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Asymptotic analysis of a class of minimization problems in a thin multidomain

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Abstract. We consider a quasilinear Neumann problem with exponent $p \in]1, +\infty[$, in a multidomain of $\mathbf{R}^N$, $N \geq 2$, consisting of two vertical cylinders, one placed upon the other: the first one with given height and small cross section, the other one with small height and given cross section. Assuming that the volumes of the two cylinders tend to zero with same rate, we prove that the limit problem is well posed in the union of the limit domains, with respective dimension 1 and $N-1$. Moreover, this limit problem is coupled if $p > N-1$ and uncoupled if $1 < p \leq N-1$.

0 Introduction

Let $N \geq 2$, let $\omega \subset \mathbf{R}^{N-1}$ be a bounded open connected set with a smooth boundary such that the origin in $\mathbf{R}^{N-1}$, denoted by $0'$, belongs to ω, and let $\{r_n\}_{n\in\mathbf{N}}$, $\{h_n\}_{n\in\mathbf{N}}$ be two sequences of positive numbers converging to 0. For every $n \in \mathbf{N}$, consider the thin multidomain $\Omega_n = \Omega_n^1 \cup \Omega_n^2$, the union of two vertical cylinders with small volumes: $\Omega_n^1 = r_n\omega \times [0, 1[$ with small cross section $r_n\omega$ and constant height, $\Omega_n^2 = \omega \times] - h_n, 0[$ with small height h_n and constant cross section (see figure next page).

This paper arises from the desire of studying the asymptotic behaviour, as $n \to +\infty$, of the following model problem:

$$\min\left\{ J_n(V) := \int_{\Omega_n} \left(|V|^p + |D_{X'}V|^p + \left|\frac{\partial V}{\partial X_N}\right|^p + FV \right) dX : \right.$$
$$\left. V \in W^{1,p}(\Omega_n) \right\},$$

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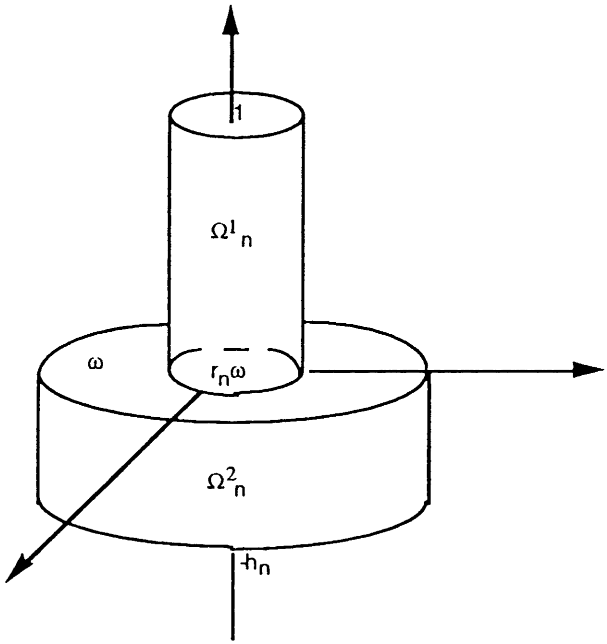

where $p \in]1,+\infty[$, $F \in L^{\frac{p}{p-1}}(\omega\times]-1,1[)$, $X = (X_1,\cdots,X_{N-1},X_N) = (X',X_N) \in \mathbf{R}^N$ and $D_{X'}V = \left(\dfrac{\partial V}{\partial X_1},\cdots,\dfrac{\partial V}{\partial X_{N-1}}\right)$.

It is well-known that this problem admits a unique solution $U_n \in W^{1,p}(\Omega_n)$. To study the asymptotic behaviour of $\{U_n\}_{n\in\mathbf{N}}$, as $n \to +\infty$, we introduce the classical transformation mapping Ω_n onto the fixed domain $\Omega = \omega\times]-1,1[$ (compare, for instance, [5], [6], [7], [14] and [17]) and set, for every $n \in \mathbf{N}$,

$$u_n(x) = \begin{cases} u_n^{(1)}(x',x_N) = U_n(r_n x',x_N), & (x',x_N) \text{ a.e. in } \Omega_1 = \omega\times]0,1[; \\ u_n^{(2)}(x',x_N) = U_n(x',h_n x_N), & (x',x_N) \text{ a.e. in } \Omega_2 = \omega\times]-1,0[. \end{cases}$$

It is easy to see that, for every $n \in \mathbf{N}$, u_n is the unique solution of the following problem:

$$\begin{aligned} &\min\Bigg\{ j_n(v) = \int_{\Omega_1}\left(|v^{(1)}|^p + \left|\frac{1}{r_n}D_{x'}v^{(1)}\right|^p + \left|\frac{\partial v^{(1)}}{\partial x_N}\right|^p + f_n v^{(1)}\right)dx + \\ &\qquad + \frac{h_n}{r_n^{N-1}}\int_{\Omega_2}\left(|v^{(2)}|^p + |D_{x'}v^{(2)}|^p + \left|\frac{1}{h_n}\frac{\partial v^{(2)}}{\partial x_N}\right|^p + f_n v^{(2)}\right)dx : \\ &v = (v^{(1)},v^{(2)}) \in W^{1,p}(\Omega_1)\times W^{1,p}(\Omega_2) \\ &\qquad v^{(1)}(x',0) = v^{(2)}(r_n x',0),\ x' \text{ a.e. in } \omega \Bigg\}, \end{aligned} \tag{0.1}$$

where $x = (x_1, \cdots, x_{N-1}, x_N) = (x', x_N) \in \mathbf{R}^N$, $D_{x'}v = \left(\frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_{N-1}}\right)$ and

$$f_n(x) = \begin{cases} F(r_n x', x_N), & (x', x_N) \text{ a.e. in } \Omega_1, \\ F(x', h_n x_N), & (x', x_N) \text{ a.e. in } \Omega_2. \end{cases}$$

This paper is devoted to the study of the asymptotic behaviour, as $n \to +\infty$, of Problem (0.1). Precisely, by assuming that the volumes of the two cylinders Ω_n^1 and Ω_n^2 tend to zero with same rate, that is

$$\lim_n \frac{h_n}{r_n^{N-1}} = q \in]0, +\infty[\tag{0.2}$$

and by assuming also that

$$f_n \rightharpoonup f \text{ weakly in } L^{\frac{p}{p-1}}(\Omega), \tag{0.3}$$

(for instance, (0.3) holds true, up to a subsequence, if $F \in L^\infty(\Omega)$), it is proved in this paper that

$$\begin{aligned} &u_n^{(1)} \to u^{(1)} \text{ strongly in } W^{1,p}(\Omega_1), u_n^{(2)} \to u^{(2)} \text{ strongly in } W^{1,p}(\Omega_2), \\ &\frac{1}{r_n} D_{x'} u_n^{(1)} \to 0 \text{ strongly in } \left(L^p(\Omega_1)\right)^{N-1}, \\ &\frac{1}{h_n} \frac{\partial u_n^{(2)}}{\partial x_N} \to 0 \text{ strongly in } L^p(\Omega_2), \end{aligned}$$

as $n \to +\infty$, where,

• if $1 < p \leq N-1$, $u^{(1)}$ and $u^{(2)}$ are the unique solutions of the following problem:

$$\min_{v^{(1)} \in W^{1,p}(]0,1[)} \left\{ j^1\left(v^{(1)}\right) = \text{meas } \omega \int_0^1 \left(|v^{(1)}(x_N)|^p + \left| \frac{\partial v^{(1)}}{\partial x_N}(x_N) \right|^p \right) dx_N \right.$$
$$\left. + \int_0^1 \left(v^{(1)}(x_N) \int_\omega f dx' \right) dx_N \right\},$$
$$\min_{v^{(2)} \in W^{1,p}(\omega)} \left\{ j^2\left(v^{(2)}\right) = \int_\omega \left(|v^{(2)}(x')|^p + \left| D_{x'} v^{(2)}(x') \right|^p \right) dx' \right.$$
$$\left. + \int_\omega \left(v^{(2)}(x') \int_{-1}^0 f dx_N \right) dx' \right\},$$

respectively;

• if $p > N-1$, $\left(u^{(1)}, u^{(2)}\right)$ is the unique solution of the following problem:

$$\min \left\{ j^1\left(v^{(1)}\right) + q\, j^2\left(v^{(2)}\right) : \right.$$
$$\left. (v^{(1)}, v^{(2)}) \in W^{1,p}(]0,1[) \times W^{1,p}(\omega), \quad v^{(1)}(0) = v^{(2)}(0') \right\}.$$

Moreover, in both cases the energies converge, that is

$$\lim_n j_n(u_n) = j^1(u^{(1)}) + q\, j^2(u^{(2)}).$$

Consequently, since $J_n(U_n) = r_n^{N-1} j_n(u_n)$, it follows that

$$\lim_n J_n(U_n) = 0.$$

We point out that the limit problem is coupled by the condition $v^{(1)}(0) = v^{(2)}(0')$ and its solution depends on q, if $p > N-1$. Otherwise, if $1 < p \leq N-1$, the limit problem is uncoupled and its solution does not depend on q. In particular, for $N = 3$, the limit exponent is $p = 2$, so that the coupling is lost at the limit for the Laplacian. Moreover we remark that the condition $p > N-1$ is necessary and sufficient for having $v^{(2)}$ continuous (and hence $v^{(2)}(0')$ meaningful) for any $v^{(2)} \in W^{1,p}(\omega)$.

Indeed, the above-mentioned result is just a corollary of a more general theorem (see Theorem 1.1) proved in this paper. Precisely, for every $n \in \mathbf{N}$, let $u_n = \left(u_n^{(1)}, u_n^{(2)}\right)$ be a solution of the following problem:

$$\min \left\{ \int_{\Omega_1} \left(A\left(x, v^{(1)}, \frac{1}{r_n} D_{x'} v^{(1)}, \frac{\partial v^{(1)}}{\partial x_N}\right) + f_n v^{(1)} \right) dx + \right.$$
$$+ \frac{h_n}{r_n^{N-1}} \int_{\Omega_2} \left(A\left(x, v^{(2)}, D_{x'} v^{(2)}, \frac{1}{h_n} \frac{\partial v^{(2)}}{\partial x_N}\right) + f_n v^{(2)} \right) dx :$$
$$v = (v^{(1)}, v^{(2)}) \in W^{1,p}(\Omega_1) \times W^{1,p}(\Omega_2) : v^{(1)}(x', 0) = v^{(2)}(r_n x', 0)$$
$$\left. x' \text{ a.e. in } \omega \right\},$$

where $A : (x, s, \xi, t) \in \Omega \times \mathbf{R} \times \mathbf{R}^{N-1} \times \mathbf{R} \to A(x, s, \xi, t) \in \mathbf{R}$ is a Caratheodory function satisfying usual convexity and p-growth conditions, with $p \in]1, +\infty[$ (see assumptions (1.1)÷ (1.4)).

Then, if (0.2) and (0.3) hold, there exists an increasing sequence $\{n_i\}_{i \in \mathbf{N}} \subset \mathbf{N}$, $(u^{(1)}, u^{(2)}) \in W^{1,p}(]0,1[) \times W^{1,p}(\omega)$, $(y^{(1)}, y^{(2)}) \in L^p\left(0,1; W_m^{1,p}(\omega)\right) \times L^p\left(\omega; W_m^{1,p}(]-1,0[)\right)$ (see (1.13) for the definition), depending possibly on the sequence $\{n_i\}_{i \in \mathbf{N}}$, such that

$$u_{n_i}^{(1)} \rightharpoonup u^{(1)} \text{ weakly in } W^{1,p}(\Omega_1), \qquad u_{n_i}^{(2)} \rightharpoonup u^{(2)} \text{ weakly in } W^{1,p}(\Omega_2),$$

$$\frac{1}{r_{n_i}} D_{x'} u_{n_i}^{(1)} \rightharpoonup D_{x'} y^{(1)} \text{ weakly in } \left(L^p(\Omega_1)\right)^{N-1},$$
$$\frac{1}{h_{n_i}} \frac{\partial u_{n_i}^{(2)}}{\partial x_N} \rightharpoonup \frac{\partial y^{(2)}}{\partial x_N} \text{ weakly in } L^p\left(\Omega_2\right),$$

as $i \to +\infty$, and $\left((u^{(1)}, u^{(2)}), (y^{(1)}, y^{(2)})\right)$ is a solution of a minimization problem which depends on q, if $p > N-1$. Otherwise, if $1 < p \leq N-1$, the limit problem is uncoupled and is decomposed in two minimization problems with respective solutions $\left(u^{(1)}, y^{(1)}\right)$ and $\left(u^{(2)}, y^{(2)}\right)$, which do not depend on q. In both cases the limit problems are given explicitly and the convergence of the energies holds.

We point out that, in this general setting, the weak limits of $\dfrac{1}{r_{n_i}} D_{x'} u_{n_i}^{(1)}$ and $\dfrac{1}{h_{n_i}} \dfrac{\partial u_{n_i}^{(2)}}{\partial x_N}$ are not necessarily equal to 0 (compare [17]), as in the model case.

The proof of this theorem is performed in Sect. 4 and 5, by making use of the basic ideas of the Γ-convergence method introduced by E. De Giorgi in [11] (see also [2] and [10] for general references about the Γ-convergence method, [1], [15] and [16] in the context of thin structures, and [9] in the context of domain with oscillating boundary). In Sect. 2 some compactness properties for the sequence $\{u_n\}_{n\in\mathbf{N}}$ are obtained. These properties are based on Proposition 2.1. In Sect. 3 a density result is proved. We emphasize that the main difficulties arise in proving Proposition 2.1 and Proposition 3.1. These difficulties originate from the junction condition connecting the two thin subdomains Ω_n^1 and Ω_n^2. Otherwise the paper is very much inspired by [17].

A preliminary version of these results, concerning the model problem, but including oscillating coefficients, was published in [12] with sketch of proofs.

We recall that [3] and [4] deal with the case of oscillating coefficients having measure limits, but with $\Omega_n^2 = \Omega^2$ independent of n and with a simpler (purely algebraic) transmission condition. For a general reference about homogenization of thin structures, the reader is refered to [8].

In a forthcoming paper we study a similar problem for equations involving monotone operators, by making use of the method of oscillating test functions introduced by L. Tartar in [18].

1 Statement of the problem and main results

Let $N \geq 2$. In the sequel, $x = (x_1, \cdots, x_{N-1}, x_N) = (x', x_N)$ denotes the generic point of $\mathbf{R}^N$. Moreover, for a real function v defined in an open subset of $\mathbf{R}^N$ and with weak derivatives, $D_{x'}v$ denotes the $(N-1)-$vector function $\left(\dfrac{\partial v}{\partial x_1}, \cdots, \dfrac{\partial v}{\partial x_{N-1}}\right)$.

Let $\omega \subset \mathbf{R}^{N-1}$ be a bounded open connected set such that the origin in $\mathbf{R}^{N-1}$, denoted by $0'$, belongs to ω, $\Omega = \omega \times]-1, 1[$, $\Omega_1 = \omega \times]0, 1[$ and $\Omega_2 = \omega \times]-1, 0[$.

Let $p \in]1, +\infty[$ and $A : (x, s, \xi, t) \in \Omega \times \mathbf{R} \times \mathbf{R}^{N-1} \times \mathbf{R} \to A(x, s, \xi, t) \in \mathbf{R}$ be a function satisfying the following conditions:

(1.1)
$A(\cdot, s, \xi, t)$ is a measurable function on Ω, for every $(s, \xi, t) \in \mathbf{R} \times \mathbf{R}^{N-1} \times \mathbf{R}$;

(1.2) $\quad A(x, \cdot, \cdot, \cdot)$ is a convex function on $\mathbf{R} \times \mathbf{R}^{N-1} \times \mathbf{R}$, for a.e. $x \in \Omega$;

$$|A(x, s, \xi, t)| \leq \alpha\,(|s|^p + |\xi|^p + |t|^p) + a(x),$$

(1.3) $\quad$ for a.e. $x \in \Omega$, for every $(s, \xi, t) \in \mathbf{R} \times \mathbf{R}^{N-1} \times \mathbf{R}$;

$$A(x,s,\xi,t) \geq \beta\left(|s|^p + |\xi|^p + |t|^p\right) + b(x), \tag{1.4}$$
$$\text{for a.e. } x \in \Omega,\ \text{for every } (s,\xi,t) \in \mathbf{R} \times \mathbf{R}^{N-1} \times \mathbf{R};$$

where α, $\beta \in]0,+\infty[$ and a, $b \in L^1(\Omega)$.

For every $n \in \mathbf{N}$, let r_n, $h_n \in]0,+\infty[$, $f_n \in L^{\frac{p}{p-1}}(\Omega)$ and consider the following problem:

$$\min_{(v^{(1)},v^{(2)})\in V_n} \left\{ K_n^{(1)}(v^{(1)}) + \frac{h_n}{r_n^{N-1}} K_n^{(2)}(v^{(2)}) \right\}, \tag{1.5}$$

where

$$V_n = \Big\{ (v^{(1)}, v^{(2)}) \in W^{1,p}(\Omega_1) \times W^{1,p}(\Omega_2) : v^{(1)}(x',0) = v^{(2)}(r_n x',0),\ x' \text{ a.e. in } \omega \Big\} \tag{1.6}$$

and

$$K_n^{(1)} : v^{(1)} \in W^{1,p}(\Omega_1) \to \int_{\Omega_1} A\left(x, v^{(1)}, \frac{1}{r_n} D_{x'} v^{(1)}, \frac{\partial v^{(1)}}{\partial x_N}\right) dx + \int_{\Omega_1} f_n v^{(1)} dx, \tag{1.7}$$

$$K_n^{(2)} : v^{(2)} \in W^{1,p}(\Omega_2) \to \int_{\Omega_2} A\left(x, v^{(2)}, D_{x'} v^{(2)}, \frac{1}{h_n} \frac{\partial v^{(2)}}{\partial x_N}\right) dx + \int_{\Omega_2} f_n v^{(2)} dx. \tag{1.8}$$

By virtue of $(1.1) \div (1.3)$, $K_n^{(1)}$ and $K_n^{(2)}$ are convex and strongly continuous and, consequently, weakly l.s.c. Moreover, V_n is convex and strongly closed and, consequently, weakly closed. Then, by making use of the coerciveness (1.4), it is easy to prove that Problem (1.5) admits a solution.

The goal is to study the asymptotic behaviour, as $n \to +\infty$, of Problem (1.5) under the following assumptions:

$$\lim_n r_n = 0 = \lim_n h_n, \tag{1.9}$$

$$\lim_n \frac{h_n}{r_n^{N-1}} = q \in]0,+\infty[, \tag{1.10}$$

and

$$f_n \rightharpoonup f \text{ weakly in } L^{\frac{p}{p-1}}(\Omega). \tag{1.11}$$

Precisely, let

(1.12)
$$V=\begin{cases}\big\{(v^{(1)},v^{(2)})\in W^{1,p}(\Omega_1)\times W^{1,p}(\Omega_2):v^{(1)}\text{ is independent of }x',\\ \qquad v^{(2)}\text{ is independent of }x_N\big\}\quad \text{if } p\le N-1,\\ \big\{(v^{(1)},v^{(2)})\in W^{1,p}(\Omega_1)\times W^{1,p}(\Omega_2):v^{(1)}\text{ is independent of }x',\\ \qquad v^{(2)}\text{ is independent of }x_N,\quad v^{(1)}(0)=v^{(2)}(0')\big\}\quad \text{if } p>N-1,\end{cases}$$

and

$$Z=L^p\left(0,1;W^{1,p}_m(\omega)\right)\times L^p\left(\omega;W^{1,p}_m(]-1,0[)\right), \tag{1.13}$$

where

$$W^{1,p}_m(\omega)=\left\{v\in W^{1,p}(\omega):\int_\omega v\,dx'=0\right\},$$
$$W^{1,p}_m(]-1,0[)=\left\{v\in W^{1,p}(]-1,0[):\int_{-1}^0 v\,dx_N=0\right\}.$$

The following is our main result:

Theorem 1.1. *Let, for every* $n\in\mathbf{N}$, $(u_n^{(1)},u_n^{(2)})\in V_n$ *be a solution of Problem (1.5) under assumptions* $(1.1)\div(1.4)$ *and* $(1.9)\div(1.11)$. *Moreover, let* V *and* Z *be defined by (1.12) and (1.13) respectively. Then, there exist an increasing sequence of positive integer numbers* $\{n_i\}_{i\in\mathbf{N}}$, $\left((u^{(1)},u^{(2)}),(y^{(1)},y^{(2)})\right)\in V\times Z$, *depending possibly on the selected subsequence* $\{n_i\}_{i\in\mathbf{N}}$, *such that, as* $i\to+\infty$,

$$u^{(1)}_{n_i}\rightharpoonup u^{(1)} \text{ weakly in } W^{1,p}(\Omega_1),\quad u^{(2)}_{n_i}\rightharpoonup u^{(2)} \text{ weakly in } W^{1,p}(\Omega_2); \tag{1.14}$$

$$\frac{1}{r_{n_i}}D_{x'}u^{(1)}_{n_i}\rightharpoonup D_{x'}y^{(1)} \text{ weakly in } \left(L^p(\Omega_1)\right)^{N-1}; \tag{1.15}$$

$$\frac{1}{h_{n_i}}\frac{\partial u^{(2)}_{n_i}}{\partial x_N}\rightharpoonup \frac{\partial y^{(2)}}{\partial x_N} \text{ weakly in } L^p(\Omega_2) \tag{1.16}$$

and $\left((u^{(1)},u^{(2)}),(y^{(1)},y^{(2)})\right)$ *is a solution of the following problem:*

$$\min_{\left((v^{(1)},v^{(2)}),(z^{(1)},z^{(2)})\right)\in V\times Z}\left\{K^{(1)}(v^{(1)},z^{(1)})+qK^{(2)}(v^{(2)},z^{(2)})\right\}, \tag{1.17}$$

where

$$K^{(1)}:(v^{(1)},z^{(1)})\in W^{1,p}(\Omega_1)\times L^p\left(0,1;W^{1,p}(\omega)\right)\longrightarrow$$
$$\int_{\Omega_1}A\left(x,v^{(1)},D_{x'}z^{(1)},\frac{\partial v^{(1)}}{\partial x_N}\right)dx+\int_{\Omega_1}fv^{(1)}dx, \tag{1.18}$$

$$K^{(2)}:(v^{(2)},z^{(2)})\in W^{1,p}(\Omega_2)\times L^p\left(\omega;W^{1,p}((-1,0))\right)\longrightarrow$$
$$\int_{\Omega_2}A\left(x,v^{(2)},D_{x'}v^{(2)},\frac{\partial z^{(2)}}{\partial x_N}\right)dx+\int_{\Omega_2}fv^{(2)}dx. \tag{1.19}$$

Moreover the energies converge in the sense that

(1.20)

$$\lim\left(K_n^{(1)}(u_n^{(1)})+\frac{h_n}{r_n^{N-1}}K_n^{(2)}(u_n^{(2)})\right)=K^{(1)}(u^{(1)},y^{(1)})+qK^{(2)}(u^{(2)},y^{(2)}).$$

Remark 1.2. The convergence (1.20) holds true for the whole sequence, because the limit is the minimum of Problem (1.17) and it is independent of the subsequence. Moreover, if one assumes that

$$A(x,\cdot,\cdot,\cdot) \text{ is a strictly convex function on } \mathbf{R}\times\mathbf{R}^{N-1}\times\mathbf{R},\ \text{for a.e. } x\in\Omega,$$

then Problem (1.17) admits a unique solution $\big((u^{(1)},u^{(2)}),(y^{(1)},y^{(2)})\big)\in V\times Z$ and, consequently, the convergences (1.14)÷(1.16) hold true for the whole sequence.

Of course in this case, also problem (1.5) admits a unique solution.

Remark 1.3. The limit problem (1.17) is coupled by the condition $v^{(1)}(0)=v^{(2)}(0')$ and its solutions depend on q, if $p>N-1$. Otherwise, that is if $p\le N-1$, the limit problem is uncoupled and its solutions do not depend on q, i.e. (u_1,y_1) and (u_2,y_2) are solutions of

$$\min_{(v^{(1)},z^{(1)})\in W^{1,p}((0,1))\times L^p\left(0,1;W_m^{1,p}(\omega)\right)} K^{(1)}(v^{(1)},z^{(1)})$$

and

$$\min_{(v^{(2)},z^{(2)})\in W^{1,p}(\omega)\times L^p\left(\omega;W_m^{1,p}((-1,0))\right)} K^{(2)}(v^{(2)},z^{(2)})$$

respectively.

Remark 1.4. If $A(x,s,\xi,t)=|s|^p+|\xi|^p+|t|^p$, Problem (1.5) admits a unique solution $(u_n^{(1)},u_n^{(2)})$. By applying Theorem 1.1, it follows easily that

$$\begin{gathered}u_n^{(1)}\rightharpoonup u^{(1)} \text{ weakly in } W^{1,p}(\Omega_1),\quad u_n^{(2)}\rightharpoonup u^{(2)} \text{ weakly in } W^{1,p}(\Omega_2);\\ \frac{1}{r_n}D_{x'}u_n^{(1)}\rightharpoonup 0 \text{ weakly in } \left(L^p(\Omega_1)\right)^{N-1},\\ \frac{1}{h_n}\frac{\partial u_n^{(2)}}{\partial x_N}\rightharpoonup 0 \text{ weakly in } L^p\left(\Omega_2\right),\end{gathered}$$

as $n\to+\infty$, where $\left(u^{(1)},u^{(2)}\right)$ is the unique solution of the following problem:

$$\begin{aligned}&\min_{(v^{(1)},v^{(2)})\in V}\left\{\operatorname{meas}\omega\int_0^1\left(|v^{(1)}(x_N)|^p+\left|\frac{\partial v^{(1)}}{\partial x_N}(x_N)\right|^p\right)dx_N\right.\\&+\int_0^1\left(v^{(1)}(x_N)\int_\omega f\,dx'\right)dx_N+\\&\left.+q\int_\omega\left(|v^{(2)}(x')|^p+\left|D_{x'}v^{(2)}(x')\right|^p\right)dx'+q\int_\omega\left(v^{(2)}(x')\int_{-1}^0 f\,dx_N\right)dx'\right\}.\end{aligned}$$

Moreover, by using the convergence of the energies (1.20) with (1.10) and (1.11), the Rellich-Kondrachov compact embedding Theorem and the uniform convexity of the space L^p for $1 < p < +\infty$, it is easy to prove that the above convergences occur in the strong sense, that is

$$u_n^{(1)} \to u^{(1)} \text{ strongly in } W^{1,p}(\Omega_1), \quad u_n^{(2)} \to u^{(2)} \text{ strongly in } W^{1,p}(\Omega_2),$$

$$\frac{1}{r_n} D_{x'} u_n^{(1)} \to 0 \text{ strongly in } \left(L^p(\Omega_1)\right)^{N-1},$$
$$\frac{1}{h_n} \frac{\partial u_n^{(2)}}{\partial x_N} \to 0 \text{ strongly in } L^p\left(\Omega_2\right),$$

as $n \to +\infty$.

We point out that in this case, since the limit problem admits a unique solution, the convergences hold true for the whole sequence.

2 Compactness properties

In this section some compactness properties for sequences of solutions of Problem (1.5) are obtained. These properties are based on the following result:

Proposition 2.1. *Let $\{h_n\}_{n\in\mathbf{N}}$ satisfy (1.9) and $\left\{v_n^{(2)}\right\}_{n\in\mathbf{N}} \subset W^{1,p}\left(\Omega_2\right)$. Assume that there exists $c \in]0,+\infty[$ such that*

$$\left\| v_n^{(2)} \right\|_{W^{1,p}(\Omega_2)} \le c, \quad \forall n \in \mathbf{N}; \tag{2.1}$$

$$\left\| \frac{\partial v_n^{(2)}}{\partial x_N} \right\|_{L^p(\Omega_2)} \le c\, h_n, \quad \forall n \in \mathbf{N}. \tag{2.2}$$

Then, there exists an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbf{N}}$, $v^{(2)} \in W^{1,p}\left(\Omega_2\right)$, depending possibly on the selected sequence $\{n_i\}_{i\in\mathbf{N}}$, such that $v^{(2)}$ is independent of x_N and

$$v_{n_i}^{(2)} \rightharpoonup v^{(2)} \text{ weakly in } W^{1,p}(\Omega_2), \tag{2.3}$$

as $i \to +\infty$. Moreover, if $\{r_n\}_{n\in\mathbf{N}}$ satisfies (1.9), if $\left\{\frac{h_n}{r_n^{N-1}}\right\}_{n\in\mathbf{N}}$ satisfies (1.10) and if $p > N-1$, then

$$\lim_i \int_\omega v_{n_i}^{(2)}(r_{n_i} x', 0)\, dx' = |\omega| v^{(2)}(0'). \tag{2.4}$$

Proof. By virtue of (2.1), there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbf{N}}$ and $v^{(2)} \in W^{1,p}\left(\Omega_2\right)$ such that (2.3) holds. Moreover (1.9), (2.2), (2.3) and a l.s.c. argument provide $v^{(2)}$ independent of x_N.

Assume now $p > N-1$. To prove (2.4), for every $n \in \mathbf{N}$ set

$$\rho_n^{(2)}(x_N) = \int_\omega \left(\left| D_{x'} v_n^{(2)}(x', x_N) \right|^p + \left| v_n^{(2)}(x', x_N) \right|^p \right) dx', \; x_N \text{ a.e. in }]-1, 0[.$$

By virtue of (2.1),

$$\int_{-1}^0 \rho_n^{(2)}(x_N) dx_N = \int_{\Omega_2} \left(\left| D_{x'} v_n^{(2)}(x) \right|^p + \left| v_n^{(2)}(x) \right|^p \right) dx \leq \left\| v_n^{(2)} \right\|_{W^{1,p}(\Omega_2)} \leq c, \quad \forall n \in \mathbf{N}.$$

Consequently, by applying Fatou Lemma, it follows that

$$\int_{-1}^0 \liminf_n \rho_n^{(2)}(x_N) dx_N \leq \liminf_n \int_{-1}^0 \rho_n^{(2)}(x_N) dx_N \leq c,$$

from which it follows that

$$0 \leq \liminf_n \rho_n^{(2)}(x_N) < +\infty, \quad x_N \text{ a.e. in }]-1, 0[. \tag{2.5}$$

Fix $\overline{x}_N \in]-1, 0[$ satisfying (2.5). Then, passing possibly to a subsequence of $\{n_i\}_{i \in \mathbf{N}}$ (depending only on $\overline{x}_N$), $\left\{ \rho_{n_i}^{(2)}(\overline{x}_N) \right\}_{i \in \mathbf{N}}$ is bounded in $[0, +\infty[$, i.e.

$$\left\{ v_{n_i}^{(2)}(\cdot, \overline{x}_N) \right\}_{i \in \mathbf{N}} \text{ is bounded in } W^{1,p}(\omega), \text{ up to a subsequence.} \tag{2.6}$$

Since (2.3) and the compactness of the trace mapping provide that $v_{n_i}^{(2)}(\cdot, \overline{x}_N) \to v^{(2)}$ strongly in $L^p(\omega)$ as $i \to +\infty$ and since $W^{1,p}(\omega)$ is compactly embedded into $C^0(\overline{\omega})$ for $p > N-1$, it follows from (2.6) that

$$v_{n_i}^{(2)}(\cdot, \overline{x}_N) \to v^{(2)} \text{ strongly in } C^0(\overline{\omega}), \text{ as } i \to +\infty. \tag{2.7}$$

Now, observe that

$$\begin{aligned} \int_\omega v_{n_i}^{(2)}(r_{n_i} x', 0) dx' &= \int_\omega \left(v_{n_i}^{(2)}(r_{n_i} x', 0) - v^{(2)}(r_{n_i} x') \right) dx' + \\ + \int_\omega v^{(2)}(r_{n_i} x') dx' &= \int_\omega \left(v_{n_i}^{(2)}(r_{n_i} x', 0) - v_{n_i}^{(2)}(r_{n_i} x', \overline{x}_N) \right) dx' + \\ &+ \int_\omega \left(v_{n_i}^{(2)}(r_{n_i} x', \overline{x}_N) - v^{(2)}(r_{n_i} x') \right) dx' + \\ &+ \int_\omega v^{(2)}(r_{n_i} x') dx', \quad \forall i \in \mathbf{N}. \end{aligned} \tag{2.8}$$

As regards the first term in the right hand side of (2.8), Hölder's inequality and assumptions (2.2), (1.9) and (1.10) give

$$\lim_i \left| \int_\omega \left(v_{n_i}^{(2)}(r_{n_i} x', 0) - v_{n_i}^{(2)}(r_{n_i} x', \overline{x}_N) \right) dx' \right| \leq$$

$$= \lim_i \left| \int_\omega \int_{\overline{x}_N}^0 \frac{\partial v_{n_i}^{(2)}}{\partial x_N}(r_{n_i}x', x_N) dx_N dx' \right| \leq$$

$$\leq (\text{meas}\,\omega)^{\frac{p-1}{p}} \lim_i \left(\int_\omega \int_{-1}^0 \left| \frac{\partial v_{n_i}^{(2)}}{\partial x_N}(r_{n_i}x', x_N) \right|^p dx_N dx' \right)^{\frac{1}{p}} = \tag{2.9}$$

$$= (\text{meas}\,\omega)^{\frac{p-1}{p}} \lim_i \left(\frac{1}{r_{n_i}^{N-1}} \int_{r_{n_i}\omega} \int_{-1}^0 \left| \frac{\partial v_{n_i}^{(2)}}{\partial x_N}(x', x_N) \right|^p dx_N dx' \right)^{\frac{1}{p}} \leq$$

$$\leq (\text{meas}\,\omega)^{\frac{p-1}{p}} \lim_i \left(\frac{1}{r_{n_i}^{N-1}} \int_{\Omega_2} \left| \frac{\partial v_{n_i}^{(2)}}{\partial x_N} \right|^p dx \right)^{\frac{1}{p}} \leq$$

$$\leq (\text{meas}\,\omega)^{\frac{p-1}{p}} \lim_i \frac{c h_{n_i}}{r_{n_i}^{\frac{N-1}{p}}} = c\,(\text{meas}\,\omega)^{\frac{p-1}{p}} \lim_i \left(\frac{h_{n_i}}{r_{n_i}^{N-1}} r_{n_i}^{\frac{(N-1)(p-1)}{p}} \right) = 0.$$

As regards the second term in the right hand side of (2.8), convergence (2.7) gives

$$\lim_i \left| \int_\omega \left(v_{n_i}^{(2)}(r_{n_i}x', \overline{x}_N) - v^{(2)}(r_{n_i}x') \right) dx' \right| \leq$$

$$\leq \lim_i \left(\frac{1}{r_{n_i}^{N-1}} \int_{r_{n_i}\omega} \left| v_{n_i}^{(2)}(x', \overline{x}_N) - v^{(2)}(x') \right| dx' \right) \leq \tag{2.10}$$

$$\leq \text{meas}\,\omega \lim_i \left\| v_{n_i}^{(2)}(\cdot, \overline{x}_N) - v^{(2)}(\cdot) \right\|_{L^\infty(\omega)} = 0.$$

As regards the last term in the right hand side of (2.8), since $v^{(2)} \in C^0(\overline{\omega})$,

$$\lim_i \int_\omega v^{(2)}(r_{n_i}x') dx' = |\omega| v^{(2)}(0'). \tag{2.11}$$

Finally (2.4) is obtained by passing to the limit, as $i \to +\infty$, in (2.8) and by using of (2.9)÷ (2.11). □

In the following lemma, some a priori norm-estimates for sequences of solutions of Problem (1.5) are obtained.

Lemma 2.2. *Let, for every* $n \in \mathbf{N}$, $(u_n^{(1)}, u_n^{(2)}) \in V_n$ *be a solution of Problem (1.5) under assumptions* $(1.1) \div (1.4)$, *(1.10) and (1.11).Then, there exists* $c \in]0, +\infty[$ *such that*

$$\left\| u_n^{(1)} \right\|_{W^{1,p}(\Omega_1)} \leq c, \quad \left\| u_n^{(2)} \right\|_{W^{1,p}(\Omega_2)} \leq c, \quad \forall n \in \mathbf{N}; \tag{2.12}$$

$$\left\| D_{x'} u_n^{(1)} \right\|_{(L^p(\Omega_1))^{N-1}} \leq c\, r_n, \quad \forall n \in \mathbf{N}; \tag{2.13}$$

$$\left\| \frac{\partial u_n^{(2)}}{\partial x_N} \right\|_{L^p(\Omega_2)} \leq c\, h_n, \quad \forall n \in \mathbf{N}. \tag{2.14}$$

Proof. Since $(0,0) \in V_n$, by virtue of (1.3) it results that

$$K_n^{(1)}\left(u_n^{(1)}\right) + \frac{h_n}{r_n^{N-1}} K_n^{(2)}\left(u_n^{(2)}\right) \leq K_n^{(1)}(0) + \frac{h_n}{r_n^{N-1}} K_n^{(2)}(0)$$

$$\leq \int_{\Omega_1} a dx + \frac{h_n}{r_n^{N-1}} \int_{\Omega_2} a dx, \quad \forall n \in \mathbf{N}. \tag{2.15}$$

On the other hand, by virtue of (1.4) it results that

$$\begin{aligned} & K_n^{(1)}\left(u_n^{(1)}\right) + \frac{h_n}{r_n^{N-1}} K_n^{(2)}\left(u_n^{(2)}\right) \geq \\ & \geq \beta\left(\left\|u_n^{(1)}\right\|^p_{L^p(\Omega_1)} + \frac{1}{r_n^p}\left\|D_{x'}u_n^{(1)}\right\|^p_{(L^p(\Omega_1))^{N-1}} + \left\|\frac{\partial u_n^{(1)}}{\partial x_N}\right\|^p_{L^p(\Omega_1)}\right) + \\ & + \frac{h_n}{r_n^{N-1}}\beta\left(\left\|u_n^{(2)}\right\|^p_{L^p(\Omega_2)} + \left\|D_{x'}u_n^{(2)}\right\|^p_{(L^p(\Omega_2))^{N-1}} + \frac{1}{h_n^p}\left\|\frac{\partial u_n^{(2)}}{\partial x_N}\right\|^p_{L^p(\Omega_2)}\right) + \\ & + \int_{\Omega_1} b dx + \frac{h_n}{r_n^{N-1}} \int_{\Omega_2} b dx - \\ & - \left(\left\|u_n^{(1)}\right\|_{L^p(\Omega_1)} + \frac{h_n}{r_n^{N-1}}\left\|u_n^{(2)}\right\|_{L^p(\Omega_2)}\right) \sup_{i \in \mathbf{N}} \|f_i\|_{L^{\frac{p}{p-1}}(\Omega)}, \quad \forall n \in \mathbf{N}. \end{aligned} \tag{2.16}$$

By combining (2.15) with (2.16) and by making use of (1.10) and (1.11), it follows that there exists $c_1 \in]0, +\infty[$ such that

$$\begin{aligned} & \left\|u_n^{(1)}\right\|^p_{L^p(\Omega_1)} + \frac{1}{r_n^p}\left\|D_{x'}u_n^{(1)}\right\|^p_{(L^p(\Omega_1))^{N-1}} + \left\|\frac{\partial u_n^{(1)}}{\partial x_N}\right\|^p_{L^p(\Omega_1)} + \\ & + \left\|u_n^{(2)}\right\|^p_{L^p(\Omega_2)} + \left\|D_{x'}u_n^{(2)}\right\|^p_{(L^p(\Omega_2))^{N-1}} + \frac{1}{h_n^p}\left\|\frac{\partial u_n^{(2)}}{\partial x_N}\right\|^p_{L^p(\Omega_2)} \\ & \leq c_1\left(1 + \left\|u_n^{(1)}\right\|_{L^p(\Omega_1)} + \left\|u_n^{(2)}\right\|_{L^p(\Omega_2)}\right), \quad \forall n \in \mathbf{N}, \end{aligned}$$

from which it is easy to obtain (2.12)÷(2.14). □

Proposition 2.1 and Lemma 2.2 provide the following compactnes result:

Corollary 2.3. *Let, for every* $n \in \mathbf{N}$, $(u_n^{(1)}, u_n^{(2)}) \in V_n$ *be a solution of Problem (1.5) under assumptions* $(1.1) \div (1.4)$ *and* $(1.9) \div (1.11)$. *Moreover, let* V *and* Z *be defined in (1.12) and (1.13) respectively.Then, there exist an increasing sequence of positive integer numbers* $\{n_i\}_{i\in\mathbf{N}}$, $(u^{(1)}, u^{(2)}) \in V$ *and* $(y^{(1)}, y^{(2)}) \in Z$, *depending possibly on the selected subsequence, such that, as* $i \to +\infty$,

$$u_{n_i}^{(1)} \rightharpoonup u^{(1)} \text{ weakly in } W^{1,p}(\Omega_1), \quad u_{n_i}^{(2)} \rightharpoonup u^{(2)} \text{ weakly in } W^{1,p}(\Omega_2); \tag{2.17}$$

$$\frac{1}{r_{n_i}} D_{x'}u_{n_i}^{(1)} \rightharpoonup D_{x'}y^{(1)} \text{ weakly in } \left(L^p(\Omega_1)\right)^{N-1}; \tag{2.18}$$

$$\frac{1}{h_{n_i}}\frac{\partial u_{n_i}^{(2)}}{\partial x_N} \rightharpoonup \frac{\partial y^{(2)}}{\partial x_N} \text{ weakly in } L^p(\Omega_2). \tag{2.19}$$

Proof. By virtue of (2.12), (2.14) and Proposition 2.1 there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbf{N}}$, and $(u^{(1)}, u^{(2)}) \in W^{1,p}(\Omega_1) \times W^{1,p}(\Omega_2)$ depending possibly on the selected subsequence, such that (2.17) holds, $u^{(2)}$ is independent of x_N and

$$\lim_i \int_\omega u_{n_i}^{(2)}(r_{n_i}x', 0)dx' = |\omega| u^{(2)}(0'), \text{ if } p > N-1. \tag{2.20}$$

Moreover, (1.9), (2.13), (2.17) and a l.s.c. argument provide $u^{(1)}$ independent of x'. Then, if $p \le N-1$, $(u^{(1)}, u^{(2)}) \in V$. To prove that $(u^{(1)}, u^{(2)}) \in V$ also in the case $p > N-1$, it remains to check that

$$u^{(1)}(0) = u^{(2)}(0'), \text{ if } p > N-1. \tag{2.21}$$

At first observe that, by (2.17),

$$\lim_i \int_\omega u_{n_i}^{(1)}(x', 0)dx' = |\omega| u^{(1)}(0). \tag{2.22}$$

Then, (2.21) is obtained by passing to the limit in the following relation:

$$\int_\omega u_{n_i}^{(1)}(x', 0)dx' = \int_\omega u_{n_i}^{(2)}(r_{n_i}x', 0)dx', \quad \forall i \in \mathbf{N}$$

and by using (2.20) and (2.22).

In order to prove the existence of $y^{(1)} \in L^p\left(0,1; W_m^{1,p}(\omega)\right)$ satisfying (2.18), for every $n \in \mathbf{N}$ set

$$m_n^{(1)}(x_N) = \frac{1}{\text{meas}\,\omega}\int_\omega u_n^{(1)}(x', x_N)dx', \quad x_N \text{ a.e. in }]0,1[.$$

By virtue of the Poincaré-Wirtinger inequality, there exists $c_1 \in]0, +\infty[$ (depending only on ω and not on x_N) such that

$$\left\| \frac{1}{r_n}\left(u_n^{(1)}(\cdot, x_N) - m_n^{(1)}(x_N)\right)\right\|_{W_m^{1,p}(\omega)} \le \frac{c_1}{r_n}\left\| D_{x'}u_n^{(1)}(\cdot, x_N)\right\|_{(L^p(\omega))^{N-1}}, \quad x_N \text{ a.e. in }]0,1[, \quad \forall n \in \mathbf{N}. \tag{2.23}$$

By combining (2.13) with (2.23), it follows that there exists $c_2 \in]0, +\infty[$ such that

$$\left\| \frac{1}{r_n}\left(u_n^{(1)} - m_n^{(1)}\right)\right\|_{L^p\left(0,1; W_m^{1,p}(\omega)\right)} \le c_2, \quad \forall n \in \mathbf{N}. \tag{2.24}$$

We notice that $L^p\left(0,1; W_m^{1,p}(\omega)\right)$ is a closed subspace of $L^p\left(0,1; W^{1,p}(\omega)\right)$. Consequently, passing eventually to a subsequence of the previous selected subsequence, still denoted by $\{n_i\}_{i\in\mathbf{N}}$, it follows from (2.24) that there exists $y^{(1)} \in L^p\left(0,1; W_m^{1,p}(\omega)\right)$ such that

$$\frac{1}{r_{n_i}}\left(u_{n_i}^{(1)} - m_{n_i}^{(1)}\right) \rightharpoonup y^{(1)} \text{ weakly in } L^p\left(0,1; W_m^{1,p}(\omega)\right)$$

as $i \to +\infty$, from which (2.18) is obtained. Similarly, the existence of $y^{(2)} \in L^p\left(\omega; W^{1,p}_m(]-1,0[)\right)$ satisfying (2.19) can be proved. $\square$

3 Density properties

Let

$$\tilde{V} = \Big\{(v^{(1)}, v^{(2)}) \in W^{1,\infty}(\Omega_1) \times W^{1,\infty}(\Omega_2) : v^{(1)} \text{ is independent of } x',$$

$$v^{(2)} \text{ is independent of } x_N, \quad v^{(1)}(0) = v^{(2)}(0')\Big\}. \tag{3.1}$$

This section is devoted to prove the following density result, which will be used in the proof of Theorem 1.1:

Proposition 3.1. *Let V and $\tilde{V}$ be defined in (1.12) and (3.1) respectively. Then $\tilde{V}$ is dense in V in $W^{1,p}$-norm.*

Proof. In the case $p > N-1$, the proof in very simple. In fact, let $(v^{(1)}, v^{(2)}) \in V \subset W^{1,p}(]0,1[) \times W^{1,p}(\omega)$. If $v^{(1)}$ and $v^{(2)}$ denote also extensions of them in $W^{1,p}(\mathbf{R})$ and $W^{1,p}(\mathbf{R}^{N-1})$ respectively, then $v^{(1)} \in C^0(\mathbf{R})$ and $v^{(2)} \in C^0(\mathbf{R}^{N-1})$. Consequently, by setting, for every $n \in \mathbf{N}$,

$$v_n^{(1)} = \rho_n^{(1)} * v^{(1)} + v^{(1)}(0) - \rho_n^{(1)} * v^{(1)}(0) \text{ in } \mathbf{R},$$
$$v_n^{(2)} = \rho_n^{(2)} * v^{(2)} + v^{(2)}(0') - \rho_n^{(2)} * v^{(2)}(0') \text{ in } \mathbf{R}^{N-1},$$

where $\left\{\rho_n^{(1)}\right\}_{n\in\mathbf{N}}$ and $\left\{\rho_n^{(2)}\right\}_{n\in\mathbf{N}}$ denote sequences of mollifiers in $\mathbf{R}$ and $\mathbf{R}^{N-1}$ respectively, it results that $(v_n^{(1)}, v_n^{(2)}) \in \tilde{V}$ and

$$(v_n^{(1)}, v_n^{(2)}) \to (v^{(1)}, v^{(2)}) \text{ strongly in } W^{1,p}(]0,1[) \times W^{1,p}(\omega),$$

as $n \to +\infty$.

In the case $p \le N-1$, the proof in more complicated. In this case, $V = W^{1,p}(]0,1[) \times W^{1,p}(\omega)$ and $C^1([0,1]) \times C^1(\overline{\omega})$ is dense in V. In order to prove the assertion of Proposition 3.1, it is enough to prove that

$$\forall (v^{(1)}, v^{(2)}) \in C^1([0,1]) C^1(\overline{\omega})$$
$$\exists \left\{(v_n^{(1)}, v_n^{(2)})\right\}_{n\in\mathbf{N}} \subset W^{1,\infty}(]0,1[) \times W^{1,\infty}(\omega) : v_n^{(1)}(0) = v_n^{(2)}(0') \forall n \in \mathbf{N},$$
$$v_n^{(1)} \to v_1 \text{ strongly in } W^{1,p}(]0,1[),\ v_n^{(2)} \to v_2 \text{ strongly in } W^{1,p}(\omega).$$

Let $(v^{(1)}, v^{(2)}) \in C^1([0,1]) \times C^1(\overline{\omega})$. For every $n \in \mathbf{N}$, define $v_n^{(1)} = v^{(1)}$ in $]0,1[$. Moreover, for every $n \in \mathbf{N}$, consider two $(N-1)$-dimensional balls $B(\varepsilon_n)$ and $B(\eta_n)$ with center $0'$ and radii to be determinated later on, and such that

$$0 < \varepsilon_n < \eta_n, \quad \forall n \in \mathbf{N} \quad \text{and} \quad \lim_n \eta_n = 0. \tag{3.2}$$

Now define $v_n^{(2)}$ in ω by

$$v_n^{(2)} = v^{(1)}(0) \text{ in } \overline{B(\varepsilon_n)}, \quad v_n^{(2)} = \varphi_n v^{(1)}(0) + (1-\varphi_n)\, v^{(2)} \text{ in } \overline{B(\eta_n)} \backslash \overline{B(\varepsilon_n)},$$
$$v_n^{(2)} = v^{(2)} \text{ in } \omega \backslash \overline{B(\eta_n)},$$

where φ_n is an interpolation function to be determined later on and such that

$$\varphi_n \in C^1\left(\overline{B(\eta_n)}\backslash B(\varepsilon_n)\right), \quad 0 \le \varphi_n \le 1 \text{ in } \overline{B(\eta_n)}\backslash B(\varepsilon_n),$$
$$\varphi_n = 1 \text{ on } \partial B(\varepsilon_n), \quad \varphi_n = 0 \text{ on } \partial B(\eta_n). \tag{3.3}$$

It is clear that $\left\{(v_n^{(1)}, v_n^{(2)})\right\}_{n\in\mathbf{N}} \subset W^{1,\infty}(]0,1[) \times W^{1,\infty}(\omega)$, $v_n^{(1)}(0) = v_n^{(2)}(0')$ for every $n \in \mathbf{N}$ and $v_n^{(1)} \to v_1$ strongly in $W^{1,p}(]0,1[)$, as $n \to +\infty$.

We prove now that, for convenient $\{\varepsilon_n\}_{n\in\mathbf{N}}$, $\{\eta_n\}_{n\in\mathbf{N}}$ and $\{\varphi_n\}_{n\in\mathbf{N}}$ satisfying (3.2) and (3.3), $v_n^{(2)} \to v^{(2)}$ strongly in $W^{1,p}(\omega)$, as $n \to +\infty$. In fact, by virtue of (3.2) and (3.3), it results that

$$\lim_n \int_\omega \left|v_n^{(2)} - v^{(2)}\right|^p dx =$$
$$= \lim_n \left(\int_{B(\varepsilon_n)} \left|v^{(1)}(0) - v^{(2)}\right|^p dx' + \int_{B(\eta_n)\backslash B(\varepsilon_n)} \varphi_n^p \left|v^{(1)}(0) - v^{(2)}\right|^p dx' \right) \le$$
$$\le \left\|v^{(1)}(0) - v^{(2)}\right\|^p_{L^\infty(\omega)} \lim_n \text{meas } B(\eta_n) = 0.$$

On the other hand, by virtue of (3.3),

$$\int_\omega \left|Dv_n^{(2)} - Dv^{(2)}\right|^p dx =$$
$$= \int_{B(\varepsilon_n)} \left|Dv^{(2)}\right|^p dx' + \int_{B(\eta_n)\backslash B(\varepsilon_n)} \left|\varphi_n Dv^{(2)} + \left(v^{(1)}(0) - v^{(2)}\right) D\varphi_n\right|^p dx' \le$$
$$\le 2^p \left\|Dv^{(2)}\right\|^p_{(L^\infty(\omega))^{N-1}} \text{meas } B(\eta_n) +$$
$$+ 2^p \left\|v^{(1)}(0) - v^{(2)}\right\|^p_{L^\infty(\omega)} \int_{B(\eta_n)\backslash B(\varepsilon_n)} |D\varphi_n|^p dx',$$
$$\forall n \in \mathbf{N}.$$

Then, since $\lim_n \text{meas } B(\eta_n) = 0$, in order to complete the proof it is enough to choose $\{\varepsilon_n\}_{n\in\mathbf{N}}$, $\{\eta_n\}_{n\in\mathbf{N}}$ and $\{\varphi_n\}_{n\in\mathbf{N}}$ satisfying (3.2) and (3.3), and such that

$$\lim_n \int_{B(\eta_n)\backslash B(\varepsilon_n)} |D\varphi_n|^p dx' = 0. \tag{3.4}$$

In the case $p < N-1$, for every n, one can take $\varepsilon_n = \frac{1}{n}$, $\eta_n = \frac{2}{n}$ and $\varphi_n(x') = n \,\text{dist}\,(x', \partial B(\eta_n)) = 2 - n|x'|$ for $x' \in B(\eta_n)\backslash B(\varepsilon_n)$. Since $|D\varphi_n| = n$, this gives

$$\lim_n \int_{B(\eta_n)\backslash B(\varepsilon_n)} |D\varphi_n|^p dx' = \text{meas } B(1)(2^{N-1} - 1) \lim_n \frac{1}{n^{N-1-p}} = 0,$$

where $B(1)$ is the $(N-1)$-dimensional ball with center $0'$ and radius 1.

In the case $p = N-1$ (remark that in this case $N \geq 3$), for every $n \in \mathbf{N}$, one can choose φ_n as the solution of the $(N-1)$-capacity problem of $\overline{B(\varepsilon_n)}$ with respect to $B(\eta_n)$, that is the solution of

$$\min\left\{ \int_{B(\eta_n)} |D\varphi|^{N-1}\, dx' : \quad \varphi \in C_0^1(B(\eta_n)), \quad \varphi = 1 \text{ in } \overline{B(\varepsilon_n)}, \quad 0 \leq \varphi \leq 1 \right\}.$$

It is well known (see [13], example 2.12. page 35) that, for $N \geq 3$,

$$\int_{B(\eta_n)\setminus B(\varepsilon_n)} |D\varphi_n|^{N-1}\, dx' = \text{meas}\, \partial B(1) \left(\log \frac{\eta_n}{\varepsilon_n} \right)^{2-N}, \quad \forall n \in \mathbf{N}.$$

Consequently, in order to obtain (3.4) with $p = N-1$, it is enough to take, for every $n \in \mathbf{N}$, $\varepsilon_n = \dfrac{1}{n^2}$ and $\eta_n = \dfrac{1}{n}$. $\square$

Recall the following known result:

Lemma 3.2. *$C^1(\overline{\Omega_1}) \times C^1(\overline{\Omega_2})$ is dense in $L^p\left(0,1; W^{1,p}(\omega)\right) \times L^p\left(\omega; W^{1,p}(]-1,0[)\right)$.*

4 The convergence result

In this section we prove the following result:

Proposition 4.1. *Let, for every $n \in \mathbf{N}$, V_n, $K_n^{(1)}$, $K_n^{(2)}$, $K^{(1)}$ and $K^{(2)}$, be defined in (1.6), (1.7), (1.8), (1.18) and (1.19) respectively, under assumptions $(1.1) \div (1.4)$ and $(1.9) \div (1.11)$. Moreover, let $\tilde{V}$ be defined in (3.1). Then, for every $(v^{(1)}, v^{(2)}) \in \tilde{V}$ and $(z^{(1)}, z^{(2)}) \in C^1(\overline{\Omega_1}) \times C^1(\overline{\Omega_2})$, there exists $\left\{(v_n^{(1)}, v_n^{(2)})\right\}_{n\in\mathbf{N}}$ with $(v_n^{(1)}, v_n^{(2)}) \in V_n$ and such that, for $i = 1, 2$,*

$$\lim_n K_n^{(i)}(v_n^{(i)}) = K^{(i)}(v^{(i)}, z^{(i)}).$$

Proof. Let $(v^{(1)}, v^{(2)}) \in \tilde{V}$ and $(z^{(1)}, z^{(2)}) \in C^1(\overline{\Omega_1}) \times C^1(\overline{\Omega_2})$.

For every $n \in \mathbf{N}$, set

$$v_n^{(1)}(x) = \begin{cases} \left(r_n z^{(1)}(x', \varepsilon_n) + v^{(1)}(\varepsilon_n)\right) \frac{x_N}{\varepsilon_n} \\ \qquad + \left(h_n z^{(2)}(r_n x', 0) + v^{(2)}(r_n x')\right) \frac{\varepsilon_n - x_N}{\varepsilon_n}, \\ \qquad\qquad x = (x', x_N) \in \omega\times]0, \varepsilon_n[, \\ r_n z^{(1)}(x) + v^{(1)}(x_N), \quad x = (x', x_N) \in \omega\times]\varepsilon_n, 1[, \end{cases}$$

$$v_n^{(2)}(x) = h_n z^{(2)}(x) + v^{(2)}(x'), \quad x \in \Omega_2,$$

where $\{\varepsilon_n\}_{n\in\mathbf{N}} \subset]0,1[$ is a sequence to be determitated later on and such that

$$\lim_n \varepsilon_n = 0. \tag{4.1}$$

It is evident that, for every $n \in \mathbf{N}$, $(v_n^{(1)}, v_n^{(2)}) \in V_n$.

Since

$$\frac{1}{h_n}\frac{\partial v_n^{(2)}}{\partial x_N} = \frac{\partial z^{(2)}}{\partial x_N} \text{ in } \Omega_2, \quad \forall n \in \mathbf{N}$$

and, by (1.9),

$$\begin{aligned} &v_n^{(2)} \to v^{(2)} \text{ strongly in } L^p(\Omega_2), \\ &D_{x'} v_n^{(2)} \to D_{x'} v^{(2)} \text{ strongly in } \left(L^p(\Omega_2)\right)^{N-1}, \text{ as } n \to +\infty, \end{aligned}$$

it results from the continuity of A with respect to (s, ξ, t), (1.3), Lebesgue Theorem and (1.11) that

$$\lim_n K_n^{(2)}(v_n^{(2)}) = K^{(2)}(v^{(2)}, z^{(2)}). \tag{4.2}$$

It remains to show that

$$\lim_n K_n^{(1)}(v_n^{(1)}) = K^{(1)}(v^{(1)}, z^{(1)}). \tag{4.3}$$

At first, remark that

$$\begin{aligned} K_n^{(1)}(v_n^{(1)}) &= \int_\omega \int_0^{\varepsilon_n} A\left(x, v_n^{(1)}, \frac{1}{r_n} D_{x'} v_n^{(1)}, \frac{\partial v_n^{(1)}}{\partial x_N}\right) dx + \\ &+ \int_\omega \int_0^{\varepsilon_n} f_n^{(1)} v_n^{(1)} dx + \\ &+ \int_\omega \int_{\varepsilon_n}^1 A\left(x, r_n z^{(1)} + v^{(1)}, D_{x'} z^{(1)}, r_n \frac{\partial z^{(1)}}{\partial x_N} + \frac{\partial v^{(1)}}{\partial x_N}\right) dx + \\ &+ \int_\omega \int_{\varepsilon_n}^1 \left(r_n z^{(1)} + v^{(1)}\right) f_n^{(1)} dx = \\ &= \int_\omega \int_0^{\varepsilon_n} A\left(x, v_n^{(1)}, \frac{1}{r_n} D_{x'} v_n^{(1)}, \frac{\partial v_n^{(1)}}{\partial x_N}\right) dx + \int_\omega \int_0^{\varepsilon_n} f_n^{(1)} v_n^{(1)} dx + \\ &+ \int_{\Omega_1} A\left(x, r_n z^{(1)} + v^{(1)}, D_{x'} z^{(1)}, r_n \frac{\partial z^{(1)}}{\partial x_N} + \frac{\partial v^{(1)}}{\partial x_N}\right) dx + \\ &+ \int_{\Omega_1} \left(r_n z^{(1)} + v^{(1)}\right) f_n^{(1)} dx + \\ &- \int_\omega \int_0^{\varepsilon_n} A\left(x, r_n z^{(1)} + v^{(1)}, D_{x'} z^{(1)}, r_n \frac{\partial z^{(1)}}{\partial x_N} + \frac{\partial v^{(1)}}{\partial x_N}\right) dx + \\ &- \int_\omega \int_0^{\varepsilon_n} \left(r_n z^{(1)} + v^{(1)}\right) f_n^{(1)} dx, \quad \forall n \in \mathbf{N}. \end{aligned} \tag{4.4}$$

The task, now, is to pass to the limit, as $n \to +\infty$, in the last six integrals of (4.4). Since, by (1.9),

$$\begin{aligned} &r_n z^{(1)} + v^{(1)} \to v^{(1)} \text{ strongly in } L^p(\Omega_1), \\ &r_n \frac{\partial z^{(1)}}{\partial x_N} + \frac{\partial v^{(1)}}{\partial x_N} \to \frac{\partial v^{(1)}}{\partial x_N} \text{ strongly in } L^p(\Omega_1), \end{aligned} \tag{4.5}$$

as $n \to +\infty$, it results as previously that

$$A\left(\cdot, r_n z^{(1)} + v^{(1)}, D_{x'} z^{(1)}, r_n \frac{\partial z^{(1)}}{\partial x_N} + \frac{\partial v^{(1)}}{\partial x_N}\right) \to A\left(\cdot, v^{(1)}, D_{x'} z^{(1)}, \frac{\partial v^{(1)}}{\partial x_N}\right) \quad \text{strongly in } L^1(\Omega_1), \quad \text{as } n \to +\infty. \tag{4.6}$$

Then, from (4.6), (4.5) and (1.11) it follows that

$$\lim_n \left(\int_{\Omega_1} A\left(x, r_n z^{(1)} + v^{(1)}, D_{x'} z^{(1)}, r_n \frac{\partial z^{(1)}}{\partial x_N} + \frac{\partial v^{(1)}}{\partial x_N}\right) dx + + \int_{\Omega_1} \left(r_n z^{(1)} + v^{(1)}\right) f_n^{(1)} dx\right) = K^{(1)}(v^{(1)}, z^{(1)}). \tag{4.7}$$

In order to prove (4.3), it is enough to show that the remaining integrals in (4.4) converge to zero.
Since, by (4.1),

$$\chi_{\omega \times]0,\varepsilon_n[} \rightharpoonup 0 \text{ in } L^\infty(\Omega_1) \text{ weak } *, \quad \text{as } n \to +\infty,$$

where $\chi_{\omega \times]0,\varepsilon_n[}$ denotes the characteristic function of $\omega \times]0, \varepsilon_n[$ in Ω_1, it follows from (4.6) that

$$\lim_n \int_\omega \int_0^{\varepsilon_n} A\left(x, r_n z^{(1)} + v^{(1)}, D_{x'} z^{(1)}, r_n \frac{\partial z^{(1)}}{\partial x_N} + \frac{\partial v^{(1)}}{\partial x_N}\right) dx = = \lim_n \int_{\Omega_1} \left(A\left(x, r_n z^{(1)} + v^{(1)}, D_{x'} z^{(1)}, r_n \frac{\partial z^{(1)}}{\partial x_N} + \frac{\partial v^{(1)}}{\partial x_N}\right) \chi_{\omega \times]0,\varepsilon_n[}\right) dx = 0. \tag{4.8}$$

As regards the last integral in (4.4), Hölder's Inequality, (1.9), (1.11) and (4.1) provide that

$$\lim_n \left| \int_\omega \int_0^{\varepsilon_n} \left(r_n z^{(1)} + v^{(1)}\right) f_n^{(1)} dx \right| \leq \leq (\text{meas } \omega)^{\frac{1}{p}} \sup_i \left\| f_i^{(1)} \right\|_{L^{\frac{p}{p-1}}(\Omega_1)} \lim_n \left(\varepsilon_n^{\frac{1}{p}} \left(r_n \left\| z^{(1)} \right\|_{L^\infty(\Omega_1)} + \left\| v^{(1)} \right\|_{L^\infty(]0,1[)} \right) \right) = 0. \tag{4.9}$$

In order to prove that

$$\lim_n \int_\omega \int_0^{\varepsilon_n} A\left(x, v_n^{(1)}, \frac{1}{r_n} D_{x'} v_n^{(1)}, \frac{\partial v_n^{(1)}}{\partial x_N}\right) dx + \int_\omega \int_0^{\varepsilon_n} f_n^{(1)} v_n^{(1)} dx = 0, \tag{4.10}$$

since, by virtue of (1.3), (1.11) and Hölder's inequality,

$$\left|\int_\omega\int_0^{\varepsilon_n} A\left(x, v_n^{(1)}, \frac{1}{r_n}D_{x'}v_n^{(1)}, \frac{\partial v_n^{(1)}}{\partial x_N}\right)dx + \int_\omega\int_0^{\varepsilon_n} f_n^{(1)}v_n^{(1)}dx\right| \le$$
$$\le \alpha\int_\omega\int_0^{\varepsilon_n}\left(\left|v_n^{(1)}\right|^p + \left|\frac{1}{r_n}D_{x'}v_n^{(1)}\right|^p + \left|\frac{\partial v_n^{(1)}}{\partial x_N}\right|^p + \frac{a(x)}{\alpha}\right)dx +$$
$$+\sup_i\left\|f_i^{(1)}\right\|_{L^{\frac{p}{p-1}}(\Omega_1)}\left(\int_\omega\int_0^{\varepsilon_n}\left|v_n^{(1)}\right|^p\right)^{\frac{1}{p}}, \quad \forall n\in N,$$

it is enough to check that

$$\lim_n\int_\omega\int_0^{\varepsilon_n}\left|v_n^{(1)}\right|^p = 0, \qquad \lim_n\int_\omega\int_0^{\varepsilon_n}\left|\frac{1}{r_n}D_{x'}v_n^{(1)}\right|^p = 0 \tag{4.11}$$

and

$$\lim_n\int_\omega\int_0^{\varepsilon_n}\left|\frac{\partial v_n^{(1)}}{\partial x_N}\right|^p = 0. \tag{4.12}$$

Since, for every $n\in\mathbf{N}$,

$$\left|v_n^{(1)}\right| \le r_n\left\|z^{(1)}\right\|_{L^\infty(\Omega_1)} + \left\|v^{(1)}\right\|_{L^\infty(]0,1[)} + h_n\left\|z^{(2)}\right\|_{L^\infty(\Omega_2)} + \left\|v^{(2)}\right\|_{L^\infty(\omega)} \text{ in } \omega\times]0,\varepsilon_n[,$$

$$\left|\frac{1}{r_n}D_{x'}v_n^{(1)}\right| \le \left\|D_{x'}z^{(1)}\right\|_{(L^\infty(\Omega_1))^{N-1}} + h_n\left\|D_{x'}z^{(2)}\right\|_{(L^\infty(\Omega_2))^{N-1}} + \left\|Dv^{(2)}\right\|_{(L^\infty(\omega))^{N-1}} \text{ a.e. in } \omega\times]0,\varepsilon_n[,$$

convergences in (4.11) follows immediately from (1.9) and (4.1).

In order to prove (4.12), remark that, since $v^{(1)}(0) = v^{(2)}(0')$, we have for any $n\in\mathbf{N}$,

$$\left|\frac{\partial v_n^{(1)}}{\partial x_N}\right| =$$
$$= \frac{1}{\varepsilon_n}\left|r_n z^{(1)}(x',\varepsilon_n) + v^{(1)}(\varepsilon_n) - h_n z^{(2)}(r_n x', 0) - v^{(2)}(r_n x')\right| =$$
$$= \frac{1}{\varepsilon_n}\left|r_n z^{(1)}(x',\varepsilon_n) - h_n z^{(2)}(r_n x', 0) + \right.$$
$$\left. + \int_0^{\varepsilon_n}\frac{\partial v^{(1)}}{\partial x_N}dx + v^{(2)}(0') - v^{(2)}(r_n x')\right| \le$$
$$\le \frac{1}{\varepsilon_n}\left(r_n\left\|z^{(1)}\right\|_{L^\infty(\Omega_1)} + h_n\left\|z^{(2)}\right\|_{L^\infty(\Omega_2)} + \right.$$
$$\left. + \varepsilon_n\left\|\frac{\partial v^{(1)}}{x_N}\right\|_{L^\infty(]0,1[)} + c_{\text{Lip}}|x'|r_n\right) \text{ a.e. in } \omega\times]0,\varepsilon_n[,$$

where c_{Lip} is the Lipschitz constant of $v^{(2)}$. Consequently, by virtue (1.9) and (4.1), convergence (4.12) is obtained as soon as $\left\{\dfrac{r_n}{\varepsilon_n}\right\}_{n\in\mathbf{N}}$ and $\left\{\dfrac{h_n}{\varepsilon_n}\right\}_{n\in\mathbf{N}}$ are bounded. By virtue of (1.10), this happens if one takes, for instance, $\varepsilon_n = r_n$.

Finally, by passing to the limit, as $n \to +\infty$, in (4.4) and by making use of (4.7)÷(4.10), convergence (4.3) is obtained. □

5 Proof of Theorem 1.1

Corollary 2.3 provides the existence of an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbf{N}}$, $(u^{(1)}, u^{(2)}) \in V$ and $(y^{(1)}, y^{(2)}) \in L^p\left(0,1; W^{1,p}_m(\omega)\right) \times L^p\left(\omega; W^{1,p}_m(]-1,0[)\right)$, depending possibly on the selected subsequence, satisfying convergences (1.14)÷(1.16).

In order to prove that $\left((u^{(1)}, u^{(2)}), (y^{(1)}, y^{(2)})\right)$ solves Problem (1.17), remark that, by virtue of (1.1)÷(1.3), the functionals

$$(v,w,t) \in L^p(\Omega_i) \times (L^p(\Omega_i))^{N-1} \times L^p(\Omega_i) \longrightarrow \int_{\Omega_i} A(x,v,w,t)\,dx,$$

for $i = 1, 2$, are convex and strongly continuous so they are weakly l.s.c.. Consequently, from (1.10), (1.11) and (1.14)÷(1.16) it follows that

(5.1)
$$K^{(1)}(u^{(1)}, y^{(1)}) + qK^{(2)}(u^{(2)}, y^{(2)}) \le \liminf_i \left(K^{(1)}_{n_i}(u^{(1)}_{n_i}) + \frac{h_{n_i}}{r^{N-1}_{n_i}} K^{(2)}_{n_i}(u^{(2)}_{n_i}) \right).$$

On the other hand, by virtue of Proposition 4.1, for every $(v^{(1)}, v^{(2)}) \in \tilde{V}$ and $(z^{(1)}, z^{(2)}) \in C^1(\overline{\Omega_1}) \times C^1(\overline{\Omega_2})$, there exists $\left\{(v^{(1)}_n, v^{(2)}_n)\right\}_{n\in\mathbf{N}}$ with $(v^{(1)}_n, v^{(2)}_n) \in V_n$ and such that

$$\text{(5.2)}\quad \lim_n K^{(1)}_n(v^{(1)}_n) = K^{(1)}(v^{(1)}, z^{(1)}), \quad \lim_n K^{(2)}_n(v^{(2)}_n) = K^{(2)}(v^{(2)}, z^{(2)}).$$

Then, since $(u^{(1)}_n, u^{(2)}_n)$ solves Problem (1.5), convergences (5.1), (5.2) and (1.10) provide that

$$\begin{aligned}
&K^{(1)}(u^{(1)}, y^{(1)}) + qK^{(2)}(u^{(2)}, y^{(2)}) \le \\
&\le \liminf_i \left(K^{(1)}_{n_i}(u^{(1)}_{n_i}) + \frac{h_{n_i}}{r^{N-1}_{n_i}} K^{(2)}_{n_i}(u^{(2)}_{n_i}) \right) \le \\
&\le \limsup_i \left(K^{(1)}_{n_i}(u^{(1)}_{n_i}) + \frac{h_{n_i}}{r^{N-1}_{n_i}} K^{(2)}_{n_i}(u^{(2)}_{n_i}) \right) \le \\
&\le K^{(1)}(v^{(1)}, z^{(1)}) + qK^{(2)}(v^{(2)}, z^{(2)}), \\
&\qquad \forall (v^{(1)}, v^{(2)}) \in \tilde{V}, \quad \forall (z^{(1)}, z^{(2)}) \in C^1(\overline{\Omega_1}) \times C^1(\overline{\Omega_2}),
\end{aligned}$$

from which, by making use of Proposition 3.1 and Lemma 3.2 and by recalling that $K^{(1)}$, $K^{(2)}$ are strongly continuous, it follows that the above inequalities are also true for any $(v^{(1)}, v^{(2)}) \in V$ and $(z^{(1)}, z^{(2)}) \in Z$. Consequently,

$\big((u^{(1)}, u^{(2)}), (y^{(1)}, y^{(2)})\big)$ solves Problem (1.17) and, by taking the infimum over $\big((v^{(1)}, v^{(2)}), (z^{(1)}, z^{(2)})\big)$, it results that

$$\lim_i \left(K^{(1)}_{n_i}(u^{(1)}_{n_i}) + \frac{h_{n_i}}{r^{N-1}_{n_i}} K^{(2)}_{n_i}(u^{(2)}_{n_i}) \right) = K^{(1)}(u^{(1)}, y^{(1)}) + qK^{(2)}(u^{(2)}, y^{(2)}).$$

Since this limit is the minimum of problem (1.17) and it is independent of the selected subsequence, the convergence holds true for the whole sequence and (1.20) is proved. □

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