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On the Curvature of the Free Boundary for the Obstacle Problem in Two Dimensions*

By

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Abstract. We give a new proof of the fact that the free boundary for the obstacle problem in two dimensions satisfies a natural and sharp inner ball condition.

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1. Introduction

In a recent paper [5], the authors gave sharp upper bounds on the curvature of the free boundary in some obstacle type problems in two dimensions. The upper bounds are equivalent to that natural inner ball conditions hold. The proof was based on conformal mapping together with a computational argument showing the positivity of a certain big polynomial in several variables. The purpose of this note is to give a different proof of the same result. The new proof is based on the maximum principle combined with a topological argument and does not involve conformal mapping.

Upper bounds for the curvature have previously been obtained by Schaeffer [10] by using techniques of conformal and quasiconformal mappings. The estimates in [10] are less sharp than ours, but are applied to more general problems.

2. Statement of Result

The obstacle problem is the problem of finding the smallest superharmonic function passing above a given obstacle. With the simplest possible assumptions on the data, the difference u between the superharmonic solution and the given

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obstacle function has the following properties, in a small ball B away from fixed boundaries and source terms (cf. [3, 7]).

$$u \in C^1(B),$$
$$u \geqslant 0 \ \text{in}\, B,$$
$$\Delta u = \chi_\Omega \ \text{in}\, B,$$

where

$$\Omega = \{z \in B : u(z) > 0\}$$

is the noncoincidence set and χ_Ω denotes the characteristic function of Ω.

By definition of Ω,

$$u = 0 \quad \text{on}\, B\setminus\Omega$$

and, since this is the minimum value of u,

$$\nabla u = 0 \quad \text{on}\, B\setminus\Omega.$$

Thus, $u = |\nabla u| = 0$ on the free boundary $\partial\Omega \cap B$.

It is well-known that u is automatically more regular than stated above, e.g. that $u \in C^{1,1}_{\mathrm{loc}}(B)$, and that the free boundary to most parts is smooth real analytic ([1–3], [7–9]). Only few results seem to be known about the curvature of $\partial\Omega$, and this is what we study here.

We are only able to handle the two-dimensional case at present. Indeed, our first treatment [5] used conformal mapping and the method in this note involves a topological argument which is not immediately applicable in higher dimension.

For convenience we shall use complex variable notations, e.g. $z = x + iy$ for points in $\mathbb{R}^2 = \mathbb{C}$. To state the result we normalize B to be centred at the origin, say:

$$B = \{z \in \mathbb{C} : |z| < R\}$$

$(R > 0)$. In general we write

$$B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$$

and, for any set $A \subset \mathbb{C}$,

$$A^+ = \{x + iy \in A : y > 0\},$$
$$A^- = \{x + iy \in A : y < 0\}.$$

The main result (previously proved in [5]) reads as follows.

Theorem 2.1. *Assume that Ω^+ is relatively compact in B. Then Ω^+ is a union of semidiscs centred on $\mathbb{R}$, i.e., there exist $r(x) > 0$ for $x \in \Omega \cap \mathbb{R}$ such that*

$$\Omega^+ = \cup_{x \in \Omega\cap\mathbb{R}} B(x, r(x))^+. \tag{2.1}$$

The relative compactness of Ω^+ means that the free boundary reaches ∂B only via the (closed) lower half-plane, so that Ω^+ is a "cap" of Ω sticking up from the lower half-plane. Clearly, starting from $\partial\Omega$ in a global version of the obstacle problem, the theorem can be applied, by choosing B properly, at any part of $\partial\Omega$

where the curvature is positive (Ω convex). Thus the theorem gives local upper bounds on the curvature of $\partial\Omega$. (At points where the curvature is not positive, zero will be the upper bound.)

For several conditions equivalent to (2.1), see Proposition 2.1 of [5]. For the proof we shall use (2.1) in the following equivalent form.

Lemma 2.2. *The open set* $\Omega^+ \subset \mathbb{C}^+$ *is of the form* (2.1) *if and only if for every semicircle*

$$C_{a,r} = \partial B(a,r)^+ \tag{2.2}$$

centred on the real axis $(a \in \mathbb{R}, r > 0)$, $C_{a,r} \backslash \Omega^+$ *is connected.*

The limiting case $r \to \infty$ (together with $a \to \pm\infty$) of the statement for $C_{a,r}$ is the previously known fact that for every vertical semiline

$$L_a = \{a + iy : y > 0\}$$

$L_a \backslash \Omega^+$ is connected. It is interesting to observe that the semicircles and semilines above are exactly the geodesics for the Poincaré metric in the upper half-plane. Thus Lemma 2.2 expresses that $(\mathbb{C}\backslash\Omega)^+$ is convex with respect to that Poincaré metric.

On the other hand, (2.1) can be written

$$(\mathbb{C}\backslash\Omega)^+ = \bigcap_{x \in \Omega\cap\mathbb{R}} (\mathbb{C}\backslash B(x, r(x))^+,$$

saying that $(\mathbb{C}\backslash\Omega)^+$ is an intersection between Poincaré half-spaces $(\mathbb{C}\backslash B(a,r))^+ (a \in \mathbb{R}, r > 0)$. Thus the equivalence stated in Lemma 2.2 is just a Poincaré metric version of the well-known fact, in ordinary convexity theory, that a closed set is convex if and only if it is an intersection between closed half-spaces (see e.g. [6], Theorem 2.1.10 with corollaries). The lemma is proved just by imitating the proof of that convexity result. (We omit the details.)

3. The Proof

We now turn to the proof of Theorem 2.1.

Proof. We shall argue by contradiction and use Lemma 2.2. We keep the notation (2.2) for semicircles.

Assume that the property in Lemma 2.2 fails. This means that there exist $a \in \mathbb{R}$ and $r > 0$ such that $C_{a,r} \backslash \Omega^+$ has at least two components. In other words, there exist $z_1, z_2 \in C_{a,r} \backslash \Omega^+$ such that the part of $C_{a,r}$ which is between z_1 and z_2 passes through Ω^+. We may choose z_1 and z_2 so that the whole segment of $C_{a,r}$ between z_1 and z_2 is in Ω^+. Then $z_1, z_2 \in \partial\Omega^+$.

Let (r, θ) be polar coordinates centred at $a \in \mathbb{R}$, so that a generic point in $\mathbb{C}^+$ is written $z = a + re^{i\theta} (r > 0, 0 < \theta < \pi)$. We shall study the function

$$\frac{\partial u}{\partial \theta} = -y \frac{\partial u}{\partial x} + (x - a) \frac{\partial u}{\partial y}$$

in Ω^+. Since $\Delta u = 1$ in Ω^+ and the coefficients in

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

do not depend on θ we have

$$\Delta \frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}\Delta u = 0 \ \text{ in}\, \Omega^+.$$

On $\partial\Omega \cap B^+$, $u = \frac{\partial u}{\partial \theta} = 0$ because u and ∇u vanish on $B\backslash\Omega$. Next it is known [4] that for all $z = x + iy \in B^+$,

$$u(x+iy) \leqslant u(x-iy). \tag{3.1}$$

The reason that (3.1) holds is, very briefly and in terms of the underlying obstacle formulation, that in case it failed then replacement of $u(x+iy)$ in $\mathbb{C}^+$ by the function $\inf\{u(x+iy), u(x-iy)\}$ would give a smaller superharmonic function passing the obstacle. From (3.1) we get $\frac{\partial u}{\partial y} \leqslant 0$ on $B \cap \mathbb{R}$, which shows that

$$\frac{\partial u}{\partial \theta} \leqslant 0 \quad \text{for } y = 0, \quad x > a,$$
$$\frac{\partial u}{\partial \theta} \geqslant 0 \quad \text{for } y = 0, \quad x < a.$$

In summary, $\frac{\partial u}{\partial \theta}$ is harmonic in Ω^+ and we know its sign on all of $\partial(\Omega^+)$: $\frac{\partial u}{\partial \theta} = 0$ on $\partial(\Omega^+)\backslash\mathbb{R}$, $\frac{\partial u}{\partial \theta} \leqslant 0$ on $\partial(\Omega^+) \cap \mathbb{R}$ to the right of a and $\frac{\partial u}{\partial \theta} \geqslant 0$ on $\partial(\Omega^+) \cap \mathbb{R}$ to the left of a. At $z = a$, $\frac{\partial u}{\partial \theta} = 0$.

According to the beginning of the proof there exist points $z_1 = a + re^{i\theta_1}$, $z_2 = a + re^{i\theta_2}$ with $0 < \theta_1 < \theta_2 < \pi$ such that $z_1, z_2 \in \partial\Omega^+$ and $z = a + re^{i\theta} \in \Omega^+$ for all $\theta_1 < \theta < \theta_2$. Since $u(z_1) = u(z_2) = 0$ and $u(z) > 0$ (by definition of Ω) we have, integrating along $C_{a,r}$,

$$\int_{z_1}^{z} \frac{\partial u}{\partial \theta} d\theta = u(z) - u(z_1) > 0,$$
$$\int_{z}^{z_2} \frac{\partial u}{\partial \theta} d\theta = u(z_2) - u(z) < 0.$$

Therefore there must be points $z = a + re^{i\theta}$ with $\theta_1 < \theta < \theta_2$ arbitrarily close to θ_1 for which $\frac{\partial u}{\partial \theta} > 0$. Similarly, there must be points with $\theta_1 < \theta < \theta_2$ arbitrarily close to θ_2 for which $\frac{\partial u}{\partial \theta} < 0$.

Now we apply the maximum principle: every component of $\{\frac{\partial u}{\partial \theta} > 0\}$ must reach (have in its closure) parts of $\mathbb{R}$ which are to the left of a, because on all other possible parts of the boundary of that component we know that $\frac{\partial u}{\partial \theta} \leqslant 0$. Similarly, every component of $\{\frac{\partial u}{\partial \theta} < 0\}$ must reach parts of $\mathbb{R}$ which are to the right of a. But it is obviously topologically impossible to have components of $\{\frac{\partial u}{\partial \theta} > 0\}$ stretching from points arbitrarily close to z_1 (the rightmost end point of the described segment of $C_{a,r}$) to parts of $\mathbb{R}$ to the left of a and simultaneously components of $\{\frac{\partial u}{\partial \theta} < 0\}$ stretching from points arbitrarily close to z_2 to parts of $\mathbb{R}$ to the right of a.

This contradiction finishes the proof. $\square$

Theorem 2.1 can be augmented to given global geometric statements concerning other free boundary problems related to the obstacle problem, e.g. Hele-Shaw flow moving boundary problems and free boundary problems in potential theory (quadrature domains, partial balayage). It also gives a new proof of the fact [5] that for any positive measure μ on the interval $(-1, 1)$ the Cauchy transform

$$f(w) = \int \frac{d\mu(t)}{t - w} \quad (|w| > 1)$$

maps the exterior of the unit disc onto a domain Ω having the inner ball property in Theorem 2.1.

We refer to [5] for details on the above matters.

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