}{\partial n}\,ds\right).\end{aligned}$$

Since the singularities of u at $z_1, \ldots, z_m$ are of finite order it follows that the right member above can be written $\langle \mu, h\rangle$ for some real-valued distribution μ (not uniquely determined in general, as a distribution) with support at $z_1, \ldots, z_m$. Thus (ii) implies (i).

It remains to prove that (ii) and (iii) are equivalent. As in Proposition 1 (or [5]) the existence of a meromorphic $S(z)$ satisfying (2.10) is equivalent to the meromorphic extension of g to $\hat{D}$. Thus we only need to show that condition (2.11) is the same as (2.12).

Pulling (2.11) back to D it becomes

$$\operatorname{Re}\int_\beta (\bar{g} - g^*)\,dg = 0 \tag{2.16}$$

for every $\beta \subset \overline{D}$ with $\partial\beta \subset \partial D$ ($\gamma = g \circ \beta$). We shall prove that (2.12) and (2.16) are equivalent. There are two kinds of curves β which need to be considered in (2.12) and (2.16): (a) closed curves in D; (b) for (2.16), curves in D joining two components of ∂D and, for (2.12), the corresponding closed curves on $\hat{D}$ obtained by going back along the same track on the back-side $\tilde{D}$.

As for (a), we note that

$$\operatorname{Re}\int_\beta \bar{g}\,dg = \frac{1}{2}\int_\beta (\bar{g}\,dg + g\,d\bar{g}) = \frac{1}{2}\int_\beta d(\bar{g}g) = 0$$

when β is a closed curve in D, so then (2.16) is really the same as (2.12).

As for (b), let β be a nonclosed curve on $\overline{D}$ with endpoints on ∂D and let $\gamma = \beta - \tilde{\beta}$ ($\tilde{\beta} = \emptyset \circ \beta$) be the corresponding (oriented) closed curve on $\hat{D}$. Then

$$\begin{aligned}
\operatorname{Re} \int_\gamma g^* \, dg &= \operatorname{Re}\left\{ \int_\beta \overline{(g \circ \emptyset)} \, dg - \int_{\tilde{\beta}} \overline{(g \circ \emptyset)} \, dg \right\} \\
&= \operatorname{Re}\left\{ \int_\beta \overline{(g \circ \emptyset)} \, dg - \int_\beta \bar{g} d(g \circ \emptyset) \right\} \\
&= \operatorname{Re}\left\{ \int_\beta \overline{(g \circ \emptyset)} \, dg - [\bar{g}(g \circ \emptyset)]_{\partial\beta} + \int_\beta (g \circ \emptyset) \, d\bar{g} \right\} \\
&= \operatorname{Re}\left\{ \int_\beta g^* \, dg + \overline{\int_\beta g^* \, dg} - [\bar{g}g]_{\partial\beta} \right\} \\
&= 2 \operatorname{Re} \int_\beta g^* \, dg - \operatorname{Re} \int_\beta d(\bar{g}g) \\
&= 2 \operatorname{Re} \int_\beta (g^* - \bar{g}) \, dg.
\end{aligned}$$

Thus (2.12) is the same as (2.16) also in case (b). This completes the proof of Proposition 2. ∎

Remark. If in (ii) or (iii) of Proposition 2, (2.11) (or (2.12) respectively) is required to hold only for all closed curves γ in Ω (or D respectively) then one gets exactly the condition that (i) of Proposition 1 holds for some real-valued μ or equivalently the condition that (i) of Proposition 2 holds for all h of the form $h = \operatorname{Re} f$, $f \in AL^1(\Omega)$.

3. PROOF OF THE MAIN RESULT

Using Proposition 2 together with an approximation argument we shall prove the following, which readily implies Theorem 1 (stated in the Introduction).

PROPOSITION 3. *With D, $a \in D$, $\varepsilon > 0$, and $r > 0$ as in Theorem* 1 *there exists a meromorphic function g on $\hat{D}$ satisfying* (2.12) *such that, moreover, $g|_D$ is univalent, $|g(z) - z| < \varepsilon$ for $z \in D$, and such that all poles of g are located in $\emptyset(B(a, r)) \subset \tilde{D}$.*

Before turning to the proof let us see how Theorem 1 follows from Proposition 3. With g as in Proposition 3 it follows from Proposition 2 that there exists a real-valued distribution μ with support in finite number of points such that (2.9) holds for $\Omega = g(D)$. The support of μ is the singular set of $S(z)$, hence is contained in $g(B(a,r)) \subset B(a, r+\varepsilon)$.

Now, everything in Theorem 1 is fulfilled except that μ may be a more general distribution than a measure. But clearly μ can be replaced by a signed measure ν with support in $B(a, r+\varepsilon)$ such that $\langle \mu, h \rangle = \int h\, d\nu$ for all $h \in HL^1(\Omega)$ by mollifying it with a radially symmetric test function with small support. Thus Theorem 1 follows from Propositions 2 and 3.

Proof of Proposition 3. Take D, $a \in D$, $\varepsilon > 0$, $r > 0$ as in the statement. We shall choose g with a pole at $\tilde{a} \in \tilde{D}$ and else regular. Let p be the number of components of ∂D minus one.

As for the condition (2.12) there are then two types of curves γ to consider:

(a) γ is homologous to one of the components of ∂D

(b) $\gamma = \gamma_j = \beta_j - \tilde{\beta}_j$ where β_j is a curve in $D \setminus \{a\}$ from the jth inner component of ∂D to the outer component $(j = 1, \ldots, p)$.

Note that, e.g., a small loop around a or $\tilde{a}$ is homologous (in $\hat{D} \setminus \{\tilde{a}\}$) to $\gamma = \pm \partial D$ and hence is covered by case (a).

Now for γ of type (a), (2.12) is automatically fulfilled. Indeed, taking γ to be a component of ∂D we have (since $g^* = \bar{g}$ on ∂D)

$$2 \operatorname{Re} \int_\gamma g^* \, dg = \int_\gamma \bar{g}\, dg + \overline{\int_\gamma \bar{g}\, dg} = \int_\gamma d(\bar{g} g) = 0.$$

For $\gamma = \gamma_j$ $(j = 1, \ldots, p)$ as in (b) we introduce the bilinear functional

$$B_j(f, h) = \int_{\gamma_j} f^* \, dh$$

for functions f and h defined on γ_j. It is readily verified that

$$B_j(f, h) = \overline{B_j(h, f)}.$$

In particular $B_j(f, f)$ is automatically real and, as at the end of the proof of Proposition 2, we have

$$B_j(f, f) = \operatorname{Re} \int_{\gamma_j} f^* \, df = 2 \operatorname{Re} \int_{\beta_j} f^* \, df - \left[|f|^2 \right]_{\partial \beta_j}. \tag{3.1}$$

Now to construct g we shall use the Mergelyan approximation theorem for compact Riemann surfaces [4]. (For the corresponding Runge theorem, see, e.g., [2].) Let U be a small neighbourhood of $\overline{D}$ in $\hat{D}$ and set

$$K = \overline{U} \cup \gamma_1 \cup \cdots \cup \gamma_p.$$

Then $\hat{D} \setminus K$ is connected and contains $\tilde{a}$ (if U is chosen properly). It is easy to see that it is possible to choose functions $f, f_1, \ldots, f_p \in C(K) \cap A(U)$ satisfying

$$f(z) = f_j(z) = z \qquad \text{for } z \in D, \tag{3.2}$$

$$B_j(f, f) = 0, \tag{3.3}$$

$$B_k(f, f_j) = \delta_{kj} \tag{3.4}$$

$(k, j = 1, \ldots, p)$. Indeed, (3.2) determines f and f_j on $\overline{U}$ and then it is clear from (3.1) that f and f_j can be adjusted on $\gamma_j \setminus \overline{U}$ so that (3.3), (3.4) hold.

By the Mergelyan approximation theorem there exist functions $h, h_1, \ldots, h_p \in M(\hat{D}) \cap A(\hat{D} \setminus \{\tilde{a}\})$ which approximate $f, f_1, \ldots, f_p$ uniformly on K, say

$$|h - f| < \varepsilon/2 \qquad \text{on } K,$$

$$|h_j - f_j| < \varepsilon/2 \qquad \text{on } K.$$

Recall that $\varepsilon > 0$ was given already in the formulation of the proposition, but we shall replace it by smaller values as necessary. We shall choose our g to be

$$g = h + \varepsilon_1 h_1 + \cdots + \varepsilon_p h_p$$

for suitable small $\varepsilon_1, \ldots, \varepsilon_p$ (and ε). Thus $g \in M(\hat{D}) \cap A(\hat{D} \setminus \{\tilde{a}\})$, and whenever ε and ε_j are small enough g will be univalent on D and satisfy $|g(z) - z| < \varepsilon$ for $z \in D$.

It remains to show that g satisfies condition (2.12), i.e., in view of the discussion at the beginning of the proof, to show that $\varepsilon, \varepsilon_j$ can be chosen so that

$$B_k(g, g) = 0, \qquad k = 1, \ldots, p. \tag{3.5}$$

We have, setting $B_k(f) = B_k(f, f)$,

$$\begin{aligned} B_k(g) &= B_k\left(h + \sum_1^p \varepsilon_i h_i\right) \\ &= B_k\left(f + \sum_1^p \varepsilon_i f_i + h - f + \sum_1^p \varepsilon_i (h_i - f_i)\right) \\ &= B_k\left(f + \sum_1^p \varepsilon_i f_i\right) + B_k\left(h - f + \sum_1^p \varepsilon_i (h_i - f_i)\right) \\ &\quad + 2\,\mathrm{Re}\, B_k\left(f + \sum_1^p \varepsilon_i f_i, h - f + \sum_1^p \varepsilon_i (h_i - f_i)\right). \end{aligned}$$

Using (3.3), (3.4) we see that this is of the form

$$B_k(g) = 2\,\mathrm{Re}\,\varepsilon_k + a_k(\varepsilon) + \mathrm{Re}\sum_{i=1}^p b_{ki}(\varepsilon)\varepsilon_i + \mathrm{Re}\sum_{i,j=1}^p c_{kij}(\varepsilon)\bar{\varepsilon}_i\varepsilon_j,$$

where $a_k(\varepsilon)$, $b_{ki}(\varepsilon) = \mathscr{O}(\varepsilon)$, and $c_{kij}(\varepsilon) = \mathscr{O}(1)$ as $\varepsilon \to 0$. From this it follows, e.g., from the implicit function theorem that if just ε is chosen small enough then the system (3.5) will have a solution in $\varepsilon_1, \dots, \varepsilon_p$ (with these small). Thus the proposition follows. ∎

Finally, we wish to justify a statement made in Example 1.

PROPOSITION 4. *Let μ be any real-valued distribution with compact support. Then for $a > 0$ large enough there exists a bounded domain Ω containing the origin and* supp μ *such that*

$$\int_\Omega h\,dx\,dy = ah(0) + \langle \mu, h \rangle$$

for all $h \in HL^1(\Omega)$.

Remark. This result is valid in any number of dimensions ≥ 2.

Proof. Choose $R > 0$ so that supp $\mu \subset B(0, R)$. Then there is a signed measure ν on $\partial B(0, R)$ such that $\langle \mu, h \rangle = \int_{\partial B(0,R)} h\,d\nu$ for all h harmonic in a neighbourhood of $\overline{B(0, R)}$. (It is just to take ν to be the measure with density equal to the normal derivative of the solution u of $-\Delta u = \mu$ in $B(0, R)$, $u = 0$ on $\partial B(0, R)$.)

Doing the same thing with $a\delta_0$ and adding it follows that there is a measure ν_a on $\partial B(0, R)$ such that

$$ah(0) + \langle \mu, h \rangle = \int_{\partial B(0, R)} h \, d\nu_a \tag{3.6}$$

for all h harmonic in a neighbourhood of $\overline{B(0, R)}$. If $a > 0$ is sufficiently large ν_a will be positive and we may even take a so large that (moreover) $\int d\nu_a > 4|B(0, R)|$ (in N dimensions: $2^N |B(0, R)|$).

Under these conditions it is known [11, 6, 7, 15, 16, 10] that there exists a domain Ω containing $\overline{B(0, R)}$ such that

$$\int_\Omega h \, dx \, dy = \int h \, d\nu_a$$

for all $h \in HL^1(\Omega)$. By combining with (3.5) the desired conclusion follows. ∎

Using this proposition we may construct, for $a > 0$ large, a domain Ω such that (2.8) in Example 1 holds. Going backwards in the example it follows that also (2.7) holds, as claimed there.

Note added in proof. Using a different method of proof, M. Sakai [18] has recently generalized the main result of this paper to higher dimensions.

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