Pergamon

Nonlinear Analysis, Theory, Methods & Applications, Vol. 30, No. 5, pp. 2557–2565, 1997
Proc. 2nd World Congress of Nonlinear Analysts

0362-546X/97 $17.00 + 0.00

PII: S0362-546X(96)00107-1

# DIRECT AND INVERSE BALAYAGE — SOME NEW DEVELOPMENTS IN CLASSICAL POTENTIAL THEORY

BJÖRN GUSTAFSSON

Department of Mathematics, Royal Institute of Technology, S-100 44 Stockholm, Sweden
*Keywords and phrases*: Balayage, free boundary problem, Hele-Shaw flow, mixed envelope, mother body.

## 1. Classical Balayage

The french verb "balayer" means "sweep", "clear dust away with a brush", "clean", and "balayage" is the corresponding noun. In potential theory (classical) balayage, or "Poincaré sweeping", refers to the operation of replacing a mass distribution $\mu \geq 0$ in a (say bounded) domain $D \subset \mathbf{R}^N$ $(N \geq 2)$ by a mass distribution $\nu$ on the boundary $\partial D$ without changing the external gravitational potential. The latter means that

$$U^\nu = U^\mu \qquad \text{outside } D,$$

where, generally speaking, $U^\mu = E * \mu$ denotes the Newtonian potential of $\mu$ (so that $-\Delta U^\mu = \mu$) and $E$ is the Newtonian kernel.

In the cleaning terminology classical balayage thus corresponds to a kind of complete cleaning of a room $D$ from its layer $\mu$ of dust. In the present survey paper we shall discuss the relatively new notion of *partial balayage*, corresponding to just partial cleaning, a residual layer of dust of a prescribed maximum thickness being allowed.

Let

$$\nu = \mathrm{Bal}(\mu)$$

denote the classical balayage $\nu$ of $\mu$ with respect to $D$. The most straightforward way of constructing $\nu$ is by noticing that the function $u = U^\mu - U^\nu$ will have to solve the Dirichlet problem

$$\begin{aligned} -\Delta u &= \mu \qquad \text{in } D, \\ u &= 0 \qquad \text{on } \mathbf{R}^N \setminus D. \end{aligned}$$

Then $\nu = \mu + \Delta u$, $\Delta$ taken in the distributional sense in all $\mathbf{R}^N$, or (if $\mu$ has no mass on $\partial D$)

$$\nu = (\Delta u)\big|_{\partial D} = -\frac{\partial u}{\partial n}\, ds,$$

where $ds$ denotes hypersurface measure on $\partial D$ and $\partial u/\partial n$ is the outward normal derivative of $u\big|_D$.

There are several other ways of getting hold of $\nu = \mathrm{Bal}(\mu)$, or $U^\nu$:

(i) $\nu$ is the unique solution of

$$\mathrm{Min}\|\mu - \nu\|^2_{\text{energy}} : \operatorname{supp}\nu \subset \mathbf{R}^N \setminus D,$$

where $\|\sigma\|^2_{\text{energy}} = \iint E(x-y)d\sigma(x)d\sigma(y)$, the competing $\nu$ ranging over all positive (or even signed) measures in $\mathbf{R}^N \setminus D$ with compact support and finite energy.

(ii) $U^\nu$ is (the lower semicontinuous regularization of) the *smallest* function $V$ in $\mathbf{R}^N$ satisfying

$$\begin{aligned} V &\geq U^\mu && \text{on } \mathbf{R}^N \setminus D, \\ -\Delta V &\geq 0 && \text{in } \mathbf{R}^N. \end{aligned}$$

In other words, $U^\nu$ is the so-called reduced function ("réduite") of $U^\mu$ with respect to $\mathbf{R}^N \setminus D$.

(iii) $U^\nu$ is the *largest* function $V$ satisfying

$$\begin{aligned} V &\leq U^\mu && \text{in } \mathbf{R}^N, \\ -\Delta V &\leq 0 && \text{in } D. \end{aligned}$$

Some mild regularity of $\partial D$ is assumed above.

The concept of balayage goes back at least to Gauss. We refer to [2] for details, both mathematical and historical.

## 2. Partial Balayage

For partial balayage one also has a prescribed density function $\rho \in L^\infty(D)$, $\rho \geq 0$. It is natural to set $\rho = +\infty$ outside $D$, and then the information of $D$ is built into $\rho$. The partial balayage of $\mu \geq 0$ with respect to $\rho$,

$$\nu = \mathrm{Bal}(\mu; \rho),$$

is defined as the solution $\nu$ of

$$\mathrm{Min}\|\mu - \nu\|^2_{\text{energy}} : \nu \leq \rho \text{ in } D \tag{2.1}$$

(we identify $\rho$ with $\rho$ times Lebesgue measure).

This definition requires that $\mu$ has finite energy. Another definition which does not assume this is:

$$\mathrm{Bal}(\mu; \rho) = -\Delta V^\mu \qquad (\text{in } \mathbf{R}^N),$$

where $V^\mu$ is the largest function $V$ satisfying

$$\begin{aligned} V &\leq U^\mu && \text{in } \mathbf{R}^N, \\ -\Delta V &\leq \rho && \text{in } D. \end{aligned} \tag{2.2}$$

It is not hard to show that such a largest function $V^\mu$ exists. It coincides with $U^\mu$ outside $D$ and by a Perron family argument one also finds that

$$-\Delta V^\mu = \rho \text{ in the open set } \{V^\mu < U^\mu\} \subset D. \tag{2.3}$$

Thus the two inequalities in (2.2) are complementary (at each point there is equality in at least one of them) for the actual largest $V^\mu$. This complementarity statement is equivalent to a variational inequality, which is the variational formulation of the minimization problem (2.1). Therefore the two definitions of partial balayage are equivalent. Since $V^\mu$ has the behaviour of a potential at infinity, we have $V^\mu = U^\nu$ with $\nu = \mathrm{Bal}(\mu; \rho)$. Details for all the above, and also for most of the subsequent, material on partial balayage can be found in [6]. See also [11].

By definition, $\mathrm{Bal}(\mu; \rho) \leq \rho$ in $D$, and one may also prove that $\min(\rho, \mu) \leq \mathrm{Bal}(\mu; \rho)$ in $D$, $\mu \leq \mathrm{Bal}(\mu; \rho)$ outside $D$. One further natural property is that partial balayage can always be performed in smaller steps, for example

$$\mathrm{Bal}(\mathrm{Bal}(\mu_1; \rho_2) + \mu_2; \rho_1) = \mathrm{Bal}(\mu_1 + \mu_2; \rho_1) \tag{2.4}$$

holds if $\rho_1 \leq \rho_2 + \mu_2$.

As to the structure of $\mathrm{Bal}(\mu; \rho)$, let $\Omega(\mu; \rho)$ denote the "saturated" part of $D$, namely

$$\Omega(\mu; \rho) = \text{ the largest open subset of } D \text{ in which } \mathrm{Bal}(\mu; \rho) = \rho.$$

Then, under some light regularity assumptions ($\mu \in L^\infty$ is enough),

$$\mathrm{Bal}(\mu; \rho) = \rho\chi_{\Omega(\mu;\rho)} + \mu\chi_{D\setminus\Omega(\mu;\rho)}$$

in $D$. In "good" cases the second term drops off and $\mathrm{Bal}(\mu; \rho)$ takes, inside $D$, the pure form

$$\mathrm{Bal}(\mu; \rho) = \rho\chi_{\Omega(\mu;\rho)}.$$

This occurs if $\mu$ is concentrated enough, e.g. if $\mu$ is singular with respect to Lebesgue measure or if $\mu \geq \rho$ on some open set and $\mu = 0$ outside it. It also holds if, for some open ball $B$, $\mu(B) \geq 2^N |B| \sup_B \rho$ and $\mu = 0$ outside $B$ [11, 15].

As with classical balayage (which is the case $\rho = 0$ in $D$) $\mu \mapsto \mathrm{Bal}(\mu; \rho)$ may also sweep some part of $\mu|_D$ onto $\partial D$. It is not necessary in partial balayage to assume that $\mathrm{supp}\, \mu \subset \overline{D}$, but we always assume that $\mu$ has compact support in $\mathbf{R}^N$. Outside $\overline{D}$, $\mathrm{Bal}(\mu; \rho) = \mu$.

Note that the set $\Omega(\mu; \rho)$ is unknown from the beginning, so partial balayage is effectively a free boundary problem. A direct statement of this free boundary problem in terms of the function $u = U^\mu - V^\mu$ is (assuming for simplicity $\mu \in L^\infty$, in which case $u$ will be continuously differentiable in $D$)

$$\begin{aligned} u &\geq 0 && \text{in } \mathbf{R}^N, \\ u &= 0 && \text{on } \mathbf{R}^N \setminus D, \\ \Delta u &= \rho - \mu && \text{in } \{u > 0\} \subset D, \\ u &= |\nabla u| = 0 && \text{on } (\partial\{u > 0\}) \cap D. \end{aligned}$$

In addition it is required that the unknown set $\{u > 0\}$ contains the set where $\mu > \rho$. Notice that $\{u > 0\} \subset \Omega(\mu; \rho)$ by (2.3), and typically these two sets actually coincide.

*Historical remark*: The idea of partial balayage goes back at least to the work in the 1960s of the Bulgarian geophysicist D. Zidarov [20], who also developed an intuitive and efficient numerical process for it, sometimes called "Zidarov bubbling" (cf. also [9]). In this process the mass distributions $\mu, \rho$ etc. are represented as functions on a numerical lattice $(\varepsilon\mathbf{Z})^N$ (replacing $\mathbf{R}^N$), and the "bubbling" process, which is a discrete version of $\mu \mapsto \mathrm{Bal}(\mu; \rho)$, goes by moving around indefinitely in the lattice and each time a lattice point at which $\mu$ is larger than $\rho$ is encountered the exceeding mass $\mu - \rho$ (or part of it) is redistributed equally on the $2^N$ closest neighbours. This process can be shown to converge.

Later on (and independently) corresponding ideas and methods, for the nondiscrete version, were invented and put on a firm mathematical basis by M. Sakai [13, 14, 6] and others in connection with construction of "quadrature domains".

Parallel to all this there has been a development in abstract potential theory with creation of concepts of *mixed envelopes* [1] which appear to be very close to partial balayage. In fact, the definition (2.2) can, in the notation of [1], be written

$$\mathrm{Bal}(\mu; \rho) = \rho \smile \mu,$$

where $\rho \smile \mu$ is the mixed lower envelope of $\rho$ and $\mu$. (We have here changed the framework of [1] slightly.)

## 3. Evolution version of partial balayage and Hele-Shaw flows

For partial balayage the set $D$ may very well be unbounded, and indeed $D = \mathbf{R}^N$ is a case of major interest. If $D$ is unbounded one has to assume that $\rho$ is not too small at infinity in order to ensure that $\mathrm{Bal}(\mu; \rho)$ has compact support ($\rho \geq$ const. $> 0$ is enough).

In the sequel we shall concentrate on the case

$$D = \mathbf{R}^N,$$
$$\rho = c = \text{constant} > 0.$$

In this case one can, using the "moving plane method", prove a beautiful geometric property of $\mathrm{Bal}(\mu; c)$: For any closed half-space $H$ which contains supp $\mu$ the part of $\mathrm{Bal}(\mu; c)$ which falls outside $H$ is of the form $c\chi_{\Omega \setminus H}$ (where $\Omega = \Omega(\mu; c)$) and $\Omega \setminus H$ is a subgraph of a real analytic function when viewed from $H$. In addition, the reflection of $\Omega \setminus H$ in $\partial H$ is contained in $\Omega$. If, e.g., $H = \{x : x_N \leq 0\}$ the first statement means that

$$\Omega \setminus H = \{x : 0 < x_N < \varphi(x_1, \ldots, x_{N-1})\}$$

with $\varphi$ real analytic.

Let $K$ be the closed convex hull of supp $\mu$. Applying the above result to all half-spaces $H$ containing $K$ it follows that, outside $K$, $\mathrm{Bal}(\mu; c)$ is of the form $c\chi_\Omega$, $(\partial\Omega) \setminus K$ is real analytic and

$$\text{for any } x \in (\partial\Omega) \setminus K \text{ the inward normal ray of } \partial\Omega \text{ at } x \text{ intersects } K. \tag{3.1}$$

Thus we have both regularity and geometric information about $\partial\Omega$ outside $K$.

Instead of regarding partial balayage as an instantaneous operation one may turn it into a continuous process by introducing a time parameter $t$. If one is interested in $\mathrm{Bal}(\mu; c)$ one might, e.g., look at $\mu(t) = \mathrm{Bal}\,(\mu; e^{-t})$ where $t$ goes from $-\infty$ to $-\log c$. For $t = -\log c$, $\mu(t) = \mathrm{Bal}(\mu; c)$ while, as $t \to -\infty$, $\mu(t) \to \mu$ in some sense. If e.g., $\mu \in L^\infty$ then $\mu(t) = \mu$ for all $t \leq -\log \|\mu\|_{L^\infty}$.

The above continuous process has several physical applications. One is Hele-Shaw flows, in which case a blob of a viscous incompressible fluid is squeezed in the narrow region between two parallel plates [12, 3]. If the fluid region at time $t$ is represented by $\omega(t) \subset \mathbf{R}^2$ (a bounded domain) and the plates are squeezed so that the distance between them at time $t$ is proportional to $e^{-t}$, then the Hele-Shaw law for the growth of $\omega(t)$ is that $\partial\omega(t)$ propagates with outward normal velocity equal to minus the outward normal derivative of $p_{\omega(t)}$, where $p = p_{\omega(t)}(x)$ is the unique solution of

$$\begin{cases} -\Delta p &= 1 \qquad \text{in } \omega(t), \\ \quad p &= 0 \qquad \text{on } \partial\omega(t). \end{cases} \tag{3.2}$$

Extending $p$ by zero outside $\omega(t)$ and taking the Laplacian in the distributional sense in all $\mathbf{R}^2$ the above gives that

$$\begin{aligned} -\Delta p_{\omega(t)} &= \chi_{\omega(t)} - \frac{d}{dt}\,\chi_{\omega(t)} \\ &= -e^t\,\frac{d}{dt}\left(e^{-t}\chi_{\omega(t)}\right). \end{aligned}$$

Integrating we find

$$\Delta u = e^{-t}\chi_{\omega(t)} - e^{-s}\chi_{\omega(s)},$$

where $u = \int_s^t e^{-\tau} p_{\omega(\tau)} d\tau$ (the "Baiocchi" transform of $p$).

For $s < t$, $\omega(s) \subset \omega(t)$, hence $u = 0$ outside $\omega(t)$ and $u \geq 0$ everywhere. From this it easily follows that

$$e^{-t}\chi_{\omega(t)} = \mathrm{Bal}\left(e^{-s}\chi_{\omega(s)}; e^{-t}\right) \tag{3.3}$$

when $s < t$. Thus the fluid region at any particular instant is obtained as a natural partial balayage of the fluid region at any previous instant. Equation (3.3) may also be written as $\chi_{\omega(t)} = \mathrm{Bal}(e^{t-s}\chi_{\omega(s)}; 1)$ or $\omega(t) = \Omega(e^{t-s}\chi_{\omega(s)}; 1)$ (up to null-sets). Taking $s = 0$ we have

$$\chi_{\omega(t)} = \mathrm{Bal}\left(e^t\chi_{\omega(0)}; 1\right)$$

for $t > 0$.

The above was one version of Hele-Shaw flow moving boundary problems. Another popular version is to not squeeze the plates but instead inject fluid at a constant rate at some point $a \in \omega(0)$. Then $p = p_{\omega(t)}$ will satisfy instead $-\Delta p = \delta_a$, i.e., it will be the Green's function of $\omega(t)$, and one ends up with the balayage formula

$$\chi_{\omega(t)} = \mathrm{Bal}\left(\chi_{\omega(0)} + t\,\delta_a; 1\right) \tag{3.4}$$

for $t > 0$. This is a (somewhat stronger) form of the moment property discovered by Richardson [12].

The Hele-Shaw interpretations above make physical sense only in two dimensions, but clearly the mathematics works equally well in any number of dimensions.

## 4. Inverse balayage and mother bodies

The above forward versions of Hele-Shaw flows can be considered as being fairly well understood by now. Less well understood, however, are the backward versions of Hele-Shaw flows (letting time go backwards or, equivalently, changing a sign in (3.2)) and the corresponding notions of *inverse balayage*.

In the backward version of (3.3) $\omega(0)$ is given and one asks for "good" domains $\omega(t)$ for $t < 0$ such that (3.3) holds whenever $s < t \leq 0$. In particular one requires

$$\chi_{\omega(0)} = \mathrm{Bal}\left(e^{-t}\chi_{\omega(t)}; 1\right)$$

for $t < 0$. This problem is ill-posed, and good solutions $\omega(t)$ (satisfying, e.g., $\overline{\omega(t)} \subset \omega(0)$ for $t < 0$) only exist if $\partial\omega(0)$ is real analytic (roughly speaking).

There always exist "bad" solutions in abundance. One example is obtained by ball-packing: first write $\omega(0) = \bigcup_{n=1}^{\infty} B(a_n, r_n) \cup$ (null set), where the $B(a_n, r_n)$ are disjoint balls, and then take $\omega(t) = \bigcup_{n=1}^{\infty} B(a_n, r_n e^{t/N})$ $(t < 0)$. Such solutions are not very interesting, and it is in fact a major problem to find additional conditions for (3.3) which single out some reasonable class of good solutions. The extreme alternative is to allow only "classical" solutions of the moving boundary problem as stated at (3.2) (with time going backwards) but these exist only rarely (when $\partial\omega(0)$ is analytic basically) and, when they do exist, usually break down rapidly because of development of singularities on the boundary [7]. Some good concept of weak solution would therefore be welcome. Cf. [10], and also below.

Another approach to backward Hele-Shaw flows, perhaps the most intuitive one, is a probabilistic approach using Brownian motion. Then it becomes a version of so-called diffusion-limited aggregation [19]: at each instant $t$ a particle is dropped with uniform probability in $\omega(t)$ and there starts moving around according to the Brownian law of motion. Sooner or later it reaches $\partial\omega(t)$ where it gets stuck and becomes part of the "aggregate" $\mathbb{R}^N \setminus \omega(t)$, which then grows by a corresponding amount (it is easiest to think of the case in which both time and space have been discretized). The above gives, except for a time scaling, exactly the law at (3.2) (with signs adjusted to get a shrinking $\omega(t)$). For stochastic versions of forward Hele-Shaw problems, see [21].

Now, assume that we are given a backward solution $\omega(t)$ of (3.3), defined for all $t \leq 0$. Then $\omega(s) \subset \omega(t)$ for $s < t$ and $e^{-t}|\omega(t)| = |\omega(0)|$ for all $t \leq 0$. It follows that, as $t \to -\infty$, $e^{-t}\chi_{\omega(t)}$ converges weakly to a measure $\mu$ which lives on a set of Lebesgue measure zero and which generates the whole flow in the sense that

$$\chi_{\omega(t)} = \mathrm{Bal}\left(e^{-t}\mu; 1\right) \tag{4.1}$$

for all $-\infty < t < \infty$.

Related to this there is a vague notion of *mother body*, or potential theoretic skeleton, for a domain $\omega \subset \mathbf{R}^N$ [20]. A mother body $\mu$ (positive measure) for $\omega$ should satisfy

$$\chi_\omega = \mathrm{Bal}(\mu; 1), \tag{4.2}$$

and moreover be sufficiently concentrated in some sense. Suggestions for making the latter requirement precise are [5]:

supp $\mu$ has Lebesgue measure zero; (4.3)

each point in $\omega \setminus \mathrm{supp}\ \mu$ can be joined to $\mathbf{R}^N \setminus \overline{\omega}$ by a curve in $\mathbf{R}^N \setminus \mathrm{supp}\ \mu$. (4.4)

It may be worth saying that (4.2) is equivalent to the two requirements

$$U^{\chi_\omega} \leq U^\mu \qquad \text{in } \mathbf{R}^N, \tag{4.5}$$

$$U^{\chi_\omega} = U^\mu \qquad \text{(a.e.) on } \mathbf{R}^N \setminus \omega. \tag{4.6}$$

This can be seen relatively easy from the second definition of partial balayage along with (2.3).

In the backward Hele-Shaw case (3.3) we see that $\mu$ will be a mother body of $\omega(0)$ in the above sense provided (4.3), (4.4) are satisfied. To ensure (4.3) we just need to require that $|\partial\omega(t)| = 0$ for all $t < 0$. The requirement (4.4) is more crucial and is, e.g., not satisfied in the ballpacking example above: we then have supp $\mu = \overline{\{a_1, a_2, \dots\}}$, which typically contains all of $\partial\omega(0)$.

One point with the above notion of mother body is that, in at least some cases, one can get close to both existence and uniqueness, and even to direct constructability [5, 17]. Note also that mother bodies provide one approach to weak solution concepts for backward Hele-Shaw problems. This is in particular true for the squeezing version, as (4.1) in this case right away defines a global weak solution out of any mother body for the initial domain.

Also the (backward of the) injection version (3.4) of Hele-Shaw flows can be handled by mother bodies to a certain extent. First one notices that (3.4) can be written

$$\chi_{\omega(t)} = \mathrm{Bal}(\mu + t\delta_a; 1) \tag{4.7}$$

($t > 0$) if $\mu$ is a mother body of $\omega = \omega(0)$. (This is a consequence of (2.4), (4.2).) In very lucky cases $\mu + t\delta_a \geq 0$ for some interval of negative values of $t$, and then (4.7) can be used directly for these $t$ to get a backward solution.

In the general case one first has to modify $\mu + t\delta_a$ a little, e.g. to $\mathrm{Bal}(\mu; c_1) + t\ \mathrm{Bal}(\delta_a; c_2)$ for suitable $c_1, c_2 \geq 1$. Then

$$\mathrm{Bal}\ (\mathrm{Bal}(\mu; c_1) + t\ \mathrm{Bal}(\delta_a; c_2); 1) = \mathrm{Bal}(\mu + t\delta_a; 1) = \chi_{\omega(t)} \tag{4.8}$$

for $t \geq 0$ by (2.4), (4.7). Next one realizes that $c_1, c_2 \geq 1$ can be chosen so that the left member of (4.8) makes sense and is of the form $\chi_{\omega(t)}$ also for some negative values of $t$. This gives our backward solution $\omega(t)$ for some, usually short, interval $-\varepsilon < t < 0$.

*Examples* of mother bodies.

1. The simplest example is any ball $\omega = B(a,r)$, which has the unique mother body $\mu = |B(a,r)|\delta_a$ (i.e., $\mu$ is unique among all measures satisfying (4.2)–(4.4)).
2. For $\omega$ an ellipsoid in $\mathbf{R}^N$ there exists a mother body supported by the so-called focal ellipsoid (which is the segment between the foci in the case of $N=2$) [8, 18]. When $N=2$ this mother body is unique [17].
3. If $\omega$ is a convex polyhedron in $\mathbf{R}^N$ then $\omega$ also has a unique mother body [5]. It is supported by the set of points $x \in \overline{\omega}$ which has at least two closest neighbours on $\partial\omega$ (the "ridge" of $\omega$).
4. For $\omega$ a general polyhedron in $\mathbf{R}^2$ there exists at least one mother body of $\omega$, but there are examples of nonuniqueness [20]. In higher dimensions existence is not clear.

Unfortunately some very good domains do not admit mother bodies in our sense. This is the case, e.g., for any domain $\omega \subset \mathbf{R}^2$ which is the conformal image of the unit disc under a polynomial of degree strictly greater than one. The same is true, more generally, for any domain $\omega \subset \mathbf{R}^N$ for which (4.6) holds for some distribution (not necessarily a measure) $\mu$ with support in a finite number of points in $\omega$, but for which (4.5) fails to hold. Such a $\mu$ will automatically satisfy (4.3), (4.4), and it is easy to see that there cannot simultaneously exist a positive measure satisfying all of (4.3)–(4.6) (i.e., (4.2)–(4.4)).

In principle one could of course relax the notion of mother body by allowing more general distributions than positive measures and by dropping requirement (4.5). It seems, however, that (4.5) is crucial for the evolution version (4.1) and for the coupling to Hele-Shaw flows. Also, without (4.5) and positivity of $\mu$ there will be no coupling whatsoever between the geometry of $\omega$ and that of supp $\mu$ [4, 16]. For positive $\mu$ satisfying (4.2) we have at least some geometric information, by (3.1) for example. Note that (4.2) implies $\omega = \Omega(\mu; 1)$ (up to null-sets).

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