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Équations aux dérivées partielles/*Partial Differential Equations*

Coupled and uncoupled limits for a N-dimensional multidomain Neumann problem

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Abstract. We consider the quasilinear Neumann problem with exponent p, in a multi-domain of $\mathbb{R}^N$ made of the union of a cylinder with given height and small cross section and a cylinder with small height and given cross section. Assuming that the volumes of the two cylinders tend to zero with same order, we prove that the limit problem is posed in the union of the limit domains, with respective dimensions 1 and $(N-1)$. Moreover this limit problem is coupled if $p > N-1$ and uncoupled otherwise. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Limite couplée ou non pour un problème de Neumann sur un multidomaine mince

Résumé. *Nous considérons le problème de Neumann quasi linéaire d'exposant p, dans un multidomaine de $\mathbb{R}^N$ constitué d'un cylindre de hauteur donnée et de petite section et d'un cylindre de section donnée et de petite hauteur. En supposant que les volumes de ces deux cylindres tendent vers zéro tout en restant du même ordre, nous montrons que le problème limite est posé sur la réunion des domaines limites, de dimensions respectives 1 et $(N-1)$, et qu'il est couplé pourvu que $p > N-1$ et découplé sinon.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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On considère le multidomaine mince $\Omega^\varepsilon = \Omega_1^\varepsilon \cup S^\varepsilon \cup \Omega_2^\varepsilon$, où $\Omega_1^\varepsilon = r^\varepsilon\omega \times (0,1)$, $\Omega_2^\varepsilon = \omega \times (-h^\varepsilon, 0)$, $S^\varepsilon = r^\varepsilon\omega \times \{0\}$, avec ω domaine borné de $\mathbb{R}^{N-1}$ ($N \geqslant 2$), ω contenant l'origine. On suppose que r^ε et h^ε sont petits, de sorte que Ω^ε est l'union d'un cylindre vertical de petite section et d'un cylindre vertical de petite hauteur.

Note présentée par Philippe G. CIARLET.

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On s'intéresse au problème de Neumann quasi linéaire d'exposant p $(1 < p < \infty)$ dans Ω^ε, avec coefficients et terme source dépendant (éventuellement) de ε, et on l'écrit comme un problème de minimisation :

$$(\mathrm{P}^\varepsilon) \qquad \operatorname{Inf}\big\{J^\varepsilon(V)\ :\ V \in \mathrm{W}^{1,p}\big(\Omega^\varepsilon\big)\big\},$$

avec $J^\varepsilon(V) = \frac{1}{p}\int_{\Omega^\varepsilon}\big[A^\varepsilon|\nabla_{X'}V|^p + B^\varepsilon\big|\frac{\partial V}{\partial X_N}\big|^p + C^\varepsilon|V|^p\big]\,\mathrm{d}X - \int_{\Omega^\varepsilon} F^\varepsilon V\,\mathrm{d}X$, en notant $X = (X', X_N) = (X_1, \ldots, X_{N-1}, X_N)$ le point générique de $\mathbb{R}^N$, $\nabla_{X'}V = \big(\partial V/\partial X_1, \ldots, \partial V/\partial X_{N-1}\big)$.

Classiquement, $F^\varepsilon \in \mathrm{L}^{p'}(\Omega^\varepsilon)$, où p' est le conjugué de p, A^ε, B^ε, C^ε sont des fonctions strictement positives de $\mathrm{L}^\infty(\Omega^\varepsilon)$ avec $1/A^\varepsilon$, $1/B^\varepsilon$, $1/C^\varepsilon$ dans $\mathrm{L}^\infty(\Omega^\varepsilon)$. Nous supposons que $B_1^\varepsilon = B^\varepsilon_{|\Omega_1^\varepsilon}$ ne dépend que de X'/r^ε et $A_2^\varepsilon = A^\varepsilon_{|\Omega_2^\varepsilon}$ ne dépend que de X_N/h^ε :

$$B_1^\varepsilon = B^\varepsilon_{|\Omega_1^\varepsilon} = b_1^\varepsilon\big(X'/r^\varepsilon\big), \quad X \in \Omega_1^\varepsilon, \quad \text{et} \quad A_2^\varepsilon = A^\varepsilon_{|\Omega_2^\varepsilon} = a_2^\varepsilon\big(X_N/h^\varepsilon\big), \quad X \in \Omega_2^\varepsilon.$$

L'exemple le plus simple est obtenu pour $p = 2$, $A^\varepsilon \equiv B^\varepsilon \equiv C^\varepsilon \equiv 1$, $F^\varepsilon \equiv F : \Omega' \to \mathbb{R}^N$, $\Omega' \supset \Omega^\varepsilon$ pour tout ε.

Il est clair que (P^ε) admet une solution unique U^ε. On s'intéresse au comportement limite de U^ε et $J^\varepsilon(U^\varepsilon)$ lorsque r^ε et h^ε tendent vers zéro. Pour cela, nous nous ramenons à un domaine fixe en introduisant la transformation :

$$\Omega_1^\varepsilon \cup \Omega_2^\varepsilon \longrightarrow \Omega_1 \cup \Omega_2, \quad \Omega_1 = \omega \times (0,1),\ \Omega_2 = \omega \times (-1,0),$$
$$X = \big(X', X_N\big) \longmapsto x = \big(x', x_N\big),$$

avec $x' = X'/r^\varepsilon$, $x_N = X_N$ pour $X \in \Omega_1^\varepsilon$ et $x' = X'$, $x_N = X_N/h^\varepsilon$ pour $X \in \Omega_2^\varepsilon$. Pour toute fonction définie sur Ω^ε et notée par une lettre majuscule (telle F^ε), nous notons avec un indice i sa restriction à Ω_i^ε ($F_i^\varepsilon = F^\varepsilon_{|\Omega_i^\varepsilon}$) et avec une minuscule son écriture sur Ω_i ($f_i^\varepsilon(x) = F_i^\varepsilon(X)$). Avec ces notations, nous montrons le :

THÉORÈME 1. – *On suppose que* : (i) $q^\varepsilon = h^\varepsilon/(r^\varepsilon)^{N-1} \to q \in (0,\infty)$ *dans* $\mathbb{R}$; (ii) $f_i^\varepsilon \rightharpoonup f_i$ *faiblement dans* $\mathrm{L}^{p'}(\Omega_i)$ $(i = 1,2)$; (iii) *les suites* $\{1/a_i^\varepsilon\}_\varepsilon$, $\{1/b_i^\varepsilon\}_\varepsilon$, $\{1/c_i^\varepsilon\}_\varepsilon$ *sont bornées dans* $\mathrm{L}^\infty(\Omega_i)$ $(i = 1,2)$ *et* $\{a_1^\varepsilon\}_\varepsilon$ *est bornée dans* $\mathrm{L}^\infty(\Omega_1)$; (iv) $b_1^\varepsilon = b_1^\varepsilon(x')$, $a_2^\varepsilon = a_2^\varepsilon(x_N)$, $b_1^\varepsilon \rightharpoonup b_1$ *faiblement* $*$ *dans* $\mathrm{L}^\infty(\omega)$, $a_2^\varepsilon \rightharpoonup a_2$ *faiblement* $*$ *dans* $\mathrm{L}^\infty(-1,0)$; (v) $c_i^\varepsilon \rightharpoonup c_i$ *faiblement* $*$ *dans* $\mathrm{L}^\infty(\Omega_i)$ $(i = 1,2)$.

Soit alors U^ε *la solution de* (P^ε) *et soit* u^ε *son écriture dans* $\Omega_1 \cup \Omega_2$. *Sous les hypothèses précédentes,* u_i^ε *converge faiblement vers* u_i *dans* $\mathrm{W}^{1,p}(\Omega_i)$ $(i = 1,2)$*, où*

– *si* $p \leqslant N-1$, u_1 *est la solution de*

$$\operatorname{Inf}\big\{K_1(v_1),\ v_1 \in \mathrm{W}^{1,p}(0,1)\big\},$$

avec $K_1(v_1) = \frac{\beta_1}{p}\int_0^1 \big|\frac{\mathrm{d}v_1}{\mathrm{d}x_N}\big|^p\,\mathrm{d}x_N + \frac{1}{p}\int_0^1 \gamma_1(x_N)|v_1(x_N)|^p\,\mathrm{d}x_N - \int_0^1 \varphi_1(x_N)v_1(x_N)\,\mathrm{d}x_N$, $\beta_1 = \int_\omega b_1(x')\,\mathrm{d}x'$, $\gamma_1(x_N) = \int_\omega c_1(x', x_N)\,\mathrm{d}x'$, $\varphi_1(x_N) = \int_\omega f_1(x', x_N)\,\mathrm{d}x'$ *et* u_2 *est la solution de*

$$\operatorname{Inf}\big\{K_2(v_2),\ v_2 \in \mathrm{W}^{1,p}(\omega)\big\},$$

avec $K_2(v_2) = \frac{\alpha_2}{p}\int_\omega |\nabla v_2|^p\,\mathrm{d}x' + \frac{1}{p}\int_\omega \gamma_2(x')|v_2(x')|^p\,\mathrm{d}x' - \int_\omega \varphi_2(x')v_2(x')\,\mathrm{d}x'$, $\alpha_2 = \int_{-1}^0 a_2(x_N) \times \mathrm{d}x_N$, $\gamma_2(x') = \int_{-1}^0 c_2(x', x_N)\,\mathrm{d}x_N$, $\varphi_2(x') = \int_{-1}^0 f_2(x', x_N)\,\mathrm{d}x_N$;

– *si* $p > N-1$, $u = (u_1, u_2)$ *est la solution de*

$$\operatorname{Inf}\big\{K_1(v_1) + qK_2(v_2),\ v_1 \in \mathrm{W}^{1,p}(0,1),\ v_2 \in \mathrm{W}^{1,p}(\omega),\ v_1(0) = v_2(0)\big\}.$$

Dans les deux cas, $J^\varepsilon(U^\varepsilon) \simeq (r^\varepsilon)^{N-1}(K_1(u_1) + qK_2(u_2))$*, qui tend vers zéro.*

Nous remarquons que la première hypothèse est vérifiée (à une sous-suite près) dès que les volumes de Ω_1^ε et Ω_2^ε sont du même ordre de grandeur. De même la seconde hypothèse est vérifiée (à une sous-suite près) dès que les suites $\{|\Omega_1^\varepsilon|^{-1}\int_{\Omega_1^\varepsilon}|F^\varepsilon|^{p'}\,\mathrm{d}X\}_\varepsilon$ et $\{|\Omega_2^\varepsilon|^{-1}\int_{\Omega_2^\varepsilon}|F^\varepsilon|^{p'}\,\mathrm{d}X\}_\varepsilon$ sont bornées et donc en particulier si F^ε est dans L^∞ et si la suite $\{\|F^\varepsilon\|_{\mathrm{L}^\infty(\Omega^\varepsilon)}\}_\varepsilon$ est bornée, ce qui est vrai si $F^\varepsilon \equiv F \in \mathrm{L}^\infty(\Omega')$, $\Omega' \supset \Omega^\varepsilon$ pour tout ε.

De plus, le problème limite est découplé et sa solution est indépendante de q si $p \leqslant N-1$. Au contraire il est couplé et sa solution dépend de q si $p > N-1$. Dans les deux cas l'« énergie » $J^\varepsilon(U^\varepsilon)$ tend vers zéro et $J^\varepsilon(U^\varepsilon)/(r^\varepsilon)^{N-1}$ tend vers une limite dépendant de q.

Enfin, nous notons que $p > N-1$ est la condition nécessaire et suffisante pour que v soit continu (et donc $v(0)$ défini) pour tout v dans $\mathrm{W}^{1,p}(\omega)$.

La démonstration détaillée de ce résultat figure dans [9] et des compléments seront présentés dans [10]. Signalons enfin que ce travail fait suite à une étude précédente où le problème limite était dégénéré, mais la condition de transmission était plus simple (purement algébrique) et $\Omega_2^\varepsilon = \Omega_2$ était indépendant de ε (*voir* [2] et [3]).

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1. Statement of the problem and of the result

For ω a bounded domain in $\mathbb{R}^{N-1}$, $N \geqslant 2$, $0 \in \omega$, let us consider the thin multidomain $\Omega^\varepsilon = \Omega_1^\varepsilon \cup S^\varepsilon \cup \Omega_2^\varepsilon$, which is the union of two vertical cylinders with small volumes: the first one $\Omega_1^\varepsilon = r^\varepsilon\omega \times (0,1)$ has small cross section $r^\varepsilon\omega$ and constant height, the second one $\Omega_2^\varepsilon = \omega \times (-h^\varepsilon, 0)$ has small height and constant cross section, the interface is $S^\varepsilon = r^\varepsilon\omega \times \{0\}$.

Assuming that r^ε and h^ε tend to zero with ε, our concern is studying the asymptotic behaviour, as ε tends to 0, of the quasilinear Neumann problem in Ω^ε, with coefficients and source term (possibly) depending on ε, which we write as a minimization problem:

$$(\mathrm{P}^\varepsilon) \qquad \mathrm{Inf}\big\{J^\varepsilon(V),\ V \in \mathrm{W}^{1,p}\big(\Omega^\varepsilon\big)\big\},$$

with $1 < p < \infty$, $J^\varepsilon(V) = \frac{1}{p}\int_{\Omega^\varepsilon}\big[A^\varepsilon|\nabla_{X'}V|^p + B^\varepsilon\big|\frac{\partial V}{\partial X_N}\big|^p + C^\varepsilon|V|^p\big]\,\mathrm{d}X - \int_{\Omega^\varepsilon}F^\varepsilon V\,\mathrm{d}X$, $X = (X', X_N) = (X_1, \ldots, X_{N-1}, X_N)$, $\nabla_{X'}V = \big(\partial V/\partial X_1, \ldots, \partial V/\partial X_{N-1}\big)$.

Classically, $F^\varepsilon \in \mathrm{L}^{p'}(\Omega^\varepsilon)$, where p' is the conjugate of p, A^ε, B^ε, C^ε are positive functions in $\mathrm{L}^\infty(\Omega^\varepsilon)$ with inverses $1/A^\varepsilon$, $1/B^\varepsilon$, $1/C^\varepsilon$ in $\mathrm{L}^\infty(\Omega^\varepsilon)$ as well. Here we assume that $B_1^\varepsilon = B^\varepsilon_{|\Omega_1^\varepsilon}$ and $A_2^\varepsilon = A^\varepsilon_{|\Omega_2^\varepsilon}$ have specific dependence upon coordinates:

$$B_1^\varepsilon = B^\varepsilon_{|\Omega_1^\varepsilon} = b_1^\varepsilon\big(X'/r^\varepsilon\big), \quad X \in \Omega_1^\varepsilon, \quad \text{and} \quad A_2^\varepsilon = A^\varepsilon_{|\Omega_2^\varepsilon} = a_2^\varepsilon\big(X_N/h^\varepsilon\big), \quad X \in \Omega_2^\varepsilon.$$

Of course the simplest example is obtained for $p = 2$, $A^\varepsilon \equiv B^\varepsilon \equiv C^\varepsilon \equiv 1$, $F^\varepsilon \equiv F : \Omega' \to \mathbb{R}^N$, $\Omega' \supset \Omega^\varepsilon$ for every ε.

Clearly, (P^ε) admits a unique solution U^ε and we want to predict the limit behaviour of U^ε and $J^\varepsilon(U^\varepsilon)$ as ε tends to zero. This question combines the double character of homogenization [15] and boundary layers, since both the coefficients and the thin multidomain depend on ε. For solving the second difficulty, and following a classical method (*see* for example [4–6,11,14]), we consider the transform which maps $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon$ onto the fixed domain $\Omega_1 \cup \Omega_2$, $\Omega_1 = \omega \times (0,1)$, $\Omega_2 = \omega \times (-1, 0)$:

$$X = \big(X', X_N\big) \mapsto x = \big(x', x_N\big),$$
$$x' = X'/r^\varepsilon,\ x_N = X_N \quad \text{for } X \in \Omega_1^\varepsilon, \quad x' = X',\ x_N = X_N/h^\varepsilon \quad \text{for } X \in \Omega_2^\varepsilon.$$

In connection with this transform, we introduce the following notations: for a general function $G^\varepsilon : \Omega^\varepsilon \to \mathbb{R}$, we write $G_i^\varepsilon = G^\varepsilon_{|\Omega_i^\varepsilon}$ and define $g_i^\varepsilon : \Omega_i \to \mathbb{R}$ by $g_i^\varepsilon(x) = G_i^\varepsilon(X)$. This allows us to write (P$^\varepsilon$) as a minimization problem on the fixed domain $\Omega_1 \cup \Omega_2$, with coefficients a_i^ε, b_i^ε, c_i^ε and source terms f_i^ε and with a constraint depending on ε, derived from the transmission condition on S^ε:

$$(\mathcal{P}^\varepsilon) \qquad \operatorname{Inf}\{K_1^\varepsilon(v_1) + q^\varepsilon K_2^\varepsilon(v_2),\ v = (v_1, v_2) \in \mathcal{V}^\varepsilon\},$$

where $q^\varepsilon = h^\varepsilon/(r^\varepsilon)^{N-1}$,

$$\mathcal{V}^\varepsilon = \{v = (v_1, v_2) \in \mathrm{W}^{1,p}(\Omega_1) \times \mathrm{W}^{1,p}(\Omega_2),\ v_1(x', 0) = v_2(r^\varepsilon x', 0)\},$$

$$K_1^\varepsilon(v_1) = \frac{1}{p(r^\varepsilon)^p} \int_{\Omega_1} a_1^\varepsilon |\nabla_{x'} v_1|^p \,\mathrm{d}x + \frac{1}{p} \int_{\Omega_1} b_1^\varepsilon \left|\frac{\mathrm{d}v_1}{\mathrm{d}x_N}\right|^p \mathrm{d}x + \frac{1}{p} \int_{\Omega_1} c_1^\varepsilon |v_1|^p \,\mathrm{d}x - \int_{\Omega_1} f_1^\varepsilon v_1 \,\mathrm{d}x,$$

$$K_2^\varepsilon(v_2) = \frac{1}{p} \int_{\Omega_2} a_2^\varepsilon |\nabla_{x'} v_2|^p \,\mathrm{d}x + \frac{1}{p(h^\varepsilon)^p} \int_{\Omega_2} b_2^\varepsilon \left|\frac{\mathrm{d}v_2}{\mathrm{d}x_N}\right|^p \mathrm{d}x + \frac{1}{p} \int_{\Omega_2} c_2^\varepsilon |v_2|^p \,\mathrm{d}x - \int_{\Omega_2} f_2^\varepsilon v_2 \,\mathrm{d}x.$$

(Note that $J^\varepsilon(V) = (r^\varepsilon)^{N-1}(K_1^\varepsilon(v_1) + q^\varepsilon K_2^\varepsilon(v_2)) = (r^\varepsilon)^{N-1} K_1^\varepsilon(v_1) + h^\varepsilon K_2^\varepsilon(v_2)$.) Of course it is clear that U^ε solves (P$^\varepsilon$) if and only if its transcription u^ε in the fixed domain solves ($\mathcal{P}^\varepsilon$).

In our study of the limit behaviour of (P$^\varepsilon$) and ($\mathcal{P}^\varepsilon$), the essential assumptions are that $q^\varepsilon = h^\varepsilon/(r^\varepsilon)^{N-1} = |\Omega_2^\varepsilon|/|\Omega_1^\varepsilon|$ is bounded away from zero and infinity and that the sequences $\{|\Omega_i^\varepsilon|^{-1} \int_{\Omega_i^\varepsilon} |F^\varepsilon|^{p'} \times \mathrm{d}X\}_\varepsilon$ are bounded (that is $\{f_i^\varepsilon\}_\varepsilon$ is bounded in $\mathrm{L}^{p'}(\Omega_i)$) for $i = 1, 2$, which occurs for example if F^ε belongs to L^∞ and if $\{\|F^\varepsilon\|_{\mathrm{L}^\infty(\Omega^\varepsilon)}\}_\varepsilon$ is bounded, and even more particularly if $F^\varepsilon \equiv F \in \mathrm{L}^\infty(\Omega')$, $\Omega' \supset \Omega^\varepsilon$ for all ε. Then up to extraction of a subsequence, we may suppose that:

$$q^\varepsilon = h^\varepsilon/(r^\varepsilon)^{N-1} \to q \in (0, \infty), \tag{1}$$

$$f_i^\varepsilon \rightharpoonup f_i \quad \text{weakly in } \mathrm{L}^{p'}(\Omega_i)\ (i = 1, 2). \tag{2}$$

As for the coefficients we assume for simplicity:

$$\{1/a_i^\varepsilon\}_\varepsilon,\ \{1/b_i^\varepsilon\}_\varepsilon, \{1/c_i^\varepsilon\}_\varepsilon \text{ bounded in } \mathrm{L}^\infty(\Omega_i), \quad \{a_1^\varepsilon\}_\varepsilon \text{ bounded in } \mathrm{L}^\infty(\Omega_1), \tag{3}$$

$$b_1^\varepsilon = b_1^\varepsilon(x'), \qquad a_2^\varepsilon = a_2^\varepsilon(x_N), \tag{4}$$

$$b_1^\varepsilon \rightharpoonup b_1,\ a_2^\varepsilon \rightharpoonup a_2,\ c_i^\varepsilon \rightharpoonup c_i \text{ weakly* respectively in } \mathrm{L}^\infty(\omega),\ \mathrm{L}^\infty(-1, 0), \mathrm{L}^\infty(\Omega_i) \text{ for } i = 1, 2. \tag{5}$$

Our result is the following one.

THEOREM 1. – *Let U^ε be the solution of* (P$^\varepsilon$) *and let $u^\varepsilon = u_1^\varepsilon$ in Ω_1, u_2^ε in Ω_2 be both its transcription in $\Omega_1 \cup \Omega_2$ and the solution of* ($\mathcal{P}^\varepsilon$). *Under the above conditions* (1) *to* (5), *u_i^ε converges weakly in $\mathrm{W}^{1,p}(\Omega_i)$ to u_i $(i = 1, 2)$, defined as follows*:

– *if $p \leqslant N - 1$, u_1 solves*

$$\operatorname{Inf}\{K_1(v_1),\ v_1 \in \mathrm{W}^{1,p}(0, 1)\},$$

with $K_1(v_1) = \frac{\beta_1}{p} \int_0^1 \left|\frac{\mathrm{d}v_1}{\mathrm{d}x_N}\right|^p \mathrm{d}x_N + \frac{1}{p} \int_0^1 \gamma_1(x_N) |v_1(x_N)|^p \,\mathrm{d}x_N - \int_0^1 \varphi_1(x_N) v_1(x_N) \,\mathrm{d}x_N$, $\beta_1 = \int_\omega b_1(x') \,\mathrm{d}x'$, $\gamma_1(x_N) = \int_\omega c_1(x', x_N) \,\mathrm{d}x'$, $\varphi_1(x_N) = \int_\omega f_1(x', x_N) \,\mathrm{d}x'$ *and u_2 solves*

$$\operatorname{Inf}\{K_2(v_2),\ v_2 \in \mathrm{W}^{1,p}(\omega)\},$$

with $K_2(v_2) = \frac{\alpha_2}{p} \int_\omega |\nabla v_2|^p \,\mathrm{d}x' + \frac{1}{p} \int_\omega \gamma_2(x') |v_2(x')|^p \,\mathrm{d}x' - \int_\omega \varphi_2(x') v_2(x') \,\mathrm{d}x'$, $\alpha_2 = \int_{-1}^0 a_2(x_N) \times \mathrm{d}x_N$, $\gamma_2(x') = \int_{-1}^0 c_2(x', x_N) \,\mathrm{d}x_N$, $\varphi_2(x') = \int_{-1}^0 f_2(x', x_N) \,\mathrm{d}x_N$,

– *if* $p > N-1$, $u=(u_1,u_2)$ *solves*

$$\operatorname{Inf}\big\{K_1(v_1)+qK_2(v_2),\ v_1\in \mathrm{W}^{1,p}(0,1),\ v_2\in \mathrm{W}^{1,p}(\omega),\ v_1(0)=v_2(0)\big\}.$$

In any case $J^\varepsilon(U^\varepsilon)=(r^\varepsilon)^{N-1}K_1^\varepsilon(u_1^\varepsilon)+h^\varepsilon K_2^\varepsilon(u_2^\varepsilon)\simeq(r^\varepsilon)^{N-1}(K_1(u_1)+qK_2(u_2))$, *which tends to zero.*

This theorem says that the limit problem is uncoupled, with solution independent of q if $p\leqslant N-1$. On the contrary it is coupled with solution depending on q if $p>N-1$. In both cases the "energy" $J^\varepsilon(U^\varepsilon)$ tends to zero and $J^\varepsilon(U^\varepsilon)/(r^\varepsilon)^{N-1}$ tends to a limit depending of q. Remark that for $N=3$, $p=2$ is the limit exponent, so that the coupling is lost at the limit for the linear Neumann problem.

Finally let us emphasize that the condition $p>N-1$ is necessary and sufficient for having v continuous (and hence $v(0)$ meaningful) for any v in $\mathrm{W}^{1,p}(\omega)$.

2. Idea of the proof

Our result is established by using a Γ-convergence method (*see* in general [1,7,8] and in the context of thin structures [12,13]). It can be easily deduced via the following steps.

LEMMA 1 (a priori estimates). – *For* $i=1,2$, *the sequence* $\{u_i^\varepsilon\}_\varepsilon$ *is bounded in* $\mathrm{W}^{1,p}(\Omega_i)$. *Moreover, the sequences*

$$\left\{\frac{1}{(r^\varepsilon)^p}\int_{\Omega_1}\left|\nabla_{x'}u_1^\varepsilon\right|^p\mathrm{d}x\right\}_\varepsilon \quad \textit{and} \quad \left\{\frac{1}{(h^\varepsilon)^p}\int_{\Omega_2}\left|\frac{\partial}{\partial x_N}u_2^\varepsilon\right|^p\mathrm{d}x\right\}_\varepsilon$$

are bounded in $\mathbb{R}$.

LEMMA 2 (compactness and limit constraint). – *Up to extraction of a subsequence,* $u_i^\varepsilon\rightharpoonup\overline{u}_i$ *weakly in* $\mathrm{W}^{1,p}(\Omega_i)$ $(i=1,2)$, *for some* $\overline{u}_i$ *in* $\mathrm{W}^{1,p}(\Omega_i)$ *such that* $\overline{u}_1\equiv\overline{u}_1(x_n)$, $\overline{u}_2\equiv\overline{u}_2(x')$. *Moreover,* $\overline{u}_1(0)=\overline{u}_2(0)$ *if* $p>N-1$.

The fact that $\overline{u}_1\equiv\overline{u}_1(x_n)$ (resp. $\overline{u}_2\equiv\overline{u}_2(x')$) follows from the boundedness of $(r^\varepsilon)^{-1}\nabla_{x'}u_1^\varepsilon$ in $\mathrm{L}^p(\Omega_1)$ (resp. $(h^\varepsilon)^{-1}\frac{\partial}{\partial x_N}u_2^\varepsilon$ in $\mathrm{L}^p(\Omega_2)$). It is more difficult to prove that $\overline{u}_1(0)=\overline{u}_2(0)$ if $p>N-1$. This is done by passing to the limit in

$$\int_\omega u_1^\varepsilon(x',0)\,\mathrm{d}x'=\int_\omega u_2^\varepsilon(r^\varepsilon x',0)\,\mathrm{d}x'=\int_\omega\big(u_2^\varepsilon(r^\varepsilon x',0)-\overline{u}_2(r^\varepsilon x')\big)\,\mathrm{d}x'+\int_\omega\overline{u}_2(r^\varepsilon x')\,\mathrm{d}x',$$

using the continuity of $\overline{u}_2$ for $p>N-1$ and proving that the first integral tends to zero for a convenient new subsequence, which follows from Fatou lemma and from the boundedness of $(h^\varepsilon)^{-1}\frac{\partial}{\partial x_N}u_2^\varepsilon$ in $\mathrm{L}^p(\Omega_2)$.

LEMMA 3 (first Γ-convergence property). – *For any* $i=1,2$ *and for any* v_i^ε, v_i *in* $\mathrm{W}^{1,p}(\Omega_i)$ *such that* $v_i^\varepsilon\rightharpoonup v_i$ *weakly in* $\mathrm{W}^{1,p}(\Omega_i)$,

$$\liminf K_i^\varepsilon(v_i^\varepsilon)\geqslant K_i(v_i).$$

The proof of Lemma 3 is quite classical (*see*, e.g., [15]): one uses the special dependence (4) of b_1^ε and a_2^ε upon coordinates.

LEMMA 4 (second Γ-convergence property). – *For any* $v=(v_1,v_2)$ *in* $\mathrm{W}^{1,\infty}(0,1)\times\mathrm{W}^{1,\infty}(\omega)$ *such that* $v_1(0)=v_2(0)$, *there exists a sequence* $\{v^\varepsilon\}_\varepsilon$ *of elements* $v^\varepsilon=(v_1^\varepsilon,v_2^\varepsilon)$ *in* $\mathcal{V}^\varepsilon$ *such that* $K_i^\varepsilon(v_i^\varepsilon)\longrightarrow K_i(v_i)$.

This lemma can be proved with: $v_2^\varepsilon\equiv v_2\equiv v_2(x')$ in Ω_2, $v_1^\varepsilon(x',x_N)\equiv v_1(x_N)$ for $\alpha^\varepsilon<x_n<1$, for some suitable α^ε (α^ε tends to zero with $\alpha^\varepsilon\geqslant Cr^\varepsilon$ for some positive constant C),

$$v_1^\varepsilon(x',x_N)=v_1(\alpha^\varepsilon)\,\frac{x_N}{\alpha^\varepsilon}+v_2(r^\varepsilon x')\,\frac{\alpha^\varepsilon-x_N}{\alpha^\varepsilon}\quad\text{for } 0<x_n<\alpha^\varepsilon.$$

LEMMA 5 (density property). – *Let*

$$\mathcal{W} = \{v = (v_1, v_2) \in \mathrm{W}^{1,\infty}(0,1) \times \mathrm{W}^{1,\infty}(\omega),\ \ v_1(0) = v_2(0)\},$$
$$\mathcal{V}^p = \{v = (v_1, v_2) \in \mathrm{W}^{1,p}(0,1) \times \mathrm{W}^{1,p}(\omega)\} \quad \textit{if } p \leqslant N-1,$$
$$\mathcal{V}^p = \{v = (v_1, v_2) \in \mathrm{W}^{1,p}(0,1) \times \mathrm{W}^{1,p}(\omega),\ v_1(0) = v_2(0)\} \quad \textit{if } p > N-1.$$

The subspace $\mathcal{W}$ is dense in $\mathcal{V}^p$.

This density result is classical for $p > N-1$. If $p \leqslant N-1$, it is enough to prove that for any v in $\mathcal{C}^1[0,1] \times \mathcal{C}^1(\overline{\omega})$, there exists a sequence of elements v^N in $\mathcal{W}$ (in particular $v_1^n(0) = v_2^n(0)$) with $v^n \to v$ for the norm of $\mathrm{W}^{1,p}(0,1) \times \mathrm{W}^{1,p}(\omega)$. One can take $v_1^n \equiv v_1$,

$$v_2^n \equiv v_2 \quad \text{in } \omega \setminus \mathrm{B}(R^n), \qquad v_2^n \equiv v_1(0) \quad \text{in } \mathrm{B}(r^n),$$
$$v_2^n = v_2^n(x') = \phi^n(x') v_1(0) + (1 - \phi^n(x')) v_2^n(x') \quad \text{in } C^n = \mathrm{B}(R^n) \setminus \mathrm{B}(r^n),$$

where $\mathrm{B}(r^n)$ and $\mathrm{B}(R^n)$ are small balls in $\mathbb{R}^{N-1}$ with respective radii r^n and R^n tending to zero ($0 < r^n < R^n$) and where $\phi^n = 0$ on $\partial\mathrm{B}(R^n)$, $\phi^n = 1$ on $\partial\mathrm{B}(r^n)$, $0 \leqslant \phi^n \leqslant 1$ in C^n and

$$\int_{C^n} |\nabla \phi^n|^p \, \mathrm{d}x \longrightarrow 0.$$

If $p < N-1$, it is enough to take $r^n = 1/n$, $R^n = 2/n$, $\phi^n(x') \equiv n d^n(x')$, where d^n is the distance to $\partial\mathrm{B}(R^n)$. If $p = N-1$, one can take for ϕ^n the solution of the p-capacity problem in C^n with R^n/r^n tending to infinity.

The detailed proofs are in [9] and some complements can be found in [10]. Finally, let us mention that in a recent work (*see* [2] and [3]), M. Boutkrida and J. Mossino have considered the case $\Omega_2^\varepsilon = \Omega_2$ independent of ε, with a simpler (purely algebraic) transmission condition, but with a degenerate limit problem.

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