Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 328 (2007) 995–1006
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Analytic continuation of the exponential transform from convex cavities

Björn Gustafsson [a,1], Mihai Putinar [b,*,2]

[a] Mathematics Department, Royal Institute of Technology, S-10044 Stockholm, Sweden
[b] Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

Received 14 February 2006

Available online 5 July 2006

Submitted by Steven G. Krantz

Abstract

The analytic continuation of the exponential transform of a domain in $\mathbf{R}^n$ is proved under some global geometric assumptions on the boundary. Two approximation schemes of the continued transform (one based on a Taylor series truncation, the other on a global eigenfunction expansion) are also discussed.

Keywords: Exponential transform; Analytic continuation; Riesz potential; Strictly convex domain

1. Introduction

The exponential transform, as defined below, is a renormalized Riesz potential $\mu * |x|^{-n-\alpha}$ at critical exponent $\alpha = 0$, on $\mathbf{R}^n$. This object has appeared in low dimensions (i.e., $n = 1, 2$) in the theory of moments of bounded densities μ, and has well served there as a tool in resolving various inverse problems, see the monograph [13] and the survey [14]. Apparently the 1D transform was discovered by A.A. Markov around 1880 in his analysis of extremal values of certain integrals appearing in probability theory, see for comments [13].

* Corresponding author.
E-mail addresses: gbjorn@math.kth.se (B. Gustafsson), mputinar@math.ucsb.edu (M. Putinar).
1 The author was partially supported by the Swedish research Council and by the Göran Gustafsson Foundation.
2 The author was partially supported by the NSF grant DMS 0350911.

doi:10.1016/j.jmaa.2006.06.022

In the case of two real variables, all quadrature domains (in the sense of Aharonov and Shapiro [1,15,16]) carry a rational exponential transform. This function encodes in an optimal way the algebraic defining equation of the boundary of the quadrature domain and has different minimal realizations (one of them in terms of the resolvent of a matrix, very much in the spirit of transfer functions appearing in linear control theory). As a byproduct of this formalism a finite step algorithm has the effect of transforming:

$$\{\text{moments of } \Omega\} \longrightarrow \{\text{defining equation of } \Omega\}.$$

This operation is exact for every planar quadrature domain Ω, see [4]. For arbitrary bounded domains, an analysis of their approximation (in moments) by quadrature domains is reflected at the level of the exponential transforms by a diagonal Padé type approximation.

The above (quintessential) transformation is of course familiar to all beneficiaries of the exact or approximate inversion formulas for integral transforms of Riesz type, such as Radon transforms and spherical mean type integrals [3,9–11]. It is also worth mentioning that there are other deviations from the standard normalization of the $\alpha = 0$ Riesz potential in the context of Marcinkiewicz integrals, see, for instance, [12].

Besides the above application, the 2D exponential transform has proved to be useful in understanding the regularity of certain free boundaries [5].

An analysis of higher-dimensional exponential transforms was started in [7]. In dimensions greater or equal than 3 some strong positivity features (such as log-subharmonicity) of the exponential transform are lost, see [7,17]. However, on quadrics, convex polyhedra or spherical shells the nD exponential transform preserves and reveals a very simple geometric feature: it selects, based solely on distant moment data, a simple defining function for the boundary of the domain. Note that Green's function or other classical reproducing kernels attached to a domain of $\mathbf{R}^n$ are not simple expressions of its moments (or equivalently of tomographic data).

The present note is concerned with a specific technical detail in the analysis of the nD exponential transform of a domain Ω possessing a strictly convex component U in its complement. Namely we prove the possibility of continuing the exponential transform across the smooth real analytic portions of ∂U, coming from inside the cavity U. There is evidence, based on examples and the better understood low-dimensional analysis, that this analytic continuation phenomenon persists on more general real analytic smooth boundaries, cf. [6].

The efficient rational approximation scheme (diagonal Padé table), well known in dimensions $d = 1, 2$, is missing in higher dimensions. As possible alternatives we discuss (and illustrate on examples) two different approximation schemes for the analytic continuation of the exponential transform. These computations, although still conceptual at this stage, might be of interest for further applications to shape reconstruction from finitely many data.

2. The exponential transform

We recall in this section the basic definitions, conventions and a few facts about the exponential transform. The reader can consult for details [5,7].

The (*exterior*) *exponential transform* of an open set $\Omega \subset \mathbb{R}^n$ is defined for $x \in \mathbb{R}^n \setminus \overline{\Omega}$ as

$$E(x) = \exp\left[-\frac{2}{|S^{n-1}|} \int_{\Omega} \frac{dy}{|x-y|^n}\right]. \tag{1}$$

For $x \in \Omega$ there is a corresponding *interior exponential transform* $H(x)$, which can, for example, be defined as the inverse of a rescaled version of the exterior exponential transform of the complementary domain

$$H(x) = \lim_{R\to\infty} \frac{1}{R^2} \exp\left[\frac{2}{|S^{n-1}|} \int_{B_R\setminus\Omega} \frac{dy}{|x-y|^n}\right] \quad (x \in \Omega).$$

Here $B_R = B_R(0)$ denotes the ball of radius R and centered at the origin. If $\Omega \subset B_R$, then there is also the more direct representation

$$H(x) = \frac{1}{R^2 - |x|^2} \exp\left[\frac{2}{|S^{n-1}|} \int_{B_R\setminus\Omega} \frac{dy}{|x-y|^n}\right].$$

If $\partial\Omega$ is smooth and Ω is bounded, the integrals above can be transformed into boundary integrals. Then the exterior and interior exponential transforms turn out to be given by exactly the same formula:

$$E(x) = \exp\left[-\frac{2}{|S^{n-1}|} \int_{\partial\Omega} \log|x-y|\, d\theta(y-x)\right] \quad (x \notin \overline{\Omega}), \tag{2}$$

$$H(x) = \exp\left[-\frac{2}{|S^{n-1}|} \int_{\partial\Omega} \log|x-y|\, d\theta(y-x)\right] \quad (x \in \Omega). \tag{3}$$

Here $d\theta$ is the solid angle differential form, i.e., the $(n-1)$-form defined in $\mathbb{R}^n \setminus \{0\}$ by

$$\begin{aligned} d\theta(x) &= \frac{*(x_1\,dx_1 + \cdots + x_n\,dx_n)}{|x|^n} \\ &= \frac{x_1\,dx_2 \cdots dx_n - x_2\,dx_1\,dx_3 \cdots dx_n + \cdots + (-1)^{n-1} x_n\,dx_1 \cdots dx_{n-1}}{|x|^n}. \end{aligned}$$

Above, and throughout this note we omit the wedge sign for the product of differential forms. The star is the Hodge star and $\partial\Omega$ is provided with its natural orientation (as a boundary of Ω). The interpretation of $d\theta$ is that if $A \subset \mathbb{R}^n \setminus \{0\}$ is a piece of a hypersurface then $\int_A d\theta$ equals the area of the radial projection of A onto S^{n-1}, with multiplicities and signs taken into account. If $A \subset S^{n-1}$ then $\int_A d\theta$ simply is the area of A.

If Ω is convex, then $d\theta(y-x)$ can, for fixed $x \in \Omega$, be considered as a positive measure of total measure $|S^{n-1}|$ on $\partial\Omega$. Hence $H(x)$ can in this case be interpreted as the geometric mean of $\frac{1}{|x-y|^2}$ over $y \in \partial\Omega$ with respect to $d\theta(y-x)$.

Example 2.1. For the ball $\Omega = B_R(0)$, we have $H(x) = \frac{1}{R^2-|x|^2}$ $(x \in \Omega)$ in all dimensions. The expression for $E(x)$ is slightly more complicated and depends on the dimension. For example, $E(x) = 1 - \frac{R^2}{|x|^2}$ for $n = 2$, $E(x) = \frac{|x|-R}{|x|+R} \exp[\frac{2R}{|x|}]$ for $n = 3$.

For further examples, see [7].

3. Main result

The aim of the present section is to prove the analytic continuation of the inverse of the interior exponential transform across strictly convex, smooth real analytic boundaries. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. By saying that Ω has a smooth real analytic boundary we mean that there exist a neighborhood V of $\partial\Omega$ and a real analytic function $q: V \to \mathbb{R}$ such that

$$(\partial\Omega) \cap V = \{x \in V\colon\ q(x) = 0\}$$

and such that $\nabla q \neq 0$ on $\partial\Omega$. When this is the case we shall always choose V and q so that q is negative in Ω and positive outside:

$$\Omega \cap V = \{x \in V\colon\ q(x) < 0\}, \tag{4}$$

and also so that q is actually real analytic in a full neighborhood of $\overline{V}$. With Ω, V and q as above, we say that Ω is *strictly convex* if Ω is convex and if the Hessian of q is positive definite on $\partial\Omega$. The latter means that for every $x \in \partial\Omega$ and every $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_{i,j=1}^{n} \frac{\partial^2 q(x)}{\partial x_i \partial x_j} \xi_i \xi_j > 0.$$

Theorem 3.1. *Let $\Omega \in \mathbb{R}^n$ be a strictly convex bounded domain with smooth real analytic boundary. Then the there exists a real analytic function F defined in some neighborhood of $\overline{\Omega}$ such that*

$$F(x) = \frac{1}{H(x)} \quad \textit{for } x \in \Omega.$$

In other words, F is a real analytic continuation of $\frac{1}{H}$.

On $\partial\Omega$, F satisfies $F = 0$, $\nabla F \neq 0$, hence serves as a global real analytic defining function of $\partial\Omega$.

Proof. Choose V and q as in the definitions preceding the theorem. We shall use the boundary integral presentation of H,

$$H(x) = \exp\left[-\frac{2}{|S^{n-1}|} \int_{\partial\Omega} \log|x - y|\, d\theta(y - x)\right] \quad (x \in \Omega),$$

and build upon the same basic idea as was used in the proof of Theorem 4.4 in [7].

Choose a point $x \in \Omega$ and a direction $\omega \in S^{n-1}$. Then the straight line

$$L(x, \omega) = \{x + t\omega\colon\ t \in \mathbb{R}\}$$

through x with direction ω intersects $\partial\Omega$ in exactly two points. Let $t = t_1(x, \omega)$ and $t = t_2(x, \omega)$ be the parameter values for these intersection points, ordered so that

$$t_1(x, \omega) < 0 < t_2(x, \omega).$$

The function $t \mapsto q(x + t\omega)$ need not be defined for all $t \in \mathbb{R}$, but it is at least defined for t close to $t_1(x, \omega)$ and $t_2(x, \omega)$, and it has simple zeros at these points. Thus we can factor $q(x + t\omega)$ as

$$\begin{aligned} q(x + t\omega) &= \big(t - t_1(x, \omega)\big)\big(t - t_2(x, \omega)\big) p(x, \omega, t) \\ &= \big(t^2 + a(x, \omega)t + b(x, \omega)\big) p(x, \omega, t), \end{aligned} \tag{5}$$

where

$$a(x, \omega) = -t_1(x, \omega) - t_2(x, \omega), \qquad b(x, \omega) = t_1(x, \omega) t_2(x, \omega),$$

and where $p(x,\omega,t)$ is a real-valued non-vanishing function defined for triples $(x,\omega,t)\in\Omega\times S^{n-1}\times\mathbb{R}$ satisfying $x+t\omega\in V$. By (4) the polynomial $t^2+a(x,\omega)t+b(x,\omega)$ has the same sign as $q(x+t\omega)$, hence we even have

$$p(x,\omega,t)>0. \tag{6}$$

The roots $t_1(x,\omega)$, $t_2(x,\omega)$ and coefficients $a(x,\omega)$, $b(x,\omega)$ are defined for all $(x,\omega)\in\Omega\times S^{n-1}$. We note that the roots are exchanged as $\omega\mapsto-\omega$:

$$t_1(x,-\omega)=t_2(x,\omega) \tag{7}$$

while the other functions are invariant under this map:

$$a(x,-\omega)=a(x,\omega),\qquad b(x,-\omega)=b(x,\omega),\qquad p(x,-\omega,t)=p(x,\omega,t).$$

If $x\in\Omega\cap V$ we may choose $t=0$ in (5), to obtain

$$t_1(x,\omega)t_2(x,\omega)=b(x,\omega)=\frac{q(x)}{p(x,\omega,0)}. \tag{8}$$

Next we note that, due to the strict convexity of Ω, the above analysis extends to $\overline{\Omega}$, i.e., we may allow $x\in\partial\Omega$. Indeed, if $x\in\partial\Omega$ and the line $L(x,\omega)$ is not tangent to $\partial\Omega$, then one of the roots $t_j(x,\omega)$ is zero, and the other non-zero. Hence $b(x,\omega)=0$, $a(x,\omega)\neq 0$. If $L(x,\omega)$ is tangent to $\partial\Omega$, then both roots are zero, hence $a(x,\omega)=b(x,\omega)=0$. Most important is that in each of these cases $p(x,\omega,t)>0$, in particular $p(x,\omega,0)>0$. For example, in the case of tangency we have

$$q(x+t\omega)=t^2p(x,\omega,t).$$

Taking two derivatives with respect to t and evaluating at $t=0$ gives

$$\sum_{i,j=1}^{n}\omega_i\omega_j\frac{\partial^2 q(x)}{\partial x_i\partial x_j}=2p(x,\omega,0),$$

and here the left member is >0 by assumption. In summary, (5), (6) hold for all $(x,\omega,t)\in\overline{\Omega}\times S^{n-1}\times\mathbb{R}$ satisfying $x+t\omega\in V$.

Now, in the boundary integral representation (3) of $H(x)$ we can write

$$y=x+t\omega$$

with $\omega\in S^{n-1}$ and t one of the roots $t_1(x,\omega)$ or $t_2(x,\omega)$. We prefer to use both roots simultaneously, and then every point $y\in\partial\Omega$ gets represented twice as ω runs through S^{n-1}. Therefore we get, using (8)

$$\begin{aligned}
H(x)&=\exp\biggl[-\frac{2}{|S^{n-1}|}\int\limits_{\partial\Omega}\log|x-y|\,d\theta(y-x)\biggr]\\
&=\exp\biggl[-\frac{1}{|S^{n-1}|}\int\limits_{S^{n-1}}\bigl(\log\bigl|t_1(x,\omega)\bigr|+\log\bigl|t_2(x,\omega)\bigr|\bigr)\,d\theta(\omega)\biggr]\\
&=\exp\biggl[-\frac{1}{|S^{n-1}|}\int\limits_{S^{n-1}}\log\bigl|b(x,\omega)\bigr|\,d\theta(\omega)\biggr]
\end{aligned}$$

$$= \exp\left[-\frac{1}{|S^{n-1}|}\int\limits_{S^{n-1}} \log\frac{-q(x)}{p(x,\omega,0)}\,d\theta(\omega)\right]$$

$$= -\frac{1}{q(x)}\exp\left[\frac{1}{|S^{n-1}|}\int\limits_{S^{n-1}} \log p(x,\omega,0)\,d\theta(\omega)\right].$$

Thus $H(x)$ equals $-\frac{1}{q(x)}$ times the geometric mean of $p(x,\omega,0)$ over $\omega \in S^{n-1}$.

Therefore the requested real analytic continuation of $\frac{1}{H(x)}$ follows if we can show that $p(x,\omega,0)$ has a continuation as a positive real analytic function to some domain $U \times S^{n-1}$, where U is a full neighborhood of $\partial\Omega$. The function F in the statement of the theorem will then be

$$F(x) = -q(x)\exp\left[-\frac{1}{|S^{n-1}|}\int\limits_{S^{n-1}} \log p(x,\omega,0)\,d\theta(\omega)\right] \tag{9}$$

for $x \in U$. The arguments used below are rather standard in the context of the Weierstrass preparation theorem, but for clarity we spell out most of the details.

That q is real analytic means that in a neighborhood of any point in its domain of definition V, q can be expressed as the sum of a convergent power series (its Taylor series at the point). Equivalently, q is the restriction to the real domain of a complex analytic function

$$q(z) = q(z_1,\dots,z_n) = q(x_1 + iy_1,\dots,x_n + iy_n)$$

defined in a neighborhood of V in $\mathbb{C}^n$. The above parameter t can therefore be allowed to take complex values, which we then denote $\tau = t + is$ ($t, s \in \mathbb{R}$, $|s|$ small).

For fixed $(x,\omega) \in \overline{\Omega} \times S^{n-1}$, let

$$D(x,\omega) = \{\tau = t + is \in \mathbb{C}\colon\ x + t\omega \in V,\ |s| < \varepsilon\}$$

with $\varepsilon > 0$ chosen so small that $q(x + i\tau)$, as a function of τ, is analytic in a neighborhood of $\overline{D(x,\omega)}$. Note that $D(x,\omega)$ may very well be disconnected, but that it in any case contains the points $t_1(x,\omega)$ and $t_2(x,\omega)$.

If $\varepsilon > 0$ is taken sufficiently small, then $q(x + \tau\omega)$ will have no other zeros than $t_1(x,\omega)$ and $t_2(x,\omega)$ in $D(x,\omega)$. Indeed, choosing $D(x,\omega)$ so that $q(x + \tau\omega)$ has no zeros on the boundary

$$\gamma = \partial D(x,\omega),$$

the number of zeros in $D(x,\omega)$ equals

$$\frac{1}{2\pi i}\int\limits_{\gamma} d\log q(x+\tau\omega) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\frac{\partial}{\partial\tau}q(x+\tau\omega)}{q(x+\tau\omega)}\,d\tau,$$

which is a local constant under variations of γ. Thus for small enough $\varepsilon > 0$ only the real roots will be in $D(x,\omega)$. Since $\overline{\Omega} \times S^{n-1}$ is compact this $\varepsilon > 0$ can be taken to be independent of $(x,\omega) \in \overline{\Omega} \times S^{n-1}$.

Now keeping $D(x,\omega)$ as above we can express the symmetric functions of $t_j(x,\omega)$, hence $a(x,\omega)$, $b(x,\omega)$ and finally also $p(x,\omega,t)$, as contour integrals over γ and with integrands only involving $q(x + \tau\omega)$. First we have

$$t_1(x,\omega)+t_2(x,\omega)=\frac{1}{2\pi i}\int_\gamma \tau\,d\log q(x+\tau\omega),$$

$$t_1(x,\omega)^2+t_2(x,\omega)^2=\frac{1}{2\pi i}\int_\gamma \tau^2\,d\log q(x+\tau\omega).$$

These expressions can be inserted into

$$a(x,\omega)=-\big(t_1(x,\omega)+t_2(x,\omega)\big),$$

$$b(x,\omega)=\frac{1}{2}\big[\big(t_1(x,\omega)+t_2(x,\omega)\big)^2-\big(t_1(x,\omega)^2+t_2(x,\omega)^2\big)\big].$$

Finally, the function (of τ)

$$\frac{q(x+\tau\omega)}{\tau^2+a(x,\omega)\tau+b(x,\omega)}$$

is analytic in $D(x,\omega)$, and by (5) it equals $p(x,\omega,t)$ when $\tau=t$ is real. Therefore,

$$p(x,\omega,t)=\frac{1}{2\pi i}\int_\gamma \frac{q(x+\tau\omega)}{\tau^2+a(x,\omega)\tau+b(x,\omega)}\,\frac{d\tau}{\tau-t}.$$

The above integrals are invariant under small deformations of the contour γ. Hence, when studying the dependence of $a(x,\omega)$, $b(x,\omega)$, $p(x,\omega,t)$ on x, ω and t we can freeze the contour γ (say $\gamma=\partial D(x_0,\omega_0)$). Then it is immediately seen from the above representations that $a(x,\omega)$, $b(x,\omega)$, $p(x,\omega,t)$ are analytic as functions of x, ω and t and can be extended to be complex analytic functions in a neighborhood of $\overline{\Omega}\times S^{n-1}$ in $\mathbb{C}^n\times\mathbb{C}^n$. Clearly, $p(x,\omega,0)$ will remain >0 on $U\times S^{n-1}$ for some neighborhood U of $\partial\Omega$ in $\mathbb{R}^n$.

The last statement of the theorem is immediate from (9).

This finishes the proof of the theorem. □

4. Applications

We record below a translation of the main result and we also discuss two appropriate approximation schemes.

4.1. Analytic continuation of E_Ω

First, the complementarity formula (see Section 2)

$$E_\Omega(x)H_U(x)=H_{\Omega\cup U}(x)\quad (x\in U)$$

yields the following result.

Proposition 4.1. *Let $\Omega\subset\mathbf{R}^n$ be a bounded domain and let U be a strictly convex, bounded connected component of $\mathbf{R}^n\setminus\overline{\Omega}$, possessing smooth, real analytic boundary. Then the function $E_\Omega(x)$ extends analytically from $x\in U$ to an open neighborhood of $\overline{U}$. Moreover, the extension, still denoted E_Ω, satisfies*

$$E_\Omega(x)=0,\quad \nabla_x E_\Omega\neq 0\quad (x\in\partial U).$$

Thus, if q is any global defining real analytic function of ∂U, non-degenerate along the boundary

$$U = \{x \in \mathbf{R}^n;\ q(x) < 0\},$$

we obtain the factorization

$$E_\Omega(x) = q(x) f(x), \quad x \in \overline{U},$$

with a real analytic, non-vanishing function f defined on a neighborhood of $\overline{U}$.

4.2. Approximation by eigenfunction expansion

In low dimension there are good approximation schemes for E_Ω, with the benefits mentioned in the Introduction. In higher dimension, the following theoretical method seems to give an appropriate theoretical method for approximating the exponential transform. The analytic continuation via global expansions (usually with respect to well chosen orthogonal systems of functions) is not new. We adapt below an idea developed in the case of a single complex variable by H.S. Shapiro and the authors of the present note in [8].

We keep the above assumptions, and consider besides the cavity U a standard system of neighborhoods of its closure, in $\mathbf{C}^n$:

$$V_\delta = \{z = x + iy \in \mathbf{C}^n;\ q(x) < \delta,\ |y| < \delta\},$$

where δ is a small positive parameter. Note that by the very definition of the exponential transform,

$$E_\Omega(x + iy) = \exp\left[-\frac{2}{|S^{n-1}|} \int_\Omega \frac{du}{(\sum_k (u_k - x_k - iy_k)^2)^{n/2}}\right]$$

is well defined and complex analytic in the domain $\{x + iy;\ q(x) < 0,\ |y| < \operatorname{dist}(x, \Omega)/2\}$. Proposition 4.1 asserts that this function admits an analytic continuation to V_δ, provided δ is small enough. Fix such a parameter δ and consider the restriction map

$$R : L^2_a(V_\delta) \longrightarrow L^2(U), \quad Rf = f|_U,$$

defined on the Bergman space associated to V_δ (and the $2n$-dimensional volume measure $dx\,dy$) and with values in the Lebesgue space $L^2(U)$ associated to the n-dimensional volume measure dx.

Since U is relatively compact in V_δ, the operator R is trace class, and hence its modulus admits a spectral decomposition

$$R^*R = \sum_{k=0}^{\infty} \lambda_n \langle \cdot, f_k \rangle f_k.$$

The system of functions $(f_k)_{k=0}^\infty$ is orthonormal and complete in $L^2_a(V_\delta)$ and in the same time the system $(Rf_k)_{k=0}^\infty$ is orthogonal and complete in $L^2(U)$. The spectrum $(\lambda_k)_{k=0}^\infty$ consists of positive eigenvalues, of at most finite multiplicity, converging exponentially to zero, as one can see from the identity

$$\|f_k\|_U^2 = \langle Rf_k, Rf_k \rangle_U = \langle R^*Rf_k, f_k \rangle_{V_\delta} = \lambda_k$$

and the min–max principle, see [8].

Knowing that $E = E_\Omega \in L^2_a(V_\delta)$ means that the Fourier series,

$$E(z) = \sum_k \langle E, f_k\rangle_{V_\delta} f_k(z)$$

is convergent in $L^2_a(V_\delta)$, hence it is uniformly convergent on compact subsets of V_δ. The main point of using such a doubly orthogonal system is that merely the values $E(x)$, $x \in U$, suffice for computing the generalized Fourier coefficients:

$$\langle E, f_k\rangle_{V_\delta} = \frac{1}{\lambda_k}\langle E, R^*Rf_k\rangle_{V_\delta} = \frac{1}{\lambda_k}\langle E, f_k\rangle_U.$$

The orthogonal decomposition

$$F(z) = \sum_{k=0}^{\infty} \frac{f_k(z)}{\lambda_k} \int_U E_\Omega(x)\overline{f_k(x)}\,dx$$

converges in $L^2_a(V_\delta)$, and by standard analyticity arguments, it converges uniformly on compact subsets of V_δ. Note that $F(z) = E_\Omega(z)$ whenever $z \in U$.

This non-local analytic continuation method is rather classical, at least in one complex variable, cf. [2,8]. There, in general, the functions f_k turned out to be special functions adapted to the geometry of the domain. We outline such and example.

Assume that the smooth, convex cavity U contains a cube Δ whose boundary may touch the boundary of U:

$$\Delta = [-1, 1] \times [-1, 1] \times \cdots \times [-1, 1].$$

Let

$$G_\rho = \left\{u + iv \in \mathbf{C};\ \frac{u^2}{\cosh^2\rho} + \frac{v^2}{\sinh^2\rho} < 1\right\}, \quad \rho > 0,$$

be the family of confocal ellipses with foci at ± 1. According to Proposition 4.1, the exponential transform E_Ω extends analytically to a neighborhood of $\overline{\Delta}$ in $\mathbf{C}^n$. Hence there exists $\rho > 0$ such that $E_\Omega(z_1, \dots, z_n)$ is analytic in the domain $z_j \in G_\rho$, $1 \leqslant j \leqslant n$.

An observation going back at least seven decades ago to Friedrichs (see for details [8]) asserts that Chebyshev polynomials of the second type,

$$U_k(\cos\zeta) = \frac{\sin(k+1)\zeta}{\sin\zeta}, \quad k \geqslant 0,$$

are mutually orthogonal on all ellipses G_t, $t > 0$, and on the segment $[-1, 1]$ endowed with the Chebyshev weight $\sqrt{1 - x^2}\,dx$. Thus, the modulus of the restriction operator

$$R : L^2_a(G_\rho \times \cdots \times G_\rho) \longrightarrow L^2\left(\Delta, \prod_j \sqrt{1 - x_j^2}\,dx_j\right)$$

admits the spectral decomposition

$$R^*R = \sum_{\alpha \in \mathbf{N}^n} \lambda_\alpha \langle\cdot, f_\alpha\rangle f_\alpha,$$

where the scalar product is taken in $L^2_a(G_\rho \times \cdots \times G_\rho)$.

The non-normalized eigenfunctions are

$$f_\alpha(z_1,\dots,z_n)=U_{\alpha_1}(z_1)\cdots U_{\alpha_n}(z_n),\quad \alpha\in\mathbf{N}^n,$$

and the corresponding eigenvalues are

$$\lambda_\alpha=\frac{\|f_\alpha\|^2_{L^2(\Delta,\prod_j\sqrt{1-x_j^2}\,dx_j)}}{\|f_\alpha\|^2_{L^2_a(G_\rho\times\cdots\times G_\rho)}}.$$

A standard (conformal mapping) argument yields

$$\int_{G_\rho}|U_j(w)|^2\,d\,\mathrm{Area}(w)=\frac{\pi\sinh(2(j+1)\rho)}{2j+2},$$

whence

$$\|f_\alpha\|^2_{L^2_a(G_\rho\times\cdots\times G_\rho)}=\prod_{k=1}^n\frac{\pi\sinh(2(\alpha_k+1)\rho)}{2\alpha_k+2},$$

and in the limiting case ($\rho\mapsto 0$)

$$\|f_\alpha\|^2_{L^2(\Delta,\prod_j\sqrt{1-x_j^2}\,dx_j)}=\pi^n.$$

This gives explicitly

$$\lambda_\alpha^{-1}=\prod_{k=1}^n\frac{\sinh(2(\alpha_k+1)\rho)}{2\alpha_k+2}.$$

In conclusion, the analytic expansion of E_Ω in a neighborhood of $\overline{\Delta}$ is given by the series

$$E_\Omega(z)=\sum_{\alpha\in\mathbf{N}^n}\prod_{k=1}^n\left[\frac{\sinh(2(\alpha_k+1)\rho)}{2\alpha_k+2}U_{\alpha_k}(z_k)\right]\int_\Delta E_\Omega(x)\prod_{k=1}^n\left[U_{\alpha_k}(x_k)\sqrt{1-x_k^2}\,\right]dx.$$

4.3. Taylor series expansion

The naive approach of using the Taylor series expansion of $E_\Omega(x)$ at an external point $x_0\in\mathbf{R}^n\setminus\overline{\Omega}$ (for continuing analytically E_Ω across portions of the boundary $\partial\Omega$) does not look very promising for arbitrary shapes Ω. There are however specific configurations when this method works. We present one of these cases.

Let $B(a,r)$ be the open ball in $\mathbf{R}^n$ centered at a and of radius $r>0$. Denote $B=B(0,1)$. We consider the domain

$$\Omega=B\setminus\bigcup_{j=1}^N\overline{B(a_j,r_j)},$$

where the closed balls $\overline{B(a_j,r_j)}$ are mutually *disjoint*.

The external exponential transform of a ball is known explicitly. Denoting $E_d(|x|)=E_B(x)$, for B the unit ball in $\mathbf{R}^d$ and $|x|>1$ we have

$$E_1(r)=\frac{r-1}{r+1},\qquad E_2(r)=1-\frac{1}{r^2},$$

and

$$E_d(r) = E_{d-2}(r)\exp\left[\frac{2}{(d-2)r^{d-2}}\right],$$

see Theorem 4.5 in [7]. The inner transform is also known:

$$H_B(x) = \frac{C_d}{|x|^2 - 1}, \quad |x| < 1,$$

where the constant C_d depends only on the ambient dimension d, cf. Theorem 4.4 in [7].

Let us choose in our example the cavity $U = B(a_1, r_1)$ and denote for simplification $B_j = B(a_j, r_j)$. Let us assume, to fix ideas, that n is even.

The already cited complementarity formula

$$E_\Omega(x)H_U(x) = H_{\Omega\cup U}(x) \quad (x \in U)$$

yields, in the above case (of Ω being the unit ball minus finitely many disjoint spherical bubbles)

$$E_\Omega(x) = \frac{H_B(x)}{H_{B_1}(x)} \frac{1}{E_{B_2}(x)\cdots E_{B_N}(x)}, \quad x \in B_1.$$

In explicit terms, for every point $x \in B_1$ we obtain

$$\begin{aligned} &E_\Omega(x)\frac{|x|^2 - 1}{|x-a_1|^2 - r_1^2} \\ &\quad = \prod_{j=2}^{N}\left[\frac{|x-a_j|^2}{|x-a_j|^2 - r_j^2}\exp\left(-\frac{1}{|x-a_j|^2} - \frac{1}{2|x-a_j|^4} - \cdots - \frac{1}{(n/2-1)|x-a_j|^{n-2}}\right)\right]. \end{aligned}$$

A similar expression holds for n odd.

In conclusion, the Taylor expansion of the exponential transform $E_\Omega(x)$ at a point $x_0 \in B_1$ converges beyond the boundary of the cavity ∂B_1, on a radius $|x - x_0| < \rho$, where ρ is the distance from x_0 to the other boundaries, specifically to $\partial[\Omega \cup B_1]$. Moreover, the above formula shows that this analytic continuation of E_Ω vanishes of the first order on ∂B_1.

References

[1] D. Aharonov, H.S. Shapiro, Domains on which analytic functions satisfy quadrature identities, J. Anal. Math. 30 (1976) 39–73.
[2] C.H. Fitzgerald, Quadratic inequalities and analytic continuation, J. Anal. Math. 31 (1977) 19–47.
[3] R.J. Gardner, Geometric Tomography, Cambridge Univ. Press, 1995.
[4] B. Gustafsson, C. He, P. Milanfar, M. Putinar, Reconstructing planar domains from their moments, Inverse Problems 16 (2000) 1053–1070.
[5] B. Gustafsson, M. Putinar, An exponential transform and regularity of free boundaries in two dimensions, Ann. Sc. Norm. Super. Pisa 26 (1998) 507–543.
[6] B. Gustafsson, M. Putinar, Analytic continuation of Cauchy and exponential transforms, in: S. Saitoh, et al. (Eds.), Analytic Extension Formulas and Their Applications, Kluwer Academic, Netherlands, 2001, pp. 47–57.
[7] B. Gustafsson, M. Putinar, The exponential transform: A renormalized Riesz potential at critical exponent, Indiana Univ. Math. J. 52 (2003) 527–568.
[8] B. Gustafsson, M. Putinar, H.S. Shapiro, Restriction operators, balayage and doubly orthogonal systems of analytic functions, J. Funct. Anal. 199 (2003) 332–378.
[9] S. Helgason, The Radon Transform, Birkhäuser, Boston, 1980.
[10] R. Harvey, B. Lawson, Boundaries of complex analytic varieties, Bull. Amer. Math. Soc. 80 (1974) 180–183.
[11] F. John, Plane Waves and Spherical Means Applied to Partial Differential Equations, Interscience, New York, 1955.

[12] P.W. Jones, N.G. Makarov, Density properties of harmonic measure, Ann. of Math. 142 (1995) 427–455.
[13] M.G. Krein, A.A. Nudelman, Markov Moment Problems and Extremal Problems, Transl. Math. Monogr., vol. 50, Amer. Math. Soc., Providence, RI, 1977.
[14] M. Putinar, A renormalized Riesz potential and applications, in: M. Neamtu, E. Saff (Eds.), Adv. Constr. Approx., Vanderbilt, 2003, Nashboro Press, Brentwood, TN, 2004, pp. 433–465.
[15] M. Putinar, in: P. Ebenfelt, et al. (Eds.), Quadrature Domains and Applications, a Harold S. Shapiro Anniversary Volume, Birkhäuser, 2005.
[16] H.S. Shapiro, The Schwarz Function and Its Generalization to Higher Dimensions, Univ. of Arkansas Lecture Notes Math., vol. 9, Wiley, New York, 1992.
[17] V. Tkachev, Subharmonicity of higher-dimensional exponential transforms, in: P. Ebenfelt, et al. (Eds.), Quadrature Domains and Applications, a Harold S. Shapiro Anniversary Volume, Birkhäuser, 2005, pp. 257–277.