CRITICAL POINTS OF GREEN'S FUNCTION AND GEOMETRIC FUNCTION THEORY

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Abstract. We study questions related to critical points of the Green's function of a bounded multiply connected domain in the complex plane. The motion of critical points, their limiting positions as the pole approaches the boundary and the differential geometry of the level lines of the Green's function are main themes in the paper. A unifying role is played by various affine and projective connections and corresponding Möbius invariant differential operators. In the doubly connected case the three Eisenstein series E_2, E_4, E_6 are used. A specific result is that a doubly connected domain is the disjoint union of the set of critical points of the Green's function, the set of zeros of the Bergman kernel and the separating boundary limit positions for these. At the end we consider the projective properties of the prepotential associated to a second order differential operator depending canonically on the domain.

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Critical point, Green's function, Neumann function, Bergman kernel, Schiffer kernel, Schottky-Klein prime form, Schottky double, weighted Bergman space, Poincaré metric, Martin boundary, projective structure, projective connection, affine connection, prepotential, Eisenstein series

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1. Introduction

The results of this paper have their origin in attempts to understand trajectories of critical points in a planar multiply connected domain. We start with studying, for a given bounded multiply connected domain Ω in the complex plane, the motion of critical points of the ordinary Green's function $G(z,\zeta)$ of Ω as the pole ζ moves around. If Ω has connectivity $\mathtt{g}+1$ and the critical points are denoted $z_1(\zeta),\dots,z_{\mathtt{g}}(\zeta)$, we show first of all that the $z_j(\zeta)$ stay within a compact subset K of Ω (this result has also recently been obtain by A. Solynin [56]) and secondly that the limiting positions of the $z_j(\zeta)$ as ζ approaches the boundary coincide with the corresponding limiting positions of the zeros of the Bergman kernel. In the doubly connected case ($\mathtt{g}=1$) it even turns out that the domain is the disjoint union of the set of critical points of the Green's function and the set of zeros of the Bergman kernel, plus the common boundary of these two sets.

The method developed to prove the existence of the compact set K is remarkably related to (a new type of) Martin compactification and uses many properties of the Bergman, Schiffer and Poisson kernels. All these functionals depend in one sense or another on critical points of the Dirichlet Green's function, and they moreover extend in a natural way to sections of suitable bundles of the Schottky double of the domain, which is a compact Riemann surface of genus $\mathtt{g}$. An important role here is played by the Schwarz function, which can be interpreted as being the coordinate transition function between the front side and the back side of the Schottky double.

It is natural in this context to study the relationship between the hyperbolic metric and level lines of the Green's function. We show, for example, that the average, taken on a level line of the Green's function, of the Green's potential G^μ of any compactly supported measure μ is proportional to the total mass of μ. We also study level lines of the Green's function, and other harmonic functions, from the point of view of hamiltonian mechanics and differential geometry. One main observation then is that such level lines are trajectories, as well as Hamilton-Jacobi geodesics, for systems with the squared modulus of the gradient of the harmonic function as hamiltonian.

Various other topics touched on are estimates of the Taylor coefficients of the regular part of the complex Green's function in terms of the distance to the boundary, and relationships between these coefficients and the Poincaré metric.

A major part of the paper is devoted to a detailed study of the doubly connected case, modeled by the annulus. In this situation all calculations can be done explicitly because we have at our disposal the theory of modular forms (theta functions) and elliptic functions. For example the dichotomy result mentioned in the beginning is proved in this section. The Ramanujan formula for the derivatives of the basic Eisenstein series E_2, E_4, E_6 are fundamental.

Some of the mentioned Taylor coefficients and Eisenstein series transform under coordinate changes as different kinds of connections. In the second half of the paper we study affine and projective connections in quite some depth. In general, projective connections play a unifying role in the paper, in fact almost all of our work turns around projective connections, and to some extent affine connections. One example is that they generate Möbius invariant differential operators, called Bol operators and denoted Λ_m, for which boundary integral formulas, of Stokes type but for higher order derivatives, can be proved. These are useful for studying weighted Bergman spaces in general multiply connected domains. There is one such Bergman space for each half-integer, and the elements of the space should be thought of as differentials of this order (actually integrals if the order is negative). The Bol operator then is an isometry $\Lambda_m : A_m(\Omega) \to B_m(\Omega)$ from a Bergman type space $A_m(\Omega)$ of differentials of order $\frac{1-m}{2}$ to a (weighted) Bergman space $B_m(\Omega)$ consisting of differentials of order $\frac{1+m}{2}$. Also reproducing kernels for these, and some other, spaces are studied.

Along with connections, several questions related to curvature of, for example, level lines are discussed. The Study formula finds here its natural meaning. A further aspect is that the operators Λ_m all turn out to be symmetric powers of a single one, namely Λ_2.

In the final section of the paper we try to clarify in our setting the meaning of a certain prepotential for second order differential operators which has arisen in some recent physics papers.

2. Generalities on function theory of finitely connected plane domains

2.1. **The Green's function and the Schottky double.** Let $\Omega \subset \mathbb{C}$ be a bounded domain of finite connectivity, each boundary component being nondegenerate (i.e., consisting of more than one point). The oriented boundary (having Ω on its left hand side) is denoted $\partial\Omega = \Gamma = \Gamma_0 + \Gamma_1 + \dots + \Gamma_{\mathtt{g}}$, where Γ_0 is the outer component. We shall in most of the paper discuss only conformally invariant questions, and then we may assume that each boundary component Γ_j is a smooth analytic curve. Alternatively, one may think of Ω as a plane bordered Riemann surface.

Much of the functions theory on Ω is conveniently described in terms of the Schottky double $\hat{\Omega}$ of Ω. This is the compact Riemann surface of genus $\mathtt{g}$ obtained, when Ω has analytic boundary, by welding Ω along $\partial\Omega$, with an identical copy $\tilde{\Omega}$. Thus, as a point set $\hat{\Omega} = \Omega \cup \partial\Omega \cup \tilde{\Omega}$. The "backside" $\tilde{\Omega}$ is provided with the opposite conformal structure. This means that if $\tilde{z} \in \tilde{\Omega}$ denotes the point opposite to $z \in \Omega$ then the map $\tilde{z} \mapsto \bar{z}$ is a holomorphic coordinate on $\tilde{\Omega}$. The construction of the Schottky double generalizes to any bordered Riemann surface and the result is always a symmetric Riemann surface, i.e., a Riemann surface provided with an antiholomorphic involution. In the Schottky double case this is the map $J:\hat{\Omega}\to\hat{\Omega}$ which exchanges z and $\tilde{z}$ and which keeps $\partial\Omega$ pointwise fixed.

The Schottky double of a plane domain Ω has a holomorphic atlas consisting of only two charts: the corresponding coordinate functions are the identity map $\phi_1 : z \mapsto z$ on Ω and the map $\phi_2 : \tilde{z} \mapsto \bar{z}$ on $\tilde{\Omega}$. When $\partial\Omega$ is analytic, as is henceforth assumed, both these maps extend analytically across $\partial\Omega$ in $\hat{\Omega}$, hence their domains of definitions overlap and the union covers all $\hat{\Omega}$. Let

$$S=\phi_2\circ\phi_1^{-1} \tag{2.1}$$

be the coordinate transition function. It is analytic and defined in a neighbourhood of $\partial\Omega$ in $\mathbb{C}$, and on $\partial\Omega$ it satisfies

$$S(z)=\bar{z}\quad (z\in\partial\Omega). \tag{2.2}$$

Thus it is the *Schwarz function* [10], [11], [54] of $\partial\Omega$.

Differentiating (2.2) gives

$$d\bar{z}=S'(z)dz\quad \text{along } \partial\Omega. \tag{2.3}$$

With s an arc-length parameter along $\partial\Omega$ such that Ω lies to the left as s increases,

$$T(z)=\frac{dz}{ds} \tag{2.4}$$

is the oriented unit tangent vector on $\partial\Omega$. By (2.3) and since $|T(z)|=1$,

$$S'(z)=\frac{1}{T(z)^2},\quad z\in\partial\Omega.$$

It follows that $1/T(z)$ extends analytically to a neighbourhood of $\partial\Omega$ and that it gives a selection of a square-root of $S'(z)$. This is an important observation because it means that on the Schottky double of any plane domain there is a canonical choice of square-root of the canonical bundle, i.e., there is canonical meaning of the concept of a differential of order one-half, and hence of a differential of any half-integer order.

A function f on $\hat{\Omega}$ is most conveniently described as a pair of functions f_1, f_2 on Ω, continuously extendable to $\partial\Omega$, such that

$$f_1(z)=\overline{f_2(z)}\quad (z\in\partial\Omega).$$

The formal relations to f in terms of the coordinates functions ϕ_1 and ϕ_2 above are

$$\begin{cases} f_1=f\circ\phi_1^{-1},\\ f_2=c\circ f\circ\phi_2^{-1}\circ c,\end{cases}$$

where c denotes complex conjugation. It follows, for example, that f is meromorphic if and only if f_1 and f_2 are meromorphic. With the same kind of identifications, a differential of order m on $\hat{\Omega}$ is represented by a pair of functions f_1, f_2 on Ω such that

$$f_1(z)dz^m=\overline{f_2(z)dz^m}\quad \text{along } \partial\Omega.$$

This is to be interpreted as $f_1(z)T(z)^m=\overline{f_2(z)T(z)^m}$, or

$$f_1(z)=\overline{f_2(z)T(z)^{2m}}\quad (z\in\partial\Omega). \tag{2.5}$$

Clearly this makes unambiguous sense for any $m\in\frac{1}{2}\mathbb{Z}$.

The Green's function $G(z,\zeta)$ of Ω is, as a function of z for fixed $\zeta\in\Omega$, defined by the properties

$$G(z,\zeta)=-\log|z-\zeta|+\text{harmonic}\quad z\in\Omega,$$

$$G(z,\zeta)=0,\quad z\in\partial\Omega.$$

It is symmetric in z and ζ, $G(z,\zeta)=G(\zeta,z)$, and it extends to the Schottky double as an"odd" function in each variable, for example, $G(J(z),\zeta)=-G(z,\zeta)$.

The above extension of the Green's function makes it a special case of a fundamental potential which exists on any compact Riemann surface, and which is a suitable starting point for discussing the classical function and differentials: for any three distinct points a,b,w on a compact Riemann surface M there exists a unique function of the form

$$V(z)=V(z,w;a,b)=-\log|z-a|+\log|z-b|+\text{harmonic}, \tag{2.6}$$

normalized by $V(w,w;a,b)=0$. It has the symmetries

$$V(z,w;a,b)=V(a,b;z,w)=-V(z,w;b,a) \tag{2.7}$$

and the transitivity property

$$V(z,w;a,b)+V(z,w;b,c)=V(z,w;a,c). \tag{2.8}$$

See below for explanations, and also [52], Ch. 4. If The Riemann surface is symmetric, with involution J, then

$$V(z,w;a,b)=V(J(z),J(w);J(a),J(b)). \tag{2.9}$$

Example 2.1. In the case of the Riemann sphere, $M=\mathbb{P}$, we have

$$V(z,w;a,b)=-\log|(z:w:a:b)|=-\log\Big|\frac{(z-a)(w-b)}{(z-b)(w-a)}\Big|,$$

$(z:w:a:b)$ denoting the classical cross-ratio.

With $M=\hat{\Omega}$, the Green function is given in terms of V by

$$G(z,\zeta)=\frac{1}{2}V(z,J(z);\zeta,J(\zeta))=V(z,w;\zeta,J(\zeta)), \tag{2.10}$$

where w is an arbitrary point on $\partial\Omega$ and where the second equality follows from (2.7), (2.8), (2.9). Cf. also [18], p. 125f.

The existence of $V(z,w;a,b)$ in general is immediate from classical potential theory, see e.g. [52], [17] or, more generally, from Hodge theory. In fact, V solves the Poisson equation $-d^*dV=2\pi(\delta_a-\delta_b)$ on M, where the star is the Hodge star and the Dirac measures in the right member are to be considered as 2-form currents; the solution exists because $\int_M 2\pi(\delta_a-\delta_b)=0$ and it is unique up to an additive constant.

From $V(z,w;a,b)$ much of the classical function theory on M can be built up. For example,

$$\begin{aligned}\upsilon_{a-b}(z)&=-2\frac{\partial V(z,w;a,b)}{\partial z}dz=-dV(z)-\mathrm{i}^*dV(z)\\&=\frac{dz}{z-a}-\frac{dz}{z-b}+\text{analytic}\end{aligned} \tag{2.11}$$

is the unique abelian differential of the third kind with poles of residues ±1 at $z=a,b$ and having purely imaginary periods. The subscript in υ_{a-b} should be thought of as a divisor, and the definition extends to any divisor of degree zero. Conversely, we retrieve V from υ_{a-b} by

$$V(z,w;a,b)=-\mathrm{Re}\int_w^z\upsilon_{a-b}, \tag{2.12}$$

from which the symmetries (2.7) and transitivity property (2.8) of V follow, using Riemann's bilinear relations for one of the symmetries.

From $V(z,w;a,b)$ one can also construct the $2\mathsf{g}$-dimensional space of harmonic differentials on M by considering the conjugate periods. By (2.12), the harmonic differentials $dV=-\mathrm{Re}\,\upsilon_{a-b}$ and ${}^*dV=-\mathrm{Im}\,\upsilon_{a-b}$ do not depend on w. The first one is exact, while the second has certain periods, which depend on a, b: if γ is a closed curve on M then the function

$$\phi_\gamma(a,b)=\frac{1}{2\pi}\int_\gamma {}^*dV(\cdot,w;a,b)$$

is, away from γ, harmonic in a and b and makes a unit jump as a (or b) crosses γ. It follows that, keeping b fixed, $d\phi_\gamma(\cdot,b)$ extends to a harmonic differential on M with periods given by

$$\int_\sigma d\phi_\gamma(\cdot,b)=\sigma\times\gamma,$$

where $\sigma\times\gamma$ denotes the intersection number of σ and γ. Choosing γ among the curves in a canonical homology basis gives the standard harmonic differentials.

As a further aspect, the right member in (2.12) can be written as a Dirichlet integral:

$$\begin{aligned}V(z,w;a,b)&=-\frac{1}{2\pi}\int_M dV(\cdot,c;z,w)\wedge^* dV(\cdot,c;a,b)\\&=\frac{1}{4\pi}\mathrm{Im}\int_M \upsilon_{z-w}\wedge\bar{\upsilon}_{a-b},\end{aligned}\tag{2.13}$$

where $c\in M$ is an arbitrary point. Thus V reproduces itself in a certain sense, and the equation also expresses that $V(z,w;a,b)$, besides being the potential of the charge distribution $\delta_a-\delta_b$ when considered as a function of merely z, also is the mutual energy of the two charge distributions $\delta_a-\delta_b$ and $\delta_z-\delta_w$. The first equality in (2.13) follows by partial integration, and the second from the definition of υ_{a-b}. For the Green's function the corresponding equation is

$$G(z,\zeta)=\frac{1}{2\pi}\int_\Omega dG(\cdot,z)\wedge^* dG(\cdot,\zeta).$$

We recall that υ_{a-b} has $2\mathsf{g}$ zeros (since it has 2 poles). With $M=\hat{\Omega}$ and $a=\zeta$, $b=J(\zeta)$, half of the zeros are on Ω, and by (2.10), (2.11) these are exactly the critical points of $G(z,\zeta)$ (the points where the gradient vanishes). Hence the Green's function has exactly g critical points on Ω.

2.2. **The Schottky-Klein prime function.** The harmonic theory on a compact Riemann surface is simple and intuitive, as indicated above. For the holomorphic theory one usually prefers to work with abelian differentials of the third kind normalized, not as υ_{a-b}, but so that, in terms of a canonical homology basis, half of the periods vanish. Following [59] and [35] we shall, in the case of a Schottky double of a planar domain Ω, choose a canonical homology basis $\{\alpha_1,\dots,\alpha_\mathsf{g},\beta_1,\dots,\beta_\mathsf{g}\}$ such that α_j goes from Γ_0 to Γ_j on Ω and back to Γ_0 along the same track on the back-side $\tilde{\Omega}$, and such that $\beta_j=\Gamma_j$ ($j=1,\dots,\mathsf{g}$). Thus the basis is symmetric with respect to the involution J, more precisely $J(\alpha_j)=-\alpha_j$, $J(\beta_j)=\beta_j$.

We denote by ω_{a-b} the abelian differential of the third kind with the same singularities as υ_{a-b} but normalized so that the α_j-periods vanish:

$$\int_{\alpha_j}\omega_{a-b}=0,\quad j=1,\dots,\mathsf{g}.\tag{2.14}$$

This makes sense only for $a,b\notin\alpha_j$, and with the α_j being fixed curves. Hence ω_{a-b} is less canonical than υ_{a-b} but it has the advantage of depending analytically on a, b, whereas for υ_{a-b} the dependence is only harmonic in general.

Remark 2.1. As to notation, the differential we denote by ω_{a-b} is in Fay [18], Schiffer-Spencer [52], Farkas-Kra [17] denoted, respectively, ω_{a-b}, $-d\omega_{ab}$, τ_{ab}. For our υ_{a-b} the corresponding list is: Ω_{a-b}, $-d\Omega_{ab}$, ω_{ab}. Also, Fay [18], and Hejhal [28] use a homology basis with switched roles between the α_j and β_j curves, but, as noticed by A. Yamada [59], the present choice has certain advantages (see Lemma 2.1 below).

The integral $\int_w^z \omega_{a-b}$, with unspecified path of integration, is locally holomorphic in all variables, but multivalued. To cope with the $2\pi\mathrm{i}$ indeterminacy coming from the poles one may form the exponential. In the case of the Riemann sphere this simply gives the cross-ratio between z, w, a, b:

$$\exp \int_w^z \omega_{a-b} = (z : w : a : b).$$

For Riemann surfaces of genus $\mathsf{g} > 0$, the exponential remains multivalued, with multiplicative periods (cf. [17]), Ch.III.9). However, one can still write it as a kind of cross-ratio if one is willing to accept further multivaluedness:

$$\exp \int_w^z \omega_{a-b} = \frac{E(z,a)E(w,b)}{E(z,b)E(w,a)}. \tag{2.15}$$

Here $E(z,\zeta)$ is the Schottky-Klein prime function (prime form), which should be regarded as a differential of order $-\frac{1}{2}$ in each of the variables, but as such still has multiplicative periods. Near the diagonal it behaves like $z-\zeta$. The exact definition of $E(z,\zeta)$ is usually given in terms of theta functions on the Jacobi variety, see [18], [28], [42] and below. For the Schottky doubles of a plane region there is also a representation in terms of the Schottky uniformization of the domain, see for example [9].

It should be remarked that, in the Schottky double case, the kind of differential of order $-\frac{1}{2}$ referred to for $E(z,\zeta)$ has no representation in the form (2.5) (with $m=-1$). The bundle of half-order differentials defined by (2.5) is "even" and does not allow any global holomorphic section, whereas the one needed for $E(z,\zeta)$ should be "odd", which does allow for a holomorphic section. The definition of such a bundle requires a finer atlas than the one consisting of only Ω and $\tilde{\Omega}$ for its representation. Of the $2^{2\mathsf{g}}$ bundles of half-order differentials, $2^{\mathsf{g}-1}(2^{\mathsf{g}}-1)$ are odd and $2^{\mathsf{g}-1}(2^{\mathsf{g}}+1)$ are even, see [28].

Now, quite remarkably, in case the Riemann surface is the Schottky double of a plane domain the two abelian differentials ω_{a-b} and υ_{a-b} coincide when a and b are symmetrically opposite points:

Lemma 2.1. *For $M=\hat{\Omega}$ and with canonical homology basis chosen as above,*

$$\omega_{\zeta-J(\zeta)} = \upsilon_{\zeta-J(\zeta)}.$$

Remark 2.2. The lemma does not hold if the homology basis is chosen in a different way, for example with switched roles between the α_j and β_j curves. Note also that even though ω_{a-b} makes a jump by $2\pi\mathrm{i}$ as a or b crosses α_j, there is no such jump for $\omega_{\zeta-J(\zeta)}$ because ζ and $J(\zeta)$ cross α_j simultaneously and the two contributions cancel.

Proof. One simply has to notice that $\upsilon_{\zeta-J(\zeta)}$ satisfies the normalization (2.14) of $\omega_{\zeta-J(\zeta)}$. Expressed in terms of V, since dV already is exact, the statement to be proven becomes, for any $w\in\partial\Omega$,

$$\int_{\alpha_j} {}^*dV(\cdot, w; \zeta, J(\zeta)) = 0, \quad (j=1,\dots,\mathsf{g}).$$

That these periods vanish follows from the symmetry properties (2.13), (2.9) of V with respect to J. $\square$

For a general Riemann surface, the differential ω_{a-b} can be recovered from the prime form as a logarithmic derivative of (2.15):

$$\omega_{a-b}(z) = d\log\frac{E(z,a)}{E(z,b)}. \tag{2.16}$$

From this one sees clearly that $E(z,\zeta)$ depends on the homology basis, since ω_{a-b} does. The Green's function of a planar domain Ω is most directly related to υ_{a-b} on $\hat{\Omega}$ as in (2.10), (2.11), but in view of Lemma 2.1 one can equally well use ω_{a-b}. Therefore, (2.16) gives the following expression of the Green's function of Ω in terms of the prime form on $M=\hat{\Omega}$.

$$G(z,\zeta) = -\log|\frac{E(z,\zeta)}{E(z,J(\zeta))}| \quad (z,\zeta\in\Omega).$$

See [59] for further details, and also [9], [35] for other aspects.

We introduce next the harmonic measures u_j, $j=1,\dots,\mathtt{g}$, i.e., the harmonic functions in Ω defined by having the boundary values $u_j=\delta_{kj}$ on Γ_k. It is easy to see that their differentials du_j extend to the Schottky double as everywhere harmonic differentials with

$$\int_{\alpha_k} du_j = 2\delta_{kj}, \quad \int_{\alpha_k} {}^*du_j = 0.$$

Here the second equation is a consequence of the symmetry of the extended differential under J. Thus, the $d\mathcal{U}_j = \frac{1}{2}(du_j + \mathrm{i}^*du_j)$ $(j=1,\dots,\mathtt{g})$ constitute the canonical basis of abelian differentials of the first kind (everywhere holomorphic differentials), the period matrix with respect to the α_k curves being the identity matrix. For the β_k-periods we have

$$\int_{\beta_k} du_j = 0, \quad \int_{\beta_k} {}^*du_j = \int_{\partial\Omega} u_k\, {}^*du_j = \int_{\Omega} du_k\wedge {}^*du_j,$$

where the latter make up a positive definite matrix. On setting

$$\tau_{kj} = \frac{1}{2}\int_{\beta_k} du_j + \mathrm{i}\,{}^*du_j$$

we thus have $\operatorname{Re}\tau_{kj}=0$ and that the matrix $(\operatorname{Im}\tau_{kj})$ is positive definite. With $\mathcal{U}_j(z) = \int^z \frac{1}{2}(du_j + \mathrm{i}^*du_j) = \int^z \frac{\partial u_j}{\partial z}dz$ denoting the corresponding abelian integrals (multivalued), the map

$$z\mapsto \mathcal{U}(z) = (\mathcal{U}_1(z),\dots\mathcal{U}_{\mathtt{g}}(z))$$

defines, up to a shift, the Abel map into the Jacobi variety $\mathbb{C}^{\mathtt{g}}/(\mathbb{Z}^{\mathtt{g}}+\tau\mathbb{Z}^{\mathtt{g}})$, $\tau=(\tau_{kj})$.

At this point we can make the definition of the Schottky-Klein prime function slightly more precise. The first order theta function with (half-integer) characteristics $\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}$, where $\delta,\epsilon\in\frac{1}{2}\mathbb{Z}^{\mathtt{g}}$ are row vectors, is defined by

$$\vartheta\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(w) = \vartheta\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(w;\tau) = \sum_{m\in\mathbb{Z}^{\mathtt{g}}} \exp\left[\mathrm{i}\pi(m+\delta)\tau(m+\delta)^t + 2\pi\mathrm{i}(w+\epsilon)(m+\delta)^t\right] \tag{2.17}$$

for $w=(w_1,\dots,w_{\mathtt{g}})\in\mathbb{C}^{\mathtt{g}}$. The superscript t denotes transposition of a matrix. To define the prime form the characteristics should first of all be chosen to be odd, i.e., so that $4\delta\epsilon^t$ is an odd number (e.g., $\delta=(\frac{1}{2},0\dots,0)$, $\epsilon=(\frac{1}{2},0\dots,0)$). For simplicity, set $\vartheta_* = \vartheta\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}$ whenever such a choice of δ, ϵ has been made, and introduce

$$c_j = \frac{\partial\vartheta_*}{\partial w_j}(0).$$

The δ and ϵ should in addition be chosen to be non-singular, i.e., so that not all of the constants c_j vanish. This can always be done, see [18]. Then the Schottky-Klein prime form is, when considered as defined on $\hat{\Omega}$,

$$E(z,\zeta)=\frac{\vartheta_*(\mathcal{U}(z)-\mathcal{U}(\zeta))}{\sqrt{\sum_{j=1}^{\mathsf{g}} c_j d\mathcal{U}_j(z)}\sqrt{\sum_{j=1}^{\mathsf{g}} c_j d\mathcal{U}_j(\zeta)}}.$$

See [35] (Appendix) for further details using the present point of view.

3. Critical points of the Green's function

3.1. **The Bergman and Poisson kernels.** The Bergman and Schiffer kernels [4], [52], $K(z,\zeta)$ and $L(z,\zeta)$ respectively, will be discussed in some detail later on. For the moment we just recall their representations in terms of the Green's function and the potential $V(z,w;a,b)$ in (2.6):

$$K(z,\zeta)=-\frac{2}{\pi}\frac{\partial^2 G}{\partial z\partial\bar{\zeta}}(z,\zeta)=\frac{1}{\pi}\frac{\partial^2 V}{\partial z\partial\bar{\zeta}}(z,J(z);\zeta,J(\zeta)), \tag{3.1}$$

$$L(z,\zeta)=-\frac{2}{\pi}\frac{\partial^2 G}{\partial z\partial\zeta}(z,\zeta)=\frac{1}{\pi}\frac{\partial^2 V}{\partial z\partial\zeta}(z,J(z);\zeta,J(\zeta)). \tag{3.2}$$

By the symmetry of $G(z,\zeta)$ (or V) we have $L(z,\zeta)=L(\zeta,z)$ and $K(z,\zeta)=\overline{K(\zeta,z)}$.

Since $G(z,\zeta)=0$ for $z\in\partial\Omega$, $\zeta\in\Omega$,

$$L(z,\zeta)dz+\overline{K(z,\zeta)dz}=0 \tag{3.3}$$

along $\partial\Omega$ (with respect to z). This means that, for any fixed ζ, the pair $K(z,\zeta)dz$, $L(z,\zeta)dz$ combines into a meromorphic differential on $\hat{\Omega}$. It has a pole of order two residing in $L(z,\zeta)$:

$$L(z,\zeta)=\frac{1}{\pi}\frac{1}{(z-\zeta)^2}-\ell(z,\zeta), \tag{3.4}$$

where $\ell(z,\zeta)$ (the "ℓ-kernel") is regular in both variables. Consequently, $K(z,\zeta)dz$, $L(z,\zeta)dz$ have altogether $2\mathsf{g}$ zeros. For $\zeta\in\partial\Omega$ the zeros are, by (3.3) with switched roles between z and ζ, equally shared: g zeros for $K(z,\zeta)dz$ and $L(z,\zeta)dz$ at the same points, and no zeros on $\partial\Omega$ as we shall see in the next section. A further consequence of (3.3) is that when $\partial\Omega$ is analytic, the kernels $K(z,\zeta)$, $L(z,\zeta)$ have analytic extensions across $\partial\Omega$ in $\mathbb{C}$.

Using the symmetries of $K(z,\zeta)$ and $L(z,\zeta)$, and (3.3) one finds that the double differentials $K(z,\zeta)dzd\bar{\zeta}$, $L(z,\zeta)dzd\zeta$ are "real" on the boundary:

$$L(z,\zeta)dzd\zeta\in\mathbb{R},\quad K(z,\zeta)dzd\bar{\zeta}\in\mathbb{R}\quad(z,\zeta\in\partial\Omega).$$

The precise meaning of for example the first equation is that $L(z,\zeta)T(z)T(\zeta)\in\mathbb{R}$, where $T(z)$ is the tangent vector at $z\in\partial\Omega$ (see (2.4)).

The Poisson kernel of Ω is the normal derivative of the Green function when one of the variables is on the boundary:

$$P(z,\zeta)=-\frac{1}{2\pi}\frac{\partial G}{\partial n_z}(z,\zeta),\quad z\in\partial\Omega,\ \zeta\in\Omega.$$

Here $\frac{\partial}{\partial n_z}$ denotes the outward normal derivative at $z\in\partial\Omega$. With $ds=ds_z$ denoting arc-length with respect to z, the definition may also be written in differential form in several ways, for example (with $d=d_z$)

$$\begin{aligned}P(z,\zeta)ds_z&=-\frac{1}{2\pi}{}^*dG(z,\zeta)=-\frac{1}{\pi}\mathrm{Im}\,\frac{\partial G}{\partial z}(z,\zeta)dz\\&=-\frac{1}{2\pi}\mathrm{Im}\,\frac{\partial V}{\partial z}(z,J(z);\zeta,J(\zeta))dz=-\frac{1}{2\pi}\mathrm{Im}\,\upsilon_{\zeta-J(\zeta)}(z).\end{aligned}$$

3.2. **Critical points.** In this subsection and the next we show that, as the pole moves around, the set of critical points of the Green's function of Ω stay within a compact subset of Ω, and that the limiting positions, as the pole tends to $\partial\Omega$, coincide with the zero set of the Bergman kernel in this boundary limit. The latter result was the main motivation for the present work. Related results have been obtained by S. Bell [3] (section 30) and A. Solynin [56].

For fixed $z \in \partial\Omega$, the map $\zeta \mapsto P(z,\zeta)$ is harmonic and strictly positive in Ω and vanishes on $\partial\Omega \setminus \{z\}$. It extends, when $\partial\Omega$ is analytic, to a harmonic function in a neighborhood (in $\mathbb{C}$) of $\partial\Omega$ except for a pole at $\zeta = z$. Since $P(z,\cdot)$ attains its minimum in $\bar{\Omega}\setminus\{z\}$ on $\partial\Omega$, the Hopf maximum principle [47] shows that $\dfrac{\partial P}{\partial n_\zeta}(z,\zeta) < 0$ for all $\zeta \in \partial\Omega \setminus \{z\}$. Thus

$$\frac{\partial^2 G}{\partial n_z \partial n_\zeta}(z,\zeta) > 0, \quad z,\zeta \in \partial\Omega, \quad z \neq \zeta. \tag{3.5}$$

The fact that this double normal derivative is nonnegative can be more directly understood by interpreting the left member of (3.5) as a double difference quotient: with differences $\triangle z$, $\triangle\zeta$ pointing in the normal direction into Ω from $z,\zeta \in \partial\Omega$, respectively, we have

$$\frac{\partial^2 G(z,\zeta)}{\partial n_z \partial n_\zeta} =$$

$$= \lim_{|\triangle z|,|\triangle\zeta|\to 0} \frac{G(z+\triangle z,\zeta+\triangle\zeta) - G(z+\triangle z,\zeta) - G(z,\zeta+\triangle\zeta) + G(z,\zeta)}{|\triangle z||\triangle\zeta|}.$$

This is obviously nonnegative because all terms in the numerator vanish except the first: $G(z+\triangle z,\zeta+\triangle\zeta) > 0$.

Complementary to (3.5) we have

$$\frac{\partial^2 G}{\partial s_\zeta \partial s_z}(z,\zeta) = \frac{\partial^2 G}{\partial s_\zeta \partial n_z}(z,\zeta) = \frac{\partial^2 G}{\partial n_\zeta \partial s_z}(z,\zeta) = 0 \quad (z,\zeta \in \partial\Omega,\ z \neq \zeta).$$

It follows that, for $z,\zeta \in \partial\Omega$, $z \neq \zeta$,

$$\frac{\partial^2 G}{\partial z \partial\zeta}(z,\zeta) \neq 0, \quad \frac{\partial^2 G}{\partial z \partial\bar{\zeta}}(z,\zeta) \neq 0, \quad \frac{\partial^2 G}{\partial z \partial n_\zeta}(z,\zeta) \neq 0,$$

because these derivatives contain derivations in the normal direction in each of the variables. In view of the singularity of type $1/(z-\zeta)^2$ at $z = \zeta$ and of the continuity of $\dfrac{\partial^2 G}{\partial z \partial\zeta}$ and $\dfrac{\partial^2 G}{\partial z \partial\bar{\zeta}}$ as mappings from a neighborhood of $\partial\Omega \times \partial\Omega \subset \mathbb{C} \times \mathbb{C}$ to the Riemann sphere it is clear that the above inequalities persist to hold for $z = \zeta$, and also that the quantities are bounded away from zero:

$$|\frac{\partial^2 G}{\partial z\, \partial\zeta}| \geq c > 0, \quad |\frac{\partial^2 G}{\partial z\, \partial\bar{\zeta}}| \geq c > 0 \tag{3.6}$$

in a full neighborhood of $\partial\Omega \times \partial\Omega$ and for some constant $c > 0$. Note that (3.6) says that $K(z,\zeta)$, $L(z,\zeta)$ are bounded away from zero in a neighborhood of $\partial\Omega \times \partial\Omega$ when $\partial\Omega$ is analytic. With z in a neighborhood of $\partial\Omega$ and $\zeta \in \partial\Omega$ we also infer

$$|\frac{\partial^2 G}{\partial z\, \partial n_\zeta}| \geq c > 0. \tag{3.7}$$

Since $\dfrac{\partial G}{\partial z}(z,\zeta) = 0$ for $\zeta \in \partial\Omega$ we obtain, on integrating (3.7) with respect to ζ in the normal direction,

$$|\frac{\partial G}{\partial z}(z,\zeta)| \geq c\, \mathrm{dist}(\zeta,\partial\Omega)$$

for z,ζ in a neighborhood of $\partial\Omega$ and for some $c>0$. In particular there exists a compact set $K\subset\Omega$ such that

$$\frac{\partial G}{\partial z}(z,\zeta)\neq 0 \tag{3.8}$$

for all $z,\zeta\in\Omega\setminus K$. Now we are ready to conclude the following, also obtained (using slightly different arguments) by A. Solynin [56].

Theorem 3.1. *Let $\Omega\subset\mathbb{C}$ be a bounded finitely connected domain such that each component of $\mathbb{C}\setminus\Omega$ consists of a least two points. Then there exists a compact set $K\subset\Omega$ such that*

$$\frac{\partial G}{\partial z}(z,\zeta)\neq 0,\quad z\in\Omega\setminus K,\quad \zeta\in\Omega, \tag{3.9}$$

$$\frac{\partial^2 G}{\partial z\partial\zeta}(z,\zeta)\neq 0,\quad z,\zeta\in\Omega\setminus K, \tag{3.10}$$

$$\frac{\partial^2 G}{\partial z\partial\bar\zeta}(z,\zeta)\neq 0,\quad z,\zeta\in\Omega\setminus K. \tag{3.11}$$

Proof. The statements are conformally invariant, so it is enough to prove them when Ω has smooth analytic boundary. By (3.6), (3.7), (3.8) the desired inequalities are valid for $z,\zeta\in\Omega\setminus K$ and it remains only to prove that, in the first inequality, we can allow all $\zeta\in\Omega$ by possibly enlarging the compact set K. If this were not true, then there would exist a sequence $\{(z_n,\zeta_n)\}$ with

$$\frac{\partial G}{\partial z}(z_n,\zeta_n)=0,\quad z_n\to z\in\partial\Omega,\quad \zeta_n\to\zeta\in\bar\Omega.$$

According to (3.8), $\zeta\in\Omega$. But then also $\dfrac{\partial G}{\partial z}(z,\zeta)=0$, which however cannot be true because $\dfrac{\partial G}{\partial z}(z,\zeta)$ is a nonzero constant factor times the Poisson kernel $P(z,\zeta)$, which is strictly positive for $\zeta\in\Omega, z\in\partial\Omega$. This contradiction finishes the proof. $\square$

Remark 3.1. With slight modifications the proof also works in several real variables: if $\Omega\subset\mathbb{R}^n$ has real analytic boundary then there exists a compact set $K\subset\Omega$ such that $\nabla_x G(x,y)\neq 0$ for all $x\in\Omega\setminus K$, $y\in\Omega$. The crucial observation is that (3.5) persists to hold, so that $\displaystyle\sum_{i,j=1}^n\frac{\partial^2 G(x,y)}{\partial x_i\partial y_j}\xi_i\eta_j\neq 0$ for $x,y\in\partial\Omega$ whenever ξ, η are nontangential vectors at x and y respectively.

Example 3.1. The boundary behavior and the nature of the pole are illustrated by the case of the unit disk: $\Omega=\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. In this case $\hat\Omega=\mathbb{P}=\mathbb{C}\cup\{\infty\}$ with involution $J:z\mapsto 1/\bar z$, and the Schwarz function is $S(z)=1/z$. The fundamental potential was given in Example 2.1, and the Green's function is

$$G(z,\zeta)=-\log\left|\frac{z-\zeta}{1-z\bar\zeta}\right|.$$

It follows that

$$\frac{\partial G(z,\zeta)}{\partial z}=-\frac{1-|\zeta|^2}{2(z-\zeta)(1-z\bar\zeta)},$$

$$\frac{\partial^2 G}{\partial z\,\partial\zeta}=-\frac12\frac{1}{(z-\zeta)^2},\quad \frac{\partial^2 G}{\partial z\,\partial\bar\zeta}=-\frac12\frac{1}{(1-z\bar\zeta)^2},$$

$$P(z,\zeta)=\frac{1-|\zeta|^2}{|z-\zeta|^2}.$$

As for the estimates (3.6), (3.7) we note that $|z-\zeta|\le 2$, $|1-z\bar\zeta|\le 2$ so that, for $z,\zeta\in\partial\mathbb{D}$,

$$|\frac{\partial^2 G}{\partial z\,\partial\zeta}|\ge\frac{1}{8},\quad |\frac{\partial^2 G}{\partial z\,\partial\bar\zeta}|\ge\frac{1}{8},\quad |\frac{\partial^2 G}{\partial z\,\partial n_\zeta}|\ge\frac{1}{8}.$$

3.3. **Critical points in the boundary limit.** When $\zeta\in\partial\Omega$, $\dfrac{\partial G}{\partial z}(z,\zeta)=0$ for all $z\in\Omega$, but by "blow-up" one can still speak of a nontrivial limit of $\dfrac{\partial G}{\partial z}(z,\zeta)$ as $\zeta\to\partial\Omega$. In a certain sense $\dfrac{\partial^2 G}{\partial z\partial n_\zeta}(z,\zeta)$, and hence the Bergman and Schiffer kernels, represents this limit, but there is also a representation in terms of a Martin type construction, which we shall discuss in the next section. Throughout this subsection we assume that $\partial\Omega$ is analytic.

Fix a point $a\in\Omega$ such that $\partial_z G(a,\zeta)\ne 0$ for all $\zeta\in\Omega$ and $K(a,\zeta)\ne 0$ for all $\zeta\in\partial\Omega$. This is possible (with a close enough to the boundary) by Theorem 3.1 and what precedes it, viz. (3.6). Then define

$$F(z,\zeta)=\frac{\partial_z G(z,\zeta)}{\partial_z G(a,\zeta)},\quad z,\zeta\in\Omega. \tag{3.12}$$

As a function of z, $F(z,\zeta)$ is meromorphic in Ω with a pole at ζ. It is normalized so that $F(a,\zeta)=1$, which prevents it from degeneration as $\zeta\to\partial\Omega$. In Example 3.1, for the unit disk, we see this from

$$F(z,\zeta)=\frac{(a-\zeta)(1-a\bar\zeta)}{(z-\zeta)(1-z\bar\zeta)}=\frac{(a-\zeta)(a-1/\bar\zeta)}{(z-\zeta)(z-1/\bar\zeta)}.$$

As a comparison,

$$\frac{K(z,\zeta)}{K(a,\zeta)}=\frac{(1-a\bar\zeta)^2}{(1-z\bar\zeta)^2}=\frac{(a-1/\bar\zeta)^2}{(z-1/\bar\zeta)^2}.$$

Since $\zeta=1/\bar\zeta$ on the boundary, it follows that $F(z,\zeta)$ and $\dfrac{K(z,\zeta)}{K(a,\zeta)}$ coincide there.

The following theorem shows that the above is what happens in general.

Theorem 3.2. *Assume $\partial\Omega$ is analytic and choose $a\in\Omega$ as above. Then the function $F(z,\zeta)$, originally defined in $\Omega\times\Omega$, extends continuously to $\Omega\times\overline{\Omega}$, and on $\Omega\times\partial\Omega$ it agrees with $\dfrac{K(z,\zeta)}{K(a,\zeta)}$. Specifically, if $(z_n,\zeta_n)\in\Omega\times\Omega$, $(z,\zeta)\in\Omega\times\partial\Omega$ and $(z_n,\zeta_n)\to(z,\zeta)$ as $n\to\infty$, then*

$$\lim_{n\to\infty}F(z_n,\zeta_n)=\frac{K(z,\zeta)}{K(a,\zeta)}.$$

Proof. Let $(z_n,\zeta_n)\to(z,\zeta)$ as in the statement and let $\eta_n\in\partial\Omega$ be closest points on the boundary to the ζ_n:

$$|\zeta_n-\eta_n|=d(\zeta_n)=d(\zeta_n,\partial\Omega).$$

With $T(\eta_n)$ the tangent vector at $\eta_n\in\partial\Omega$, we have

$$\frac{\partial G}{\partial z}(z_n,\eta_n)=0,$$

$$\frac{\partial^2 G}{\partial z\partial\zeta}(z_n,\eta_n)\,T(\eta_n)+\frac{\partial^2 G}{\partial z\partial\bar\zeta}(z_n,\eta_n)\,\overline{T(\eta_n)}=0.$$

Therefore, Taylor expansion with respect to ζ of $\frac{\partial G}{\partial z}(z_n,\zeta)$ at $\zeta=\eta_n$ gives

$$\frac{\partial G}{\partial z}(z_n,\zeta_n)=\frac{\partial^2 G}{\partial z\partial\zeta}(z_n,\eta_n)(\zeta_n-\eta_n)+\frac{\partial^2 G}{\partial z\partial\bar\zeta}(z_n,\eta_n)(\bar\zeta_n-\bar\eta_n)+\mathcal{O}(|\zeta_n-\eta_n|^2)$$

$$= 2\mathrm{i}\frac{\partial^2 G}{\partial z \partial \zeta}(z_n, \eta_n)\, T(\eta_n) \operatorname{Im} \frac{\zeta_n - \eta_n}{T(\eta_n)} + \mathcal{O}(d(\zeta_n)^2)$$

$$= 2\mathrm{i}\frac{\partial^2 G}{\partial z \partial \zeta}(z_n, \eta_n)\, T(\eta_n) d(\zeta_n) + \mathcal{O}(d(\zeta_n)^2).$$

Note that $\operatorname{Im} \frac{\zeta_n - \eta_n}{T(\eta_n)} = d(\zeta_n)$. The remainder term $\mathcal{O}(d(\zeta_n)^2)$ in principle depends on z_n, but it is uniformly small as $z_n \to z \in \Omega$.

We now obtain

$$F(z_n, \zeta_n) = \frac{\partial_z G(z, \zeta_n)}{\partial_z G(a, \zeta_n)} = \frac{2\mathrm{i}\frac{\partial^2 G}{\partial z \partial \zeta}(z_n, \eta_n) T(\eta_n) d(\zeta_n) + \mathcal{O}(d(\zeta_n)^2)}{2\mathrm{i}\frac{\partial^2 G}{\partial z \partial \zeta}(a, \eta_n) T(\eta_n) d(\zeta_n) + \mathcal{O}(d(\zeta_n)^2)}$$

$$= \frac{K(z_n, \eta_n) + \mathcal{O}(d(\zeta_n))}{K(a, \eta_n) + \mathcal{O}(d(\zeta_n))}.$$

Since $K(a, \zeta)$ is bounded away from zero for ζ close to $\partial\Omega$ the statements of the theorem follow. $\square$

As a consequence we have

Corollary 3.3. *The limit set of the set of critical points of $G(z, \zeta)$ as $\zeta \to \partial\Omega$ is exactly the set of zeros of $K(z, \zeta)$ for $\zeta \in \partial\Omega$. More precisely, for $z \in \Omega, \zeta \in \partial\Omega$, the following statements are equivalent.*

(i) $K(z, \zeta) = 0$.
(ii) $L(z, \zeta) = 0$.
(iii) *There exist $(z_n, \zeta_n) \in \Omega \times \Omega$, $(n = 1, 2, \cdots)$ with $\partial_z G(z_n, \zeta_n) = 0$ such that $(z_n, \zeta_n) \to (z, \zeta)$ as $n \to \infty$.*
(iv) *For each sequence $\{\zeta_n\}_{n \geq 1} \subset \Omega$ such that $\zeta_n \to \zeta$, there exist $z_n \in \Omega$ with $z_n \to z$ such that $\partial_z G(z_n, \zeta_n) = 0$.*

Proof. By (3.3) (with switched roles of z and ζ), (i) and (ii) are equivalent, and clearly (iv) implies (iii).

If $\partial_z G(z_n, \zeta_n) = 0$, i.e., $F(z_n, \zeta_n) = 0$, then it is immediate from Theorem 3.2 that $K(z, \zeta) = 0$. Thus (iv) implies (i). Conversely, assume for instance that $K(z, \zeta) = 0$, $\frac{\partial}{\partial z} K(z, \zeta) \neq 0$ and let $\zeta_n \in \Omega$, $\zeta_n \to \zeta$. The functions $F(\cdot, \zeta_n)$ and $K(\cdot, \zeta)$ have equally many poles and zeros (namely 2 poles and $2\mathtt{g}$ zeros seen in the Schottky double), and since $F(\cdot, \zeta_n)$ tends to a constant times $K(\cdot, \zeta)$, $F(\cdot, \zeta_n)$ must have a zero z_n near the zero z of $K(\cdot, \zeta)$. In fact, let $\gamma = \partial\mathbb{D}(z, \epsilon)$ with $\epsilon > 0$ small enough. Then, as $n \to \infty$,

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma d \log F(\cdot, \zeta_n) \to \frac{1}{2\pi \mathrm{i}} \int_\gamma d \log K(\cdot, \zeta) = 1,$$

the last integral being de number of zeros of $K(\cdot, \zeta)$ inside γ. Hence $F(\cdot, \zeta_n)$ has exactly one zero z_n inside γ. Letting ϵ tend to 0 as n tends to ∞, we conclude that there are zeros z_n with $z_n \to z$. Thus (i) implies (iv), and the proof is complete. $\square$

Remark 3.2. The following example illustrates the fact that critical points of Green's function are not necessarily simple. Let $\omega = e^{2\pi \mathrm{i}/3}$, $D_j = \mathbb{D}(\omega^j, \rho)$, $j = 1, 2, 3$, and $0 < \rho < \frac{\sqrt{3}}{2}$. The Green's function $G(z)$ of $\Omega = \mathbb{C} \setminus (\bar{D}_1 \cup \bar{D}_2 \cup \bar{D}_3)$ with pole at infinity has two critical points ($\mathtt{g} = 2$). Since Ω is invariant under $z \to \omega z$, the two critical points must be a double point located at the origin. By means of a Möbius transformation, one obtains an example of a 2-holed disk which has a Green's function with a multiple critical point.

4. The Martin and gradient boundaries

4.1. **The Martin compactification.** In this section we shall give the preceding considerations their right meaning. We adapt the construction of Martin compactification as presented for instance in I.S. Gal [21]. Let us first recall a general theorem of Constantinescu-Cornea (see [7], p. 97).

Theorem 4.1. *Let Ω be a non-compact locally compact Hausdorff space and let Φ be a family of continuous functions $\Omega \to [-\infty, +\infty]$. Then there exists a compact topological space Ω^M, unique up to homeomorphisms, such that*

(i) *Ω is an open and dense subset of Ω^M.*
(ii) *Every $f \in \Phi$ can be extended to a continuous function f^M on Ω^M.*
(iii) *The functions f^M separate points on $\partial^M\Omega := \Omega^M \setminus \Omega$.*

For example, when Ω is a multiply connected domain in the plane one may, for a fixed $a \in \Omega$, consider the family Φ of functions $\zeta \mapsto M(z,\zeta) = \dfrac{G(z,\zeta)}{G(a,\zeta)}$, parametrized by $z \in \Omega$ and with the convention that $M(a,a) = 1$. Each function $z \mapsto M(z,\zeta)$ is continuous, even at a. The space Ω^M obtained by the theorem of Constantinescu-Cornea is, up to a homeomorphism, independent of a. It is the Martin compactification of Ω, and the Martin boundary is $\partial^M\Omega = \Omega^M \setminus \Omega$.

As discovered by M. Brelot and I. S. Gal (see [21]), the Martin boundary can also be introduced via uniform structures. For generalities on such, see [32]. Let X be a nonempty set, let $(Y, \mathcal{V})$ be a uniform space and let Φ be a family of functions $\phi : X \to Y$. Then there is a weakest uniform structure on X making the functions in Φ uniformly continuous. It is the uniformity $\mathcal{U}$ generated by all the $\phi^{-1}(V) = \{(x, y) \in X \times X : (\phi(x), \phi(y)) \in V\}$, for $V \in \mathcal{V}$, $\phi \in \Phi$, as a subbase.

If $\mathcal{V}$ is precompact (totally bounded), so is $\mathcal{U}$. Now the space $Y = [-\infty, +\infty]$ is compact, hence has a unique uniform structure, and this is precompact. With Ω a multiply connected domain in the complex plane, and Φ the set of functions $M(z,\cdot)$ $(z \in \Omega)$ above, this gives a uniform structure the completion of which is the Martin compactification. The same remark also applies, with $Y = \mathbb{P}$, to the family of functions of $\zeta \mapsto F(z,\zeta) = \partial_z G(z,\zeta)/\partial_z G(a,\zeta)$ for $z \in \Omega$, $F(a,a) = 1$. This gives a uniform structure $\mathcal{U}_G$, which we call the *gradient structure*. Since $\mathbb{P}$ is compact we have

Theorem 4.2. *The gradient structure $\mathcal{U}_G$ is precompact.*

To make the link with the Constantinescu-Cornea theorem we recall that as soon as a uniform structure is introduced on a set X, there automatically arises a complete uniform space $(\bar{X},\bar{\mathcal{U}})$ and an injection map $f : X \to \bar{X}$ such that $f(X)$ is dense in $\bar{X}$ and $\mathcal{U} = f^{-1}\left(\bar{\mathcal{U}}\right)$. The triple $\left(f, \bar{X},\bar{\mathcal{U}}\right)$ is the completion of X with respect to $\mathcal{U}$. Moreover the space $\bar{X}$ is compact if and only if the uniform structure $\mathcal{U}$ is precompact. In our setting, with a multiply connected domain $\Omega \subset \mathbb{C}$, we can therefore formulate Theorem 4.2 in a more precise way as follows.

Theorem 4.3. *The completion of the space Ω with respect to its gradient structure $\mathcal{U}_G$ yields a compact space $\Omega^{\mathcal{G}} = \Omega \cup \partial^{\mathcal{G}}\Omega$, the gradient compactification of Ω.*

In analogy with the Martin compactification, we call the set $\partial^{\mathcal{G}}\Omega = \Omega^{\mathcal{G}} \setminus \Omega$ the *gradient boundary.* In Theorem 3.2 we showed that for multiply connected domains with analytic boundary, the gradient boundary is identical with the Euclidian one.

4.2. **An estimate of P. Levy.** Thinking of the Martin boundary as a local concept we may, in view of Example 3.1 and beginning of subsection 3.3, ask for local approximation of the Green's function of a general domain with sufficiently smooth boundary by the Green's function of the upper half-plane. Sharp estimates in this direction have been obtained by P. Levy [38]. Below we review Levy's method.

Let $z \in \Omega$, $d(z) = \text{dist}\,(z, \partial\Omega)$, c a point on $\partial\Omega$ on distance $d(z)$ from z and let z' be the symmetric point of z with respect to the boundary $\partial\Omega$, so that $c = \dfrac{1}{2}(z+z')$. We assume that $z' \notin \bar{\Omega}$, which is the case for instance if $\partial\Omega$ satisfies an exterior ball condition (which we henceforth assume) and $d(z)$ is sufficiently small. Let R the radius of the largest circle in Ω which is tangent to $\partial\Omega$ at c and R' the radius of the largest circle outside $\bar{\Omega}$ which is tangent to $\partial\Omega$ at c. We denote by a, a' the centers of the respective circles, assuming that $R' < \infty$; the case $R' = \infty$ can be easily covered by a limit argument. The points a, z, c, z', a' lie along a straight line.

Theorem 4.4. *In the above notation,*

$$\log\left(1 - \frac{2d}{2R'+d}\right) < \log|\frac{z'-\zeta}{z-\zeta}| - G(z,\zeta) < \log\left(1 + \frac{2d}{2R-d}\right)$$

for every $\zeta \in \Omega$.

Proof. The function $\zeta \mapsto \log|\frac{z'-\zeta}{z-\zeta}| - G(z,\zeta)$ is harmonic in Ω and equals $\log|\dfrac{z'-\zeta}{z-\zeta}|$ on $\partial\Omega$. The level lines of the latter function are circles, namely Apollonius circles with respect to the points z and z'. Recall that the Apollonius circles with centers z and z' are $\Gamma_k = \{\zeta \in \mathbb{C} : |z'-\zeta| = k|z-\zeta|\}$ for $k > 0$.

We conclude from the above that we get upper and lower bounds for $\log|\dfrac{z'-\zeta}{z-\zeta}|$ on $\partial\Omega$ by considering one Apollonius circle which lies entirely inside Ω and one which lies entirely outside Ω. Optimal choices with respect to the given data are those Apollonius circles which are tangent to the largest interior and exterior balls. The points of tangency are denoted by b and b'. It is easily seen that

$$\log|\frac{z'-b'}{z-b'}| < \log|\frac{z'-\zeta}{z-\zeta}| < \log|\frac{z'-b}{z-b}|$$

for $\zeta \in \partial\Omega$. Since

$$|z'-b| = 2R+d, \quad |z-b| = 2R-d,$$

$$|z'-b'| = 2R'-d, \quad |z-b'| = 2R'+d$$

we get, for $\zeta \in \partial\Omega$,

$$\log\left(1 - \frac{2d}{2R'+d}\right) < \log|\frac{z'-\zeta}{z-\zeta}| < \log\left(1 + \frac{2d}{2R-d}\right),$$

and hence

$$\log\left(1 - \frac{2d}{2R'+d}\right) < \log|\frac{z'-\zeta}{z-\zeta}| - G(z,\zeta) < \log\left(1 + \frac{2d}{2R-d}\right).$$

Now the assertion of the theorem follows from the maximum principle. □

5. The Poincaré metric, Taylor coefficients and level lines

5.1. **The Poincaré metric on level lines for the Green function.** Here we shall discuss some properties of level lines of Green's function and in particular their connections with the Poincaré metric. Later on we shall interpret these level lines as geodesics for a different metric.

Consider a simply connected domain Ω, let $d\sigma_z = \rho(z)|dz|$ be the Poincaré metric and $G(z,\zeta)$ the Green's function of Ω.

Proposition 5.1. *For every $a \in \Omega$, the density $\rho(z)$ of the Poincaré metric is*

$$\rho(z) = \frac{|\frac{\partial G}{\partial z}(z,a)|}{\sinh G(z,a)}. \tag{5.1}$$

In particular, on each level line $G(z,a) = c$,

$$\rho(z) = -\frac{1}{2\sinh c}\frac{\partial G}{\partial n_z}(z,a),$$

i.e., ρ is proportional to the harmonic measure with respect to a, or equivalently to the Poisson kernel of the enclosed domain.

Proof. The proof is classical and we recall it for the convenience of the reader. With $\mathcal{G}(z,a) = G(z,a) + \mathrm{i}G^*(z,a)$ an analytic completion of $G(z,a)$ with respect to z, the map $z \mapsto \zeta = e^{-\mathcal{G}(z,a)}$ sends Ω onto the unit disk, for which the Poincaré metric is

$$d\sigma_\zeta = \frac{|d\zeta|}{1-|\zeta|^2}.$$

By conformal invariance

$$d\sigma_z = \rho(z)|dz| = \frac{|d(e^{-\mathcal{G}(z,a)})|}{1-|e^{-\mathcal{G}(z,a)}|^2}$$

$$= \frac{e^{-G(z,a)}\,|2\frac{\partial G}{\partial z}(z,a)|}{1-e^{-2G(z,a)}}|dz| = \frac{|\frac{\partial G}{\partial z}(z,a)|}{\sinh\left(G(z,a)\right)}|dz|,$$

which is the desired result. □

Remark 5.1. If Ω is multiply connected the above expression for the Poincaré metric still holds if $G(z,\zeta)$ is interpreted as the Green's function for the universal covering surface of Ω. This is not single-valued in Ω, but the combination appearing in the expression for ρ is single-valued.

As an application, we reprove a result due to T. Kubo [36]. With Ω simply connected as above, let $a \in \Omega$, let μ be a positive measure with compact support in Ω and choose $c > 0$ so that the support of μ is contained in $K = \{z \in \Omega : G(z,a) \geq c\}$. We define the Green's potential G^μ of μ by

$$G^\mu(z) = \int G(z,\zeta)\,d\mu(\zeta).$$

The functions G, G^μ are both harmonic in $\Omega \setminus K$ and vanish on $\partial\Omega$. Hence, with $d\sigma_z$ the Poincaré metric as above,

$$\int_{\partial K} G^\mu(z)d\sigma_z = -\frac{1}{2\sinh c}\int_{\partial K} G^\mu(z)\frac{\partial G}{\partial n}(z,a)ds$$

$$= \frac{1}{2\sinh c}\int_{\partial(\Omega\setminus K)} G^\mu(z)\frac{\partial G}{\partial n}(z,a)ds = \frac{1}{2\sinh c}\int_{\partial(\Omega\setminus K)} G(z,a)\frac{\partial G^\mu}{\partial n}ds$$

$$= -\frac{c}{2\sinh c}\int_{\partial K}\frac{\partial G^\mu}{\partial n}ds = \frac{\pi c}{\sinh c}\mu(K).$$

Thus, under the above assumptions,

Theorem 5.2. *The average of the Green potential G^μ, with respect to the Poincaré metric, on a level line $G(\cdot,a) = c$ equals a constant (independent of μ) times the total mass of μ.*

For the Poincaré metric in a simply connected domain we have the estimates (see, for example, [40])

$$\frac{1}{4d(z)} \leq \rho(z) \leq \frac{1}{d(z)}, \tag{5.2}$$

where the lower bound is a consequence of the Koebe one-quarter theorem. Compare also (5.8) below. By (5.1) this gives the following estimate of the distance to the boundary directly in terms of the Green function:

$$\frac{\sinh G(z,a)}{4|\frac{\partial G}{\partial z}(z,a)|} \leq d(z) \leq \frac{\sinh G(z,a)}{|\frac{\partial G}{\partial z}(z,a)|}.$$

See [40] for possible applications of such estimates to computer graphics.

Also the Bergman kernel can provide estimates for the distance to the boundary, even in the multiply connected case. In fact, setting

$$K^{(m,n)}(z,\zeta) = \frac{\partial^{m+n}}{\partial z^m \partial \bar{\zeta}^n} K(z,\zeta)$$

we have, according to P. Davis and H. Pollak [10], the Cauchy-Hadamard type formula

$$\frac{1}{d(z)} = \limsup_{n\to\infty} \frac{e}{n} \left(K^{(n,n)}(z,z) \right)^{\frac{1}{2n}} = \limsup_{n\to\infty} \left(\frac{1}{(n!)^2} K^{(n,n)}(z,z) \right)^{\frac{1}{2n}} \tag{5.3}$$

for any $z \in \Omega$.

In the simply connected case, this gives an interesting formula for the distance to the boundary. Let $\phi(\zeta) = \exp(-\mathcal{G}(\zeta, z)) = a_1(\zeta - z) + a_2(\zeta - z)^2 + \cdots$, $a_1 > 0$, be the conformal map from Ω to the unit disk which takes a given point z to the origin. Then

$$\frac{1}{d(z)} = \limsup_{n\to\infty} \left[\sum_{k=0}^{n} (k+1) \left| \sum_{j=k}^{n} \frac{(n-j+1)}{j!} a_{n-j+1} \frac{d^j}{d\zeta^j}\Big|_{\zeta=z} \phi(\zeta)^k \right|^2 \right]^{\frac{1}{2n}}. \tag{5.4}$$

5.2. **Taylor coefficients.** Above we saw how the distance to the boundary controls the Poincaré metric. Below we shall see more generally how this distance controls the Taylor coefficients of the Green's function, which in the simply connected case embody the Poincaré metric. For any multiply connected domain Ω, let $H(z,\zeta)$ be the regular part of the Green's function, defined by

$$G(z,\zeta) = -\log|z-\zeta| + H(z,\zeta), \tag{5.5}$$

and let $\mathcal{G}(z,\zeta)$ and $\mathcal{H}(z,\zeta) = H(z,\zeta) + \mathrm{i}H^*(z,\zeta)$ denote analytic completions of $G(z,\zeta)$ and $H(z,\zeta)$ with respect to z, for fixed ζ. Then $\mathcal{G}(z,\zeta)$ is multivalued, but $\mathcal{H}(z,\zeta)$ is a perfectly well-defined analytic function (in z), uniquely determined after the normalization $\operatorname{Im}\mathcal{H}(\zeta,\zeta) = 0$, henceforth assumed. Thus $\mathcal{H}(z,\zeta)$ can be expanded in a power series around $z = \zeta$:

$$\mathcal{H}(z,\zeta) = c_0(\zeta) + c_1(\zeta)(z-\zeta) + c_2(\zeta)(z-\zeta)^2 + \dots, \tag{5.6}$$

where c_0 is real by the normalization chosen.

The first few of the coefficients $c_n(\zeta)$ are domain function which have geometric and physical relevance. The constant term, $c_0(\zeta)$, is sometimes called the Robin constant and $e^{-c_0(\zeta)}$ is a kind of capacity (if one allows $\infty \in \Omega$ then $c_0(\infty)$ is the ordinary logarithmic capacity of $\mathbb{C} \setminus \Omega$), cf. [49].

As follows from (3.1), $c_0(z)$ is related to the Bergman kernel by

$$-\Delta c_0(z) = 4\pi K(z,z).$$

Under conformal mappings $c_0(z)$ transforms in such a way that

$$d\sigma = e^{-c_0(z)}|dz|$$

is a conformally invariant metric (see further Section 7). When Ω is simply connected this metric coincides with the Poincaré metric and with the Bergman metric, normalized to be $d\sigma = \sqrt{\pi K(z,z)}|dz|$, but for multiply connected domains none of these metrics are the same. Note also that comparison with (5.1) shows that

$$c_0(z) = -\log \frac{|\frac{\partial G}{\partial z}(z,a)|}{\sinh G(z,a)}$$

for all $z, a \in \Omega$ when Ω is simply connected. In the multiply connected case we still have that the limit, as $z \to a$, of the right member equals $c_0(a)$.

In a simply connected domain any conformally invariant metric has constant Gaussian curvature, because the curvature transforms as a scalar and the conformal group acts transitively on a simply connected domain. For the above metrics the constant curvature is negative: $\kappa_{\text{Gauss}} = -4$. This means that c_0 satisfies the Liouville equation $-\Delta c_0 = 4e^{-2c_0}$, and for the Bergman and Schiffer kernels it means that

$$\frac{\partial^2 \log K(z,\zeta)}{\partial z \partial \bar{\zeta}} = 2\pi K(z,\zeta), \quad \frac{\partial^2 \log L(z,\zeta)}{\partial z \partial \zeta} = -2\pi L(z,\zeta). \tag{5.7}$$

We emphasize that these relations only hold for simply connected domains. For multiply connected domains there are counterparts of for example (5.7) involving also the zeros of the Bergman kernel, see [51], [28], [29] and Remark 9.2 in the present paper.

The function $c_0(z)$ can be estimated in terms of the distance $d(z) = d(z, \partial\Omega)$ to the boundary by

$$\log d(z) \le c_0(z) \le \log d(z) + A \tag{5.8}$$

for some constant A. The lower bound is an elementary consequence of the maximum principle combined with monotonicity properties of $c_0(z)$ with respect to the domain. The upper bound depends on the nature of the domain. If for example the domain is convex, one can take $A = \log 2$, and for a general simply connected domain (5.8) is the same as (5.2), i.e., $A = \log 4$ works. See [49], [1] for further discussions.

For the higher coefficients we have the following estimates, one of which will be used in Section 8.

Lemma 5.3. *For $n \ge 1$,*

$$|c_n(\zeta)| \le \frac{1}{n d(\zeta, \partial\Omega)^n} \quad (z \in \Omega).$$

Proof. The derivative of the Green's function has the Taylor expansion, with respect to z,

$$2\frac{\partial G(z,\zeta)}{\partial z} = -\frac{1}{z-\zeta} + \sum_{n=1}^{\infty} n c_n(\zeta)(z-\zeta)^{n-1}.$$

Thus, for any region $D \subset \Omega$ containing ζ,

$$n c_n(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} 2\frac{\partial G(z,\zeta)}{\partial z} \frac{dz}{(z-\zeta)^n} = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{dG(z,\zeta) + \mathrm{i}^* dG(z,\zeta)}{(z-\zeta)^n}$$

On taking $D = \{z \in \Omega : G(z,\zeta) > \epsilon\}$, where $\epsilon > 0$, we have $dG(\cdot,\zeta) = 0$ along ∂D, while $-{}^*dG(\cdot,\zeta)$ can be considered as a positive measure of total mass 2π on ∂D.

By letting $\epsilon \to 0$ this gives the desired estimates:

$$|nc_n(\zeta)| = |\frac{1}{2\pi}\int_{\partial\Omega}\frac{{}^*dG(z,\zeta)}{(z-\zeta)^n}| \leq \frac{1}{d(\zeta,\partial\Omega)^n}.$$

□

5.3. **The Green's function by domain variations.** The coefficient c_1 in (5.6) can be directly obtained via the formula

$$c_1(\zeta) = \frac{\mathrm{i}}{2}\int_{\partial\Omega}(\frac{\partial G(z,\zeta)}{\partial z})^2 dz = \frac{1}{2\mathrm{i}}\int_{\partial\Omega}|\frac{\partial G(z,\zeta)}{\partial n}|^2 d\bar{z},$$

which follows from the residue theorem. Instead of the residue theorem one may use the fact that $2\bar{c}_1$ is the gradient of c_0, as is easily checked (see (7.23)), combined with the Hadamard variational formula,

$$\delta G_\Omega(z,\zeta) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{\partial G(\cdot,z)}{\partial n}\frac{\partial G(\cdot,\zeta)}{\partial n}\delta n\, ds,$$

which gives the infinitesimal change of the Green function under an infinitesimal deformation δn of $\partial\Omega$ in the outward normal direction. Since $\delta H_\Omega(z,\zeta) = \delta G_\Omega(z,\zeta)$, one obtains $\delta c_0(\zeta)$ by choosing $z=\zeta$ above, and then the gradient of c_0 is obtained by choosing δn suitably. See [19], Lemma 8.4, for further details.

Here we wish to expand slightly on another use of the Hadamard formula. To avoid infinitesimals one may replace δ by $\frac{\delta}{\delta t}$, where t is a time parameter, so that $\frac{\delta n}{\delta t}$ means the velocity of $\partial\Omega$ in the normal direction. Of special interest is to take this normal velocity proportional to the normal derivative of the Green's function itself, say

$$\frac{\delta n}{\delta t} = -\frac{\partial G(\cdot,a)}{\partial n},$$

where $a \in \Omega$ is a fixed point. This is called Laplacian growth, or Hele-Shaw flow with a point source (see, e.g., [25] for further information), and if $\nabla(a) = \frac{\delta}{\delta t}$ denotes the corresponding derivative, acting on domain functionals, the Hadamard formula gives

$$\nabla(a)G_\Omega(b,c) = -\frac{1}{2\pi}\int_{\partial\Omega}\frac{\partial G(\cdot,a)}{\partial n}\frac{\partial G(\cdot,b)}{\partial n}\frac{\partial G(\cdot,c)}{\partial n}ds,$$

hence that $\nabla(a)G_\Omega(b,c)$ is totally symmetric in a, b, c.

This remarkable fact appears in a series of articles by M. Mineev, P. Wiegmann, A. Zabrodin, I. Krichever, A. Marshakov, L. Takhtajan, for example [41], [35], [57], from a more general perspective to be a consequence of two things. The first is that the regular part $H(z,\zeta)$ of the Green's function is a double variational derivative of an energy functional $F = F(\Omega)$, which can be identified with the logarithm of a certain τ-function [34]. Precisely, $H_\Omega(a,b) = \nabla(a)\nabla(b)F(\Omega)$, hence

$$G_\Omega(a,b) = -\log|a-b| + \nabla(a)\nabla(b)F(\Omega),$$

where

$$F(\Omega) = \frac{1}{8\pi^2}\int_{\mathbb{D}(0,R)\setminus\Omega}\int_{\mathbb{D}(0,R)\setminus\Omega}\log|z-\zeta|dA(z)dA(\zeta),$$

dA denoting area measure and where R is large enough, so that $\Omega \subset \mathbb{D}(0,R)$. The second ingredient is that the Dirchlet problem is "integrable" in the sense that

$$\nabla(a)\nabla(b) = \nabla(b)\nabla(a).$$

Clearly these two facts embody the total symmetry of $\nabla(a)G_\Omega(b,c)$.

5.4. **Level lines of harmonic functions as geodesics and trajectories.** Here we shall interpret level lines of harmonic functions as geodesics in riemannian manifolds and trajectories of hamiltonian systems. Curvature of level lines and geodesics will be discussed in Section 8.

Proposition 5.4. *Let u be harmonic in some domain, u^* a harmonic conjugate of u and let $\varphi > 0$ be any smooth function in one real variable. Then, away from critical points of u, the level lines of u are geodesics for the metric*

$$d\sigma = \varphi(u^*)|\nabla u||dz|.$$

Proof. Let Φ be a primitive function of φ and let γ be a level line of u, with $\nabla u \neq 0$ on γ. By the Cauchy-Riemann equations, this level line is simultaneously an integral curve of ∇u^*, and $|\nabla u| = |\nabla u^*|$. Thus, if z_0 and z_1 denote the end points of γ ordered so that $u^*(z_0) < u^*(z_1)$,

$$\int_\gamma d\sigma = \int_\gamma \Phi'(u^*)|\nabla u^*||dz| = \int_\gamma \Phi'(u^*)du^* = \Phi(u^*(z_1)) - \Phi(u^*(z_0)).$$

Since, along a curve in general, $|\nabla u^*||dz| \geq du^*$, integration of $d\sigma$ along any curve from z_0 to z_1 will give a value $\geq \Phi(u^*(z_1)) - \Phi(u^*(z_0))$. Thus γ is a geodesic. $\square$

As a simple remark on trajectories, the level lines of any (smooth) function u in a domain $\Omega \subset \mathbb{C}$ are trajectories of the hamiltonian system with phase space Ω, symplectic form $\omega = dx \wedge dy$ and hamiltonian function u (see [2] for the terminology). The Hamilton equations then are $\frac{dx}{dt} = \frac{\partial u}{\partial y}$, $\frac{dy}{dt} = -\frac{\partial u}{\partial x}$ or, in complex form,

$$\frac{dz}{dt} = -2\mathrm{i}\frac{\partial u}{\partial \bar{z}}. \tag{5.9}$$

Such a kind of hamiltonian system describes for example the motion of a point vortex in a plane domain, see [37], [43] and Section 8 below.

The above equations guarantee that the motion is along the level lines of u. Assume now that u is harmonic. Then (5.9) gives also

$$\frac{d^2 z}{dt^2} = -2\mathrm{i}\frac{\partial^2 u}{\partial \bar{z}^2}\frac{d\bar{z}}{dt} = 4\frac{\partial^2 u}{\partial \bar{z}^2}\frac{\partial u}{\partial z} = -2\frac{\partial V}{\partial \bar{z}}, \tag{5.10}$$

where

$$V = -\frac{1}{2}|\nabla u|^2 = -2\frac{\partial u}{\partial z}\frac{\partial u}{\partial \bar{z}}.$$

Notice that V is real-valued and that the right-hand side of (5.10) is minus the gradient of V. Therefore we can think of (5.10) as an ordinary newtonian system for the motion of a unit point mass in the potential V. Thus

Proposition 5.5. *The level lines of any harmonic function u are, away from critical points, trajectories for the newtonian system with potential energy $-\frac{1}{2}|\nabla u|^2$.*

Remark 5.2. The proposition generalizes to the case that $\Delta u = c$, c constant, with V changed to

$$V = -\frac{1}{2}|\nabla u|^2 + cu.$$

If we want to put Proposition 5.5 into a hamiltonian formulation, the domain Ω takes the role of configuration space, while phase space is $\Omega \times \mathbb{C}$. The kinetic energy is $T = \frac{1}{2}|\frac{dz}{dt}|^2$ and the hamiltonian

$$H = T + V,$$

to be considered as a function of position $q = z$ and momentum $p = \frac{dz}{dt}$. The Hamilton equations, written in complex notation, are

$$\frac{dq}{dt} = 2\frac{\partial H}{\partial \bar{p}}, \quad \frac{dp}{dt} = -2\frac{\partial H}{\partial \bar{q}}. \tag{5.11}$$

Along a trajectory $t \mapsto (p(t), q(t))$ in phase space, $H(p, q)$ is constant, say $H = E$, where E is the total energy. Now, the principle of least action in the form of Maurpertuis, Euler, Lagrange and Jacobi (see [2], [20]), states that the trace in configuration space of a hamiltonian trajectory of constant energy E is a geodesic for the Jacobi metric,

$$d\rho = \sqrt{T} ds = \sqrt{E - V(z)} ds.$$

Here ds denotes the ordinary euclidean metric in Ω, hence arc-length along trajectories; in a more general context it would be the metric in configuration space induced by the kinetic energy.

In our case $V = -\frac{1}{2}|\nabla u|^2$, and since we derived the motion from (5.9), $T = \frac{1}{2}|\nabla u|^2$. Thus $E = 0$ and the Jacobi metric becomes

$$d\rho = \frac{1}{\sqrt{2}}|\nabla u|\,|dz|,$$

i.e., the least action principle becomes an instance of Proposition 5.4.

The hamiltonian system (5.11) or (5.10) has of course many more trajectories (in configuration space) than the level lines of u. In fact, through any point z there is one trajectory starting out with any prescribed speed $\frac{dz}{dt}$. (Even if we ignore the parametrization, the modulus $|\frac{dz}{dt}|$ of the speed affects the trajectory as a point set). Some of these other trajectories (one in each direction) can covered by the above analysis by mixing u with its harmonic conjugate, as follows.

Let u^* be a harmonic conjugate of u and let

$$w = f(z) = u + \mathrm{i}u^*,$$

so that f is conformal away from critical points of u. Then, for any $\theta \in \mathbb{R}$, $|\nabla u| = |2\frac{\partial u}{\partial z}| = |f'(z)| = |e^{\mathrm{i}\theta} f'(z)| = |\nabla(\cos\theta\, u - \sin\theta\, u^*)|$, so that V remains unchanged if u is replaced by $\cos\theta\, u - \sin\theta\, u^*$. Thus the level lines of all such functions are also trajectories, and it is easily seen that they represent all trajectories with total energy $E = 0$. In terms of $f(z)$ we can express the conclusion as follows.

Proposition 5.6. *The trace in configuration space Ω of the trajectories on the energy surface $E = 0$ of the hamiltonian system (5.11) or (5.10) are exactly the inverse images under the conformal map $w = f(z) = u + \mathrm{i}u^*$ of the straight lines in the w-plane.*

As an application of the above we may take u to be the Green's function of a domain $\Omega \subset \mathbb{C}$: $u(z) = G(z, a)$, $a \in \Omega$ fixed. With $\Omega' = \Omega \setminus \{z_1(a), \dots, z_{\mathsf{g}}(a)\}$, where $z_1(a), \dots, z_{\mathsf{g}}(a)$ are the critical points of $G(z, a)$, we conclude from Proposition 5.4 that that level sets of $G(z, a)$ are geodesics for

$$d\sigma_1 = |\nabla G(z, a)||dz|,$$

and

$$d\sigma_2 = \frac{|\nabla G(z, a)|}{G^*(z, a)}|dz|.$$

And, by changing the roles of G and G^* we see that the level lines of the harmonic conjugate $G^*(z, a)$ are geodesics for

$$d\sigma_3 = \frac{|\nabla G(z, a)|}{G(z, a)}|dz|$$

(and for $d\sigma_1$). In the image region under $f(z) = G(z,a) + \mathrm{i}G^*(z,a)$, the above geodesics for $d\sigma_1$ correspond to geodesics of the euclidean metric and the geodesics for $d\sigma_3$ correspond to geodesics for the Poincaré metric in the right half-plane.

6. Doubly connected domains

6.1. **The general domain functions for an annulus.** This example is fundamental, it reveals in many respect the essential ideas. Our exposition is based on ideas which have been elaborated in [53], and previously in [39]. Since we shall only discuss conformally invariant questions it will be enough work with annuli. We shall use the notation

$$A_{a,b} = \{z \in \mathbb{C} : a < |z| < b\},$$

$0 < a < b$, for an annulus centered at the origin. The conformal type is determined by the quotient b/a (the modulus), so we can fix either a or b. Note also the conformal symmetries of $A_{a,b}$: $z \mapsto e^{\mathrm{i}\theta} z$ for any $\theta \in \mathbb{R}$ and $z \mapsto ab/z$, the latter being the conformal reflection about the symmetry line $|z| = \sqrt{ab}$, which exchanges the outer and inner boundary components.

Choosing the annulus to be $A_{1,R}$, where $R > 1$, we can represent the main domain functions in terms of elliptic functions. The multivalued function $z \mapsto t = \log z$ lifts the annulus to the strip $\{t \in \mathbb{C} : 0 < \operatorname{Re} t < \log R\}$, which then represents the universal covering surface of $A_{1,R}$. The lift map extends to the Schottky double of the annulus, which is a torus, and takes it onto the universal covering surface of the torus, namely $\mathbb{C}$. The covering transformations on $\mathbb{C}$ are generated by $t \mapsto t + 2\log R$ and $t \mapsto t + 2\pi\mathrm{i}$, hence we have a period lattice with half-periods

$$\begin{cases} \omega_1 = \log R, \\ \omega_2 = \mathrm{i}\pi. \end{cases} \tag{6.1}$$

One fundamental domain, symmetric around the origin, is

$$F_0 = \{t \in \mathbb{C} : -\log R < \operatorname{Re} t < \log R,\ -\pi < \operatorname{Im} t < \pi\}.$$

Here the right half of this corresponds to the annulus $A_{1,R}$ itself, while the left half corresponds to the copy of the annulus which makes up the back-side of the Schottky double. The anti-holomorphic involution is the reflection in the imaginary axis: $J(t) = -\bar{t}$. A homology basis, as chosen in Section 2, can be taken to be $\alpha_1 = [-\log R, \log R]$, $\beta_1 = [-\mathrm{i}\pi, \mathrm{i}\pi]$, as point sets. The orientations of α_1 and β_1 will however be opposite to the ordinary orientations of the real and imaginary axes.

Sometimes it is advantageous to work in a fundamental domain whose boundary is made up of preimages of the curves in the homology basis chosen. Such a fundamental domain is

$$F_1 = \{t \in \mathbb{C} : 0 < \operatorname{Re} t < 2\log R,\ 0 < \operatorname{Im} t < 2\pi\}. \tag{6.2}$$

It is often convenient to scale the period lattice so that it is generated by 1 and τ, with $\operatorname{Im} \tau > 0$, in place of $2\omega_1$, $2\omega_2$. Then $\tau = \omega_2/\omega_2$, and in our case we have

$$\tau = \frac{\mathrm{i}\pi}{\log R}. \tag{6.3}$$

We recall the standard elliptic functions associated to the given period lattice. The Weierstrass $\wp$-function is

$$\wp(z) = \wp(z; 2\omega_1,\, 2\omega_2) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right),$$

and the ζ- and σ-functions are

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\neq 0}\Big(\frac{1}{z+\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\Big),$$

$$\sigma(z) = z\prod_{\omega\neq 0}\Big(1-\frac{z}{\omega}\Big)\exp\Big(\frac{z}{\omega}+\frac{z^2}{2\omega^2}\Big),$$

where in all cases $\omega = 2m_1\omega_1 + 2m_2\omega_2$ and summations and products are taken over all $(m_1, m_2) \in \mathbb{Z}\times\mathbb{Z}\setminus(0,0)$.

The above functions are related by

$$\wp(z) = -\zeta'(z), \quad \zeta(z) = \frac{\sigma'(z)}{\sigma(z)},$$

$\wp(z)$ is doubly periodic, $\zeta(z)$ acquires constants along the periods,

$$\zeta(z+2\omega_j) = \zeta(z) + 2\eta_j,$$

and for $\sigma(z)$ we have

$$\sigma(z+2\omega_j) = -\sigma(z)e^{2\eta_j(z+\omega_j)}$$

$(j = 1, 2)$. The constants η_j are given by $\eta_j = \zeta(\omega_j)$ and they satisfy the Legendre relation

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{\mathrm{i}\pi}{2}.$$

In our case η_1 is real and positive, η_2 is purely imaginary and the Legendre relation becomes

$$\frac{\eta_1}{\log R} - \frac{\eta_2}{\mathrm{i}\pi} = \frac{1}{2\log R}.$$

We record also the differential equation satisfied by $\wp(z)$:

$$\wp'(z)^2 = 4\wp^3 - g_2\wp(z) - g_3,$$

where

$$g_2 = 60\sum_{\omega\neq 0}\frac{1}{\omega^4}, \quad g_3 = 140\sum_{\omega\neq 0}\frac{1}{\omega^6}. \tag{6.4}$$

As a first step we shall make some of the functions and differentials appearing in Section 2 explicit in the annulus case, and then (in a later subsection) we shall study the critical points of the Green's function in some detail. When we work directly in the annulus, and with the Schottky double of it realized in the same plane by reflection, we shall denote points by letters, z, w, a, b and similar. When we work on the universal covering surface $\mathbb{C}$, with its period lattice, we shall denote the corresponding points t, s, u, v. Thus $t = \log z$, $s = \log w$, $u = \log a$, $v = \log b$.

The two versions of the abelian differentials of the third kind discussed in Section 2 are, on the universal covering surface, given by

$$\upsilon_{u-v}(t) = (\zeta(t-u) - \zeta(t-v) + A)dt, \tag{6.5}$$

$$\omega_{u-v}(t) = (\zeta(t-u) - \zeta(t-v) + B)dt, \tag{6.6}$$

where A and B are constants (depending on u, v) chosen so that υ_{u-v}, ω_{u-v} get the desired periods. This means that

$$\int_{-\log R}^{\log R}(\zeta(t-u) - \zeta(t-v) + A)dt, \quad \int_{-\pi\mathrm{i}}^{\pi\mathrm{i}}(\zeta(t-u) - \zeta(t-v) + A)dt$$

are both purely imaginary, and

$$\int_{-\log R}^{\log R}(\zeta(t-u) - \zeta(t-v) + B)dt = 0.$$

By straightforward calculations this gives

$$A = \frac{\eta_1(u-v)}{\log R} + \frac{\mathrm{Im}\,(u-v)}{2\mathrm{i}\log R}, \tag{6.7}$$

$$B = \frac{\eta_1(u-v)+\pi \mathrm{i} m}{\log R}, \tag{6.8}$$

where the integer m depends on the location of u, v relative to the preimage of α_1 in the period lattice. With $u, v \in F_0$, then $m = 0$ if u and v are in the same component of $F_0 \setminus [-\log R, \log R]$, which is the case, for example, if $\mathrm{Im}\,(u-v) = 0$. Thus we see, in accordance with Lemma 2.1, that $A = B$ when $v = J(u)$. One can also achieve $m = 0$ in more general situations by working in the fundamental domain F_1, where α_1 is represented by $[0, 2\log R]$, hence is part of the boundary.

Next we compute $V(t,s;u,v)$ by integrating υ_{u-v}, see (2.12). The calculation is straightforward and the result is

$$V(t,s;u,v) = -\log|\frac{\sigma(t-u)\sigma(s-v)}{\sigma(t-v)\sigma(s-u)}|$$
$$-\frac{\eta_1 \mathrm{Re}\,((t-s)(u-v))}{\log R} - \frac{\mathrm{Im}\,(t-s)\mathrm{Im}\,(u-v)}{2\log R}.$$

Pulling this back to the plane of the annulus, i.e., on substituting $t = \log z$ etc., we get

$$V(z,w;a,b) = -\log|\frac{\sigma(\log\frac{z}{a})\sigma(\log\frac{w}{b})}{\sigma(\log\frac{z}{b})\sigma(\log\frac{w}{a})}| - \frac{\eta_1 \mathrm{Re}\,(\log\frac{z}{w}\log\frac{a}{b})}{\log R} - \frac{\arg\frac{z}{w}\arg\frac{a}{b}}{2\log R}.$$

We record also the final expressions for υ_{a-b} and ω_{a-b} in the plane of the annulus:

$$\upsilon_{a-b} = (\zeta(\log\frac{z}{a}) - \zeta(\log\frac{z}{b}) + \frac{\eta_1\log\frac{a}{b}}{\log R} + \frac{\arg\frac{a}{b}}{2\mathrm{i}\log R})\frac{dz}{z},$$

$$\omega_{a-b} = (\zeta(\log\frac{z}{a}) - \zeta(\log\frac{z}{b}) + \frac{\eta_1\log\frac{a}{b}+\pi\mathrm{i}m}{\log R})\frac{dz}{z}.$$

The Green's function is by (2.10) just a special case of V. Choosing $s = J(t) = -\bar{t}$, $v = J(u) = -\bar{u}$ and using the first alternative in (2.10) gives, on the universal covering surface,

$$G(t,u) = -\frac{1}{2}\log|\frac{\sigma(t-u)\sigma(-\bar{t}+\bar{u})}{\sigma(t+\bar{u})\sigma(-\bar{t}-u}| - \frac{\eta_1 \mathrm{Re}\,t\,\mathrm{Re}\,u}{\log R}.$$

Hence, in the annulus ($z, a \in A_{1,R}$),

$$G(z,a) = -\frac{1}{2}\log|\frac{\sigma(\log\frac{z}{a})\sigma(\log\frac{\bar{a}}{\bar{z}})}{\sigma(\log(z\bar{a}))\sigma(\log\frac{1}{\bar{z}a})}| - \frac{2\eta_1\log|z|\log|a|}{\log R}.$$

Expanding the σ-function in an infinite product gives us what we would have obtained by the method of images (automorphization). The result is

$$G(z,a) = -\log|\frac{R(z-a)}{R^2-z\bar{a}}| - \log|\prod_{n=1}^{\infty}\frac{(R^{2n}-\frac{z}{a})(R^{2n}-\frac{a}{z})}{(R^{2n}-\frac{z\bar{a}}{R^{2n}})(R^{2n}-\frac{R^{2n}}{z\bar{a}})}|, \tag{6.9}$$

see [8], [9].

The Bergman kernel is in general

$$K(z,\zeta)dzd\bar{\zeta} = \frac{1}{\pi}\frac{\partial\omega_{\zeta-J(\zeta)}(z)}{\partial\bar{\zeta}}d\bar{\zeta}.$$

Using (6.6), (6.8), this gives in the present case

$$K(t,u) = \frac{1}{\pi}(\wp(t+\bar{u}) + \frac{\eta_1}{\log R}) \tag{6.10}$$

for $t, u \in \mathbb{C}$, and

$$K(z,a) = \frac{1}{\pi z\bar{a}}(\wp(\log(z\bar{a})) + \frac{\eta_1}{\log R}) \tag{6.11}$$

in the annulus ($z, a \in A_{1,R}$). This agrees with expressions derived by Zarankiewicz [60]. See also [18] (p.133) and [4].

Finally, we elaborate the Schottky-Klein prime function in terms of elliptic functions (see [9] for representations in terms of Poincaré series). It is obtained by combining (2.15) with (6.6) and (6.8). The result of that is

$$\exp\int_s^t \omega_{u-v} = \frac{\sigma(t-u)\sigma(s-v)}{\sigma(t-v)\sigma(s-u)}\exp\left[\frac{\eta_1(t-s)(u-v)}{\log R} + \frac{2\pi\mathrm{i}m(t-s)}{\log R}\right],$$

which is to be identified with $\dfrac{E(t,a)E(s,v)}{E(t,v)E(s,u)}$. Working in the fundamental domain F_1 in (6.2), which allows $m = 0$, this gives

$$E(t,u) = \sigma(t-u)\exp\left[\frac{\eta_1 tu}{\log R}\right]f(t)f(u),$$

for some one variable function f. To identify $f(t)$ we differentiate with respect to t at $t = u$:

$$\frac{\partial E}{\partial t}|_{t=u}E(t,u) = \sigma'(0)\exp\left[\frac{\eta_1 u^2}{\log R}\right]f(u)^2.$$

Since $E(t,u)$ is to behave like $t-u$ at $t = u$ the above derivative must be $= 1$. This gives $f(u) = \pm\exp\left[\dfrac{\eta_1 u^2}{2\log R}\right]$. Therefore actually

$$E(t,u) = \sigma(t-u)\exp\left[-\frac{\eta_1(t-u)^2}{2\log R}\right],$$

and in the plane of the annulus,

$$E(z,a) = \sigma(\log\frac{z}{a})\exp\left[-\frac{\eta_1}{2\log R}(\log\frac{z}{a})^2\right].$$

6.2. **Eisenstein series.** For further need we recall the following classical Eisenstein series and other arithmetical functions [33].

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n, \tag{6.12}$$

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{6.13}$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{6.14}$$

where

$$\sigma_k(n) = \sum_{d|n} d^k,$$

and the "nome" q always is related to τ by

$$q = e^{2\pi\mathrm{i}\tau}.$$

Thus $|q| < 1$.

It is well known that $E_4(\tau)$ and $E_6(\tau)$ are modular forms of weights 4 and 6 for the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. This is a consequence of Lipschitz' formula, which asserts that for any integer $k \geq 2$, and with B_{2k} being the k:th Bernoulli number,

$$E_{2k}(\tau) = 1 - \frac{2k}{B_{2k}} \sum_{n=1}^{\infty} \frac{n^{k-1}q^n}{1-q^n}$$

$$= 1 - \frac{2k}{B_{2k}} \sum_{m,n=1}^{\infty} \sigma_{k-1}(n)q^n = 1 - \frac{2k}{B_{2k}} \sum_{m,n=1}^{\infty} \frac{1}{(m\tau+n)^{2k}}.$$

$E_2(\tau)$ is not modular but

$$E_2^*(\tau) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - \frac{3}{\pi \mathrm{Im}\,\tau} q^3 - 168q^4 - \cdots \tag{6.15}$$

$$= 1 - \frac{3}{\pi \mathrm{Im}\,\tau} - 24q - 72q^2 - 96$$

is a non-holomorphic Eisenstein series of weight 2 for Γ. Actually the series $E_2(\tau)$ satisfies

$$E_2(\frac{a\tau+b}{c\tau+d}) = E_2(\tau)(c\tau+d)^2 + \frac{6c(c\tau+d)}{\mathrm{i}\pi} \tag{6.16}$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ. This is due to the fact that the series

$$\sum_{m,n=1}^{\infty} \frac{1}{(m\tau+n)^k}.$$

is not absolutely convergent for $k=2$. To overcome that difficulty one considers, following Hecke, the series, defined for $s>0$,

$$E_{2k}^*(\tau;s) = 1 - \frac{2k}{B_{2k}} \sum_{m,n=1}^{\infty} \frac{1}{(m\tau+n)^k|(m\tau+n|^s}.$$

and takes the limit as $s \to 0$. For $k=1$ this gives (6.15). We will use a similar idea in subsection 6.4 to recover the Green's function via an eigenvalue problem.

We recall the classical discriminant function Δ, defined as the infinite product

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots.$$

It is a cusp form for Γ of weight 12 and is related to the Eisenstein series by

$$1728\Delta(\tau) = E_4(\tau)^3 - E_6(\tau)^2.$$

Following Ramanujan [48] we also introduce the functions

$$\Phi_{rs}(q) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^r n^s q^{mn}. \tag{6.17}$$

Then,

$$\begin{aligned} \Phi_{rs}(q) &= \Phi_{sr}(q), \\ \Phi_{rs}(q) &= \left(q\frac{d}{dq}\right)^r \Phi_{0,s-r}(q). \end{aligned}$$

We may consider $\Phi_{rs}(q)$ as a function of τ and write $\Phi_{rs}(\tau)$. For this we have an algebraic relation

$$\Phi_{rs}(\tau) = \sum K_{lmn} E_2(\tau)^l E_4(\tau)^m E_6(\tau)^n,$$

for suitable coefficients K_{lmn}, and where the sum is taken over all positive integers l, m, n satisfying $2l + 4m + 6n = r + s + 1$, $l \le \inf(r, s) + 1$.

It is a fundamental fact [48] that every modular form on Γ is uniquely expressible as a polynomial in E_4 and E_6 and that the extension $\mathbb{C}[E_2, E_4, E_6]$ of $\mathbb{C}[E_4, E_6]$ is closed under differentiation, with the following dynamical system of Ramanujan:

$$\begin{cases} E_2' = \dfrac{1}{12}(E_2^2 - E_4), \\ E_4' = \dfrac{1}{3}(E_2E_4 - E_6), \\ E_6' = \dfrac{1}{2}(E_2E_6 - E_4^2), \end{cases} \tag{6.18}$$

where the prime means the differentiation $\dfrac{1}{2\pi \mathrm{i}}\dfrac{d}{d\tau}$.

The identity (6.16) can be expressed by saying that, up to a constant factor, E_2 defines an affine connection, and the first relation in (6.18) similarly says that E_4 is the projective connection which is the curvature of the affine connection E_2 (see Section 7 for the terminology). Hence the Eisenstein series E_2 defines a covariant derivative sending weight k modular forms f into weight $k + 2$ modular forms $f' - \dfrac{k}{12}E_2 f$.

From the system (6.18) we obtain

$$\begin{cases} E_4 &= E_2^2 - 12E_2', \\ E_6 &= E_2^3 - 18E_2E_2' + 36E_2''. \end{cases} \tag{6.19}$$

An important consequence is that the function E_2 is a solution of the Chazy equation

$$E_2''' = E_2E_2'' - \frac{3}{2}{E_2'}^2. \tag{6.20}$$

This means that we may use E_2, E_2', E_2'' instead of E_2, E_4, E_6 to express modular forms, that is $\mathbb{C}[E_2, E_4, E_6] = \mathbb{C}[E_2, E_2', E_2'']$. We shall, in a later subsection, give an illustration of this fact to determine the modulus of doubly connected domains.

6.3. **Critical points of the Green's function and zeros of the Bergman kernel.** The critical points of the Green's function $G(z, a)$ are by (2.10), (2.11) exactly the zeros of $\upsilon_{a-J(a)}$. For the annulus there is exactly one critical point, $z = z_G(a)$, and by symmetry this is located on the same diameter as a. For the more detailed investigations one may pass to the universal covering, via $t = \log z$, $u = \log a$, $-\bar{u} = J(u) = \log J(a)$. Then, by (6.5), (6.7),

$$\upsilon_{u-J(u)}(t) = (\zeta(t-u) - \zeta(t+\bar{u}) + \frac{2\eta_1 \mathrm{Re}\, u}{\log R})dt. \tag{6.21}$$

Thus the equation for the representation $t = t_G(u)$ of the critical point is

$$\zeta(t_G(u) - u) - \zeta(t_G(u) + \bar{u}) = -\frac{2\eta_1 \mathrm{Re}\, u}{\log R}, \tag{6.22}$$

where $0 < \mathrm{Re}\, u < \log R$. In the plane of the annulus the corrseponding equation, for $z = z_G(a)$, becomes

$$\zeta(\log \frac{z_G(a)}{a}) - \zeta(\log z_G(a)\bar{a}) = -\frac{2\eta_1 \log |a|}{\log R}. \tag{6.23}$$

These equations have been analyzed by A. Maria [39], and the results are summarized in Theorem 6.1 below.

The zeros of the Bergman kernel can be treated fairly explicitly. By (6.10), the zeros of $K(t,u)$ on the universal covering surface are those points $t = t_K(u)$ for which

$$\wp(t_K(u)+\bar{u}) = -\frac{\eta_1}{\log R}. \tag{6.24}$$

Thus

$$t_K(u) = s - \bar{u},$$

where s solves

$$\wp(s) = -\frac{\eta_1}{\log R}. \tag{6.25}$$

This equation has two solutions in each period parallelogram. None of them are real because $\wp$ is positive on the real axis, while the right member is negative. In fact, it is easy to see that every solution has imaginary part π, modulo multiples of 2π.

Let $s = s(R)$ denote that solution of (6.25) which satisfies $0 \le \operatorname{Re} s < \log R$, $\operatorname{Im} s = \pi$. It depends on R or, equivalently, on τ via (6.3). Define also

$$\rho = \rho(R) = e^{\operatorname{Re} s} = |e^s|. \tag{6.26}$$

According to [14], s can be explicitly computed to be

$$s(R) = \mathrm{i}\pi + \frac{3\log R}{2} + \frac{3456\sqrt{3}\pi\log R}{5}\int_\tau^{\mathrm{i}\infty}\frac{\Phi_{5,6}(t)-\Delta(t)}{\Phi_{2,3}(t)^{\frac{3}{2}}}(t-\tau)dt,$$

where the integration is vertically upwards starting at $\tau = \frac{\mathrm{i}\pi}{\log R}$.

The following theorem summarizes results in [39] and [53] and combines these with our findings for the zeros of the Bergman kernel.

Theorem 6.1. *With ρ defined by (6.26) we have*

(i) $1 < \rho < \sqrt{R}$.

(ii) *The Green's function $G(z,a)$ of $A_{1,R}$ has, for any given $a \in A_{1,R}$, a unique critical point $z = z_G(a)$. This is located on the same diameter as a but on the opposite side of the hole. More precisely,*

$$z_G(a) = -g(|a|)\frac{a}{|a|},$$

where $g : (1,R) \to (1,R)$ is an increasing function which maps $(1,R)$ onto the relatively compact subinterval $(\rho, R/\rho)$. It satisfies $g(x)g(R/x) = R$ for $1 < x < R$, in particular $g(\sqrt{R}) = \sqrt{R}$, and

$$\lim_{a\to 1} g'(a) = \lim_{a\to R} g'(a) = 0.$$

(iii) *When $\rho \le |a| \le R/\rho$ the Bergman kernel $K(z,a)$ has no zeros.*

(iv) *When $1 < |a| < \rho$ or $R/\rho < |a| < R$, $K(z,a)$ has exactly one zero, $z = z_K(a)$, and this is explicitly given by*

$$z_K(a) = \begin{cases} -\frac{\rho}{\bar{a}} & \text{when } 1 < |a| < \rho, \\ -\frac{R^2}{\rho\bar{a}} & \text{when } R/\rho < |a| < R. \end{cases}$$

This theorem is special to the annulus but is effective. We do not know of any such precise results in higher connectivity.

Note that the theorem confirms in a precise way the assertion of Corollary 3.3 that the limiting set for the critical points of the Green's function is exactly the zeros of the Bergman kernel when the parameter variable is on the boundary. In fact, choosing a on the positive real axis (for simplicity of notation) the theorem shows that

$$\lim_{a\to 1} z_G(a) = \lim_{a\to 1} z_K(a) = -\rho,$$

$$\lim_{a\to R} z_G(a) = \lim_{a\to R} z_K(a) = -\frac{R}{\rho}.$$

In addition we deduce from the theorem the following remarkable dichotomy result.

Corollary 6.2. *The annulus $A_{1,R}$ is the disjoint union of the set of critical points $\{z_G(a)\}$ of the Green's function, the set of zeros $\{z_K(a)\}$ of the Bergman kernel and the two circles $\{|z| = \rho\}$ and $\{|z| = \frac{R}{\rho}\}$.*

Proof. (of theorem)

By rotational symmetry we may assume that a is real and positive, namely $1 < a < R$. It has been shown in [39] that

$$z_G(a) = -g(a)$$

for some differentiable function $g : (1, R) \to (1, R)$, which is increasing, more precisely $g' > 0$, and satisfies

$$0 < \lim_{a\to 1} g(a) < \lim_{a\to R} g(a) < R,$$

(the latter follows also from [56] and from our Theorem 3.1) and

$$\lim_{a\to 1} g'(a) = \lim_{a\to R} g'(a) = 0.$$

Slightly more explicitly we have $g(e^u) = e^{f(u)}$, where the function $f : (0, \log R) \to (0, \log R)$ is defined as the unique solution of

$$\zeta(f(u) + \mathrm{i}\pi - u) - \zeta(f(u) + \mathrm{i}\pi + u) = -\frac{2\eta_1 u}{\log R},$$

or

$$\frac{1}{2u}\zeta(f(u) + \mathrm{i}\pi - u) - \zeta(f(u) + \mathrm{i}\pi + u) = -\frac{\eta_1}{\log R}. \tag{6.27}$$

Since $g' > 0$, and hence $f' > 0$, $f(0) = \lim_{u\to 0} f(u)$ exists, and letting $u \to 0$ in (6.27) gives (in view of $\zeta' = -\wp$)

$$\wp(f(0) + \mathrm{i}\pi) = -\frac{\eta_1}{\log R}.$$

It follows that

$$f(0) + \mathrm{i}\pi = s(R)$$

(see (6.25)–(6.26)). In other words,

$$\lim_{a\to 1} g(a) = -e^{\mathrm{Re}\, s(R)} = -\rho(R).$$

Similarly,

$$\lim_{a\to 1} g(a) = -e^{\mathrm{Re}\, s(R)} = -\frac{R}{\rho(R)}.$$

We conclude that $\rho < R/\rho$, i.e., that $1 < \rho < \sqrt{R}$. By this parts (i)and (ii) of the theorem are proven.

The assertions (iii) and (iv), about the Bergman zeros, are easy consequences of the analysis made before the statement of the theorem. In fact, since $\wp$ is an even function, the two solutions of (6.25) are $\pm s$, modulo the period lattice. It follows from (6.10) that, given $u \in F_0$, the (possible) solutions $t = t_K(u)$ of $K(t, u) = 0$ are represented by

$$t_K(u) = \begin{cases} s - \bar{u}, \\ -s - \bar{u}, \end{cases} \tag{6.28}$$

modulo the period lattice. Both u and $t_K(u)$ shall correspond to points in $A_{1,R}$, i.e., $0 < \mathrm{Re}\, u < \log R$ and $0 < \mathrm{Re}\, t_K(u) < \log R$. This occurs if and only if $0 < \mathrm{Re}\, u < s$

or $\log R - s < \mathrm{Re}\, u < \log R$, and then we have the same inequalities for the zero $t_K(u)$ itself: $0 < \mathrm{Re}\, t_K(u) < s$ and $\log R - s < \mathrm{Re}\, t_K(u) < \log R$, respectively.

In the plane of the annulus (6.28) becomes

$$z_K(a) = \begin{cases} \frac{\exp s}{\bar{a}}, \\ \frac{\exp(-s)}{\bar{a}}, \end{cases}$$

and since this $z_K(a)$ is in $A_{1,R}$ if and only if $1 < |a| < \rho$ or $R/\rho < |a| < R$ we have now proved statements (iii) and (iv) in the theorem.

□

It follows from (i) and (ii) of Theorem 6.1 that the critical points of the Green's function for any annulus of modulus $R > 1$ are located in the annulus of modulus R/ρ^2 symmetrically centered in the original annulus. Since $\rho = \rho(R) > 1$ for all $R > 1$ and ρ depends smoothly on R one easily concludes the following.

Corollary 6.3. *Let $R > 1$. There exist a sequence $\{\rho_n\}$, $1 = \rho_0 < \rho_1 < \rho_2 < \cdots < \sqrt{R}$ with $\lim_{n\to\infty} \rho_n = \sqrt{R}$ such that, on setting $A_n = \{z \in \mathbb{C} : \rho_n < |z| < R/\rho_n\}$, the critical points for the Green's function of A_n are all contained in A_{n+1}.*

One might ask what would be corresponding statement in higher connectivity.

6.4. **Spectral point of view.** The Green's function can also be obtained via spectral problems for the Laplace operator. This requires a choice of a metric, or at least a volume form. For a metric $d\sigma = \rho|dz|$, the volume form is $\rho^2 dxdy$ and the invariant Laplacian is $\frac{1}{\rho^2}\Delta$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Thus a natural spectral problem in a domain Ω is

$$\begin{cases} -\Delta u = \lambda \rho^2 u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

If $\lambda_1, \lambda_2, \ldots$ are the eigenvalues and $u_1, u_2, \ldots$ the eigenfunctions, normalized by

$$\int_\Omega |u_n|^2 \rho^2 dxdy = 1,$$

then the Green's function of Ω is given formally by

$$G(z,w) = 2\pi \sum_{n=1}^{\infty} \frac{u_n(z)u_n(w)}{\lambda_n},$$

where however the sum may converge only in a weak sense. See for example [8], [13]. (The factor 2π appears because we have no factor $1/2\pi$ in front of the singularity of the Green's function.) One way to cope with the convergence problem is to consider the corresponding Dirichlet series

$$G^s(z,w) = 2\pi \sum_{n=1}^{\infty} \frac{u_n(z)u_n(w)}{\lambda_n^s},$$

which is an analytic function in s for $\mathrm{Re}\, s > 1$. One then studies the behaviour as $s \to 1$. To continue analytically $G^s(z,w)$ we shall use the second Kronecker limit formula which is basically a connection between Epstein zeta functions and Eisenstein series [55].

The eigenfunctions u_n form an orthonormal set both with respect to the L^2-inner product $\int_\Omega uv\rho^2 dxdy$ and with respect to the Dirichlet inner product $D(u,v) = \int_\Omega \nabla u \cdot \nabla v dxdy$. Note that the λ_n and u_n depend on ρ, but $G(z,w)$ does not.

We shall elaborate the above approach to the Green's function in the annulus case, $\Omega = A_{1,R}$, and with metric coming from interpreting $A_{1,R}$ as a cylinder. This metric is equivalent to the natural flat metric on the universal covering surface of $A_{1,R}$. Thus we consider the lift map $z \mapsto t = \log z$, by which $A_{1,R}$ gets identified with

$$C_R = \{t \in \mathbb{C} : 0 < \operatorname{Re} t < \log R\}/2\pi \mathrm{i}\mathbb{Z}.$$

This can be thought of as a cylinder, with boundary ∂C_R represented by the vertical lines $\{\operatorname{Re} t = 0\}$ and $\{\operatorname{Re} t = \log R\}$ and provided with the euclidean metric $d\sigma^2 = |dt|^2$.

On $A_{1,R}$ itself the metric is given by $\rho(z) = 1/|z|$ ($z \in A_{1,R}$). We shall keep some previous notations, for example (6.3), and sometimes use τ as a parameter in place of R. When working in C_R directly, the eigenvalue problem becomes (since $\rho = 1$ in C_R)

$$\begin{cases} -\Delta u = \lambda u & \text{in } C_R, \\ u = 0 & \text{on } \partial C_R. \end{cases}$$

This problem can be solved by standard separation of variables techniques. The eigenvalues then come with two indices, namely

$$\lambda_{mn} = \frac{m^2\pi^2}{(\log R)^2} + n^2 = n^2 - \tau^2 m^2,$$

for $m = 1, 2, \ldots,\ n = 0, 1, 2, \ldots$. Using also negative values of n one can list the eigenfunctions as follows (now deviating from a previous notational convention and using the variables x, y, $z = x + \mathrm{i}y$ on the universal covering surface).

$$u_{mn}(z) = \begin{cases} \dfrac{1}{\sqrt{\pi \log R}} \sin \dfrac{m\pi x}{\log R}, & (m \geq 1,\ n = 0), \\ \dfrac{2}{\sqrt{\pi \log R}} \sin \dfrac{m\pi x}{\log R} \cos ny, & (m \geq 1,\ n \geq 1), \\ \dfrac{2}{\sqrt{\pi \log R}} \sin \dfrac{m\pi x}{\log R} \sin ny, & (m \geq 1,\ n \leq -1). \end{cases}$$

On using exponentials in place of trigonometric functions the above gives, after simplifications and still working on the universal covering surface.

$$G^s(z, w) = 2\pi \sum_{m \geq 1, n \in \mathbb{Z}}^{\infty} \frac{u_{mn}(z) u_{mn}(w)}{\lambda_{mn}^s}$$

$$= \frac{1}{2 \log R} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{(e^{m\tau(x-u)} - e^{m\tau(x+u)}) e^{\mathrm{i}n(y-v)}}{(n^2 - m^2\tau^2)^s},$$

$$= \frac{1}{2 \log R} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{(e^{m\tau(x-u)} - e^{m\tau(x+u)}) e^{\mathrm{i}n(y-v)}}{|m\tau + n|^{2s}},$$

where $z = x + \mathrm{i}y$, $w = u + \mathrm{i}v$. To go on further we shall need Kronecker's second limit formula, contained in the following lemma.

Lemma 6.4. *Let $\tau \in \mathbb{C}$, $\operatorname{Im} \tau > 0$. Let $u, v \in \mathbb{R} \setminus \mathbb{Z}$ be fixed and define for $\operatorname{Re} s > 1$ the zeta function*

$$\zeta_\tau(s; u, v) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left(\frac{\operatorname{Im} \tau}{|m\tau + n|^2} \right)^s e^{2\pi \mathrm{i}(mu + nv)}.$$

Then the analytic continuation at $s = 1$ of $\zeta_\tau(s; u, v)$ is

$$\zeta_\tau(1; u, v) = 2\pi^2 v^2 \operatorname{Im} \tau - 2\pi \log \left| \frac{\vartheta_1(u - v\tau; \tau)}{\eta(\tau)} \right|,$$

where the theta function ϑ_1 is defined (cf. (2.17)) by

$$\vartheta_1(z) = \vartheta_1(z;\tau) = \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z;\tau) = \sum_{n\in\mathbb{Z}} e^{\mathrm{i}\pi(n+\frac{1}{2})^2\tau + 2\pi\mathrm{i}(n+\frac{1}{2})(z+\frac{1}{2})}$$

$$= -2\sum_{n=0}^{\infty}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2}\sin(2n+1)\pi z$$

$$= -2q^{\frac{1}{8}}\sin\pi z\prod_{n=1}^{\infty}(1-q^n)(1-2q^n\cos 2\pi z + q^{2n}),$$

and where η is the Dedekind eta function,

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n) = \Delta(\tau)^{\frac{1}{24}}.$$

We remark that for $u, v\in\mathbb{Z}$, the function $\zeta_\tau(1;u,v)$ reduces to the Epstein zeta function

$$\zeta_\tau(s) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(\frac{\operatorname{Im}\tau}{|m+n\tau|^2}\right)^s,$$

which is no longer analytic at $s=1$ but only meromorphic. Kronecker's first limit formula (below) gives the meromorphic continuation [55].

Lemma 6.5. *The Epstein zeta function $\zeta_\tau(s)$ has a meromorphic continuation to all $\mathbb{C}$ with a simple pole at $s=1$, given by*

$$\zeta_\tau(s) = \frac{\pi}{s-1} + 2\pi\left(\gamma - \ln(2\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2)\right) + \mathcal{O}(s-1)$$

where γ is the Euler constant. In particular $\zeta_\tau(0) = -1$.

Continuing the computation of the Green's function we have, using Lemma 6.4,

$$G(z,w) = \lim_{s\to 1} G^s(z,w)$$

$$= \lim_{s\to 1}\frac{\operatorname{Im}\tau}{2\pi}\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(\frac{e^{-(x-u)\operatorname{Im}\tau}e^{\mathrm{i}n(y-v)}}{|m\tau+n|^{2s}} - \frac{e^{-(x+u)\operatorname{Im}\tau}e^{\mathrm{i}n(y-v)}}{|m\tau+n|^{2s}}\right)$$

$$= \lim_{s\to 1}\frac{(\operatorname{Im}\tau)^{1-s}}{2\pi}\left(\zeta_\tau(s;-\frac{(x-u)\operatorname{Im}\tau}{2\pi},\frac{y-v}{2\pi}) - \zeta_\tau(s;-\frac{(x+u)\operatorname{Im}\tau}{2\pi},\frac{y-v}{2\pi})\right)$$

$$= -\log|\vartheta_1(-\frac{(x-u)\operatorname{Im}\tau}{2\pi} - \frac{(y-v)\tau}{2\pi})| + \log|\vartheta_1(\frac{-(x+u)\operatorname{Im}\tau}{2\pi} - \frac{(y-v)\tau}{2\pi})|$$

$$= -\log|\vartheta_1(\frac{(z-w)\tau}{2\pi\mathrm{i}})| + \log|\vartheta_1(\frac{(z+\bar{w})\tau}{2\pi\mathrm{i}})|$$

$$= -\log|\vartheta_1(\frac{z-w}{2\log R})| + \log|\vartheta_1(\frac{z+\bar{w}}{2\log R})|.$$

This was all on the universal covering. To get back to $A_{1,R}$ one simply substitutes $\log z$, $\log w$ for z, w:

$$G(z,w) = -\log|\vartheta_1(\frac{\log z - \log w}{2\log R})| + \log|\vartheta_1(\frac{\log z + \log\bar{w}}{2\log R})|.$$

A similar formula is given in [8]. Compare also (6.9).

6.5. **A remark on the modulus of a doubly connected domain.** The connection between the preceeding topics and the Ramanujan partial differential system (6.18) or the Chazy equation (6.20) can be explored in other directions. To give an example, we quote a result in [15] showing that the Bergman minimum integrals [4] completely determines the modulus of a doubly connected domain.

In any domain Ω, consider the problem of minimizing $\|f\|^2 = \int_\Omega |f|^2 dxdy$ among functions satisfying

$$\begin{cases} f(a) = \cdots = f^{(n-1)}(a) = 0, \\ f^{(n)}(a) = 1, \end{cases}$$

where $a \in \Omega$ is a fixed point and $n \geq 1$. Let $f_n = f_{n,a}$ be the unique minimizer and let $\lambda_n = \lambda_n(a) = \|f_{n,a}\|^2$ be the minimum value. Then it is easy to see that $\{f_n/\sqrt{\lambda_n} : n = 0, 1, \dots\}$ forms an orthonormal basis in the Bergman space, and hence that

$$K(z, w) = \sum_{n=0}^{\infty} \frac{f_n(z)\overline{f_n(w)}}{\lambda_n}. \tag{6.29}$$

Here the dependence on the point $a \in \Omega$ has disappeared, even though each individual f_n and λ_n depend on a.

Let $d\sigma = \sqrt{\pi K(z, z)}|dz|$ be the Bergman metric of Ω. It has Gaussian curvature $\kappa = \kappa_{\mathrm{Gauss}} < 0$, given in terms of the Bergman kernel and the Bergman minimum integrals by

$$\kappa(z) = -\frac{2}{\pi K(z, \bar{z})} \frac{\partial^2 \log K(z, \bar{z})}{\partial z \partial \bar{z}} = -2\frac{\lambda_0^2(z)}{\lambda_1(z)}.$$

The last expression follows from (6.29) (see [15] for details). Set also

$$\gamma(z) = \frac{1}{\pi K(z, \bar{z})} \frac{\partial^2 \log(-\kappa(z))}{\partial z \partial \bar{z}} = \frac{\lambda_0(z)\lambda_1(z)}{\lambda_2(z)} - 3\frac{\lambda_0^2(z)}{\lambda_1(z)}.$$

Both κ and γ are conformally invariant scalar functions.

Now we specialize to doubly connected domains. Taking in this subsection $\Omega = A_{r,1}$ $(0 < r < 1)$ as a model, the conformal invariance implies that $\kappa(z)$ and $\gamma(z)$ depend only on $|z|$ and that they are constant on $\partial A_{r,1}$ (the same constant on both boundary components). These constants, $\kappa_r = \kappa|_{\partial A_{r,1}}$ and $\gamma_R = \gamma|_{\partial A_{r,1}}$, depend on the modulus R. It is shown in [15] that

$$\frac{\frac{2}{3}\gamma_r + 2\kappa_r + 8}{(-\kappa_r - 4)^{\frac{3}{2}}} = \frac{6g_3 - 14\alpha g_2 + 120\alpha^3}{(g_2 - 12\alpha^2)^{\frac{3}{2}}}, \tag{6.30}$$

where

$$\tau = \frac{\log r}{\mathrm{i}\pi}, \quad \alpha = -\frac{1}{12}E_2(\tau),$$

$$g_2 = \frac{1}{12}E_4(\tau), \quad g_3 = -\frac{1}{216}E_6(\tau).$$

The quantity on the right-hand side of (6.30) depends on r or, equivalently, on τ. Denote it by $f(\tau)$:

$$f(\tau) = \frac{6g_3 - 14\alpha g_2 + 120\alpha^3}{(g_2 - 12\alpha^2)^{\frac{3}{2}}}.$$

It is shown in [15] that $f(\tau)$ is monotone as a function of r. Thus $f(\tau)$ determines the modulus. By using (6.19) and Chazy equation, we also find the explicit expressions

$$
\begin{aligned}
f(\tau) =& \frac{3E_2{}'' + 2E_2E_2{}'}{E_2'^{\frac{3}{2}}}, \\
f'(\tau) =& \frac{-9E_2''^2 + 4E_2E_2{}'E_2{}'' - 5E_2'^3}{2E_2'^{\frac{5}{2}}},
\end{aligned}
\tag{6.31}
$$

where the prime denotes differentiation $\frac{1}{2\pi \mathrm{i}}\frac{d}{d\tau}$, as before.

Thus the Eisenstein series E_2, which is an affine connection, is useful to study the modulus of a doubly connected domain. In the next section we will study affine, and projective, connections in more generality.

7. Projective and affine connections

7.1. **Definitions.** We shall review some aspects of the theory of affine and projective connections. General references here are [51], [22], [23].

Let $z = f(t)$, where z, t are complex variables and f holomorphic. We introduce the nonlinear differential operators

$$
\begin{aligned}
\{z,t\}_0 &= \log\frac{dz}{dt} = -2\log\frac{1}{\sqrt{f'}}, \\
\{z,t\}_1 &= \frac{d}{dt}\log\frac{dz}{dt} = -2\sqrt{f'}\,(\frac{1}{\sqrt{f'}})', \\
\{z,t\}_2 &= \frac{d^2}{dt^2}\log\frac{dz}{dt} - \frac{1}{2}(\frac{d}{dt}\log\frac{dz}{dt})^2 = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2 = -2\sqrt{f'}\,(\frac{1}{\sqrt{f'}})''.
\end{aligned}
$$

The latter is the *schwarzian derivative* of f. For $\{z,t\}_0$ there is an additive indeterminancy of $2\pi\mathrm{i}$, so actually only its real part is completely well-defined. With s an intermediate complex variable, so that $z = g(s)$, $s = h(t)$ for some holomorphic functions g and h, we have

$$
\{z,t\}_k(dt)^k = \{z,s\}_k(ds)^k + \{s,t\}_k(dt)^k \quad (k = 0,1,2),
\tag{7.1}
$$

see [51]. In addition, one notes that

$$
\begin{aligned}
\{z,t\}_0 = 0 \quad &\text{iff} \quad z = t + b \quad (f \text{ is a translation}), \\
\{z,t\}_1 = 0 \quad &\text{iff} \quad z = at + b \quad (f \text{ is an affine map}), \\
\{z,t\}_2 = 0 \quad &\text{iff} \quad z = \frac{at+b}{ct+d} \quad (f \text{ is a Möbius transformation}).
\end{aligned}
\tag{7.2}
$$

The above differential operators can also be obtained as limits of derivatives of logarithms of difference quotients of the holomorphic function f. To be precise, let z, w and t, u be related by $z = f(t)$, $w = f(u)$. Then

$$
\begin{cases}
\{z,t\}_0 = \lim_{u\to t}\log\frac{w-z}{u-t}, \\
\{z,t\}_1 = 2\lim_{u\to t}\frac{\partial}{\partial t}\log\frac{w-z}{u-t}, \\
\{z,t\}_2 = 6\lim_{u\to t}\frac{\partial^2}{\partial t\partial u}\log\frac{w-z}{u-t}.
\end{cases}
\tag{7.3}
$$

Alternatively, for the last case, one may introduce a polarized version of the Schwarzian derivative by

$$
[z,w;t,u] = 6\frac{\partial^2}{\partial t\partial u}\log\frac{w-z}{u-t}.
$$

Then

$$
\{z,t\}_2 = [z,z;t,t].
$$

Now, let M be a Riemann surface. An *affine structure* on M is a choice of holomorphic atlas such that all transition functions between coordinates within this atlas are affine maps, i.e., such that for any two coordinates z and $\tilde{z}$ for which the domains of definition overlap, the relation $\{\tilde{z}, z\}_1 = 0$ holds. This is a quite demanding requirement, and not every Riemann surface admits an affine structure. Of the compact Riemann surfaces only those of genus one do.

Less demanding is a *projective structure*. It is a choice of holomorphic atlas on M such that all coordinate changes are Möbius transformations, i.e., such that $\{\tilde{z}, z\}_2 = 0$ whenever z and $\tilde{z}$ are coordinates with overlapping domains of definition. By the uniformization theorem every Riemann surface admits a projective structure.

Let $\phi(z)(dz)^m = \tilde{\phi}(\tilde{z})(d\tilde{z})^m$ be local expressions for a differential of order m on M. Thus the coefficients transform under holomorphic change of coordinates according to $\phi(z) = \tilde{\phi}(\tilde{z})(\frac{d\tilde{z}}{dz})^m$. We shall allow m not only to be an integer, but also a half-integer. This requires a consistent choice of signs in the multipliers $(\frac{d\tilde{z}}{dz})^m$, i.e., requires a choice of a square root of the canonical bundle (which has transition functions $\frac{d\tilde{z}}{dz}$). Such a square root always exists, in fact there are in general several inequivalent choices. We refer to [22], [26], [28] for details about this. Recall however (Section 2) that in case M is the Schottky double of a plane domain then there is a canonical choice of square root of the canonical bundle, namely obtained by choosing the square root of the Schwarz function to be $1/T(z)$, where $T(z)$ the tangent vector of the oriented boundary. This is the choice of square root which will be used in the sequel.

An *affine connection* on M is an object which is represented by local differentials $r(z)dz$, $\tilde{r}(\tilde{z})d\tilde{z}$,... (one in each coordinate variable) glued together according to the rule

$$\tilde{r}(\tilde{z})d\tilde{z} = r(z)dz - \{\tilde{z}, z\}_1 \, dz. \tag{7.4}$$

In the presence of an affine connection it is possible to define, for every $k \in \frac{1}{2}\mathbb{Z}$, a *covariant derivative* ∇_k from k:th order differentials to $(k+1)$:th order differentials by $\phi(dz)^k \mapsto (\nabla_k \phi)(dz)^{k+1}$, where

$$\nabla_k \phi = \frac{\partial \phi}{\partial z} - kr\phi. \tag{7.5}$$

The covariance means that if $\phi(dz)^k = \tilde{\phi}(d\tilde{z})^k$ then $\nabla_k \phi(dz)^{k+1} = \tilde{\nabla}_k \tilde{\phi}(d\tilde{z})^{k+1}$.

Similarly, a *projective connection* on M consists of local quadratic differentials $q(z)(dz)^2$, $\tilde{q}(\tilde{z})(d\tilde{z})^2$, ..., glued together according to

$$\tilde{q}(\tilde{z})(d\tilde{z})^2 = q(z)(dz)^2 - \{\tilde{z}, z\}_2 \, (dz)^2. \tag{7.6}$$

From (7.1) it follows that this law (as well as (7.4)) is associative. In general, we do not require r and q to be holomorphic, although our main interest is in the holomorphic (or meromorphic) case.

In addition to the above one may consider also 0-connections, quantities defined up to multiples of $2\pi\mathrm{i}$ and which transform according to

$$\tilde{p}(\tilde{z}) = p(z) - \{\tilde{z}, z\}_0. \tag{7.7}$$

This means exactly that $e^{p(z)}$ is well-defined and transforms as differential of order one.

A projective connection is less powerful than an affine one, but it still allows for certain covariant derivatives: for each $m = 0, 1, 2, \dots$ there is, in the presence of a projective connection q, a well-defined linear differential operator Λ_m taking differentials of order $\frac{1-m}{2}$ to differentials of order $\frac{1+m}{2}$: $\phi(dz)^{\frac{1-m}{2}} \mapsto (\Lambda_m \phi)(dz)^{\frac{1+m}{2}}$.

The first few look, in a local coordinate z,

$$\Lambda_0(\phi) = \phi,$$
$$\Lambda_1(\phi) = \frac{\partial \phi}{\partial z},$$
$$\Lambda_2(\phi) = \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{2} q\phi,$$
$$\Lambda_3(\phi) = \frac{\partial^3 \phi}{\partial z^3} + 2q\frac{\partial \phi}{\partial z} + \frac{\partial q}{\partial z}\phi,$$
$$\Lambda_4(\phi) = \frac{\partial^4 \phi}{\partial z^4} + 10q\frac{\partial^2 \phi}{\partial z^2} + 10\frac{\partial q}{\partial z}\frac{\partial \phi}{\partial z} + (9q^2 + 3\frac{\partial^2 q}{\partial z^2})\phi.$$

The covariant derivative Λ_m will be called the m:th order *Bol operator*, for reasons to be explained further on.

Any affine connection r gives rise to a projective connection q by

$$q = \frac{\partial r}{\partial z} - \frac{1}{2} r^2. \tag{7.8}$$

This q is sometimes called the "curvature" of r (cf. [12]). In fact, its definition is analogous to that of the curvature form in ordinary differential geometry, see [20]. Slightly more generally than (7.8), any two affine connections $r_j(z)dz$, $j = 1, 2$, combine into a projective connection by

$$q = \frac{1}{2}(\frac{\partial r_1}{\partial z} + \frac{\partial r_2}{\partial z} - r_1 r_2).$$

Not every projective connection is the curvature of an affine connection. In the case that a projective connection comes from an affine connection, as in (7.8), the corresponding covariant derivatives are related by

$$\Lambda_1 = \nabla_0, \quad \Lambda_2 = \nabla_{\frac{1}{2}} \nabla_{-\frac{1}{2}}, \quad \Lambda_3 = \nabla_1 \nabla_0 \nabla_{-1}, \quad \text{etc.} \tag{7.9}$$

See [23] for the proof.

The difference between two projective connections is a quadratic differential, and the difference between two affine connections is an ordinary differential. Hence, if $q(z)(dz)^2$ is one projective connection the most general one is $q(z)(dz)^2$ plus a quadratic differential. Similarly for affine connections.

Now, a central fact is that there is a one-to-one correspondence between holomorphic projective connections and projective structures: given a projective connection, represented by a holomorphic function $q(z)$ in a general coordinate z, a projective coordinate t is obtained by solving the differential equation

$$\{t, z\}_2 = q(z). \tag{7.10}$$

It follows from (7.1), (7.2) that the set of coordinates obtained in this way are related by Möbius transformations. In the other direction, given a projective structure, a projective connection is obtained by simply setting $q(t) = 0$ in any projective coordinate t.

One way to solve (7.10), when $q(z)$ is holomorphic, is to consider the second order linear differential equation

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{2} q(z) u = 0, \tag{7.11}$$

i.e., $\Lambda_2(u) = 0$, for u considered as a differential of order minus one-half. The solutions to (7.10) are exactly the functions

$$t(z) = \frac{u_2(z)}{u_1(z)},$$

where u_1, u_2 are linearly independent solutions of (7.11).

In terms of a projective structure, the meaning of the covariant derivatives Λ_m, for holomorphic q, is that in any projective coordinate t corresponding to the connection, Λ_m simply is the k:th order derivative with respect to t: $\Lambda_m(\phi(t)(dt)^{\frac{1-m}{2}}) = \frac{\partial^m \phi(t)}{\partial t^m}(dt)^{\frac{1+m}{2}}$. The fact that the right member here is covariant under Möbius transformation is sometimes called "Bol's lemma" [6], [23]. The precise statement of this lemma is as follows.

Lemma 7.1. [6] *Let* $z = f(t) = \frac{at+b}{ct+d}$, $ad - bc = 1$, $\lambda(t) = (ct+d)^{-1}$, *so that* $f'(t) = \lambda(t)^2$. *Then, for any smooth function* $F(z)$ *and any positive integer* m,

$$\frac{\partial^m}{\partial t^m}(F(f(t))\lambda(t)^{1-m}) = \frac{\partial^m F}{\partial z^m}(f(t))\lambda(t)^{1+m}.$$

For $m = 1$ this is the ordinary chain rule, holding for any change of coordinate f, whereas for $m \geq 2$ the formula holds only for Möbius transformations.

In any projective coordinate t, a natural fundamental set (basis) of solutions of the equation $\Lambda_m u = 0$ is $\{1, t, \dots, t^{m-1}\}$. Considering these functions as $\frac{1-m}{2}$-forms (namely $(dt)^{\frac{1-m}{2}}$, $t(dt)^{\frac{1-m}{2}}$ etc.) and turning to a general coordinate z, this basis transforms into $\{u_1^{m-1}, u_1^{m-2}u_2, \dots, u_2^{m-1}\}$, where $u_1 = \sqrt{\frac{dz}{dt}}$, $u_2 = t\sqrt{\frac{dz}{dt}}$ are $-\frac{1}{2}$-forms. The transformation property of Λ_m then shows that $\{u_1^{m-1}, u_1^{m-2}u_2, \dots, u_2^{m-1}\}$ is a fundamental solution set for $\Lambda_m u = 0$ when considered as a differential equation in the z variable. It follows that the operator Λ_m agrees with the $(m-1)$-fold symmetric product of Λ_2: $\Lambda_m = S^{m-1}(\Lambda_2)$.

In summary, the Bol operators Λ_m, $m \geq 2$, are all generated by Λ_2, and their solutions are generated by two solutions of $\Lambda_2 u = 0$. In the last section of the paper we shall discuss a method, via a prepotential, of finding all solutions of $\Lambda_2 u = 0$ from a single one.

Turning to affine connections there are analogous statements as for projective connections: there is a one-to-one correspondence between affine structures and holomorphic affine connections, the correspondence between an affine (or "flat") coordinate τ and an affine connection $r(z)$ given in a general coordinate z being

$$\{\tau, z\}_1 = r(z). \tag{7.12}$$

This equation can be directly integrated as

$$\tau(z) = \int^z \exp(\int^w r(\zeta)d\zeta)dw.$$

One may also think of (7.12) as

$$\nabla_1\nabla_0\,\tau = 0,$$

or $\nabla_1(d\tau) = 0$, cf. [12]. With τ used as coordinate, $\nabla_m(\phi) = \frac{\partial\phi}{\partial\tau}$ for any m-differential $\phi(\tau)(d\tau)^m$.

Along with Λ_m and ∇_m it is natural to introduce their conjugates $\bar{L}_m$ and $\overline{\nabla}_m$, defined by replacing $\frac{\partial}{\partial z}$ by $\frac{\partial}{\partial \bar{z}}$ and q, r by $\bar{q}$, $\bar{r}$.

Even more powerful than an affine connection is a hermitean metric, which we in the present section prefer to write in either of the following two ways:

$$d\sigma = \frac{|dz|}{\omega(z)} = e^{p(z)}|dz|. \tag{7.13}$$

Here $\omega > 0$ transforms as the coefficient of a form of bidegree $(-\frac{1}{2}, -\frac{1}{2})$ (i.e., $\omega(dz)^{-1/2}(d\bar{z})^{-1/2}$ is invariant), whereas $p(z) = -\log\omega(z)$, in analogy with (7.4), (7.6), transforms as the real part of a 0-connection:

$$\tilde{p}(\tilde{z}) = p(z) - \operatorname{Re}\{\tilde{z}, z\}_0. \tag{7.14}$$

Any hermitean metric gives rise to a, not necessarily holomorphic, affine connection by

$$r = -2\frac{\partial}{\partial z}\log\omega = 2\frac{\partial p}{\partial z}. \tag{7.15}$$

This relationship says that the covariant derivative of the metric vanishes: $\nabla_{\frac{1}{2}}(1/\omega) = \overline{\nabla}_{\frac{1}{2}}(1/\omega) = 0$, or, which turns out to be the same,

$$\nabla_{-\frac{1}{2}}\omega = \overline{\nabla}_{-\frac{1}{2}}\omega = 0. \tag{7.16}$$

Clearly (7.9), (7.16) imply that

$$\Lambda_2\omega = 0, \tag{7.17}$$

and also that $\bar{\Lambda}_2\omega = 0$. Moreover, it is easy to check that $\Lambda_m\omega^{m-1} = \bar{\Lambda}_m\omega^{m-1} = 0$ for any $m \geq 2$.

The Gaussian curvature of the metric (7.13) is

$$\kappa_{\text{Gauss}} = -e^{-2p}\Delta p = 4(\omega\frac{\partial^2\omega}{\partial z\partial\bar{z}} - |\frac{\partial\omega}{\partial z}|^2), \tag{7.18}$$

which is a real and scalar quantity. In view of (7.15), it follows that r is holomorphic if and only if $\kappa_{\text{Gauss}} = 0$. The corresponding projective connection is

$$q = \frac{\partial r}{\partial z} - \frac{1}{2}r^2 = 2(\frac{\partial^2 p}{\partial z^2} - (\frac{\partial p}{\partial z})^2) = -2\,\frac{\partial^2\omega/\partial z^2}{\omega}. \tag{7.19}$$

From (7.18), (7.19) follows $\frac{\partial\kappa}{\partial z} = -2\omega^2\frac{\partial q}{\partial\bar{z}}$ or, since κ is real,

$$d\kappa_{\text{Gauss}} = -2\omega^2(\frac{\partial q}{\partial\bar{z}}dz + \frac{\partial\bar{q}}{\partial z}d\bar{z}).$$

Cf. [23]. It follows that q is holomorphic if and only if the curvature κ_{Gauss} is constant. When this is the case there is a corresponding projective structure, and working in any projective coordinate t, equation (7.17) and its conjugate can be directly integrated to give $\omega(t) = at\bar{t} + bt + \bar{b}\bar{t} + c$, a, c real, b complex. The value of the curvature comes out to be $\kappa_{\text{Gauss}} = 4(ac - |b|^2)$. It also follows that if Ω is complete for the metric, meaning roughly that $\omega = 0$ on $\partial\Omega$, then in any projective coordinate $\partial\Omega$ is a circle or a straight line.

Being complete and having constant negative curvature characterizes the Poincaré metric up to a constant factor. Starting from the domain, first the Poincaré metric (normalized so that $\kappa_{\text{Gauss}} = -4$), then the projective connection $q(z)$ and finally projective coordinates, can be obtained directly by solving differential equations in Ω. In fact, $p(z)$ satisfies by (7.18) a Liouville equation and blows up on the boundary:

$$\begin{cases}\Delta p = 4e^{2p} & \text{in } \Omega,\\ p = +\infty & \text{on } \partial\Omega.\end{cases}$$

It can be shown that this boundary value problem, when properly formulated, has a unique solution p (see, e.g., [1], Ch.1). From this solution, q is obtained via (7.19), and finally projective coordinates t are gotten by solving (7.10), more precisely by taking the quotient between two solutions of (7.11). The above indicates one of the early attempts to solve the uniformization theorem. It was proposed by H. A. Schwarz and later brought to an end by Picard, Poincaré and Bieberbach; see [44], [5].

This approach has recently gained new interest in some versions of string theory, in which the above Schwarzian derivative $q(z) = \{t, z\}_2$ of the lift map t to the universal covering surface takes the role of being the energy-momentum tensor, which hence is a projective connection. See for example [45], [58].

7.2. **Examples of projective connections.** If Ω is a plane domain bounded by finitely many analytic curves, several projective structures and connections can be naturally associated to Ω.

Example 7.1. The trivial one, $q(z) = 0$, i.e., with the coordinate variable z in $\mathbb{C}$ as a projective coordinate.

Example 7.2. Let $t : \Omega \to \Omega_{\mathrm{circ}}$ be a conformal map of Ω onto a domain Ω_{circ} bounded by circles. This map is not uniquely determined, but any two such maps are related by a Möbius transformation. Hence it defines a unique projective structure for which t is a projective coordinate. This projective structure on Ω extends to a projective structure on the Schottky double $\hat{\Omega}$, as will be seen more exactly in the next section. The associated connection coefficient is obtained from (7.10).

Example 7.3. The universal covering surface of Ω is conformally equivalent to the unit disk. Let $\pi : \mathbb{D} \to \Omega$ be a universal covering map. The inverse of π is multivalued, unless Ω is simply connected, but the different branches of π^{-1} are related by Möbius transformations. Hence a unique projective structure on Ω is obtained by using local branches of π^{-1} as projective coordinates. The connection coefficients $q(z)$ are related to the (branches of the) multivalued liftings π^{-1} by (7.10), that is

$$\{\pi^{-1}(z), z\}_2 = q(z).$$

Example 7.4. Let $\hat{\Omega}$ be the Schottky double of Ω and let $\pi : U \to \hat{\Omega}$ be a universal covering map for $\hat{\Omega}$. This U can be taken to be either the Riemann sphere (if Ω is simply connected), the complex plane (if Ω is doubly connected) or the unit disk (if Ω has at least three boundary components). In any case, $\hat{\Omega}$, and hence Ω, is provided with a unique projective structure by using local inverses of the uniformization map as projective coordinates. If Ω is doubly connected, it even gets an affine structure.

In general, the projective structures in the above examples are all different. For example, if Ω is the annulus $A_{1,R}$, then by straight-forward computations one finds that the projective connections are given by

$$q(z) = \frac{a}{z^2},$$

where $a = 0$ in the cases 1) and 2) above, $a = \frac{1}{4}$ in case 3) and $a = \frac{1}{4}(1 + \frac{\pi^2}{(\log R)^2})$ in case 4). If Ω is replaced by a noncircular doubly connected domain also the cases 1) and 2) will be unequal.

Example 7.5. Connections also come up from Taylor coefficients of regular parts of certain domain functions. For example, for the Taylor coefficients of the regular part of the complex Green's function, see Section 5.2, we have the following.

Proposition 7.2. *In the notation of Section 5.2, plus (3.4), the quantities*

$$p(\zeta) = -c_0(\zeta), \tag{7.20}$$

$$r(\zeta) = -2c_1(\zeta) = 2\frac{\partial p}{\partial z}, \tag{7.21}$$

$$q(\zeta) = -6(\frac{\partial c_1(\zeta)}{\partial \zeta} - 2c_2(\zeta)) = 6\pi\ell(\zeta, \zeta) \tag{7.22}$$

transform under conformal mapping as, respectively, the real part of a 0*-connection, an affine connection and a projective connection.*

Proof. The Green's function transforms under conformal mappings $f : \Omega \to \tilde{\Omega}$, $\tilde{z} = f(z)$, as $G(z, \zeta) = \tilde{G}(\tilde{z}, \tilde{\zeta})$, hence

$$\mathcal{H}(z, \zeta) = -\log \frac{\tilde{z} - \tilde{\zeta}}{z - \zeta} + \tilde{\mathcal{H}}(\tilde{z}, \tilde{\zeta}).$$

Now the assertions follow by easy computations, using the formulas (7.3):

$$c_0(\zeta) = \mathcal{H}(\zeta, \zeta) = H(\zeta, \zeta),$$

$$c_1(\zeta) = \frac{\partial \mathcal{H}(z, \zeta)}{\partial z}\Big|_{z=\zeta} = 2\frac{\partial H(z, \zeta)}{\partial z}\Big|_{z=\zeta} = \frac{\partial}{\partial \zeta} H(\zeta, \zeta) = \frac{\partial}{\partial \zeta} c_0(\zeta), \tag{7.23}$$

$$\frac{\partial c_1(\zeta)}{\partial \zeta} - 2c_2(\zeta) = \frac{\partial^2 \mathcal{H}(z, \zeta)}{\partial z \partial \zeta}\Big|_{z=\zeta} = 2\frac{\partial^2 H(z, \zeta)}{\partial z \partial \zeta}\Big|_{z=\zeta} = -\pi \ell(\zeta, \zeta).$$

□

We conclude from (7.22) that $d\sigma = e^{-c_0(\zeta)}|d\zeta|$ is a conformally invariant metric and that $r(z)$ is the associated affine connection. However, $q(z)$ in (7.22) is in general not the same as the projective connection associated to $r(z)$ by the general receipt (7.8), namely

$$Q(\zeta) = \frac{\partial r(\zeta)}{\partial \zeta} - \frac{1}{2} r(\zeta)^2 = -2\frac{\partial c_1(\zeta)}{\partial \zeta} - 2c_1(\zeta)^2.$$

In the multiply connected case $Q(\zeta)$ is not holomorphic, whereas $q(\zeta)$ in (7.22) is always holomorphic.

Example 7.6. Coefficients of linear differential equation in Ω transform in complicated ways under changes of coordinates, and in some cases exactly as connections. For example, it is well-known that a second order equation always can be written on the form (7.11), that is $u'' + \frac{1}{2}Q(z)u = 0$. Then $Q(z)$ works as a projective connection, and the corresponding projective structure uniformizes the equation. This differential equation will be further discussed in Section 12.

Example 7.7. Let us spell out the relevant quantities for the unit disk and upper half-plane provided with the Poincaré metric:

(i) Unit disk:

$$\omega(z) = 1 - |z|^2, \quad r(z) = \frac{2\bar{z}}{1 - |z|^2}, \quad q(z) = 0, \quad \kappa_{\text{Gauss}} = -4.$$

(ii) Upper half-plane:

$$\omega(z) = 2y, \quad r(z) = \frac{\mathrm{i}}{y}, \quad q(z) = 0, \quad \kappa_{\text{Gauss}} = -4.$$

Note that $r(z)$ is singular on the boundary, while $q(z)$ is not.

8. Connections on the Schottky double of a plane domain

8.1. **Generalities.** If the Riemann surface is the Schottky double of a plane domain Ω, then the connection coefficients can be described in terms of pairs of functions on Ω, in analogy with the previous description (2.5) of differentials of any half-integer order. Recall (2.1) that the transition function between the coordinate $\tilde{z} = \bar{z}$ on the back-side $\tilde{\Omega}$ and z on the front side Ω is given by the Schwarz function. Writing the unit tangent vector as $T(z) = \frac{dz}{ds} = e^{\mathrm{i}\theta}$, the curvature of $\partial\Omega$ is $\kappa = \frac{d\theta}{ds} = -\mathrm{i}T'(z)$, where the prime denotes the complex derivative $\frac{\partial}{\partial z}$ for the analytically extended $T(z)$. Then, along $\partial\Omega$,

$$\{S(z), z\}_0 = -2\log T(z) = -2\mathrm{i}\theta,$$

$$\{S(z),z\}_1 dz = -2\frac{T'(z)}{T(z)}dz = -2\mathrm{i}d\theta = -2\mathrm{i}\kappa ds,$$

$$\{S(z),z\}_2 dz^2 = -2\frac{T''(z)}{T(z)}dz^2 = -2\mathrm{i}\frac{d\kappa}{ds}ds^2.$$

Therefore, a 0-connection on $\hat{\Omega}$ can be described as a pair of functions, $p_1(z)$ (for the front side) and $p_2(z)$ (for the backside), on Ω, satisfying on $\partial\Omega$ the matching condition

$$p_1(z) = \overline{p_2(z)} - 2\mathrm{i}\theta.$$

Similarly, an affine connection is represented by a pair $r_1(z)$, $r_2(z)$ satisfying the matching condition $r_1(z)dz = \overline{r_2(z)dz} - 2\mathrm{i}\kappa ds$, or

$$r_1(z)T(z) = \overline{r_2(z)T(z)} - 2\mathrm{i}\kappa \quad \text{on } \partial\Omega, \tag{8.1}$$

and a projective connection by a pair $q_1(z)$, $q_2(z)$ with

$$q_1(z)T(z)^2 = \overline{q_2(z)T(z)^2} - 2\mathrm{i}\frac{d\kappa}{ds} \quad \text{on } \partial\Omega.$$

Note that if z and $\tilde{z}$ are both projective coordinates then $q_1 = q_2 = 0$ and it follows that $\frac{d\kappa}{ds} = 0$ on $\partial\Omega$, i.e., that Ω is a circular domain.

We may notice that (8.1) is consistent with the general fact [51] that the sum of residues of a meromorphic affine connection on a compact Riemann surface of genus g equals $2(\mathsf{g}-1)$ (our definition of connection differs from that in [51] by a minus sign). In fact, in the case of the Schottky double the sum is, by (8.1),

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial\Omega} r_1(z)dz + \frac{1}{2\pi\mathrm{i}}\int_{-\partial\Omega}\overline{r_2(z)}d\bar{z} = -\frac{1}{\pi}\int_{\partial\Omega}\kappa ds = 2(\mathsf{g}-1)$$

since there is one outer component and g inner ones.

Naturally, one may be particularly interested in symmetric, or "real", connections, namely those for which $p_1 = p_2$, $r_1 = r_2$ or $q_1 = q_2$ (respectively). The above matching conditions then become, for the single representatives $p(z)$, $r(z)$, $q(z)$ on Ω,

$$\operatorname{Im} p(z) = -\theta \quad \text{on } \partial\Omega,$$

$$\operatorname{Im}(r(z)T(z)) = -\kappa \quad \text{on } \partial\Omega, \tag{8.2}$$

$$\operatorname{Im}(q(z)T(z)^2) = -\frac{d\kappa}{ds} \quad \text{on } \partial\Omega.$$

8.2. **Formulas for the curvature of a curve.** The differential parameter $\{z,w\}_1$ turns out to be a useful tool for summarizing various formulas for the curvature of a curve in the complex plane.

The definition of $\{z,w\}_1$, with $z = f(w)$ analytic, can be written in differential form as

$$\{z,w\}_1 dw = d\log\frac{dz}{dw} = d(\log dz - \log dw).$$

Hence

$$\operatorname{Im}(\{z,w\}_1 dw) = d\arg dz - d\arg dw = {}^*d(\log|dz| - \log|dw|).$$

To interpret this formula one should let w run along a curve Γ, say with parametrization $t \mapsto w(t)$, $t \in \mathbb{R}$. Then $z(t) = f(w(t))$ runs along $f(\Gamma)$ and one just replaces dz and dw by $z'(t)$ and $w'(t)$, respectively.

Denoting the curvature of Γ by κ_w and the curvature of $f(\Gamma)$ by κ_z we thus have the following formula.

Proposition 8.1. *Under a conformal mapping $f : z \mapsto w$, the curvature of a curve Γ and its image curve $f(\Gamma)$ are related by*

$$\mathrm{Im}\,(\{z, w\}_1 dw) = \kappa_z ds_z - \kappa_w ds_w. \tag{8.3}$$

Here $ds_z = |dz|$, $ds_w = |dw|$.

Example 8.1. (The curvature in terms of a real parameter.) With $\Gamma = \mathbb{R}$ and $z = f(w)$, so that $z = f(t)$ with $t = \mathrm{Re}\, w$ parametrizes $f(\Gamma)$, we have $\kappa_w = 0$, which gives the well-known formula

$$\kappa_z = \mathrm{Im}\,(\frac{d}{dw}(\log\frac{dz}{dw})\frac{dw}{ds_z}) + \kappa_w\frac{ds_w}{ds_z}$$

$$= \mathrm{Im}\,(\frac{f''(w)}{f'(w)}\frac{dw}{|f'(w)||dw|}) = \frac{1}{|f'(t)|}\mathrm{Im}\,\frac{f''(t)}{f'(t)}.$$

Example 8.2. (The curvature in terms of an angular parameter; essentially Study's formula [46], p.125.) With $\Gamma = \partial\mathbb{D}$, $w = e^{\mathrm{i}t}$, $dw = \mathrm{i}wdt$ we have,

$$\kappa_z = \mathrm{Im}\,(\{z, w\}_1\frac{dw}{ds_z}) + \kappa_w\frac{ds_w}{ds_z} = \mathrm{Im}\,(\frac{f''(w)}{f'(w)}\frac{\mathrm{i}wdt}{|f'(w)||dt|}) + \frac{1}{|f'(w)|}$$

$$= \frac{1}{|f'(w)|}(\mathrm{Re}\,\frac{wf''(w)}{f'(w)} + 1).$$

Example 8.3. (The curvature of a curve given as a level line of a harmonic function.) Let Γ be any level line of a harmonic function u in the z-plane. Assume that the curve is nonsingular, so that $\nabla u \neq 0$ on Γ. With v any harmonic conjugate of u, $w = u + iv = g(z)$ maps Γ into a vertical line in the w-plane. Thus $\kappa_w = 0$ and the curvature of Γ is obtained from

$$\kappa_z ds_z = \mathrm{Im}\,(\{z, w\}_1 dw) + \kappa_w ds_w = -\mathrm{Im}\,(\{w, z\}_1 dz)$$

$$= \mathrm{Im}\,(\frac{\partial}{\partial z}(\log\frac{\partial w}{\partial z})dz) = \mathrm{Im}\,(2\frac{\partial}{\partial z}\log(|2\frac{\partial u}{\partial z}|)dz)$$

$$= -\frac{\partial}{\partial x}\log|\nabla u|dy + \frac{\partial}{\partial y}\log|\nabla u|dx = -\frac{\partial}{\partial n}\log|\nabla u|ds_z.$$

We may may also extract from the above the formula (cf. [31])

$$\kappa_z = |g'(z)|\mathrm{Re}\,\frac{g''(z)}{g'(z)^2}.$$

In summary, for a level curve Γ of a harmonic function u the curvature is given by

$$\kappa_z = -\frac{\partial}{\partial n}\log|\nabla u| = -\frac{d}{dn}\log|\frac{du}{dn}| = -\frac{d^2u}{dn^2}/\frac{du}{dn}. \tag{8.4}$$

In the last expressions we used $\frac{d}{dn}$ in place of $\frac{\partial}{\partial n}$ because the latter becomes obscure when applied twice ($\frac{\partial}{\partial n}$ is not really a partial derivative since n is not a coordinate of a coordinate system; $\frac{d}{dn}$ should be interpreted as a directional derivative along the straight line in the normal direction).

In higher dimensions, (8.4) gives the mean curvature of a hypersurface given as the level surface of a harmonic function. This can easily be deduced from the more general (and well-known, cf. [20]) formula $\kappa_{\mathrm{mean}} = \mathrm{div}\,\frac{\nabla u}{|\nabla u|}$, for the mean curvature of a level surface of any smooth function u.

8.3. **Geodesics.** The equation for geodesic curves $z = z(t)$ for the metric (7.13) is

$$\frac{d^2z}{dt^2} + r(z)(\frac{dz}{dt})^2 = 0, \tag{8.5}$$

where $r(z)$ is the corresponding affine connection (7.15) and the parameter t measures arc-length with respect to the metric. This equation is the usual geodesic equation in differential geometry [20], just written in complex analytic language. The classical Christoffel symbols Γ^i_{jk} $(i, j, k = 1, 2)$ turn out to coincide with the components $\pm\mathrm{Re}\, r(z), \pm\mathrm{Im}\, r(z)$. An intuitive direct derivation of (8.5) goes as follows.

The tangent vector along the curve is $\frac{dz}{dt}$ and the geodesic equation is supposed to say that this propagates by parallel transport, i.e., has covariant derivative zero along the curve. Since a vector in one complex variable can be thought of as a differential of order minus one the relevant covariant derivative (7.5) will be $\nabla_{-1} = \frac{\partial}{\partial z} + r(z)$. This is the covariant version of $\frac{\partial}{\partial z}$, and the covariant version of $\frac{d}{dt}$ then is $\frac{\partial}{\partial t} + r(z)\frac{dz}{dt}$. Applying this to $\frac{dz}{dt}$ gives (8.5).

It is convenient to write (8.5) on the form

$$\frac{d}{dt}\log\frac{dz}{dt} + r(z)\frac{dz}{dt} = 0 \tag{8.6}$$

and to decompose it into real and imaginary parts. The real part just contains internal information about how the curve is parametrized, namely saying that t measures arc-length with respect to the metric. The information about the geometry of the curve (that it is a geodesic) is entirely contained in the imaginary part,

$$\frac{d}{dt}\arg\frac{dz}{dt} + \mathrm{Im}\,(r(z)\frac{dz}{dt}) = 0. \tag{8.7}$$

In equation (8.7), t can be taken to be any parameter, for example euclidean arc-length s. Then the first term equals the euclidean curvature of the geodesic and $\frac{dz}{ds}$ is the unit tangent vector. Thus we have

Proposition 8.2. *The curvature $\kappa = \kappa(z, \frac{dz}{ds})$ of the geodesic passing through a point z and having direction $\frac{dz}{ds}$ (a unit vector) is given by*

$$\kappa(z, \frac{dz}{ds}) = -\mathrm{Im}\,(r(z)\frac{dz}{ds}) = -\mathrm{Im}\,(2\frac{\partial p}{\partial z}\frac{dz}{ds}) = -\frac{\partial p}{\partial n}, \tag{8.8}$$

where $\frac{\partial p}{\partial n}$ denotes the derivative in the rightward normal direction of the curve. In particular, the sharp bound

$$|\kappa(z, \frac{dz}{ds})| \le |r(z)| \tag{8.9}$$

holds for all geodesics passing through z, and equality is attained if and only $\frac{dz}{ds}$ is tangent to the level line of p at z.

The above estimate (8.9) can be combined with geometric estimates of $r(z)$. For example, for the coefficients of the Green's function, as in Section 5 and Example 7.5, we have by (7.20), (7.21) that $r(z) = -2c_1(z)$ is the affine connection for the metric $d\sigma = e^{-c_0}|dz|$. For $c_1(z)$ we have the estimate in Lemma 5.3, so the above proposition shows that

$$|r(z)| \le \frac{2}{d(z, \partial\Omega)}$$

for this connection. When Ω is simply connected the metric in question is the Poincaré metric. Combining with (8.9) we therefore have the following.

Corollary 8.3. *For a simply connected domain Ω provided with its Poincaré metric, the curvature for any geodesics through a point z is subject to the estimate*

$$|\kappa(z, \frac{dz}{ds})| \leq \frac{2}{d(z, \partial\Omega)}.$$

It is allowed here that $\Omega \subset \mathbb{P}$ contains the point of infinity, and at least in this generality the corollary is sharp. Indeed, with $\Omega = \mathbb{P} \setminus \overline{\mathbb{D}(0, \epsilon)}$, $\epsilon > 0$ small, the circle with center at any finite point $c \in \Omega$, and having radius $|c|$, is almost a geodesic for the Poincaré metric in Ω. The curvature for this circle is $1/|c|$, hence the bound in the corollary is essentially attained at the point $2c$ on the circle.

8.4. **Some applications in physics.** Projective and affine connections come up naturally in both classical and modern physics. As to modern physics, e.g., conformal field theory and string theory, we have already mentioned the example with the energy-momentum tensor as a projective connection. For further examples, see [12], [58] and references therein.

For classical physics, we shall briefly mention one example from vortex dynamics. Consider in Ω an incompressible inviscid fluid which is irrotational except for a point vortex of unit strength at a point z_0. If Ω is simply connected this makes the flow uniquely determined. The stream function ψ will, at each instant of time, coincide with the Green's function, $\psi(z) = G(z, z_0)$. However, the flow will not be stationary because the vortex will move with the speed obtained by subtracting off, from the general flow, the rotationally symmetric singular part corresponding to $-\log|z - z_0|$ in the stream function. Thus in fact $\psi = \psi(z, t) = G(z, z_0(t))$ and one deduces, in the notation of Example 7.5, that the vortex moves along a level line of $c_0(z)$ and that the velocity is given by

$$\frac{dz_0(t)}{dt} = -\mathrm{i}\overline{c_1(z_0(t))} = -\mathrm{i}\frac{\partial c_0}{\partial \bar{z}}(z_0(t)). \tag{8.10}$$

This means that $c_0(z)$ is a kind of stream function for the vortex motion, called the *Routh stream function*, [37], [43]. It also follows that the vortex motion is a hamiltonian motion, determined by the symplectic form $dx \wedge dy$ (area 2-form) and hamiltonian function $H = \frac{1}{2}c_0$. Note that (8.10) is a special case of (5.9).

If Ω is multiply connected one need to prescribe the circulations γ_j around the holes to make the flow uniquely determined. These circulations will be preserved in time (Kelvin's theorem) and everything will be as in the simply connected case but with the ordinary Green's function replaced by the hydrodynamic Green's function $G_\gamma(z, z_0)$, to be discussed in Section 9.

A related physical application comes from electrostatics. Think of the complement $K = \mathbb{C} \setminus \Omega$ as a perfect conductor and let it be grounded, so that its potential is zero. Then consider a unit charge located at $z_0 \in \Omega$. This will induce charges of the opposite sign on ∂K, namely distributed so that the density with respect to arc-length is given by the normal derivative $\frac{\partial G(\cdot, z_0)}{\partial n_z}$. These charges will exert a force on the charge at z_0, and this force is

$$F(z_0) = -\frac{1}{2\pi}\overline{c_1(z_0(t))}.$$

If Ω is multiply connected and the components of K are not grounded to a common zero, but are isolated from each other, then the hydrodynamic Green function shall be used in place of the ordinary one, with the γ_j proportional to the respective total charges isolated on the different components of K.

9. On Neumann functions and the hydrodynamic Green's function

9.1. **Definitions.** Recall that the Bergman kernel and the Schiffer kernel are given in terms of the ordinary Green's function by (3.1), (3.2). Decomposing the Green's function as in (5.5) we also have, for the "ℓ-kernel" (3.4),

$$\ell(z,\zeta)=\frac{2}{\pi}\frac{\partial^2 H(z,\zeta)}{\partial z\partial\zeta}.$$

The reduced Bergman kernel (see Proposition 9.2) and its adjoint have several similar representations in terms of Neumann type functions and hydrodynamic Green's functions. Here we shall briefly review these matters. See [49], [51] for more details.

By a *Neumann function* we mean a domain function $N_a(z,\zeta)$ with a logarithmic singularity at a given point $\zeta\in\Omega$ and satisfying Neumann boundary data given by a boundary function a subject to

$$\int_{\partial\Omega} a ds = 2\pi. \tag{9.1}$$

The requirements on $N_a(z,\zeta)$ are, more exactly,

$$N_a(z,\zeta)=-\log|z-\zeta|+\text{harmonic}\quad\text{in }\Omega, \tag{9.2}$$

$$-\frac{\partial N_a(\cdot,\zeta)}{\partial n}=a\quad\text{on }\partial\Omega, \tag{9.3}$$

$$\int_{\partial\Omega} N_a(\cdot,\zeta) a ds = 0. \tag{9.4}$$

The final condition (9.4) can also be written

$$\int_{\partial\Omega} N_a(\cdot,\zeta)^* dN_a(\cdot,z)=0,$$

and is a normalization which guarantees that N_a is symmetric:

$$N_a(z,\zeta)=N_a(\zeta,z).$$

A *hydrodynamic Green's function* (or "modified Green's function") [37], [19], [9] is defined in terms of $\mathsf{g}+1$ prescribed circulations, $\gamma_0,\dots\gamma_{\mathsf{g}}$ subject to the consistency condition

$$\gamma_0+\dots+\gamma_{\mathsf{g}}=2\pi. \tag{9.5}$$

Let $\gamma=(\gamma_0,\dots\gamma_{\mathsf{g}})$ denote the entire vector of periods. The defining properties of the hydrodynamic Green function, denoted $G_\gamma(z,\zeta)$, are

$$G_\gamma(z,\zeta)=-\log|z-\zeta|+\text{harmonic}\quad\text{in }\Omega, \tag{9.6}$$

$$G_\gamma(z,\zeta)=b_j(\zeta)\quad\text{for }z\in\Gamma_j, \tag{9.7}$$

$$-\int_{\Gamma_j}\frac{\partial G_\gamma(z,\zeta)}{\partial n}ds(z)=\gamma_j\quad(j=0,\dots,\mathsf{g}), \tag{9.8}$$

$$\sum_{j=0}^{\mathsf{g}}\gamma_j\beta_j(\zeta)=0. \tag{9.9}$$

Here $b_j(\zeta)$ denote "floating constants" (they cannot be preassigned), i.e., (9.7) really means

$$dG_\gamma(\cdot,\zeta)=0\quad\text{along }\partial\Omega.$$

Condition (9.9) is a normalization which can be written

$$\int_{\partial\Omega} G_\gamma(\cdot,\zeta)\,{}^*dG_\gamma(\cdot,z)=0$$

and which guarantees the symmetry,

$$G_\gamma(z,\zeta)=G_\gamma(\zeta,z).$$

The hydrodynamic Green's function can be constructed from the ordinary Green's function by

$$G_\gamma(z,\zeta)=G(z,\zeta)+\sum_{k,j=0}^{\mathtt{g}} c_{kj}u_k(z)u_j(\zeta),$$

where u_k, $k=0,\dots,\mathtt{g}$ are the harmonic measures. With the above 'Ansatz', the requirements (9.6) and (9.7) are automatically satisfied, and (9.8), (9.9) give a system of equations which determine the coefficients c_{kj} uniquely. The matrix (c_{kj}) is symmetric and positive semidefinite.

9.2. **Reproducing kernels for Dirichlet and Bergman spaces.** The Neumann and (hydrodynamic) Green's functions have logarithmic singularities, hence cannot themselves be reproducing kernels for any Hilbert spaces of harmonic functions. However, the singularities disappear when subtracting them, and also after application of the the differential operator $\frac{\partial^2}{\partial z\partial\bar\zeta}$. In these cases we do obtain reproducing kernels for important spaces. Below we elaborate on these matters, slightly extending the analysis in [4] and [51].

Let $D(u,v)$ denote the Dirichlet inner product:

$$D(u,v)=\int_\Omega \nabla u\cdot\nabla v\,dxdy=\int_\Omega du\wedge^* dv,$$

let a be boundary data as above, satisfying (9.1), and define the period vector γ by

$$\gamma_k=\int_{\Gamma_k} ads. \tag{9.10}$$

Then (9.5) holds, by (9.1). Define $H(\Omega)$ to be the Hilbert space of all harmonic functions u in Ω which satisfy $D(u,u)<\infty$ and

$$\int_{\partial\Omega} u\,ads=0, \tag{9.11}$$

and let $H_e(\Omega)$ be the subspace consisting of those functions which in addition satisfy

$$\int_{\Gamma_j}\frac{\partial u}{\partial n}ds=0\quad (j=0,\dots,\mathtt{g}). \tag{9.12}$$

Note that (9.11) just fixes the additive constant in u which the inner product leaves free. Except for this additive constant $H(\Omega)$ does not depend on a. The following is a slight extension of results in Ch.V:3 of [4].

Proposition 9.1. *The reproducing kernels for $H(\Omega)$ and $H_e(\Omega)$ are, respectively,*

$$k(z,\zeta)=\frac{1}{2\pi}(N_a(z,\zeta)-G(z,\zeta)),$$

$$k_e(z,\zeta)=\frac{1}{2\pi}(N_a(z,\zeta)-G_\gamma(z,\zeta)).$$

In other words, $k(\cdot,\zeta)\in H(\Omega)$ and

$$u(\zeta)=D(u,k(\cdot,\zeta))\quad (u\in H(\Omega)), \tag{9.13}$$

and similarly for $k_e(\cdot,\zeta)$.

Proof. All verifications are straightforward. Let us just show, for example, that (9.13) holds for $k_e(\cdot,\zeta)$ and $u \in H_e(\Omega)$). We may assume that u is smooth up to the boundary. The proof amounts to standard applications of Green's formula to functions with a singularity. For the Neumann function we get

$$D(u, N_a(\cdot,\zeta)) = \int_\Omega du \wedge^* dN_a(\cdot,\zeta)$$

$$= \lim_{\epsilon\to 0} \int_{\Omega\setminus\mathbb{D}(\zeta,\epsilon)} d(u \wedge^* dN_a(\cdot,\zeta)) = -\int_{\partial\Omega} u\, a ds + 2\pi u(\zeta) = 2\pi u(\zeta),$$

and for the hydrodynamic Green's function,

$$D(u, G_\gamma(\cdot,\zeta)) = D(G_\gamma(\cdot,\zeta), u) = \lim_{\epsilon\to 0} \int_{\Omega\setminus\mathbb{D}(\zeta,\epsilon)} d(G_\gamma(\cdot,\zeta) \wedge {}^*du)$$

$$= \sum_{j=0}^{\mathtt{g}} b_j(\zeta) \int_{\Gamma_j} \frac{\partial u}{\partial n} ds - \lim_{\epsilon\to 0} \int_{\partial\mathbb{D}(\zeta,\epsilon)} G_\gamma(\cdot,\zeta) \frac{\partial u}{\partial n} ds = 0.$$

Now (9.13) follows. □

By definition of $H_e(\Omega)$, all functions $u \in H_e(\Omega)$ have single-valued harmonic conjugates. Thus we can form the space of analytic functions $f = u + \mathrm{i}u^*$ with $u \in H_e(\Omega)$ and with also u^* normalized by (9.11). This is simply the ordinary *Dirichlet space* with normalization (9.11), i.e., the Hilbert space $\mathcal{A}(\Omega)$ of analytic functions in Ω provided with the hermitean inner product

$$(f, g)_{-1} = \int_\Omega f' \bar{g}'\, dxdy = -\frac{1}{2\mathrm{i}} \int_\Omega df \wedge d\bar{g},$$

and subject to the normalization

$$\int_{\partial\Omega} f\, a ds = 0. \tag{9.14}$$

The reason for indexing the inner product by -1 is that $\mathcal{A}(\Omega)$ shortly will be identified as one space in a sequence of weighted Bergman spaces with inner products in general denoted $(f, g)_m$, $m \in \mathbb{Z}$. We note that, with $u = \mathrm{Re}\, f$, $v = \mathrm{Re}\, g$,

$$D(u, v) = \int_\Omega du \wedge {}^*dv = -\mathrm{Re}\, \frac{1}{2\mathrm{i}} \int_\Omega df \wedge d\bar{g} = \mathrm{Re}\,(f, g)_{-1}.$$

The reproducing kernel for $\mathcal{A}(\Omega)$ is the analytic completion $\mathcal{K}(z,\zeta)$ of $k_e(z,\zeta)$, normalized by (9.14). In terms of the multivalued analytic completions $\mathcal{N}_a(z,\zeta)$, $\mathcal{G}_\gamma(z,\zeta)$ of $N_a(z,\zeta)$ and $G_\gamma(z,\zeta)$ we therefore have

$$\mathcal{K}(z,\zeta) = \frac{1}{2\pi}(\mathcal{N}_a(z,\zeta) - \mathcal{G}_\gamma(z,\zeta)).$$

An adjoint kernel may be introduced as

$$\mathcal{L}(z,\zeta) = \frac{1}{2\pi}(\mathcal{N}_a(z,\zeta) + \mathcal{G}_\gamma(z,\zeta)).$$

Remark 9.1. $\mathcal{N}_a(z,\zeta)$, $\mathcal{G}_\gamma(z,\zeta)$ are not symmetric with respect to z, ζ and are not analytic or antianalytic with respect to ζ. However, $\mathcal{K}(z,\zeta)$ is (of course) hermitean symmetric and is antianalytic in ζ. The adjoint kernel, $\mathcal{L}(z,\zeta)$ is multivalued analytic in both z and ζ. See [51] for more details.

Example 9.1. For the unit disk, $\Omega = \mathbb{D}$, with $a = 1$ and (necessarily) $\gamma = (\gamma_0) = (2\pi)$ we have

$$N_a(z,\zeta) = -\log|z-\zeta| - \log|1 - z\bar{\zeta}|,$$
$$G_\gamma(z,\zeta) = -\log|z-\zeta| + \log|1 - z\bar{\zeta}|,$$
$$k(z,\zeta) = -\frac{1}{\pi}\log|1 - z\bar{\zeta}|,$$

$$\mathcal{K}(z,\zeta) = -\frac{1}{\pi}\log(1-z\bar{\zeta}),$$
$$\mathcal{L}(z,\zeta) = -\frac{1}{\pi}\log(z-\zeta)$$
$$\mathcal{N}_a(z,\zeta) = -\log(z-\zeta) - \log(1-z\bar{\zeta}),$$
$$\mathcal{G}_\gamma(z,\zeta) = -\log(z-\zeta) + \log(1-z\bar{\zeta}),$$

Of course, $G_\gamma(z,\zeta) = G(z,\zeta)$, $\mathcal{G}_\gamma(z,\zeta) = \mathcal{G}(z,\zeta)$ in the simply connected case.

We shall denote the ordinary Bergman space by $B(\Omega)$. It is the the Hilbert space of analytic functions in Ω with $(f,f)_1 < \infty$, where

$$(f,g)_1 = \int_\Omega f\bar{g}dxdy$$

is the inner product. The exact Bergman space, $B_e(\Omega)$, is the subspace consisting of those functions of the form F', where F is (single-valued) analytic in Ω. The Dirichlet space is related to the exact Bergman space by differentiation: $\Lambda_1 : f \mapsto f'$ (or $f \mapsto df$) is an isometric isomorphism $\Lambda_1 : \mathcal{A}(\Omega) \to B_e(\Omega)$.

Proposition 9.2. *The reduced Bergman kernel, i.e., the reproducing kernel for $B_e(\Omega)$, is given by*

$$K_e(z,\zeta) = \frac{2}{\pi}\frac{\partial^2 N_a(z,\zeta)}{\partial z\partial\bar{\zeta}} = -\frac{2}{\pi}\frac{\partial^2 G_\gamma(z,\zeta)}{\partial z\partial\bar{\zeta}} = \frac{\partial^2\mathcal{K}(z,\zeta)}{\partial z\partial\bar{\zeta}}$$

for any choices of a and γ as above. Similarly, the corresponding adjoint kernel is

$$L_e(z,\zeta) = \frac{2}{\pi}\frac{\partial^2 N_a(z,\zeta)}{\partial z\partial\zeta} = -\frac{2}{\pi}\frac{\partial^2 G_\gamma(z,\zeta)}{\partial z\partial\zeta} = \frac{\partial^2\mathcal{L}(z,\zeta)}{\partial z\partial\zeta}.$$

Proof. The proof is essentially well-known and straight-forward. Cf. [4], [52]. ☐

Remark 9.2. A beautiful argument, due to M. Schiffer [50], shows that the function $F(z) = \int^z L_e(t,\zeta)dt$ is univalent and maps Ω onto a domain $D \subset \mathbb{P}$ with $\infty \in D$ and such that each component of $\mathbb{P} \setminus D$ is convex. See [49] for further information and several related issues.

A particular consequence is that $L_e(z,\zeta)$ has no zeros in Ω, hence that $K_e(z,\zeta)$ has $2\mathtt{g}$ zeros, $m_1(\zeta), \ldots, m_{2\mathtt{g}}(\zeta)$, in Ω. In terms of these the following generalization [51], [28] of (5.7) to the multiply connected case holds

$$\frac{\partial^2}{\partial z\partial\zeta}\log K_e(z,\zeta) = 2\pi K_e(z,\zeta) + \pi\sum_{k=1}^{2\mathtt{g}} L_e\left(z,\overline{m_k(\zeta)}\right)\overline{m_k'(\zeta)}. \tag{9.15}$$

For the full Bergman kernel $K(z,\zeta)$ there are similar results [29], but they are slightly more complicated because the distribution of the $2\mathtt{g}$ zeros between $K(z,\zeta)$ and $L(z,\zeta)$ is less clear in this case. For example, Theorem 6.1 shows that the number of zeros of $K(z,\zeta)$ in general depends on the location of $\zeta \in \Omega$.

9.3. **Behavior of Neumann function under conformal mapping.** To describe how an object (e.g., the Green's function) on a domain Ω transforms under conformal mapping is equivalent to telling what kind of object it is (function, differential, connection etc.) when the domain is considered as a Riemann surface. In fact, a conformal map onto another domain can be considered simply as a change of holomorphic coordinate on the Riemann surface. Many objects associated to a domain actually extend to the Schottky double, and since this Riemann surface is described by an atlas with two charts overlapping on the boundary of the domain, complete information of the conformal behavior of the object then is contained in the transition formula on the boundary.

The ordinary Green's function $G(z,\zeta)$ extends directly to the Schottky double as an odd function in each variable, see (2.10), and its differential is the real part of normalized abelian differentials of the third kind on the double. Precisely, by (2.10), (2.11), Lemma 2.1,

$$dG(\cdot,\zeta) = -\mathrm{Re}\, \upsilon_{\zeta-J(\zeta)} = -\mathrm{Re}\, \omega_{\zeta-J(\zeta)}. \tag{9.16}$$

When trying to extend the hydrodynamic Green function as an odd function on the double, there appear jumps ($=2b_j$) on $\partial\Omega$. However, the differential $dG_\gamma(\cdot,\zeta)$ extends perfectly well, along with the conjugate differential ${}^*dG_\gamma(\cdot,\zeta)$. Therefore $dG_\gamma(\cdot,\zeta) + \mathrm{i}\,{}^*dG_\gamma(\cdot,\zeta)$ is an abelian differential of the third kind with poles at ζ and $J(\zeta)$. Computing the periods gives

$$\int_{\alpha_j} (dG_\gamma(z,\zeta) + \mathrm{i}\,{}^*dG_\gamma(z,\zeta)) = 2(b_j(\zeta) - b_0(\zeta)),$$

$$\int_{\beta_j} (dG_\gamma(z,\zeta) + \mathrm{i}\,{}^*dG_\gamma(z,\zeta)) = \mathrm{i}\gamma_j,$$

$j = 1, \dots, \mathtt{g}$.

If we in particular choose the period vector to be

$$\gamma = (2\pi, 0, \dots, 0)$$

we see that the β_j-periods ($j = 1, \dots, \mathtt{g}$) of $dG_\gamma(\cdot,\zeta)+\mathrm{i}\,{}^*dG_\gamma(\cdot,\zeta)$ vanish. Therefore (in this case)

$$dG_\gamma(\cdot,\zeta) + \mathrm{i}\,{}^*dG_\gamma(\cdot,\zeta) = -\tilde{\omega}_{\zeta-J(\zeta)},$$

where $\tilde{\omega}_{a-b}$ in general denotes the abelian differential of the third kind with the same singularities as ω_{a-b} (and υ_{a-b}), but normalized so that the β_j-periods vanish. Thus, in analogy with (9.16) we have

$$dG_\gamma(\cdot,\zeta) = -\mathrm{Re}\, \tilde{\omega}_{\zeta-J(\zeta)}$$

for γ as above.

For the Neumann function the situation is slightly more complicated than for the Green's functions. We shall consider $N_a(z,\zeta)$ and $\mathcal{N}_a(z,\zeta) = N_a(z,\zeta) + \mathrm{i}N_a^*(z,\zeta)$ together with the meromorphic differential

$$\Gamma_a(z,\zeta)dz = 2\frac{\partial N_a(z,\zeta)}{\partial z}dz = \frac{\partial \mathcal{N}_a(z,\zeta)}{\partial z}dz.$$

Along the boundary (with respect to z),

$$\mathrm{Im}\,(\Gamma_a(z,\zeta)dz) = \mathrm{Im}\,(d\mathcal{N}_a(z,\zeta)dz) =^* dN_a(z,\zeta) = -a ds.$$

In order to discuss questions of conformal invariance the boundary function a has to be linked to the domain Ω. The most naive choice, a = constant, turns out not to give any good behavior under conformal mapping. More promising is to relate a to the curvature κ of the boundary. If we simply take $a = \kappa$ then (8.2) shows that $\Gamma_a(z,\zeta)dz$ extends to the Schottky double as a symmetric meromorphic affine connection. Moreover, on integrating we see that

$$\mathrm{Im}\,\mathcal{N}_a(z,\zeta) = -\theta \quad (+\,\text{local constant}),$$

i.e., that the analytic completion of the Neumann function behaves essentially as a symmetric 0-connection.

Unfortunately, the choice $a = \kappa$ is allowed only for simply connected domains because in the multiply connected case it violates (9.1). In fact,

$$\int_{\partial\Omega} \kappa ds = 2\pi(1 - \mathtt{g}).$$

On the other hand, we are allowed to take $a = \frac{\kappa}{1-\mathtt{g}}$ for any $\mathtt{g} \neq 1$ and then $(1-\mathtt{g})\Gamma_a(z,\zeta)dz$ becomes a symmetric meromorphic affine connection on $\hat{\Omega}$:

$$\operatorname{Im}((1-\mathtt{g})\Gamma_a(z,\zeta)dz) = -\kappa ds \tag{9.17}$$

along $\partial\Omega$. Moreover, it is an affine connection with simplest possible pole structure: simple poles with residue $1-\mathtt{g}$ at each of the points $z=\zeta$ and $z=\tilde{\zeta}$. For the Neumann function we get

$$\operatorname{Im}(1-\mathtt{g})\mathcal{N}_a(z,\zeta) = -\theta \quad (+\,\text{local constant}) \tag{9.18}$$

on $\partial\Omega$.

For the case $\mathtt{g}=1$ the situation is actually better, because in this case there exists a holomorphic affine connection (in agreement with the fact that in the genus one case the Riemann surface admits an affine structure). Just let v be the regular harmonic function in Ω with Neumann boundary data $\frac{\partial v}{\partial n} = -\kappa$. This is consistent since $\int_{\partial\Omega}\kappa ds = 0$ when $\mathtt{g}=1$, and v is determined up to an additive constant. Now, $\Gamma(z) = 2\frac{\partial v}{\partial z}$ is the required holomorphic affine connection.

A slightly different point of view on $N_a(z,\zeta)$ and $\Gamma_a(z,\zeta)$ is taken in [51], where exterior domains are considered, i.e., domains $\Omega \subset \mathbb{P}$ with $\infty \in \Omega$. In this case

$$\int_{\partial\Omega}\kappa ds = -2\pi(1+\mathtt{g})$$

and hence it is possible, for any $\mathtt{g}$, to choose

$$a = -\frac{\kappa}{1+\mathtt{g}}$$

in the definition of the Neumann function. In place of (9.17), (9.18) one then gets

$$\operatorname{Im}((1+\mathtt{g})\Gamma_a(z,\zeta)dz) = \kappa ds, \tag{9.19}$$

$$\operatorname{Im}(1+\mathtt{g})\mathcal{N}_a(z,\zeta) = \theta \quad (+\,\text{constant}). \tag{9.20}$$

Thus, $-(1+\mathtt{g})\Gamma_a(z,\zeta)dz$ is now a symmetric affine connection on $\hat{\Omega}$. (In [51] this connection is denoted $\Gamma(z,\zeta)dz$.) However, it has not simplest possible pole structure. Besides the pole with residue $1+\mathtt{g}$ at $z=\zeta$ it has on Ω also a pole with residue -2 at $z=\infty$. Similarly on the backside $\tilde{\Omega}$.

To discover the pole at infinity one has to introduce a regular (holomorphic) coordinate there, for example $w=\frac{1}{z}$. Then

$$\{w,z\}_1 dz = \frac{d}{dz}\log\frac{dw}{dz}dz = -2\frac{dz}{z} = 2\frac{dw}{w}.$$

By (7.4) the representative of $-(1+\mathtt{g})\Gamma_a(z,\zeta)dz$ with respect to w is

$$-(1+\mathtt{g})\Gamma_a(\frac{1}{w},\zeta)d(\frac{1}{w}) - 2\frac{dw}{w},$$

which has residue -2 at $w=0$ (the first term is regular at $w=0$).

As an example, consider the exterior disk: $\Omega = \{z\in\mathbb{C} : |z|>1\}\cup\{\infty\}$. For this we have, with $a = -\frac{\kappa}{1+\mathtt{g}} = 1$,

$$N_a(z,\zeta) = -\log|z-\zeta| - \log|1-z\bar{\zeta}| + 2\log|z| + 2\log|\zeta|,$$

$(1<|z|,|\zeta|<\infty)$ which gives, for the affine connection $-(1+\mathtt{g})\Gamma_a(z,\zeta)dz$,

$$-\Gamma_a(z,\zeta)dz = \frac{dz}{z-\zeta} + \frac{dz}{z-\frac{1}{\bar{\zeta}}} - 2\frac{dz}{z}.$$

In terms of $w=\frac{1}{z}$ the same connection is given by

$$-\Gamma_a(\frac{1}{w},\zeta)d(\frac{1}{w}) - 2\frac{dw}{w} = \frac{dw}{w-\frac{1}{\zeta}} + \frac{dw}{w-\bar{\zeta}} - 2\frac{dw}{w},$$

valid actually in all Ω ($|w| < 1$). Here one sees clearly the pole of residue -2 at $w = 0$ (i.e., $z = \infty$), along with the pole with residue one at $w = \frac{1}{\zeta}$ (i.e., $z = \zeta$).

10. Weighted Bergman spaces

We have discussed so far the Dirichlet space $\mathcal{A}(\Omega)$ and the Bergman spaces $B(\Omega)$ and $B_e(\Omega)$, and remarked that $\mathcal{A}(\Omega)$ and $B_e(\Omega)$ are connected via the simplest Bol operator, namely $\Lambda_1 : F \mapsto dF$. In this section we shall extend the discussion to a sequence of weighted Dirichlet spaces $\mathcal{A}_m(\Omega)$ and Bergman spaces $B_m(\Omega)$ and $B_{e,m}(\Omega)$.

Let

$$d\sigma = \frac{|dz|}{\omega(z)}$$

be the Poincaré metric of Ω, characterized by its constant curvature and being complete:

$$\begin{cases} \kappa_{\text{Gauss}} = -4, \\ \omega = 0 \quad \text{on } \partial\Omega. \end{cases}$$

As discussed previously it gives rise to a holomorphic projective connection $q(z)$ and a projective structure, namely that for which the projective coordinates are the coordinate on the universal covering surface of Ω when this is taken to be a disk or a half-plane. Let Λ_m be the corresponding Bol operator, given in a projective coordinate t by $\Lambda_m = \frac{\partial^m}{\partial t^m}$.

For $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, and f, g analytic in Ω, smooth up to $\partial\Omega$, define

$$(f, g)_\alpha = \int_\Omega f \bar{g} \omega^{\alpha-1} dxdy. \tag{10.1}$$

When $\alpha > 0$ is real this is an inner product on a space of analytic functions, a weighted Bergman space (taken to be complete, hence a Hilbert space). We denote this space $B_\alpha(\Omega)$. It will mainly be used for $\alpha = m$ an integer. Clearly, $B_1(\Omega)$ is the ordinary Bergman space, and it is well-known that as $\alpha \to 0$ one gets the Hardy space, with the Szegö inner product given by

$$\lim_{\alpha \to 0} \alpha (f, g)_\alpha = \frac{1}{\pi} \int_{\partial\Omega} f \bar{g} |dz|$$

Remark 10.1. Our notations for weighted Bergman spaces deviate from some standard notations used in the case of the unit disk. In for example [27] the notation A^2_α is used for the Bergman space with weight $\frac{\alpha+1}{\pi}\omega^\alpha dxdy$, in place of our $\omega^{\alpha-1}dxdy$, in the definition (10.1) of the inner product. In the context of automorphic forms the inner product $(f, g)_\alpha$ is known as the Petersson inner product.

If F, G are analytic in a neighborhood of $\overline{\Omega}$ and are kept fixed, then the inner product, regarded as a function of α,

$$\alpha \mapsto (F, G)_\alpha,$$

is analytic for $\operatorname{Re}\alpha > 0$ and has a meromorphic extension to all $\mathbb{C}$, with simple poles at $\alpha = 0, -1, -2, \dots$. The proof of this is implicit in the proof of Proposition 10.1 below. We shall then extend the inner product $(F, G)_\alpha$ to such values of α by setting

$$(F, G)_{-m} = \operatorname{Res}_{\alpha=-m} \int_\Omega F \bar{G} \omega^{\alpha-1} dxdy,$$

$m = 0, 1, 2, \dots$. For example, for $m = 0$ we retrieve the Szegö inner product.

Now, the main issue is that, for any $m = 0, 1, 2, \dots$, $(F, G)_{-m}$ is an inner product on a generalized Dirichlet space $\mathcal{A}_m(\Omega)$ of analytic functions and that the Bol operator is an isometric isomorphism of $\mathcal{A}_m(\Omega)$ onto the subspace $B_{e,m}(\Omega)$ of 'exact' differentials in $B_m(\Omega)$):

$$\Lambda_m : \mathcal{A}_m(\Omega) \to B_{e,m}(\Omega).$$

In order to elaborate the details of the above we need certain (in principle known) integral formulas for the Λ_m.

Proposition 10.1. *With F, G (regarded as $\frac{1-m}{2}$-forms) and g (regarded as a $\frac{1+m}{2}$-form) analytic in Ω, smooth up to $\partial\Omega$, we have*

$$\int_\Omega (\Lambda_m F)\bar{g}\omega^{m-1} dz d\bar{z} = \mathrm{i}^{m-1}(m-1)! \int_{\partial\Omega} F\bar{g}(dz)^{\frac{1-m}{2}}(d\bar{z})^{\frac{1+m}{2}}, \tag{10.2}$$

$$\operatorname{Res}_{\alpha=-m} \int_\Omega F\bar{G}\omega^{\alpha-1} dz d\bar{z} = \frac{(-\mathrm{i})^{m+1}}{m!} \int_{\partial\Omega} F\overline{\Lambda_m G}(dz)^{\frac{1-m}{2}}(d\bar{z})^{\frac{1+m}{2}}. \tag{10.3}$$

In other words,

$$(\Lambda_m F, g)_m = \frac{\mathrm{i}^m m!}{2} \int_{\partial\Omega} F\bar{g}(dz)^{\frac{1-m}{2}}(d\bar{z})^{\frac{1+m}{2}}, \tag{10.4}$$

$$(F, G)_{-m} = \frac{(-\mathrm{i})^m}{2m!} \int_{\partial\Omega} F\overline{\Lambda_m G}(dz)^{\frac{1-m}{2}}(d\bar{z})^{\frac{1+m}{2}}. \tag{10.5}$$

Equation (10.2) is valid only for $m = 1, 2, \ldots$, while (10.3), (10.4), (10.5) hold also for $m = 0$ (with $0! = 1$).

By choosing $g = \Lambda_m G$ in (10.4) we obtain

Corollary 10.2. *For $m = 0, 1, 2, \ldots$,*

$$(F, G)_{-m} = (\Lambda_m F, \Lambda_m G)_m, \tag{10.6}$$

hence $(F, F)_{-m} \geq 0$ with equality if and only if $\Lambda_m F = 0$.

Proof. The proof of the proposition is based on ideas of Jaak Peetre, and parts of it have previously been outlined in [24].

Let $\{\varphi_j\}$ be a smooth partition of unity (i.e., $\sum_j \varphi_j = 1$) on $\overline{\Omega}$ such that each individual φ_j has support within the domain of definition of a projective coordinate t_j. In such a coordinate ω is of the form $\omega_j = a|t_j|^2 + bt_j + \bar{b}\bar{t_j} + c$, with a, c real and $|b|^2 = ac+1$ (to have curvature $\kappa = -4$). After an additional Möbius transformation we may even assume that $\omega_j = 2\operatorname{Im} t_j$ and that t_j takes values in the closed upper half plane. Note then that $\partial\Omega$ necessarily is mapped into the real line (since $\omega = 0$ on $\partial\Omega$).

In the coordinates t_j, the coefficients F, G, g correspond to (say) F_j, G_j, g_j (i.e., $F(dz)^{\frac{1-m}{2}} = F_j(dt_j)^{\frac{1-m}{2}}$ etc.) and Λ_m becomes $\frac{\partial^m}{\partial t_j^m}$. On setting $\psi_j = \varphi_j \circ t_j^{-1}$ we have

$$\int_\Omega \Lambda_m F\bar{g}\omega^{m-1} dz d\bar{z} = \sum_j \int_\Omega \Lambda_m F\varphi_j\bar{g}\omega^{m-1} dz d\bar{z}$$
$$= \sum_j \int_{\operatorname{Im} t_j > 0} \frac{\partial^m F_j}{\partial t_j^m} \psi_j \bar{g}_j (2\operatorname{Im} t_j)^{m-1} dt_j d\bar{t_j}.$$

After an m-fold partial integration this becomes

$$\sum_j (-1)^m \int F_j \frac{\partial^m}{\partial t_j^m}(\psi_j \bar{g}_j (2\operatorname{Im} t_j)^{m-1}) dt_j d\bar{t_j}$$
$$+ \sum_j (-1)^m \sum_{k=0}^{m-1} (-1)^k \int_{\mathbb{R}} \frac{\partial^{m-k-1} F_j}{\partial t^{m-1-k}} \frac{\partial^k}{\partial t_j^k}(\psi_j \bar{g}_j (2\operatorname{Im} t_j)^{m-1}) dt_j d\bar{t_j}.$$

Here all the boundary terms except the last one (with $k = m-1$), and with $\frac{\partial^k}{\partial t_j^k} = \frac{\partial^{m-1}}{\partial t_j^{m-1}}$ acting only on $(2\mathrm{Im}\, t_j)^{m-1}$, vanish because $\mathrm{Im}\, t_j = 0$ on $\mathbb{R}$. Moving back to the z-plane and taking covariance (Lemma 7.1) into account, the above expression becomes

$$\sum_j (-1)^m \int_\Omega F\Lambda_m(\varphi_j \bar{g}\omega^{m-1}) dz d\bar{z} + \mathrm{i}^{m-1}(m-1)! \sum_j \int_{\partial\Omega} F\varphi_j \bar{g}(dz)^{\frac{1-m}{2}}(d\bar{z})^{\frac{1+m}{2}}$$

$$= (-1)^m \int_\Omega F\bar{g}\Lambda_m(\omega^{m-1}) dz d\bar{z} + \mathrm{i}^{m-1}(m-1)! \int_{\partial\Omega} F\bar{g}(dz)^{\frac{1-m}{2}}(d\bar{z})^{\frac{1+m}{2}}$$

$$= \mathrm{i}^{m-1}(m-1)! \int_{\partial\Omega} F\bar{g}(dz)^{\frac{1-m}{2}}(d\bar{z})^{\frac{1+m}{2}}.$$

This proves (10.2). A different proof is given in [23].

Now we turn to (10.3). In order to show that $\int_\Omega F\bar{G}\omega^{\alpha-1} dz d\bar{z}$ has a meromorphic extension and to compute $\mathrm{Res}_{\,\alpha=-m} \int_\Omega F\bar{G}\omega^{\alpha-1} dz d\bar{z}$ we first observe that, with partition of unity and notations as above,

$$d\Big(\sum_{k=0}^m (-1)^k \frac{\partial^k}{\partial \bar{t_j}^k}(F_j(2\mathrm{Im}\, t_j)^{\alpha+m}) \frac{\partial^{m-k}}{\partial \bar{t_j}^{m-k}}(\psi_j \bar{G}_j) dt_j\Big)$$

$$= F_j \frac{\partial^{m+1}}{\partial \bar{t_j}^{m+1}}(\psi \bar{G}_j)(2\mathrm{Im}\, t_j)^{\alpha+m} d\bar{t}_j dt_j + (-1)^m \frac{\partial^{m+1}}{\partial \bar{t_j}^{m+1}}(F_j(2\mathrm{Im}\, t_j)^{\alpha+m})\psi_j \bar{G}_j d\bar{t}_j dt_j.$$

Therefore,

$$\sum_{k=0}^m (-1)^k \int_{\mathbb{R}} \frac{\partial^k}{\partial \bar{t}^k}(F_j(2\mathrm{Im}\, t_j)^{\alpha+m}) \frac{\partial^{m-k}}{\partial \bar{t}^{m-k}}(\psi_j \bar{G}_j) dt_j$$

$$= \int_{\mathrm{Im}\, t_j > 0} F_j \frac{\partial^{m+1}}{\partial \bar{t}_j^{m+1}}(\psi_j \bar{G}_j)(2\mathrm{Im}\, t_j)^{\alpha+m} d\bar{t}_j dt_j$$

$$-(-\mathrm{i})^{m+1}(\alpha+m)\cdots(\alpha+1)\alpha \int_{\mathrm{Im}\, t_j>0} F_j \bar{G}_j \psi_j (2\mathrm{Im}\, t_j)^{\alpha-1} d\bar{t}_j dt_j.$$

The boundary terms vanish whenever $\mathrm{Re}\,\alpha > 0$ and it follows that for such α,

$$\int_{\mathrm{Im}\, t_j>0} F_j \bar{G}_j \psi_j (2\mathrm{Im}\, t_j)^{\alpha-1} d\bar{t}_j dt_j$$

$$= \frac{\mathrm{i}^{m+1}}{(\alpha+m)\cdots(\alpha+1)\alpha} \int_{\mathrm{Im}\, t_j>0} F_j \frac{\partial^{m+1}}{\partial \bar{t}_j^{m+1}}(\psi_j \bar{G}_j)(2\mathrm{Im}\, t_j)^{\alpha+m} d\bar{t}_j dt_j.$$

Here the integral in the right member is an analytic function in α for $\mathrm{Re}\,\alpha > -m-1$, hence the left member has a meromorphic extension to this range, with the residue at $\alpha = -m$ given by

$$\mathrm{Res}_{\,\alpha=-m} \int F_j \bar{G}_j \psi_j (2\mathrm{Im}\, t_j)^{\alpha-1} d\bar{t}_j dt_j = -\frac{(-\mathrm{i})^{m+1}}{m!} \int F \frac{\partial^{m+1}}{\partial \bar{t}_j^{m+1}}(\psi_j \bar{G}_j) d\bar{t}_j dt_j.$$

Putting the pieces together it follows that also $\mathrm{Res}_{\,\alpha=-m} \int_\Omega F\bar{G}\omega^{\alpha-1} dz d\bar{z}$ has a meromorphic continuation to $\mathrm{Re}\,\alpha > -m-1$ with residue at $\alpha = -m$ given by

$$\mathrm{Res}_{\,\alpha=-m} \int_\Omega F\bar{G}\omega^{\alpha-1} dz d\bar{z}$$

$$= -\sum_j \mathrm{Res}_{\,\alpha=-m} \int_{\mathrm{Im}\, t_j>0} F_j \bar{G}_j \psi_j (2\mathrm{Im}\, t_j)^{\alpha-1} d\bar{t}_j dt_j$$

$$= \frac{(-\mathrm{i})^{m+1}}{m!} \sum_j \int_{\mathrm{Im}\, t_j>0} F_j \frac{\partial^{m+1}}{\partial \bar{t}_j^{m+1}}(\psi_j \bar{G}_j) d\bar{t}_j dt_j$$

$$= \frac{(-\mathrm{i})^{m+1}}{m!}\sum_j \int_{\operatorname{Im} t_j>0} d(F_j \frac{\partial^m}{\partial \bar t_j^m}(\psi_j \bar G_j) dt_j)$$

$$= \frac{(-\mathrm{i})^{m+1}}{m!}\sum_j \int_{\mathbb{R}} F_j \frac{\partial^m}{\partial \bar t_j^m}(\psi_j \bar G_j) dt_j$$

$$= \frac{(-\mathrm{i})^{m+1}}{m!}\sum_j \int_{\mathbb{R}} F_j \frac{\partial^m}{\partial \bar t_j^m}(\psi_j \bar G_j)(dt_j)^{\frac{1-m}{2}}(d\bar t_j)^{\frac{1+m}{2}}$$

$$= \frac{(-\mathrm{i})^{m+1}}{m!}\sum_j \int_{\partial\Omega} F \bar\Lambda_m(\varphi_j \bar G)(dz)^{\frac{1-m}{2}}(d\bar z)^{\frac{1+m}{2}}$$

$$= \frac{(-\mathrm{i})^{m+1}}{m!}\int_{\partial\Omega} F \bar\Lambda_m \bar G (dz)^{\frac{1-m}{2}}(d\bar z)^{\frac{1+m}{2}}.$$

This finishes the proof of the proposition. □

The corollary shows that the hermitean form $(F,G)_{-m}$ is positive semidefinite. Set

$$A_m(\Omega) = \{F \text{ analytic in } \Omega : (F,F)_{-m} < \infty\}.$$

The spaces $B_m(\Omega)$, with inner product $(f,g)_m$, were defined after (10.1), and by (10.6) the Bol operator Λ_m is an isometry

$$\Lambda_m : A_m(\Omega) \to B_m(\Omega).$$

However, this map is neither injective nor surjective when $m > 0$. We shall say something both about its kernel and its cokernel.

As to the kernel, recall that Λ_m in terms of any projective coordinate t is simply $\Lambda_m = \frac{\partial^m}{\partial t^m}$. With $F(z)(dz)^{\frac{1-m}{2}}$ a holomorphic differential of order $\frac{1-m}{2}$, it follows that $\Lambda_m F = 0$ if and only if F becomes a polynomial of degree $\le m-1$ when expressed in any projective coordinate. Accordingly, we denote by

$$P_m(\Omega) = \{F \in A_m(\Omega) : F \text{ holomorphic in } \Omega,\ \Lambda_m F = 0\}$$

the kernel of Λ_m. The image of Λ_m is by definition

$$B_{m,e}(\Omega) = \{\Lambda_m F \in B_m(\Omega) : F \in A_m(\Omega)\}.$$

Then we have the exact sequence

$$0 \to P_m(\Omega) \to A_m(\Omega) \stackrel{\Lambda_m}{\to} B_{m,e}(\Omega) \to 0.$$

In other words,

$$A_m(\Omega)/P_m(\Omega) \cong B_{m,e}(\Omega)$$

and $(\cdot,\cdot)_{-m}$ is positive definite on the quotient space $A_m(\Omega)/P_m(\Omega)$.

Instead of having $(\cdot,\cdot)_{-m}$ defined on a quotient space of $A_m(\Omega)$ it might be desirable to have it defined on a subspace of $A_m(\Omega)$. One advantage then is that the elements in the space become functions, hence it will be possible to discuss reproducing kernels. There seems to be no canonical choice of such a subspace, but in principle it is obtained by imposing normalization conditions. To this purpose, choose a linear operator

$$\mathtt{a} : A_m(\Omega) \to \mathbb{C}^m$$

which does not degenerate on $P_m(\Omega)$, i.e., such that $\mathtt{a}(P_m(\Omega)) = \mathbb{C}^m$. Then we define

$$\mathcal{A}_m(\Omega) = \{F \in A_m(\Omega) : \mathtt{a}(F) = 0\}$$

(the dependence on $\mathtt{a}$ is suppressed in the notation). Then $\mathcal{A}_m(\Omega) \cong A_m(\Omega)/P_m(\Omega)$ and Λ_m is an isometric isomorphism

$$\Lambda_m : \mathcal{A}_m(\Omega) \to B_{e,m}(\Omega),$$

as desired.

In the special case $m = 1$ we may choose a on the form

$$\mathsf{a}(F) = \int_{\partial\Omega} F\, ads$$

for some boundary function a with $\int_{\partial\Omega} ads \neq 0$ and then we retrieve the previously discussed (see after (9.14)) Dirichlet space, i.e., $\mathcal{A}_1(\Omega) = \mathcal{A}(\Omega)$.

The cokernel $B_m(\Omega)/B_{m,e}(\Omega)$ of Λ_m can be identified with the orthogonal complement of $B_{m,e}(\Omega)$ in $B_m(\Omega)$, which we write simply as $B_{m,e}(\Omega)^\perp$. And for this finite dimensional space (see [24] for the dimension) we have the following description.

Proposition 10.3. *$B_{m,e}(\Omega)^\perp$ consists of those elements in $B_m(\Omega)$ which extend to the Schottky double $\hat{\Omega}$ as holomorphic differentials of order $\frac{1+m}{2}$.*

Proof. By definition, $g \in B_{m,e}(\Omega)^\perp$ if and only if

$$(\Lambda_m F, g)_m = 0 \quad \text{for all} \quad F \in A_m(\Omega).$$

Using (10.4) this becomes

$$\int_{\partial\Omega} F\bar{g}(dz)^{\frac{1-m}{2}}(d\bar{z})^{\frac{1+m}{2}} = 0$$

for all $F \in A_m(\Omega)$. In terms of the function

$$h(z) = \overline{g(z)}(dz)^{\frac{-1-m}{2}}(d\bar{z})^{\frac{1+m}{2}} = \overline{g(z)}T(z)^{-1-m}, \tag{10.7}$$

defined on $\partial\Omega$, this becomes

$$\int_{\partial\Omega} Fhdz = 0$$

for all $F \in A_m(\Omega)$, which implies that h has a holomorphic extension to Ω. This proves the proposition because h represents the continuation of g to the back-side of the Schottky double. Note that (10.7) is an instance of (2.5). $\square$

Example 10.1. The case $m = 1$ is the well-known [52] fact that $B_{1,e}(\Omega)^\perp$ consists of the abelian differentials of the first kind.

11. Reproducing kernels

11.1. Reproducing kernels for weighted Bergman spaces.

Each of the weighted Bergman spaces $B_m(\Omega)$ and $B_{m,e}(\Omega)$, $m \geq 0$, have reproducing kernels $K_m(z,\zeta)$, $K_{m,e}(z,\zeta)$, which extend to the Schottky double as differentials of order $\frac{1-m}{2}$ in each variable. The continuations to the backside are represented by the adjoint kernels $L_m(z,\zeta)$, $L_{m,e}(z,\zeta)$, which have singularities

$$L_m(z,\zeta) = \frac{(-\mathrm{i})^{m-1}}{\pi(z-\zeta)^{m+1}} + \text{less singular terms}$$

(similarly for $L_{m,e}(z,\zeta)$). See [24] for details and proofs. Recall also that $K_1(z,\zeta)$ is the ordinary Bergman kernel and $K_0(z,\zeta)$ the Szegö kernel.

For the spaces $A_m(\Omega)$ and $\mathcal{A}_m(\Omega)$ the situation is not quite that good. But at least $\mathcal{A}_m(\Omega)$ is a Hilbert space of functions for which all point evaluations are continuous linear functionals, and hence it has a reproducing kernel, which we denote $\mathcal{K}_m(z,\zeta)$. This should be thought of as a differential of order $\frac{1-m}{2}$ in each of z and $\bar{\zeta}$, but its extension to the Schottky double is cumbersome because of appearance of branch points and multi-valuedness on the backside, as have already been observed in the case of Dirichlet space, $m = -1$, Example 9.1. Still it is possible to define indirectly a kind of (multi-valued) adjoint kernel, which we denote $\mathcal{L}_m(z,\zeta)$. In fact, we simply define $\mathcal{L}_m(z,\zeta)$ to be any solution of equation (11.2) below.

Theorem 11.1. *Let $\mathcal{K}_m(z,\zeta)$ denote the reproducing kernel for $\mathcal{A}_m(\Omega)$ and $\mathcal{L}_m(z,\zeta)$ the adjoint kernel. Then,*

$$\Lambda_m\bar{\Lambda}_m\mathcal{K}_m(z,\zeta)=K_{m,e}(z,\zeta), \tag{11.1}$$

$$\Lambda_m\Lambda_m\mathcal{L}_m(z,\zeta)=L_{m,e}(z,\zeta), \tag{11.2}$$

where the first Λ_m acts on the z-variable and the second one on ζ. The leading term in the singularity of $\mathcal{L}_m$ is given by

$$\mathcal{L}_m(z,\zeta)=-\frac{\mathrm{i}^{m-1}}{\pi m!(m-1)!}(z-\zeta)^{m-1}\log(z-\zeta)+\text{less singular terms.}$$

Proof. The formula (11.1) follows immediately by letting Λ_m act on both members, as functions of ζ, in the defining equation (reproducing property)

$$F(\zeta)=(F,\mathcal{K}_m(\cdot,\zeta))_{-m}\quad (F\in\mathcal{A}_m(\Omega))$$

for $\mathcal{K}_m(z,\zeta)$ and then applying (10.6).

As (11.2) was simply taken as the definition of $\mathcal{L}_m(z,\zeta)$ it just remains to prove the form of the singularity. This is a matter of computation, which we omit. One has to check that the claimed singularity for $\mathcal{L}_m(z,\zeta)$ match with that of $L_{m,e}(z,\zeta)$ (which is the same as that of $L_m(z,\zeta)$) under (11.2). The computation may be performed in a projective coordinate because the assertion only concerns the leading term of the singularity, which is the same in any coordinate. □

Example 11.1. For the unit disk we have

$$\mathcal{K}_m(z,\bar{\zeta})=-\frac{\mathrm{i}^{m-1}}{\pi m!(m-1)!}(1-z\bar{\zeta})^{m-1}\log(1-z\zeta)+\sum_{k=0}^{m-1}a_k(z)\bar{\zeta}^k+\sum_{k=0}^{m-1}\overline{a_k(\zeta)}z^k,$$

$$\mathcal{L}_m(z,\zeta)=-\frac{\mathrm{i}^{m-1}}{\pi m!(m-1)!}(z-\zeta)^{m-1}\log(z-\zeta)+\sum_{k=0}^{m-1}a_k(z)\zeta^k+\sum_{k=0}^{m-1}a_k(\zeta)z^k,$$

where the analytic functions a_k depend on the normalization chosen in the definition of $\mathcal{A}_m(\Omega)$. If the normalization for example is that $F(0)=\cdots=F_{m-1}(0)=0$ for $F\in\mathcal{A}_m(\Omega)$, then all the a_j are zero and the kernels consist of just the first term.

11.2. **The reproducing kernel as a resolvant.** Interchanging the roles of F and G in (10.5) gives, after conjugation,

$$(F,G)_{-m}=-\frac{(-\mathrm{i})^m}{2m!}\int_{\partial\Omega}(\Lambda_mF)\overline{G}(dz)^{\frac{1+m}{2}}(d\bar{z})^{\frac{1-m}{2}}.$$

With $F\in\mathcal{A}_m(\Omega)$ and $G=\mathcal{K}_m(z,\zeta)$ we get

$$F(\zeta)=-\frac{(-\mathrm{i})^m}{2m!}\int_{\partial\Omega}\Lambda_mF(z)\overline{\mathcal{K}_m(z,\zeta)}(dz)^{\frac{1+m}{2}}(d\bar{z})^{\frac{1-m}{2}}. \tag{11.3}$$

This formula says that $\mathcal{K}_m$ represents the inverse, or resolvant, of Λ_m, in terms of the boundary values. In the case $m=2$, (11.3) becomes

$$F(\zeta)=\frac{1}{4}\int_{\partial\Omega}\Lambda_2F(z)\overline{\mathcal{K}_2(z,\zeta)}(dz)^{\frac{3}{2}}(d\bar{z})^{-\frac{1}{2}}\quad(\zeta\in\Omega), \tag{11.4}$$

which resembles a formula of R. J. V. Jackson [30].

Example 11.2. For the unit disk (11.4) becomes

$$F(\zeta)=\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathbb{D}}F''(z)(1-\bar{z}\zeta)\log(1-\bar{z}\zeta)zdz,$$

if $\mathcal{A}_2(\mathbb{D})$ is normalized by $F(0)=F'(0)=0$, (i.e., $\mathsf{a}(F)=(F(0),F'(0))$).

Equation (11.4) is valid for $\zeta \in \Omega$. For $\zeta \neq \overline{\Omega}$ it does not make sense because, as is seen clearly in Examples 9.1 and 11.1, even though the kernel extends analytically across $\partial\Omega$, the extended kernel is not single-valued in any neighborhood of $\overline{\Omega}$. However, it is possible to let $\zeta \in \Omega$ approach $\partial\Omega$ from inside and get a sensible version of (11.4) for $\zeta \in \partial\Omega$. This is what Jackson [30] does. The equation (11.4) then solely deals with objects defined on $\partial\Omega$ and it is appropriate to consider the kernel $\mathcal{K}_2(z,\zeta)$ as a resolvent of Λ_2. This kernel is not smooth on the boundary, as is seen in the example of the unit disk, there is a discontinuity in the first derivative caused by the jump in the imaginary part of $\log(1-\bar{z}\zeta)$ as z passes ζ.

Remark 11.1. Since $\mathcal{K}_2(z,\zeta)$ is to be considered as a form of degree $-1/2$ in each of z and $\bar{\zeta}$ it is natural to compare it with $1/K_0(z,\zeta)$, $1/\sqrt{K_1(z,\zeta)}$ and, on the diagonal, compare all these with $\omega(z)$. Certainly, $1/\pi K_0(z,z) = 1/\sqrt{\pi K_1(z,z)} = \omega(z)$ in the simply connected case. However, $\mathcal{K}_2(z,\zeta)$ is of a different nature because its continuation $\mathcal{L}_2(z,\zeta)$ to the back-side of the Schottky double is not single-valued.

Similar remarks as above apply to the equation (11.3) for all values of m.

12. The prepotential of a second order linear DE

12.1. **The method of Faraggi and Matone.** Here we shall discuss a further topic related to projective structures. Consider a differential equation of the form (7.11) in general. We write it as

$$u'' + \frac{1}{2}Q(z)u = 0, \tag{12.1}$$

where $Q(z)$ is a holomorphic function, say in a neighborhood of $z=0$. Note that the Wronskian

$$W = W(u_1,u_2) = u_1u_2' - u_1'u_2 \tag{12.2}$$

of any pair u_1, u_2 of solutions is constant. If $u_1(0)\neq 0$, then any other solution u_2 is obtained from u_1 by

$$u_2(z) = u_1(z)\Big(\int_0^z \frac{W d\zeta}{u_1(\zeta)^2} + C\Big) \tag{12.3}$$

for suitable constants W (which then becomes the Wronskian) and C.

An interesting approach to the problem of producing further solutions from a given one has been considered for instance by A. E. Faraggi and M. Matone [16]. The authors introduce a "prepotential" $\mathcal{F}(u) = \mathcal{F}_{u_1}(u)$, a function of a complex variable u, which functionally depends also on the first solution u_1 (alternatively, depends on Q if initial conditions for u_1 are specified), to the effect that a second solution u_2 is obtained by taking the derivative with respect to u:

$$u_2(z) = \frac{d\mathcal{F}_{u_1}(u)}{du}\Big|_{u=u_1(z)}. \tag{12.4}$$

As noted in [16], $\mathcal{F}(u)$ is essentially a Legendre transform, namely of the independent variable z considered as a function of any projective coordinate t associated to the projective structure given by $Q(z)$. We proceed to explain briefly these issues, going slightly beyond [16].

Fixing a value $W\neq 0$ of the Wronskian, consider pairs $u_1 = u_1(z)$, $u_2 = u_2(z)$ of solutions, holomorphic in a neighborhood of $z=0$ and subject to (12.2), i.e.,

$$u_1du_2 - u_2du_1 = Wdz. \tag{12.5}$$

We assume that $u_1(0)\neq 0$ and set

$$s = u_1^2, \quad t = \frac{u_2}{u_1}. \tag{12.6}$$

Then

$$\frac{dt}{dz} = \frac{W}{s}, \quad \{t, z\}_2 = Q(z),$$

and also, for example, $\{z, t\}_1 = \frac{1}{W}\frac{ds}{dz}$. In terms of s and t, (12.5) becomes

$$s\,dt = Wdz. \tag{12.7}$$

We note also that, in terms of s and t, the $(m-1)$-fold symmetric product $S^{m-1}(L)$ of the operator $L = \frac{d^2}{dz^2} + \frac{1}{2}Q(z)$, mentioned briefly after Lemma 7.1, will have solutions generated by $s^{\frac{m-1}{2}}t^k$, $k = 0, 1, \dots, m-1$.

In a neighborhood of $z = 0$ we can invert $t = t(z)$ to consider z as a function of t: $z = z(t)$. Then $\frac{dz}{dt} = \frac{s}{W}$. Assuming for a moment that $\frac{d^2z}{dt^2} \neq 0$ we can form the Legendre transform of $Wz(t)$. It is

$$\mathcal{L}(s) = st - Wz(t) \quad \text{with } t \text{ chosen so that } s = W\frac{dz(t)}{dt}. \tag{12.8}$$

Note that the final equation assures that the variable t keeps its meaning as $t = \frac{u_2}{u_1}$ when $s = u_1^2$.

By a direct computation, or by using that the Legendre transform is involutive, one realizes that

$$\frac{d\mathcal{L}(s)}{ds} = t,$$

with s, t related to u_1, u_2 as above. This shows that

$$u_2 = u_1 \cdot t = \frac{1}{2}\frac{ds}{du_1}\frac{d\mathcal{L}(s)}{ds} = \frac{1}{2}\frac{d\mathcal{L}(u_1^2)}{du_1},$$

hence establishes the Legendre transform as essentially the desired prepotential:

$$\mathcal{F}(u) = \frac{1}{2}\mathcal{L}(u^2).$$

An alternative and slightly more general approach, which makes sense also if $\frac{d^2z}{dt^2} = 0$, is to consider u_1, u_2 and z as independent variables, or coordinates, in a three dimensional space. Then (12.6) is simply a coordinate transformation (in two of the variables) and (12.5), (12.7) should be thought of as defining a contact structure (see, e.g., [2], Appendix 4). With

$$\mathcal{L} = st - Wz = u_1u_2 - Wz, \tag{12.9}$$

now considered as a function in the three dimensional space, we have

$$d\mathcal{L} = tds = 2u_2du_1$$

when the contact structure is taken into account. This gives again

$$t = \frac{d\mathcal{L}}{ds}\Big|_{sdt=Wdz}, \quad u_2 = \frac{1}{2}\frac{d\mathcal{L}}{du_1}\Big|_{u_1du_2-u_2du_1=Wdz}.$$

To derive explicit formulas for $\mathcal{F}(u)$ it is convenient to assume that the first solution u_1 is invertible near $z = 0$, i.e., to assume that $u_1'(0) \neq 0$. Since the independent variable u in $\mathcal{F}(u)$ will finally, in (12.4), be assigned to have the value $u_1(z)$, it is natural to invert this relation to $z = u_1^{-1}(u)$. Substituting into (12.3), (12.9) and recalling that $\mathcal{F}$ is half of $\mathcal{L}$ one arrives at

$$\mathcal{F}_{u_1}(u) = \frac{1}{2}(u_1u_2 - Wz) = \frac{1}{2}(u^2\int_0^{u_1^{-1}(u)}\frac{Wd\zeta}{u_1(\zeta)^2} + Cu^2 - Wu_1^{-1}(u)). \tag{12.10}$$

After a partial integration and a change of variable this gives the formula presented in [16]:

$$\mathcal{F}_{u_1}(u) = u^2(\int_{u_1(0)}^{u}\frac{Wu_1^{-1}(\eta)d\eta}{\eta^3} + \frac{C}{2}) = Wu^2\int\frac{u_1^{-1}(u)du}{u^3}. \tag{12.11}$$

Here the final integral is an "indefinite integral". One readily verifies that (12.4) indeed holds.

As a final issue one may notice (as in [16]) that the prepotential satisfies a third order differential equation. This is obtained by considering u_2 as a function of u_1 (in place of z). This renders (12.1) of the form

$$\frac{d^2u_2}{du_1^2}=\frac{Q}{2W^2}(u_1\frac{du_2}{du_1}-u_2)^3.$$

Replacing u_1 by u and u_2 by $\frac{d\mathcal{F}(u)}{du}$ gives

$$\frac{d^3\mathcal{F}(u)}{du^3}=\frac{Q}{2W^2}(u\frac{d^2\mathcal{F}(u)}{du^2}-\frac{d\mathcal{F}(u)}{du})^3.$$

12.2. **Example.** Consider the differential equation

$$u''+u=0,$$

i.e., (12.1) with $Q=2$, and choose

$$u_1(z)=\cos z.$$

Then

$$u_2(z)=\cos z(\int_0^z\frac{Wd\zeta}{\cos^2\zeta}+C)=W\sin z+C\cos z.$$

Using (12.10) the prepotential becomes

$$\mathcal{F}_{u_1}(u)=\frac{1}{2}(u^2\int_0^{\arccos u}\frac{Wd\zeta}{\cos^2\zeta}+Cu^2-W\arccos u)=\frac{W}{2}(u\sqrt{1-u^2}-\arccos u)+\frac{C}{2}u^2.$$

Somewhat surprisingly perhaps, we start with a very simple differential equation and arrive at a rather complicated prepotential. Note that $u_1(z)=\cos z$ violates the assumption of being invertible at $z=0$. This causes $\mathcal{F}_{u_1}(u)$ to have a branch point at $u=u_1(0)=1$.

13. Glossary of notations

- $\mathbb{D}(a,r)=\{z\in\mathbb{C}:|z-a|<r\}$, $\mathbb{D}=\mathbb{D}(0,1)$.
- $\mathbb{P}=\mathbb{C}\cup\{\infty\}$, the Riemann sphere.
- $A_{r,R}=\{z\in\mathbb{C}:r<|z|<R\}$.
- Ω usually denotes a finitely connected domain in $\mathbb{P}$, with boundary components denoted $\Gamma_0,\dots\Gamma_{\mathtt{g}}$, $\mathtt{g}\geq 0$ (each Γ_j consisting of more than one point).
- $\hat{\Omega}=\Omega\cup\partial\Omega\cup\tilde{\Omega}$, the Schottky double of Ω, a symmetric compact Riemann surface of genus $\mathtt{g}$.
- $J:\hat{\Omega}\to\hat{\Omega}$, the anticonformal involution on $\hat{\Omega}$.
- α_j, β_j ($j=1,\dots,\mathtt{g}$), canonical homology basis on a compact Riemann surface of genus $\mathtt{g}$.
- $d(z,A)=\operatorname{dist}(z,A)$, distance from a point z to a set A; $d(z)=d(z,\partial\Omega)$ if $z\in\Omega$.
- ${}^*\omega$, the Hodge star of a differential form ω, for example ${}^*(adx+bdy)=-bdx+ady$, ${}^*dz=-\mathrm{i}dz$. If u is a harmonic function then ${}^*du=d(u^*)$ where u^* is a harmonic conjugate of u.
- $V(z,w;a,b)$, a fundamental potential on a compact Riemann surface.
- $E(z,\zeta)$, the Schottky-Klein prime function.
- $G(z,\zeta)$, the Green's function of a domain $\Omega\subset\mathbb{P}$.
- $H(z,\zeta)$, the regular part of the Green's function
- $\mathcal{G}(z,\zeta)=G(z,\zeta)+\mathrm{i}G^*(z,\zeta)$, the analytic completion of $G(z,\zeta)$ with respect to z. Similarly for $\mathcal{H}(z,\zeta)$.

- $P(z,\zeta) = -\frac{\partial G(z,\zeta)}{\partial n_z}$, the Poisson kernel ($\frac{\partial}{\partial n}$ outward normal derivative).
- $G_\gamma(z,\zeta)$, the hydrodynamic Green function with circulations γ.
- $\mathcal{G}_\gamma(z,\zeta)$, the analytic completion of $G_\gamma(z,\zeta)$ with respect to z.
- $N_a(z,\zeta)$, the Neumann function with Neumann data $-a$ on $\partial\Omega$.
- $\mathcal{N}_a(z,\zeta)$, the analytic completion of $N_a(z,\zeta)$ with respect to z
- υ_{a-b}, the abelian differential of the third kind with poles at a, b and having purely imaginary periods.
- ω_{a-b}, the abelian differential of the third kind with poles at a, b and with vanishing α_j-periods.
- $\tilde{\omega}_{a-b}$, the abelian differential of the third kind with poles at a, b and with vanishing β_j-periods.
- $A_m(\Omega)$, $A(\Omega) = A_1(\Omega)$: spaces of analytic functions in Ω ($m = 0,1,2,\dots$). Essentially weighted Bergman spaces of negative index. $A_0(\Omega)$ is Hardy space.
- $\mathcal{A}_m(\Omega)$, $\mathcal{A}(\Omega) = \mathcal{A}_1(\Omega)$: subspace of $A_m(\Omega)$ defined by a normalization (to make the inner product positive definite). $\mathcal{A}(\Omega)$ is Dirichlet space.
- $B_m(\Omega)$, $B(\Omega) = B_1(\Omega)$: Weighted Bergman spaces of positive index $m = 1,2,\dots$. $B(\Omega)$ ordinary Bergman space.
- $B_e(\Omega)$, $B_{m,e}(\Omega)$ ($m = 1,2,\dots$), the subspaces of $B(\Omega)$, $B_m(\Omega)$ consisting of "exact" differentials with respect to Λ_m.
- $P_m(\Omega) = \{F \in A_m(\Omega) : \Lambda_m F = 0\} = \{F \in A_m(\Omega) : (f,f)_{-m} = 0\}$, the space of $\frac{1-m}{2}$:s order differentials in Ω which expressed in any projective coordinate are polynomials of degree $\leq m-1$ ($m = 1,2,\dots$).
- $H(\Omega)$, $H_e(\Omega)$: spaces of harmonic functions.
- $(f,g)_m$, the inner product on $B_m(\Omega)$ for $m \geq 1$, on $\mathcal{A}_{-m}(\Omega)$ for $m \leq 0$.
- $D(f,g)$, Dirichlet inner product.
- $\mathcal{K}_m(z,\zeta)$, reproducing kernel for $\mathcal{A}_m(\Omega)$; $\mathcal{K}(z,\zeta) = \mathcal{K}_1(z,\zeta)$.
- $\mathcal{L}_m(z,\zeta)$, adjoint kernel for $\mathcal{A}_m(\Omega)$; $\mathcal{L}(z,\zeta) = \mathcal{L}_1(z,\zeta)$.
- $K(z,\zeta) = -\frac{2}{\pi}\frac{\partial^2 G(z,\zeta)}{\partial z \partial \bar{\zeta}}$, the Bergman kernel, reproducing kernel for $B(\Omega)$.
- $K_e(z,\zeta) = -\frac{2}{\pi}\frac{\partial^2 G_\gamma(z,\zeta)}{\partial z \partial \bar{\zeta}}$, the reduced Bergman kernel, reproducing kernel for $B_e(\Omega)$.
- $K_m(z,\zeta)$, the reproducing kernel for $B_m(\Omega)$.
- $K_{m,e}(z,\zeta)$, the reproducing kernel for $B_{m,e}(\Omega)$.
- $L(z,\zeta) = \frac{2}{\pi}\frac{\partial^2 G(z,\zeta)}{\partial z \partial \zeta}$, the Schiffer kernel, or adjoint Bergman kernel. (Similarly for $L_m(z,\zeta)$, $L_{m,e}(z,\zeta)$.)
- $\ell(z,\zeta)$, the regular part of $L(z,\zeta)$.
- $k(z,\zeta)$, $k_e(z,\zeta)$, reproducing kernels for $H(\Omega)$ and $H_e(\Omega)$ respectively.
- $M(z,\zeta)$, the ordinary Martin kernel.
- $F(z,\zeta)$, the Martin kernel for the gradient structure.
- ds, arc-length differental along a curve.
- κ, the curvature of a curve in the complex plane (and sometimes short for $\kappa_{\rm Gauss}$).
- $d\sigma = \rho(z)|dz| = \frac{|dz|}{\omega(z)} = e^{p(z)}|dz|$, a hermitean metric, e.g., the Poincaré metric.
- $\kappa_{\rm Gauss}$, the Gaussian curvature of a hermitean metric.
- $S(z)$, the Schwarz function of an analytic curve ($S(z) = \bar{z}$ on the curve, $S(z)$ analytic in a full neighborhood of the curve).
- $T(z) = \frac{dz}{ds}$, the unit tangent vector on a curve (and its analytic extension to a neighborhood of the curve, if the curve is analytic).
- $\{z,t\}_k$, a differential expression appearing in the definition of a k-connection ($k = 0,1,2$). (The Schwarzian derivative if $k = 2$.)

- ∇_k, the covariant derivative for an affine connection, when acting on k:th order differentials.
- Λ_m, the m:th order Bol operator, the covariant derivative for a projective connection, acting on differentials of order $\frac{1-m}{2}$.
- ∇, the gradient.
- $\mathcal{L}$, the Legendre transform.
- S^m, the m-fold symmetric product (of a differential operator).
- $\partial = \partial_z = \frac{\partial}{\partial z}$, $\bar{\partial} = \bar{\partial}_z = \frac{\partial}{\partial \bar{z}}$.

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