A Distortion Theorem for Quadrature Domains for Harmonic Functions

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We prove that any finitely connected domain in the plane can be distorted so that it becomes “graviequivalent” to a signed measure with arbitrarily small support. Precisely: if $D \subset \mathbb{C}$ is a bounded, finitely connected domain with analytic boundary then for any $a \in D$ and $r > 0$, $\epsilon > 0$ with $B(a, r + \epsilon) \subset D$ there exists a univalent function $g$ in $D$ with $|g(z) - z| < \epsilon$ ($z \in D$) and a signed measure $\mu$ with support in $B(a, r)$ such that for every integrable harmonic function $h$ in $\Omega = g(D)$ we have $\int_{\Omega} h\, dx\, dy = \int h\, d\mu$. © 1996 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to prove the following theorem, stated, but not proved, in [8].

(Notation. $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.)

THEOREM 1. Let $D$ be a bounded domain in the complex plane $\mathbb{C}$ with $\partial D$ consisting of finitely many disjoint analytic Jordan curves, let $a \in D$, and let $r > 0$, $\epsilon > 0$ be any numbers satisfying $B(a, r + \epsilon) \subset D$. Then there exists a univalent function $g$ in $D$ with $|g(z) - z| < \epsilon$ ($z \in D$) such that the image domain $\Omega = g(D)$ has the following property: there exists a signed measure $\mu$.

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169
with support in $B(a, r)$ such that
\[ \int_{\Omega} h \, dx \, dy = \int h \, d\mu. \tag{1.1} \]
for every integrable harmonic function $h$ in $\Omega$.

That an equality of type (1.1) holds for (e.g.) all harmonic integrable $h$ is often referred to by saying that $\Omega$ is a "quadrature domain" for the measure $\mu$ (see, e.g., [11, 17] and references therein). By taking $h(z) = \log|z - \zeta|$ with $\zeta \not\in \Omega$ one sees that (1.1) means (or at least implies) that $\Omega$ regarded as a body with density one produces the same gravitational (or logarithmic) potential outside $\Omega$ as does the signed mass distribution $\mu$.

Thus the theorem roughly says that the set of bodies graviequivalent to signed mass distributions supported in any fixed ball $B(a, r)$ are dense in the set of all bodies containing $B(a, r)$. One main point with this result is that if one instead asks for positive mass distribution doing the same job, i.e., if one in the theorem requires also $\mu \geq 0$, then it becomes tremendously false. Indeed, one may show that if (1.1) holds with $\mu \geq 0$ and with area $|\Omega| \geq 4\pi^3 r^3$ then $\Omega$ is essentially ball shaped (e.g., $\Omega$ is simply connected, $\partial \Omega$ is analytic, and for any $z \in \partial \Omega$ the inward normal of $\partial \Omega$ at $z$ intersects $B(a, r)$). This follows from results in [11, 6, 7, 16, 10].

The corresponding theorem with (1.1) required to hold for all integrable analytic functions $h$ was (essentially) proved in [5]. From this follows the theorem for harmonic $h$ in case $\Omega$ is simply connected. A short direct proof for this case is also given in [8]. In the present paper we shall extend the results of [5] to prove Theorem 1.

Notation.

\begin{align*}
C(\Omega) &= \{ f : \Omega \to \mathbb{C} : f \text{ is continuous} \}, \\
A(\Omega) &= \{ f : \Omega \to \mathbb{C} : f \text{ is analytic} \}, \\
M(\Omega) &= \{ f : \Omega \to \mathbb{C} \cup \{\infty\} : f \text{ is meromorphic} \}, \\
H(\Omega) &= \{ h : \Omega \to \mathbb{R} : h \text{ is harmonic} \}, \\
AL^1(\Omega) &= \left\{ f \in A(\Omega) : \int_{\Omega} |f| \, dx \, dy < \infty \right\}, \\
HL^1(\Omega) &= \left\{ h \in H(\Omega) : \int_{\Omega} |h| \, dx \, dy < \infty \right\}, \\
\delta_a &= \text{Dirac measure at the point } a \in \mathbb{C}, \\
\delta_{kj} &= \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases} \\
|\Omega| &= \text{area of } \Omega.
\end{align*}
2. QUADRATURE DOMAINS FOR ANALYTIC AND HARMONIC FUNCTIONS

Let $D \subset \mathbb{C}$ be a finitely connected bounded domain with $\partial D$ analytic, as in Section 1. Then $D$ can be completed to a compact Riemann surface $\hat{D}$, called the “Schottky double” of $D$. It consists of $D$, a copy $\tilde{D}$ provided with the opposite conformal structure, and their common boundary $\partial D$ through which they are welded together. See [9, 5]. If $z \in D$ we denote by $\tilde{z}$ the corresponding point on $\tilde{D}$.

There is a natural anticonformal involution $\varnothing$ on $D$ given by $\varnothing(z) = \bar{z}$, $\varnothing(\bar{z}) = z$ if $z \in D$ ($\bar{z} \in D$) and by $\varnothing(z) = z$ if $z \in \partial D$. Together, the pair $(\hat{D}, \varnothing)$ constitutes a so-called symmetric Riemann surface. If $f$ is any meromorphic function on $D$ then

$$f^* = \overline{f \circ \varnothing}$$

is also meromorphic on $\hat{D}$. On $\partial D$, $f^* = \bar{f}$. Setting $f_1 = f|_{\pi}$, $f_2 = f^*|_{\pi}$ it follows that $f_1$ and $f_2$ are meromorphic functions in $D$ continuous (in the extended sense) up to $\partial D$ and satisfying

$$f_1 = \bar{f}_2 \quad \text{on } \partial D. \quad (2.1)$$

Conversely, any pair of meromorphic functions $f_1, f_2$ on $\hat{D}$ satisfying (2.1) represents a meromorphic function on $D$. Note that if $f$ is represented as above by $(f_1, f_2)$, then $f^*$ is represented by $(f_2, f_1)$.

We shall first recall a result for quadrature domains for analytic functions which has been proved in various parts in [3, 1, 5]. See also, e.g., [17, 13, 14].

**Proposition 1.** With $D$ and $\hat{D}$ as above, let $\Omega \subset \mathbb{C}$ be a bounded domain conformally equivalent to $D$ and let $g : D \to \Omega$ be a conformal map. Then the following assertions are equivalent.

(i) There exists a complex-valued distribution $\mu$ with support in a finite number of points in $\Omega$ such that

$$\int_{\Omega} f dx dy = \langle \mu, f \rangle \quad \text{for all } f \in AL^2(\Omega). \quad (2.2)$$

(ii) There exists a meromorphic function $S(z)$ in $\Omega$, continuously extendible up to $\partial \Omega$ with

$$S(z) = \bar{z} \quad \text{on } \partial \Omega. \quad (2.3)$$

(iii) $g$ extends to a meromorphic function on $\hat{D}$. 

When (i)–(iii) hold, $\mu$, $S$, and $g$ are related by

$$
\langle \mu, f \rangle = \pi \sum_{z \in \Omega} \text{Res}(S(z)f(z)) \quad (f \in \mathcal{AL}^1(\Omega))
$$

and $S(z)$ is called the Schwarz function of $\partial \Omega$ [3, 17].

As to the proof we just mention that (ii) $\Rightarrow$ (i) follows from partial integration and the residue theorem,

$$
\int_{\Omega} f dx dy = \frac{1}{2i} \int_{\partial \Omega} f dz \, dz
$$

and that (iii) simply is the statement (ii) pulled back to $D$ by means of $g$.

Note that, in terms of the representation (2.1), Eq. (2.3) says that the pair $(z, S(z))$ represents a meromorphic function on $\Omega$.

Any distribution $\mu \in \mathcal{D}'(\Omega)$ with support in a finite number of points is of the form

$$
\langle \mu, \varphi \rangle = \sum_{k=1}^m \sum_{i,j=0}^n a_{kij} \frac{\partial^{i+j}\varphi}{\partial x^i \partial y^j}(z_k) \quad (\varphi \in C^\infty(\Omega))
$$

for suitable $m$, $n$ and $z_1, \ldots, z_m \in \Omega$, $a_{kij} \in \mathbb{C}$. Since $\partial f/\partial x = -i(\partial f/\partial y)$ when $f$ is analytic, many different distributions $\mu$ of the form (2.5) have the same action on $\mathcal{AL}^1(\Omega)$. It is always possible to take $a_{kij}$ to be real whenever $i + j \geq 1$, but this still does not make $\mu$ uniquely determined in general, since, e.g., $\partial^2 f/\partial x^2 = -\partial^2 f/\partial y^2$.

One way to write the action of $\mu$ on $\mathcal{AL}^1(\Omega)$ in a canonical form is to write (2.5) as

$$
\langle \mu, f \rangle = \sum_{k=1}^m \sum_{j=0}^{2n} c_k f^{(j)}(z_k)
$$

for $f \in \mathcal{AL}^1(\Omega)$.

When (2.2) holds we see, by taking $f = 1$, that $\sum_{k=1}^m \theta_{k0} = \sum_{k=1}^m \epsilon_{k0}$ is real and positive, but the individual $a_{k00} = c_{k0}$ may very well be nonreal. To be precise, we have the following. Suppose that (2.2) holds and choose
$a_{kij}$ for $i + j \geq 1$ to be real. Then $a_{k00} \in \mathbb{R}$ for all $k$ if and only if the identity

$$\int_{\Omega} u \, dx \, dy = \langle \mu, u \rangle$$

holds for every harmonic function $u$ which is the real part of some $f \in AL^{1}(\Omega)$.

Indeed, if $a_{k00} \in \mathbb{R}$ then $\mu$ is real-valued (i.e., takes real values on real-valued test functions) and (2.6) follows from (2.2) by taking real parts. Conversely, (2.6) alone implies (2.2) and then for every $f \in AL^{1}(\Omega)$, $\langle \mu, \text{Re} \, f \rangle = \int_{\Omega} (\text{Re} \, f) \, dx \, dy = \text{Re} \int_{\Omega} f \, dx \, dy = \text{Re} \langle \mu, f \rangle$, which implies that $a_{k00} \in \mathbb{R}$ when $a_{kij} \in \mathbb{R}$, $i + j \geq 1$. (If, e.g., $a_{100} \notin \mathbb{R}$ then we get a contradiction by taking $f$ to be a polynomial with $f(z_1) = 1$, $f(z_k) = 0$ for $k \neq 1$.)

**Example 1.** It is possible to construct $\Omega$ such that for a suitably large $a > 0$

$$\int_{\Omega} f \, dx \, dy = af(0) + i(f(1) - f(-1))$$

for all $f \in AL^{1}(\Omega)$, (See Proposition 4 below.) Here the right member is $\langle \mu, f \rangle$ with $\mu = a \delta_0 + i \delta_1 - i \delta_{-1}$. This $\mu$ is nonreal and its action on $AL^{1}(\Omega)$ does not coincide with that of any real-valued distribution with support in a finite number of points. Therefore, the identity (2.6) does not hold for all $u = \text{Re} \, f$, $f \in AL^{1}(\Omega)$.

In order to get a quadrature identity for this $\Omega$ and holding for all $u = \text{Re} \, f$, $f \in AL^{1}(\Omega)$, write $f = u + iv$ and let $\gamma$ be the straight line segment from $-1$ to $+1$. Then

$$i(f(1) - f(-1)) = i \int_{\gamma} \frac{\partial f}{\partial x} \, dx$$

$$= i \int_{\gamma} \frac{\partial u}{\partial x} \, dx - \int_{\gamma} \frac{\partial v}{\partial x} \, dx$$

$$= i \int_{\gamma} \frac{\partial u}{\partial x} \, dx + \int_{\gamma} \frac{\partial u}{\partial y} \, dx$$

so that, taking real parts in (2.7),

$$\int_{\Omega} u \, dx \, dy = au(0) + \int_{\gamma} \frac{\partial u}{\partial y} \, dx$$

(2.8)
for all \( u = \text{Re} f, f \in AL^3(\Omega) \). Thus (2.6) holds, but with \( \mu \) a distribution of a more general form.

Next we wish to discuss quadrature identities

\[
\int_{\Omega} h \, dx \, dy = \langle \mu, h \rangle \tag{2.9}
\]

with \( \mu \) of the form (2.5) holding for all \( h \in HL^3(\Omega) \). Clearly, if (2.9) holds with \( \mu \) complex-valued it also holds for \( \text{Re} \mu \) (and \( \text{Im} \mu \) annihilates \( HL^3(\Omega) \)), so we may as well assume from the beginning that \( \mu \) is real-valued. Recall from the previous discussion that (2.9) then holds for all \( h \in HL^3(\Omega) \) of the form \( h = \text{Re} f, f \in AL^3(\Omega) \) if and only if the equivalent conditions in Proposition 1 hold.

**Proposition 2.** With \( D \) and \( \hat{D} \) as in the beginning of this section, let \( \Omega \subset \mathbb{C} \) be a bounded domain conformally equivalent to \( D \) and let \( g : D \to \Omega \) be a conformal map. Then the following assertions are equivalent.

(i) There exists a real-valued distribution \( \mu \) with support in a finite number of points in \( \Omega \) such that (2.9) holds for all \( h \in HL^3(\Omega) \).

(ii) There exists a meromorphic function \( S(z) \) in \( \Omega \), continuously extendible up to \( \partial \Omega \) with

\[
S(z) = \bar{z} \quad \text{on } \partial \Omega \tag{2.10}
\]

and satisfying in addition

\[
\text{Re} \int_{\gamma} (\bar{z} - S(z)) \, dz = 0 \tag{2.11}
\]

for every smooth curve \( \gamma \subset \overline{\Omega} \) avoiding the poles of \( S(z) \) and satisfying \( \partial \gamma \subset \partial \Omega \) (i.e., \( \gamma \) either is closed or joins two components of \( \partial \Omega \)).

(iii) \( g \) extends to a meromorphic function on \( \hat{D} \) satisfying the additional condition that

\[
\text{Re} \int_{\gamma} g^* \, dg = 0 \tag{2.12}
\]

for every closed curve \( \gamma \) in \( \hat{D} \) avoiding the poles of \( g^* dg \).

**Proof.** We first prove that (i) and (ii) are equivalent. To large parts this is actually well-known (see [12, 14], e.g.), but since there are some subtle points we prefer to outline the full proof.
Let $E(z) = -(1/2\pi)\log|z|$ be the standard fundamental solution of $-\Delta$ (so that $-\Delta E = \delta_0$). If (i) holds, set

$$u = (\mu - \chi_\Omega) * E.$$  

Then $u$ is a real-valued distribution, continuously differentiable outside supp $\mu$ and satisfying

$$\Delta u = \chi_\Omega - \mu \quad \text{in } \mathbb{C},$$  

$$u = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \quad \text{on } \Omega^c.$$  

The latter equation follows by taking $h(z) = E(z - \zeta)$, $(\partial E/\partial x)(z - \zeta)$, $(\partial E/\partial y)(z - \zeta)$ for $\zeta \in \Omega^c$ in (2.9).

Define $S(z)$ by

$$S(z) = \bar{z} - 4\frac{\partial u}{\partial z}, \quad z \in \overline{\Omega}.$$  

Then $\partial S/\partial \bar{z} = 1 - \Delta u = \mu$ in $\Omega$ so that $S(z)$ is meromorphic in $\Omega$. By (2.14), $S(z)$ is continuous up to $\partial \Omega$ with (2.10) holding there. Finally, (2.11) holds since

$$\Re \int_\gamma (\bar{z} - S(z)) \, d\gamma = 4 \Re \int_\gamma \frac{\partial u}{\partial z} \, d\gamma$$  

$$= 2\int_\gamma du,$$

which is zero since $u = 0$ on $\partial \gamma$ (or $\partial \gamma$ is empty). Thus (i) implies (ii).

Conversely, suppose (ii) holds. Let $z_1, \ldots, z_m \in \Omega$ be the poles of $S(z)$, pick a point $z_0 \in \partial \Omega$, and define $u(z)$ on $\overline{\Omega} \setminus \{z_1, \ldots, z_m\}$ by

$$u(z) = \frac{1}{2} \Re \int_{z_0}^{z} (\bar{\zeta} - S(\zeta)) \, d\zeta$$  

and extend it by zero outside $\overline{\Omega}$. Then, due to (2.10), (2.11), $u$ is a well-defined continuously differentiable real-valued function in $\mathbb{C} \setminus \{z_1, \ldots, z_m\}$ satisfying (2.14). Moreover,

$$\Delta u = \chi_\Omega \quad \text{in } \mathbb{C} \setminus \{z_1, \ldots, z_m\}.$$  

(2.15)

This follows from the definition of $u$ together with the known fact [1, 13] that $\partial \Omega$ is piecewise smooth (algebraic) when (2.10) holds.
At $z_1,\ldots,z_m$, $u$ has polar singularities. Unfortunately, $u$ need not be integrable over these, so $u$ need not, in a canonical way, be a distribution on $\mathbb{C}$. Therefore one cannot define $\mu$ right away by (2.13), as one is tempted to do, with $\Delta u$ the distributional Laplacian of $u$ on $\mathbb{C}$. Nevertheless, with $B_\varepsilon = B(z_1, \varepsilon) \cup \cdots \cup B(z_m, \varepsilon)$ for $\varepsilon > 0$ small it follows, using (2.14), (2.15), and the cutting-off technique in [11, p. 60], that for $h \in HL^1(\Omega)$

$$\int_\Omega h \, dx \, dy = \lim_{\varepsilon \to 0} \int_{\Omega \setminus B_\varepsilon} h \Delta u \, dx \, dy$$

$$= \lim_{\varepsilon \to 0} \left( \int_{\partial B_\varepsilon} u \frac{\partial h}{\partial n} \, ds - \int_{\partial B_\varepsilon} h \frac{\partial u}{\partial n} \, ds \right).$$

Since the singularities of $u$ at $z_1,\ldots,z_m$ are of finite order it follows that the right member above can be written $\langle \mu, h \rangle$ for some real-valued distribution $\mu$ (not uniquely determined in general, as a distribution) with support at $z_1,\ldots,z_m$. Thus (ii) implies (i).

It remains to prove that (ii) and (iii) are equivalent. As in Proposition 1 (or [5]) the existence of a meromorphic $S(z)$ satisfying (2.10) is equivalent to the meromorphic extension of $g$ to $\hat{D}$. Thus we only need to show that condition (2.11) is the same as (2.12).

Pulling (2.11) back to $D$ it becomes

$$\Re \int_\beta (\bar{g} - g^*) \, dg = 0 \quad (2.16)$$

for every $\beta \subset \overline{D}$ with $\partial \beta \subset \partial D$ ($\gamma = g \circ \beta$). We shall prove that (2.12) and (2.16) are equivalent. There are two kinds of curves $\beta$ which need to be considered in (2.12) and (2.16): (a) closed curves in $D$; (b) for (2.16), curves in $D$ joining two components of $\partial D$ and, for (2.12), the corresponding closed curves on $D$ obtained by going back along the same track on the back-side $\hat{D}$.

As for (a), we note that

$$\Re \int_\beta \bar{g} \, dg = \frac{1}{2} \int_\beta (\bar{g} \, dg + g \, d\bar{g}) = \frac{1}{2} \int_\beta d(\bar{g}g) = 0$$

when $\beta$ is a closed curve in $D$, so then (2.16) is really the same as (2.12).
As for (b), let \( \beta \) be a nonclosed curve on \( \overline{D} \) with endpoints on \( \partial D \) and let \( \chi = \beta - \tilde{\beta} \) (where \( \tilde{\beta} = \partial \circ \beta \)) be the corresponding (oriented) closed curve on \( D \). Then

\[
\Re \int_{\gamma} g^* dg = \Re \left( \int_{\beta} (g \circ \partial) \, dg - \int_{\beta} (g \circ \partial) \, dg \right)
\]

\[
= \Re \left( \int_{\beta} (g \circ \partial) \, dg - \int_{\beta} \bar{g} d(g \circ \partial) \right)
\]

\[
= \Re \left( \int_{\beta} (g \circ \partial) \, dg - \bar{g} \left( g \circ \partial \right) + \int_{\beta} (g \circ \partial) \, dg \right)
\]

\[
= \Re \left( \int_{\beta} g^* dg + \int_{\beta} g^* dg - \bar{g} \right) \bigg|_{\beta}
\]

\[
= 2 \Re \int_{\beta} g^* dg - \Re \int_{\beta} d(g \bar{g})
\]

\[
= 2 \Re \int_{\beta} (g^* - \bar{g}) \, dg.
\]

Thus (2.12) is the same as (2.16) also in case (b). This completes the proof of Proposition 2.

Remark. If in (ii) or (iii) of Proposition 2, (2.11) (or (2.12) respectively) is required to hold only for all closed curves \( \gamma \) in \( \Omega \) (or \( D \) respectively) then one gets exactly the condition that (i) of Proposition 1 holds for some real-valued \( \mu \) or equivalently the condition that (i) of Proposition 2 holds for all \( h \) of the form \( h = \Re f, f \in AL^1(\Omega) \).

3. PROOF OF THE MAIN RESULT

Using Proposition 2 together with an approximation argument we shall prove the following, which readily implies Theorem 1 (stated in the Introduction).

**Proposition 3.** With \( D, a \in D, \rho > 0, \) and \( r > 0 \) as in Theorem 1 there exists a meromorphic function \( g \) on \( D \) satisfying (2.12) such that, moreover, \( g \mid_D \) is univalent, \( |g(z) - z| < \rho \) for \( z \in D \), and such that all poles of \( g \) are located in \( \bigcap (B(a, r)) \subset D \).

Before turning to the proof let us see how Theorem 1 follows from Proposition 3. With \( g \) as in Proposition 3 it follows from Proposition 2 that
there exists a real-valued distribution $\mu$ with support in finite number of points such that (2.9) holds for $\Omega = g(D)$. The support of $\mu$ is the singular set of $S(z)$, hence is contained in $g(B(a, r)) \subset B(a, r + \varepsilon)$.

Now, everything in Theorem 1 is fulfilled except that $\mu$ may be a more general distribution than a measure. But clearly $\mu$ can be replaced by a signed measure $\nu$ with support in $B(a, r + \varepsilon)$ such that $\langle \mu, h \rangle = \int h d\nu$ for all $h \in H^2_\alpha(\Omega)$ by mollifying it with a radially symmetric test function with small support. Thus Theorem 1 follows from Propositions 2 and 3.

Proof of Proposition 3. Take $D, a \in D$, $\varepsilon > 0$, $r > 0$ as in the statement. We shall choose $g$ with a pole at $\tilde{a} \in D$ and else regular. Let $p$ be the number of components of $\partial D$ minus one.

As for the condition (2.12) there are then two types of curves $\gamma$ to consider:

(a) $\gamma$ is homologous to one of the components of $\partial D$

(b) $\gamma = \gamma_j = \beta_j - \tilde{\beta}_j$, where $\beta_j$ is a curve in $D \setminus \{a\}$ from the $j$th inner component of $\partial D$ to the outer component ($j = 1, \ldots, p$).

Note that, e.g., a small loop around $a$ or $\tilde{a}$ is homologous (in $\hat{D} \setminus \{\tilde{a}\}$) to $\gamma = \pm \partial D$ and hence is covered by case (a).

Now for $\gamma$ of type (a), (2.12) is automatically fulfilled. Indeed, taking $\gamma$ to be a component of $\partial D$ we have (since $g^* = \bar{g}$ on $\partial D$)

$$2 \text{Re} \int_\gamma g^* \, dg = \int_\gamma \bar{g} \, dg + \int_\gamma g \, dg = \int_\gamma d(\bar{g} g) = 0.$$

For $\gamma = \gamma_j$ ($j = 1, \ldots, p$) as in (b) we introduce the bilinear functional

$$B_j(f, h) = \int_{\gamma_j} f^* \, dh$$

for functions $f$ and $h$ defined on $\gamma_j$. It is readily verified that

$$B_j(f, h) = \overline{B_j(h, f)}.$$

In particular $B_j(f, f)$ is automatically real and, as at the end of the proof of Proposition 2, we have

$$B_j(f, f) = \text{Re} \int_{\gamma_j} f^* \, df = 2 \text{Re} \int_{\beta_j} f^* \, df - \left[ |f|^2 \right]_{\beta_j}.$$ (3.1)
Now to construct \( g \) we shall use the Mergelyan approximation theorem for compact Riemann surfaces [4]. (For the corresponding Runge theorem, see, e.g., [2].) Let \( U \) be a small neighbourhood of \( \overline{D} \) in \( D \) and set

\[
K = \overline{U} \cup \gamma_1 \cup \ldots \cup \gamma_p.
\]

Then \( \hat{D} \setminus K \) is connected and contains \( \hat{a} \) (if \( U \) is chosen properly). It is easy to see that it is possible to choose functions \( f, f_1, \ldots, f_p \in C(K) \cap A(U) \) satisfying

\[
f(z) = f_j(z) = z \quad \text{for} \ z \in D, \quad (3.2)
\]

\[
B_j(f, f) = 0, \quad (3.3)
\]

\[
B_k(f, f_j) = \delta_{k,j} \quad (3.4)
\]

\((k, j = 1, \ldots, p)\). Indeed, (3.2) determines \( f \) and \( f_j \) on \( \overline{U} \) and then it is clear from (3.1) that \( f \) and \( f_j \) can be adjusted on \( \gamma_j \setminus \overline{U} \) so that (3.3), (3.4) hold.

By the Mergelyan approximation theorem there exist functions \( h, h_1, \ldots, h_p \in M(\hat{D}) \cap A(\hat{D} \setminus \{\hat{a}\}) \) which approximate \( f, f_1, \ldots, f_p \) uniformly on \( K \), say

\[
|h - f| < \varepsilon/2 \quad \text{on} \ K,
\]

\[
|h_j - f_j| < \varepsilon/2 \quad \text{on} \ K.
\]

Recall that \( \varepsilon > 0 \) was given already in the formulation of the proposition, but we shall replace it by smaller values as necessary. We shall choose our \( g \) to be

\[
g = h + \varepsilon_1 h_1 + \cdots + \varepsilon_p h_p
\]

for suitable small \( \varepsilon_1, \ldots, \varepsilon_p \) (and \( \varepsilon \)). Thus \( g \in M(\hat{D}) \cap A(\hat{D} \setminus \{\hat{a}\}) \), and whenever \( \varepsilon \) and \( \varepsilon_j \) are small enough \( g \) will be univalent on \( D \) and satisfy \( |g(z) - z| < \varepsilon \) for \( z \in D \).

It remains to show that \( g \) satisfies condition (2.12), i.e., in view of the discussion at the beginning of the proof, to show that \( \varepsilon, \varepsilon_j \) can be chosen so that

\[
B_k(g, g) = 0, \quad k = 1, \ldots, p. \quad (3.5)
\]
We have, setting \( B_\varepsilon(f) = B_\varepsilon(f, f) \),

\[
B_\varepsilon(g) = B_\varepsilon \left( h + \sum_{i=1}^{p} \varepsilon_i h_i \right) \\
= B_\varepsilon \left( f + \sum_{i=1}^{p} \varepsilon_i f_i + h - f + \sum_{i=1}^{p} \varepsilon_i (h_i - f_i) \right) \\
= B_\varepsilon \left( f + \sum_{i=1}^{p} \varepsilon_i f_i \right) + B_\varepsilon \left( h - f + \sum_{i=1}^{p} \varepsilon_i (h_i - f_i) \right) \\
+ 2 \Re \left( eB_\varepsilon \left( f + \sum_{i=1}^{p} \varepsilon_i f_i, h - f + \sum_{i=1}^{p} \varepsilon_i (h_i - f_i) \right) \right).
\]

Using (3.3), (3.4) we see that this is of the form

\[
B_\varepsilon(g) = 2 \Re \varepsilon_k + a_k(\varepsilon) + \Re \sum_{i=1}^{p} b_{\varepsilon_i}(\varepsilon) \varepsilon_i + \Re \sum_{i,j=1}^{p} c_{\varepsilon_i \varepsilon_j}(\varepsilon) \overline{\varepsilon_i} \varepsilon_j,
\]

where \( a_1(\varepsilon), b_{\varepsilon_1}(\varepsilon) = \mathcal{O}(\varepsilon) \), and \( c_{\varepsilon_i \varepsilon_j}(\varepsilon) = \mathcal{O}(1) \) as \( \varepsilon \to 0 \). From this it follows, e.g., from the implicit function theorem that if just \( \varepsilon \) is chosen small enough then the system (3.5) will have a solution in \( \varepsilon_1, \ldots, \varepsilon_p \) (with these small). Thus the proposition follows.

Finally, we wish to justify a statement made in Example 1.

**Proposition 4.** Let \( \mu \) be any real-valued distribution with compact support. Then for \( a > 0 \) large enough there exists a bounded domain \( \Omega \) containing the origin and \( \text{supp } \mu \) such that

\[
\int_{\Omega} h \, dx \, dy = ah(0) + \langle \mu, h \rangle
\]

for all \( h \in \mathcal{H}^1(\Omega) \).

**Remark.** This result is valid in any number of dimensions \( \geq 2 \).

**Proof.** Choose \( R > 0 \) so that \( \text{supp } \mu \subset B(0, R) \). Then there is a signed measure \( \nu \) on \( \partial B(0, R) \) such that \( \langle \mu, h \rangle = \int_{\partial B(0, R)} h \, d\nu \) for all \( h \) harmonic in a neighbourhood of \( B(0, R) \). (It is just to take \( \nu \) to be the measure with density equal to the normal derivative of the solution \( u \) of \( -\Delta u = \mu \) in \( B(0, R) \), \( u = 0 \) on \( \partial B(0, R) \).)
Doing the same thing with $a\delta_0$ and adding it follows that there is a measure $\nu_a$ on $\partial B(0, R)$ such that

$$ah(0) + \langle \mu, h \rangle = \int_{\partial B(0, R)} h \, d\nu_a$$

for all $h$ harmonic in a neighbourhood of $\overline{B(0, R)}$. If $a > 0$ is sufficiently large $\nu_a$ will be positive and we may even take $a$ so large that (moreover) $\int d\nu_a > 4|B(0, R)|$ (in $N$ dimensions: $2^N |B(0, R)|$).

Under these conditions it is known [11, 6, 7, 15, 16, 10] that there exists a domain $\Omega$ containing $\overline{B(0, R)}$ such that

$$\int_{\Omega} h \, dx \, dy = \int h \, d\nu_a$$

for all $h \in HL^2(\Omega)$. By combining with (3.5) the desired conclusion follows.

Using this proposition we may construct, for $a > 0$ large, a domain $\Omega$ such that (2.8) in Example 1 holds. Going backwards in the example it follows that also (2.7) holds, as claimed there.

Note added in proof. Using a different method of proof, M. Sakai [18] has recently generalized the main result of this paper to higher dimensions.

REFERENCES