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PROPERTIES OF SOME BALAYAGE OPERATORS, WITH APPLICATIONS TO QUADRATURE DOMAINS AND MOVING BOUNDARY PROBLEMS

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1. INTRODUCTION

IN THIS PAPER we define and derive some properties of a fairly wide class of balayage operators. We also apply our results to obtain geometric information about quadrature domains and solutions of certain moving boundary problems (e.g. for Hele–Shaw flows). Actually it was our interest in these applications that was the starting point for the present investigations and the balayage operators were developed rather as a tool.

Our balayage operators are denoted $F = F_\rho = F_{\rho,R}$ and they depend on an open set $R \subset \mathbb{R}^N$ and a measure ρ in R; actually we always assume that ρ has a density (also denoted ρ) in $L^\infty(R)$. What F does is replace a given measure μ by the nearest one (in the energy norm) $\nu = F(\mu)$ which satisfies $\nu \le \rho$ in R. This presupposes that μ has finite energy. If this is not the case, there is another definition, which is the one we will actually use: among all distributions u in $\mathbb{R}^N$ satisfying $u \le U^\mu$ (the Newtonian potential of μ) in $\mathbb{R}^N$ and $-\Delta u \le \rho$ in R there is a largest one, $u = V^\mu$. Then $F(\mu) = -\Delta V^\mu$ (by definition).

The operator of interest in the above-mentioned applications is F with $R = \mathbb{R}^N$, $\rho \equiv 1$. The definition of $F(\mu)$ when μ has finite energy can be formulated as an elliptic variational inequality (or linear complementarity problem), and this is one of the standard tools when dealing with free boundary problems of the above kind (see [1–4]). However, we think that the balayage point of view is quite natural in our context.

The reason for considering the balayage operators for more general R and ρ than $R = \mathbb{R}^N$, $\rho = 1$ is, firstly, that we need them in the proof of one of our theorems (theorem 4.1). Secondly, and perhaps more important, is that we think that the more general operators have quite a lot of intrinsic interest, in particular as they turn out to contain "classical" balayage (of positive measures) as a special case (namely with R bounded and $\rho = 0$). Recently, we also learnt that discrete (or numerical) versions of these operators have been developed and applied in geophysics during the last three decades by Dimiter Zidarov, who calls the balayage process "(partial) gravi-equivalent mass scattering". See [5], and also, for example, [6].

The paper is organized as follows. In Section 2 we define and establish the basic properties of our operators. At the end we also explain their relation to quadrature domains and certain moving boundary problems. Section 3 is devoted to one single theorem (theorem 3.1) and its corollaries. These results concern the F with $R = \mathbb{R}^N$, $\rho = 1$ and give rather good geometric information about $F(\mu)$ outside the convex hull of supp μ when $\mu \geq 0$. The implications for the applications are quite striking in some cases.

The proof of theorem 3.1 is based on a kind of reflexion argument which is similar to methods which have earlier been used in [7–10]. (These methods seem to be due to Alexandroff, however.) In Section 4 we generalize part of theorem 3.1 and use a different method of proof. For the moment we have no concrete applications of the result there, theorem 4.1, but we hope that it can be used as a tool in the study of $F(\mu)$ ($R = \mathbb{R}^N$, $\rho = 1$) inside the convex hull of supp μ.

Notation

$B(a; r) = \{x \in \mathbb{R}^N: |x - a| < r\}$ $(N \geq 2)$;
$\Omega^c = \mathbb{R}^N \backslash \Omega$;
$\Omega^e = \mathbb{R}^N \backslash \bar{\Omega}$;
χ_Ω = characteristic function of Ω;
h_ε: mollifiers; see (2.8);
m = Lebesgue measure;
$[\Omega] = \{x \in \mathbb{R}^N: m(B(x; r)\backslash\Omega) = 0$ for some $r > 0\}$ = the completion of Ω with respect to Lebesgue measure (if $\Omega \subset \mathbb{R}^N$ is open);
δ_x = Dirac measure (point mass) at $x \in \mathbb{R}^N$;
$\delta = \delta_0$;
Δ = Laplacian operator;
∇ = gradient;
$*$: convolution;
$\mathfrak{D}'(\mathbb{R}^N)$: the distributions in $\mathbb{R}^N$;
$\mathcal{E}'(\mathbb{R}^N)$: those with compact support;
$M_c = \mathcal{E}'(\mathbb{R}^N)_+ - \mathcal{E}'(\mathbb{R}^N)_+$: the Radon measures with compact support in $\mathbb{R}^N$;
$L^\infty = L^\infty(\mathbb{R}^N)$;
$L^\infty_c = L^\infty_c(\mathbb{R}^N) = \{f \in L^\infty(\mathbb{R}^N): f = 0$ outside a compact set$\}$;
$M' = \{\mu \in M_c: \mu_- \in L^\infty\} = \mathcal{E}'(\mathbb{R}^N)_+ + L^\infty_c$;
$\langle \mu, \varphi \rangle$: the action of a distribution μ on a test function φ, a measure μ on a continuous function φ ($\langle \mu, \varphi \rangle = \int \varphi \, d\mu$) etc.;
$H^1_0(B), H^{-1}(B)$: Sobolev spaces;
E = the spherically symmetric fundamental solution of $-\Delta$ (so that $-\Delta E = \delta$);
$U^\mu = E * \mu$ = the Newtonian potential of $\mu \in M_c$ (or $\mu \in \mathcal{E}'(\mathbb{R}^N)$);
$V^\mu = V^\mu_\rho$: defined in theorem 2.1;
$\omega(\mu) = \{x \in R: V^\mu(x) < U^\mu(x)\}$ (see theorem 2.1);
$\Omega(\mu)$: see (2.31);
$F = F_\rho = F_{\rho,R}$: defined before theorem 2.2;
$\mathfrak{F}^\mu_\rho$: see (2.5);
$SL^1(\Omega), HL^1(\Omega)$: see after example 2.2;
$Q(\mu, SL^1), Q(\mu, HL^1)$: see after example 2.2.

Primarily, measures etc. are regarded as distributions, therefore we usually do not distinguish notationally between an absolutely continuous measure and its density function with respect to Lebesgue measure, e.g. m (Lebesgue measure) and 1 denote the same thing, and two absolutely continuous measures μ and ν coincide if and only if $\mu = \nu$ a.e. as density functions.

2. THE OPERATORS F_ρ

In this section we define and establish some basic properties of our balayage operators $F = F_\rho = F_{\rho,R}$. For a given measure μ, $\nu = F(\mu)$ will turn out to be the measure closest (in the energy norm) to μ among all measures satisfying $\nu \le \rho$ in R. This $\nu = F(\mu)$ will be equivalent to μ in the sense that $U^\nu = U^\mu$ holds wherever $\nu < \rho$ in R and also everywhere outside R.

The data needed are

$$\text{an open set } R \subset \mathbb{R}^N \ (N \ge 2); \tag{2.1}$$

$$\text{a function } \rho \in L^\infty(R). \tag{2.2}$$

ρ will be regarded as a measure (identified with ρm, where m denotes Lebesgue measure), and in principle we could allow ρ to be a fairly arbitrary measure, but unfortunately this leads to technical difficulties which we have not been able to handle. For R we assume that it is regular for Dirichlet's problem, i.e. that

$$\text{there exists a barrier (see [11]) at each point of } \partial R. \tag{2.3}$$

Moreover, if R is unbounded we assume that

$$\rho \ge \text{constant} > 0 \text{ outside a bounded subset of } R. \tag{2.4}$$

It is sometimes convenient to have ρ defined in all $\mathbb{R}^N$. We then set (formally) $\rho = +\infty$ outside R. Thus, $R = \{\rho < +\infty\}$ and in our notation (e.g. writing F_ρ instead of $F_{\rho,R}$) the information about R will usually be thought of as being built into ρ.

For $\mu \in M_c$ (see the list of notations before this section), we set

$$\mathfrak{F}_\rho^\mu = \{u \in \mathfrak{D}'(\mathbb{R}^N) \colon u \le U^\mu \text{ in } \mathbb{R}^N,\ -\Delta u \le \rho \text{ in } R\}. \tag{2.5}$$

We shall see that this family $\mathfrak{F}_\rho^\mu$ always contains a largest element $V^\mu = \sup \mathfrak{F}_\rho^\mu$ (this is true even under more general assumptions on ρ and R than stated above) and the balayage operator F_ρ will be defined by $F_\rho(\mu) = -\Delta V^\mu$. Then $F_\rho(\mu) \in \mathcal{E}'(\mathbb{R}^N)$ if $\mu \in M_c$.

It is crucial for our theory that $F_\rho(\mu)$ is a measure. If $R = \mathbb{R}^N$ this is automatically the case, but in general we will have to make additional assumptions on μ to ensure this (as well as some other properties we need). What we need is that the negative part of μ is not too bad. For convenience we shall in most of the paper simply assume that $\mu_- \in L^\infty$, although that certainly is far from the best possible assumption (what we will be using is simply that U^{μ_-} is a continuous function). The class of measures we will usually work with is thus

$$M' = \{\mu \in M_c \colon \mu_- \in L^\infty\} = \mathcal{E}'(\mathbb{R}^N)_+ + L_c^\infty.$$

Since our main results in this paper just concern positive measures, the above limitations on μ will be of minor importance.

It is immediate from the definition (2.5) of $\mathfrak{F}_\rho^\mu$ that

$$\mathfrak{F}_{\rho+\sigma}^{\mu+\sigma} = \mathfrak{F}_\rho^\mu + U^\sigma \qquad \text{for } \sigma \in M' \cap L^\infty(R), \tag{2.6}$$

$$\mathfrak{F}_{t\rho}^{t\mu} = t\mathfrak{F}_\rho^\mu \qquad \text{for } t > 0, \tag{2.7}$$

where in both formulas it is understood that R is unchanged when going from ρ to $\rho + \sigma$ or $t\rho$ (as is consistent with writing $R = \{\rho < +\infty\}$).

Radial mollifiers will be needed in the paper. Let $h \in C_c^\infty(\mathbb{R}^N)$ be a function depending only on $r = |x|$ and satisfying $h \geq 0$, $\operatorname{supp} h \subset B(0; 1)$ and $\int h \, dm = 1$. For any $\varepsilon > 0$ set

$$h_\varepsilon(x) = \varepsilon^{-N} h(\varepsilon^{-1} x). \tag{2.8}$$

Then $h_\varepsilon \geq 0$, $\operatorname{supp} h_\varepsilon \subset B(0; \varepsilon)$ and $\int h_\varepsilon \, dm = 1$. Recall that if φ is (say) a subharmonic function then $\varphi * h_\varepsilon \geq \varphi$ within the domain of definition of the convolution $\varphi * h_\varepsilon$, and $\varphi * h_\varepsilon \searrow \varphi$ pointwise as $\varepsilon \searrow 0$.

An important remark is the following. A distribution u satisfying $-\Delta u \leq \rho \in L^\infty(R)$ in an open set R has a canonical (= smallest here) representative in the form of an upper semicontinuous function with values in $\mathbb{R} \cup \{-\infty\}$. In fact, if $\rho = 0$, this is the representative in the form of a subharmonic function, and in the general case the difference to a subharmonic function is a function φ satisfying $\Delta\varphi \in L^\infty(R)$ and such a φ has a unique continuous representative. Another characterization of the canonical representative u above is that, with $\{h_\varepsilon\}_{\varepsilon > 0}$ mollifiers as in (2.8),

$$u * h_\varepsilon \to u \qquad \text{pointwise as } \varepsilon \to 0. \tag{2.9}$$

If $-\Delta u \geq \rho \in L^\infty(R)$ similar statements hold of course (there is a canonical lower semicontinuous representative etc.). Thus, any distribution u with Δu bounded from above or below has a canonical representative which we will often refer to in the sequel. For $\mu \in M'$, U^μ always refers to the canonical lower semicontinuous representative, and this coincides with the function defined pointwise by

$$U^\mu(x) = \int E(x - y) \, d\mu(y).$$

Note, as a consequence of (2.9), that if an inequality $u_1 \leq u_2$ holds in the distribution sense between two distributions u_1 and u_2 which have canonical representatives in the above sense, then the same inequality holds pointwise (at every point) for their canonical representatives.

THEOREM 2.1. With ρ, R as above ((2.1)–(2.4)) and with $\mu \in M'$, $\mathfrak{F}_\rho^\mu$ contains a largest element, denoted V^μ (or, if necessary, V_ρ^μ). Moreover,

(a) V^μ coincides with U^μ outside a compact set,

(b) $-\Delta V^\mu \geq \lambda$, where λ is the measure defined by

$$\lambda = \begin{cases} \min(\rho, \mu) & \text{in } R, \\ \mu & \text{on } R^c. \end{cases}$$

It follows in particular that V^μ is the potential of a measure in M' (namely $-\Delta V^\mu$) and that V^μ has a canonical lower semicontinuous representative (which we will always refer to in the sequel).

(c) V^μ is continuously differentiable in R (in fact $-\Delta V^\mu \in L^\infty(R)$) and satisfies $V^\mu = U^\mu$ everywhere on R^c.

(d) $-\Delta V^\mu = \rho$ in the (bounded) open set $\omega(\mu) = \{x \in R: V^\mu(x) < U^\mu(x)\}$.

Thus V^μ satisfies a kind of complementarity system.

$$V^\mu \leq U^\mu \qquad \text{in } \mathbb{R}^N, \tag{2.10}$$

$$-\Delta V^\mu \leq \rho \qquad \text{in } R, \tag{2.11}$$

$$V^\mu = U^\mu \qquad \text{on } R^c, \tag{2.12}$$

$$-\Delta V^\mu = \rho \qquad \text{in } \omega(\mu) = \{x \in R: V^\mu(x) < U^\mu(x)\}. \tag{2.13}$$

We shall see later that, at least if μ has finite energy, (2.10)–(2.13) characterize V^μ (among potentials of finite energy).

It is sometimes more convenient to work with the function

$$u = U^\mu - V^\mu \tag{2.14}$$

instead of V^μ. For u we then have

$$u \geq 0 \qquad \text{in } \mathbb{R}^N, \tag{2.15}$$

$$\Delta u \leq \rho - \mu \qquad \text{in } R, \tag{2.16}$$

$$u = 0 \qquad \text{on } R^c, \tag{2.17}$$

$$\Delta u = \rho - \mu \qquad \text{in } \omega(\mu) = \{x \in R\colon u(x) > 0\}. \tag{2.18}$$

Moreover, u has compact support and is the smallest function (or distribution) satisfying (2.15)–(2.16). In (2.18) we are referring to the canonical lower semicontinuous representative of u in R. Outside R, the right-hand side of (2.14) may take the form $(+\infty)$–$(+\infty)$ at certain points. By definition we set $u = 0$ at such points (and then (2.17) follows from (2.12)). Note that working with u instead of V^μ, one automatically takes care of the "covariance" (2.6).

Proof of theorem 2.1. First observe that all statements of the theorem are invariant under transformations as in (2.6). We will use freely this possibility of replacing (μ, ρ) by $(\mu + \sigma, \rho + \sigma)$ whenever convenient. This means that we can always assume, for example, that $\mu \geq 0$, or that $\rho = 0$ in any bounded set (but not both these things at the same time).

We start by assuming that

$$\mu \geq 0, \tag{2.19}$$

$$\rho \geq \text{constant} > 0 \qquad \text{in } R. \tag{2.20}$$

Set

$$u_\varepsilon = U^\mu * h_\varepsilon.$$

Then $u_\varepsilon \leq U^\mu$ in $\mathbb{R}^N$ by (2.19) and, because of (2.20),

$$-\Delta u_\varepsilon = \mu * h_\varepsilon \leq \rho \qquad \text{in } R \tag{2.21}$$

if just $\varepsilon > 0$ is large enough.

Fix an $\varepsilon > 0$ such that (2.21) holds. Then $u_\varepsilon \in \mathfrak{F}_\rho^\mu$ and $u_\varepsilon = U^\mu$ outside an ε-neighbourhood of $\operatorname{supp}\mu$. Also note that U^μ, $u_\varepsilon \in L^1_{\text{loc}}(\mathbb{R}^N)$.

The above shows that $\mathfrak{F}_\rho^\mu$ is nonempty , and that in forming the supremum of $\mathfrak{F}_\rho^\mu$ we may restrict ourselves to elements $u \in \mathfrak{F}_\rho^\mu$ which satisfy $u_\varepsilon \leq u \leq U^\mu$, hence, are locally integrable, locally bounded from below and coincide with U^μ outside some compact set K. This also shows that nothing is changed if we cut off R slightly outside K. It is easy to see that this cutting off can be done without violating (2.3), and that when going back to the original R the statements of the theorem remain valid. Thus we assume from now on that R is bounded.

With R bounded and regular for the Dirichlet problem, we shall (for later use) improve u_ε a little: for arbitrary $\varepsilon > 0$, let v_ε be the solution of $\Delta v_\varepsilon = 0$ in R, $v_\varepsilon = u_\varepsilon$ on ∂R and define

$$w_\varepsilon = \begin{cases} v_\varepsilon & \text{in } R \\ u_\varepsilon & \text{on } R^c. \end{cases}$$

Then w_ε is superharmonic and continuous, $w_\varepsilon \le u_\varepsilon$ and $w_\varepsilon \in \mathfrak{F}_\rho^\mu$. As $\varepsilon \searrow 0$, $u_\varepsilon \nearrow U^\mu$ at every point and it follows [11, theorem 4.15] that $w_\varepsilon \nearrow w$, where $w = \sup_{\varepsilon>0} w_\varepsilon$ is a superharmonic function satisfying $w = U^\mu$ on R^c. Moreover, $w \in \mathfrak{F}_\rho^\mu$.

Next we replace assumptions (2.19), (2.20) by

$$\rho = 0 \qquad \text{in } R \tag{2.22}$$

(with R bounded and with (2.19) possibly violated). Define $V^\mu \in L^1_{\text{loc}}(\mathbb{R}^N)$ by

$$V^\mu = \begin{cases} U^\mu & \text{on } R^c \\ u & \text{in } R, \end{cases}$$

where u is the largest subharmonic function in R satisfying $u \le U^\mu$ (in R). Observing that the family $\mathfrak{F}_\rho^\mu$ is locally bounded from above in R, it follows from standard facts about subharmonic functions [11, theorem 4.16] that this u exists; it is contructed as the upper regularization of the pointwise supremum of all subharmonic functions φ in R satisfying $\varphi \le U^\mu$. It is immediate from this construction that $V^\mu \in \mathfrak{F}_\rho^\mu$ and that $\varphi \le V^\mu$ for all $\varphi \in \mathfrak{F}_\rho^\mu$.

Thus, $V^\mu = \sup \mathfrak{F}_\rho^\mu$ is now constructed and it satisfies $V^\mu \in L^1_{\text{loc}}(\mathbb{R}^N)$ and (a) in the theorem. Moreover, V^μ never takes the value $-\infty$ and in R it, moreover, never attains $+\infty$.

Set $\omega(\mu) = \{x \in R: V^\mu(x) < U^\mu(x)\}$. For every $x \in \omega(\mu)$ there is, by the semicontinuities of V^μ and U^μ, a neighbourhood $B \subset\subset R$ of x and a number $\alpha \in \mathbb{R}$ such that $V^\mu \le \alpha \le U^\mu$ in $\bar{B}$. It follows that $-\Delta V^\mu = 0$ in B, for otherwise the replacement of V^μ in B by its Poisson integral with boundary values V^μ on ∂B would give a larger element in $\mathfrak{F}_\rho^\mu$, contradicting the maximality of V^μ (cf. [11, lemma 4.17] and also "Perron's method"). We conclude that $-\Delta V^\mu = 0$ $(=\rho)$ in $\omega(\mu)$. This proves (d) in the theorem.

Next we want to prove (b). From (2.6), we get $\mathfrak{F}_\rho^\mu = \mathfrak{F}_{\rho-\lambda}^{\mu-\lambda} + U^\lambda$, and this shows that, in order to prove (b), it is enough (and necessary) to prove that

$$-\Delta V^\mu \ge 0$$

under the assumption that

$$\mu \ge 0, \qquad \rho \ge 0.$$

For this purpose set

$$\mathcal{G} = \{v \in \mathfrak{D}'(\mathbb{R}^N): v \ge V^\mu, -\Delta v \ge 0\}.$$

Then $U^\mu \in \mathcal{G}$ and in a standard manner (as in the construction of V^μ), it follows that $\mathcal{G}$ contains a smallest element, which has a representative W^μ in the form of a superharmonic (hence, lower semicontinuous) function. Thus $V^\mu \le W^\mu \le U^\mu$ (a.e.) and $-\Delta W^\mu \ge 0$. We claim that

$$-\Delta W^\mu \le \rho \qquad \text{in } R. \tag{2.23}$$

If this is true, then $W^\mu \in \mathfrak{F}_\rho^\mu$. Since $V^\mu \le W^\mu$, it follows that $W^\mu = V^\mu$ (a.e.). Hence, $-\Delta V^\mu \ge 0$, which is the desired conclusion.

Consider the function

$$v = W^\mu - V^\mu \qquad \text{in } R. \tag{2.24}$$

Then it is enough to prove that v is subharmonic, for since $-\Delta V^\mu \le \rho$ in R this would give (2.23). We have

$$v \ge 0 \qquad \text{in } R, \tag{2.25}$$

$$\Delta v \le \rho \qquad \text{in } R, \tag{2.26}$$

where the first inequality holds pointwise (by an earlier remark), if we take v to be its canonical lower semicontinuous representative.

Set

$$I = \{x \in R: v(x) = 0\},$$

$$\Omega = \{x \in R: v(x) > 0\}.$$

Then I is closed (in R), Ω is open and $I \cup \Omega = R$. Clearly $\Omega \subset \omega(\mu)$. Thus, by what we have already proved, $-\Delta V^{\mu} = \rho$ in Ω, and using the same argument as for V^{μ} we find that $-\Delta W^{\mu} = 0$ in Ω. Thus,

$$\Delta v = \rho \qquad \text{in } \Omega. \tag{2.27}$$

Since v is to be proved to be subharmonic, we have to prove that it is upper semicontinuous, hence, that it actually is continuous. This can be deduced from (2.26), (2.27): let $B \subset\subset R$ be an open ball (say) and consider the (positive) measure

$$\nu = \rho - \Delta v \qquad \text{in } B.$$

By (2.27)

$$\operatorname{supp} \nu \subset \bar{B} \cap I. \tag{2.28}$$

Since $-\Delta(U^{\nu} - v) = \rho \in L^{\infty}$ in B, $U^{\nu} - v$ is continuous in B. Hence, it is enough to prove that U^{ν} is continuous.

But by (2.28) $U^{\nu}|_{\operatorname{supp}\nu} = (U^{\nu} - v)|_{\operatorname{supp}\nu}$. Hence, $U^{\nu}|_{\operatorname{supp}\nu}$ is continuous, and by a "continuity principle" for potentials [12, p. 16; 11, Chapter 6, Section 5] this implies that U^{ν} itself is continuous. Since B was arbitrary, this proves that v is continuous in R.

Now it is easy to prove that v is subharmonic: let $K \subset R$ be compact and let h be a continuous function on K which is harmonic in $\operatorname{int} K$ and satisfies $v \le h$ on ∂K. What we have to prove is that $v \le h$ on all K.

By (2.25) $h \ge 0$ on ∂K, hence, $h \ge 0$ on all K. This proves that $v \le h$ on $K \cap I$. In $(\operatorname{int} K) \cap \Omega$, $\Delta v \ge 0$ by (2.27) and on $\partial((\operatorname{int} K) \cap \Omega) \subset \partial K \cup (K \cap I)$, $v \le h$ by the above. Hence, $v \le h$ also in $(\operatorname{int} K) \cap \Omega$. Thus, $v \le h$ on all K as required.

The above finishes the proof that v is subharmonic in R. Hence, (2.23) follows and (b) of the theorem is proved. Note that it actually follows that $v \equiv 0$, $\Omega = \emptyset$, and that W^{μ} as a superharmonic function is the canonical representative of V^{μ}.

We now turn to (c). Since $-\Delta V^{\mu} \le \rho \in L^{\infty}(R)$ the first statement follows immediately from (b). In the first part of the proof we constructed a superharmonic function $w \in \mathfrak{F}_{\rho}^{\mu}$ satisfying $w = U^{\mu}$ on R^c. From this it follows that $W^{\mu} = U^{\mu}$ on R^c, hence, that $V^{\mu} = U^{\mu}$ on R^c, proving (c).

This finishes the proof of the theorem. ■

For fixed ρ (and R) satisfying (2.1)–(2.4), we now define $F = F_{\rho} = F_{\rho,R} : M' \to M'$ by

$$F(\mu) = -\Delta V^{\mu} \qquad (\mu \in M')$$

with $V^{\mu} = V_{\rho}^{\mu}$ as in the theorem. It follows from (a) and (b) that $F(\mu) \in M'$ and that V^{μ} is the potential of $F(\mu)$:

$$V^{\mu} = U^{F(\mu)}.$$

By definition $F(\mu) \le \rho$ in R. If already $\mu \le \rho$ in R then $U^\mu \in \mathfrak{F}_\rho^\mu$, hence, $V^\mu = U^\mu$ and $F(\mu) = \mu$. Thus, F is a projection operator, i.e. $F(F(\mu)) = F(\mu)$ for $\mu \in M'$.

By (d) there is a bounded open set $\omega(\mu) \subset R$ such that

$$F(\mu) = \rho \qquad \text{in } \omega(\mu), \tag{2.29}$$

$$U^{F(\mu)} = U^\mu \qquad \mathbb{R}^N \backslash \omega(\mu). \tag{2.30}$$

It is convenient to also introduce the open set

$$\begin{aligned}\Omega(\mu) &= (\text{the largest open subset of } R \text{ in which } F(\mu) = \rho)\\ &= R\backslash \operatorname{supp}(\rho - F(\mu)).\end{aligned} \tag{2.31}$$

Then $\Omega(\mu)$ is bounded and, by (2.29),

$$\omega(\mu) \subset \Omega(\mu). \tag{2.32}$$

Moreover, it is obvious that

$$F(\mu) = \rho \qquad \text{in } D,$$

$$U^{F(\mu)} = U^\mu \qquad \text{on } \mathbb{R}^N \backslash D$$

hold for an open set $D \subset R$ if and only if $\omega(\mu) \subset D \subset \Omega(\mu)$.

Another way of expressing (2.30) is by saying that

$$\langle F(\mu), \varphi \rangle = \langle \mu, \varphi \rangle \tag{2.33}$$

holds for any function φ of the form $\varphi(x) = E(x - y)$ with $y \notin \omega(\mu)$. This easily gives that (2.33) holds for any smooth function φ in $\mathbb{R}^N$ which is harmonic in $\omega(\mu)$. (Compare the application to quadrature domains in example 2.2.) In particular, we have

$$\langle F(\mu), 1 \rangle = \langle \mu, 1 \rangle, \tag{2.34}$$

i.e. μ and $F(\mu)$ have the same total mass.

THEOREM 2.2. Let R, R_j, $j = 1, 2, \ldots$, be open sets in $\mathbb{R}^N$, let $\rho \in L^\infty(R)$, $\rho_j \in L^\infty(R_j)$ and let μ, μ_j, $\nu \in M'$. Then

(i) $$F_{\rho+\sigma}(\mu + \sigma) = F_\rho(\mu) + \sigma \qquad \text{for } \sigma \in M' \cap L^\infty(R); \tag{2.35}$$

(ii) $$F_{t\rho}(t\mu) = tF_\rho(\mu) \qquad \text{if } t > 0;$$

(iii) $$\begin{gathered} F_{\rho_1}(F_{\rho_2}(\mu_1) + \mu_2) = F_{\rho_1}(\mu_1 + \mu_2)\\ \text{if } \rho_1 \le \rho_2 + \mu_2 \text{ (i.e. } R_2 \subset R_1 \text{ and } \rho_1 \le \rho_2 + \mu_2 \text{ in } R_2); \end{gathered} \tag{2.36}$$

(iv) $$\min(\rho, \mu) \le F_\rho(\mu) \le \rho \text{ in } R, \qquad \mu \le F_\rho(\mu) \text{ on } R^c; \tag{2.37}$$

(v) $$\text{if } \mu \le \nu \text{ then } F_\rho(\mu) \le F_\rho(\nu); \tag{2.38}$$

(vi) if $\mu_n \nearrow \mu$ weakly (i.e. $\langle \mu - \mu_n, \varphi \rangle \searrow 0$ for any continuous $\varphi \ge 0$) as $n \to \infty$ then $F_\rho(\mu_n) \nearrow F_\rho(\mu)$ weakly.

Proof. (i) and (ii) follow immediately from (2.6) and (2.7), respectively, and (iv) follows directly from the definition of $F_\rho(\mu)$ and (b) of theorem 2.1.

(iii) We have to show that

$$V_{\rho_1}^{F_{\rho_2}(\mu_1)+\mu_2} = V_{\rho_1}^{\mu_1+\mu_2}$$

if $\rho_1 \leq \rho_2 + \mu_2$. Using that for general ρ and μ, $V_\rho^\mu = U^{F_\rho(\mu)}$ is the largest function $\leq U^\mu$ satisfying $-\Delta V_\rho^\mu \leq \rho$ in R, we get $V_{\rho_1}^{F_{\rho_2}(\mu_1)+\mu_2} \leq U^{F_{\rho_2}(\mu_1)+\mu_2} = U^{F_{\rho_2}(\mu_1)} + U^{\mu_2} = V_{\rho_2}^{\mu_1} + U^{\mu_2} \leq U^{\mu_1} + U^{\mu_2} = U^{\mu_1+\mu_2}$ and, hence, since $-\Delta V_{\rho_1}^{F_{\rho_2}(\mu_1)+\mu_2} \leq \rho_1$ in R_1, $V_{\rho_1}^{F_{\rho_2}(\mu_1)+\mu_2} \leq V_{\rho_1}^{\mu_1+\mu_2}$.

Conversely, $V_{\rho_1}^{\mu_1+\mu_2} - U^{\mu_2} \leq U^{\mu_1}$ and $-\Delta(V_{\rho_1}^{\mu_1+\mu_2} - U^{\mu_2}) \leq \rho_1 - \mu_2 \leq \rho_2$ in $R_2 \subset R_1$ showing that $V_{\rho_1}^{\mu_1+\mu_2} - U^{\mu_2} \leq V_{\rho_2}^{\mu_1}$ and, hence, that $V_{\rho_1}^{\mu_1+\mu_2} \leq V_{\rho_2}^{\mu_1} + U^{\mu_2} = U^{F_{\rho_2}(\mu_1)} + U^{\mu_2} = U^{F_{\rho_2}(\mu_1)+\mu_2}$. Since $-\Delta V_{\rho_1}^{\mu_1+\mu_2} \leq \rho_1$ in R_1, this yields $V_{\rho_1}^{\mu_1+\mu_2} \leq V_{\rho_1}^{F_{\rho_2}(\mu_1)+\mu_2}$. This completes the proof of (iii).

(v) Setting

$$\lambda = \begin{cases} \min(F_\rho(\mu) + \nu - \mu, \rho) & \text{in } R, \\ F_\rho(\mu) + \nu - \mu & \text{on } R^c \end{cases}$$

we have, using (iii) and (iv),

$$F_\rho(\mu) \leq \lambda \leq F_\rho(F_\rho(\mu) + \nu - \mu) = F_\rho(\mu + \nu - \mu) = F_\rho(\nu).$$

(vi) By (v), $F(\mu_1) \leq F(\mu_2) \leq \cdots \leq F(\mu)$, and since $\langle \mu_n, 1\rangle \to \langle \mu, 1\rangle$, (2.34) shows that $\langle F(\mu_n), 1\rangle \to \langle F(\mu), 1\rangle$. From this (vi) follows. In fact, if $\varphi \geq 0$ is continuous $0 \leq \langle F(\mu) - F(\mu_n), \varphi\rangle \leq \|\varphi\|_\infty \cdot \langle F(\mu) - F(\mu_n), 1\rangle \to 0$. ■

Formula (2.36) is a kind of principle of partial balayage, saying that the balayage $F_{\rho_1}(\mu)$ of $\mu = \mu_1 + \mu_2$ can always be effected via first balayaging part, μ_1, of μ "a little" (to $F_{\rho_2}(\mu_1)$). By taking in (2.36), $\rho_1 = \rho_2 = \rho$ and $\mu_1 = \mu|_R$, $\mu_2 = \mu|_{R^c}$ for any $\mu \in M'$ we obtain $F_\rho(\mu) = F_\rho(\mu_1 + \mu_2) = F_\rho(F_\rho(\mu_1) + \mu_2) = F_\rho(\mu_1) + \mu_2$ (since $F_\rho(\mu_1) + \mu_2 \leq \rho$ in R). Thus

$$F_\rho(\mu) = F_\rho(\mu|_R) + \mu|_{R^c}$$

and in particular

$$F_\rho(\mu)|_R = F_\rho(\mu|_R).$$

It is also clear (from (2.12) for example) that

$$F_\rho(\mu)|_{R^e} = \mu|_{R^e}. \tag{2.39}$$

Thus F_ρ does not change any thing outside $\bar{R}$. About $F_\rho(\mu)|_{\partial R}$ we cannot in general say anything more than that it is $\geq \mu|_{\partial R}$ by (2.37).

As to $F_\rho(\mu)|_R$, we have, roughly speaking, that

$$F_\rho(\mu) = \rho\chi_{\Omega(\mu)} + \mu\chi_{R\setminus\Omega(\mu)} \qquad \text{in } R. \tag{2.40}$$

This formula can be proved, for example, under the assumption that $\mu \in L_c^\infty$: in $\Omega(\mu)$ (2.40) holds by definition (2.31) of $\Omega(\mu)$ and in $R\setminus\Omega(\mu)$ $V^\mu = U^\mu$ by (2.32), hence, $F_\rho(\mu) = -\Delta V^\mu = -\Delta U^\mu = \mu$ a.e. in $R\setminus\Omega(\mu)$ in view of the regularity of U^μ and V^μ when $\mu \in L_c^\infty$.

In many applications it is desirable that the second term in (2.40) drops off, i.e. that (2.40) takes the pure form

$$F_\rho(\mu) = \rho\chi_{\Omega(\mu)} \qquad \text{in } R. \tag{2.41}$$

(2.41) holds, for example, if $\mu \geq 0$ and μ is singular with respect to Lebesgue measure or if there exists an open set $D \subset R$ such that $\mu \geq \rho$ in D, $\mu = 0$ on $R \backslash D$. See, for example, [13, theorem 2.4] for proofs of the corresponding statements when $R = \mathbb{R}^N$ and $\rho = 1$.

If $\mu \in M'$, the energy of μ can be defined as

$$\|\mu\|_e^2 = \iint E(x-y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y),$$

which is possibly $= +\infty$. If μ and ν both have finite energy then

$$(\mu, \nu)_e = \iint E(x-y)\, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y)$$

makes sense and equals $\int U^\mu \, \mathrm{d}\nu = \int U^\nu \, \mathrm{d}\mu$. We now give an alternative description of F when acting on measure of finite energy.

PROPOSITION 2.3. If $\mu \in M'$ has finite energy, then $F_\rho(\mu)$ has finite energy and moreover can be characterized as the unique minimizer of $\|\mu - \nu\|_e$ among all measures $\nu \in M'$ of finite energy satisfying $\nu \leq \rho$ in R.

Proof. In proving that $F_\rho(\mu)$ has finite energy we may assume that $\mu, \rho \geq 0$ (because of (2.35) and the fact that any $\sigma \in L_c^\infty$ has finite energy). Setting $\nu = F_\rho(\mu)$, we have $\nu \geq 0$ and $U^\nu = V^\mu \leq U^\mu$, hence,

$$\|\nu\|_e^2 = \int U^\nu \, \mathrm{d}\nu \leq \int U^\mu \, \mathrm{d}\nu = \int U^\nu \, \mathrm{d}\mu \leq \int U^\mu \, \mathrm{d}\mu = \|\mu\|_e^2 < \infty.$$

To prove that $\nu = F_\rho(\mu)$ minimizes $\|\mu - \nu\|_e$, it is enough to prove that

$$(\mu - F_\rho(\mu), F_\rho(\mu) - \nu)_e \geq 0$$

for all $\nu \in M'$ of finite energy and satisfying $\nu \leq \rho$ in R. But using (2.10)–(2.13), we have

$$\begin{aligned}(\mu - F_\rho(\mu), F_\rho(\mu) - \nu)_e &= \int (U^\mu - V^\mu)\, \mathrm{d}(F_\rho(\mu) - \nu) \\ &= \int_{\omega(\mu)} (U^\mu - V^\mu)\, \mathrm{d}(F_\rho(\mu) - \nu) \\ &= \int_{\omega(\mu)} (U^\mu - V^\mu)\, \mathrm{d}(F_\rho(\mu) - \rho) + \int_{\omega(\mu)} (U^\mu - V^\mu)\, \mathrm{d}(\rho - \nu),\end{aligned}$$

where the first term vanishes and the second is ≥ 0. This finishes the proof. ■

In the second part of the proof we only used that V^μ satisfies (2.10)–(2.13). Since the minimizer $\nu = F(\mu)$ of $\|\mu - \nu\|_e$ is unique, it follows that (2.10)–(2.13) uniquely characterizes V^μ among potentials of finite energy when $\mu \in M'$ has finite energy.

In terms of $u = U^\mu - V^\mu$ we have

$$F_\rho(\mu) = \mu + \Delta u$$

and u satisfies (2.15)–(2.18). If μ has finite energy then Δu has finite energy (by proposition 2.3), hence, $u \in H_0^1(B)$ if B is a ball (say) chosen so large so that $\operatorname{supp}\mu$ and $\operatorname{supp} F_\rho(\mu)$ are contained in B. Moreover, (2.15)–(2.18) uniquely characterizes u among all functions in $H_0^1(B)$ in this case.

If we endow the dual space $H^{-1}(B)$ of $H_0^1(B)$ with its natural energy inner product (corresponding to the inner product $\int_B \nabla u \cdot \nabla v \, dm$ on $H_0^1(B)$) and define

$$\tilde{F}_\rho : H^{-1}(B) \to H^{-1}(B)$$

as the orthogonal projection onto the closed convex set $K = \{v \in H^{-1}(B) \colon v \le \rho \text{ in } R \cap B\}$, then it follows from proposition 2.3 and the above discussion that

$$F_\rho(\mu) = \tilde{F}_\rho(\mu)$$

whenever $\mu \in M' \cap H^{-1}(B)$, $\operatorname{supp}\mu \subset B$, $\operatorname{supp} F_\rho(\mu) \subset B$.

The above finishes the basic description of the operators F_ρ. We now give two examples.

Example 2.1. Take R bounded and $\rho = 0$. Then $F = F_\rho$ reduces to classical balayage, i.e. sweeping positive measures on $\bar{R}$ out to ∂R (see [11, 14]). In fact, if $\mu \ge 0$ then $\nu = F(\mu)$ satisfies $\nu = 0$ in R (by (2.37)) and $U^\nu = U^\mu$ on R^c (in particular, $\operatorname{supp}\nu \subset \partial R$ if $\operatorname{supp}\mu \subset \bar{R}$).

If μ is not positive then $F(\mu)$ differs in general from the classical balayage measure ν. If, for example, $\mu \le 0$, then $F(\mu) = \mu$ so that the mass in R is not swept out.

Example 2.2. Take $R = \mathbb{R}^N$ and $\rho = 1$. Then we get a balayage operator $F = F_\rho$ of interest in the theory of quadrature domains and certain free and moving boundary problems arising in physics (e.g. Hele–Shaw flow). Below (up to the end of this section), we expand a little on these less well-known items, referring to [3, 13, 15–22] and references therein for further details.

When $R = \mathbb{R}^N$, $\rho = 1$, (2.40) takes the form

$$F(\mu) = \chi_\Omega + \mu\chi_{\Omega^c} \qquad (\text{in } \mathbb{R}^N) \tag{2.42}$$

($\Omega = \Omega(\mu)$) and (2.41)

$$F(\mu) = \chi_\Omega \qquad (\text{in } \mathbb{R}^N). \tag{2.43}$$

Equation (2.42) is true if, for example, $\mu \in L_c^\infty$, while (2.43) is not true in general (but is often the desired result of applying F).

Note that if (2.43) holds for some open set Ω then Ω equals $\Omega(\mu)$ up to a null set. More precisely, since $\Omega(\mu)$ by definition (2.31) is chosen to be maximal, $\Omega(\mu) = [\Omega]$, the completion of Ω with respect to Lebesgue measure (see the list of notations in Section 1).

Suppose that (2.43) holds and let U^Ω denote the potential of $\chi_\Omega m$ (so that $-\Delta U^\Omega = \chi_\Omega$). Then $U^\Omega = V^\mu$ and, therefore, by (2.10)–(2.13), (2.32),

$$U^\Omega \le U^\mu \qquad \text{in } \mathbb{R}^N, \tag{2.44}$$

$$U^\Omega = U^\mu \qquad \text{outside } \Omega \tag{2.45}$$

(with $\Omega = \Omega(\mu)$).

Conversely suppose that (2.44)–(2.45) hold for some bounded open set Ω. Then $U^\Omega \le V^\mu$ because $U^\Omega \in \mathfrak{F}_1^\mu$ by (2.44). On the other hand, the continuous (if $\mu \in M'$) function $u = V^\mu - U^\Omega$ is subharmonic in Ω and, by (2.45), is ≤ 0 outside Ω. Thus $u \le 0$ everywhere by the maximum principle, hence, $V^\mu \le U^\Omega$. (Note, for later use, that this was a consequence of (2.45) alone.) Thus $V^\mu = U^\Omega$ and it follows that (2.43) holds.

Thus, (2.43) is equivalent to (2.44)–(2.45), with just the qualification that if Ω is allowed to be an arbitrary open set, then we can from (2.43) only infer (2.45) outside the completion $[\Omega]$ of Ω. Now, with Ω any bounded open set and assuming for simplicity that $\mu \in L_c^\infty$, (2.44)–(2.45) can on the other hand be seen to be equivalent to that

$$\mu = 0 \qquad \text{outside } \Omega, \tag{2.46}$$

$$\int_\Omega \varphi \, d\mu \le \int_\Omega \varphi \, dm \qquad \text{for all } \varphi \in SL^1(\Omega), \tag{2.47}$$

where $SL^1(\Omega)$ denotes the set of integrable subharmonic functions in Ω (see [13, 17, 18]).

Moreover, assuming that Ω is taken to be complete with respect to Lebesgue measure (i.e. that $\Omega = [\Omega]$), (2.45) alone is equivalent to that

$$\mu = 0 \qquad \text{outside } \Omega, \tag{2.46}$$

$$\int_\Omega \varphi \, d\mu = \int_\Omega \varphi \, dm \qquad \text{for all } \varphi \in HL^1(\Omega), \tag{2.48}$$

where $HL^1(\Omega)$ denotes the set of integrable harmonic functions in Ω. (Without the assumption that Ω is complete (2.46), (2.48) are equivalent to (2.45) together with $\nabla U^\Omega = \nabla U^\mu$ outside Ω.)

Equations (2.46), (2.47) (or (2.46), (2.48)) mean, with the definitions used in [13, 17–19] and when $\mu \in L_c^\infty$, that the bounded open set Ω is a quadrature domain (or "quadrature open set") for μ with respect to subharmonic functions (or harmonic functions, respectively). The set of such quadrature domains is denoted $Q(\mu, SL^1)$ (or $Q(\mu, HL^1)$, respectively).

We have $Q(\mu, SL^1) \subset Q(\mu, HL^1)$ and $Q(\mu, SL^1)$ is either empty or consists (up to null sets) of just $\Omega(\mu)$, depending on whether $F(\mu)$ is or is not of the form (2.43). $Q(\mu\, HL^1)$ often contains several different elements, but an interesting open question is whether $Q(\mu, HL^1)$ can contain more than one solid open set (for some μ). (A bounded open set Ω is called solid if $\partial\Omega = \partial(\Omega^e)$ and Ω^e is connected.) As an example of the usefulness of the balayage operator F in the study of $Q(\mu, HL^1)$, we have the following result, related to [17, corollary 4.10; 13, corollary 3.3], for example.

PROPOSITION 2.4. Let $\mu \in M'$ and suppose that $\Omega \in Q(\mu, HL^1)$ (or simply that (2.45) holds). Then $U^\Omega \ge V^\mu$ and $\partial\Omega \subset \overline{\Omega(\mu)}$.

Proof. Set $u = V^\mu - U^\Omega$. The inequality $u \le 0$ has already been proved (after (2.45)).

Next we prove $\partial\Omega \subset \overline{\Omega(\mu)}$ by contradiction. Suppose $x \in \partial\Omega \backslash \overline{\Omega(\mu)}$ and let $B \subset \Omega(\mu)^e$ be an open ball with centre x. Since $\Omega(\mu)^e \subset \omega(\mu)^e$, $V^\mu = U^\mu$ in B, hence $u = 0$ in $B\backslash\Omega$ (using (2.45)). By the regularity of u ($|\Delta u|$ bounded), this implies that $\Delta u = 0$ a.e. in $B\backslash\Omega$ (see [1, lemma A.4, p. 53]). In $B \cap \Omega$, $\Delta u = 1 - F(\mu) \ge 0$. Hence, u is subharmonic in B. Recalling that $u \le 0$ this gives $0 = u(x) \le (1/m(B)) \int_B u \, dm \le 0$, hence, $u \equiv 0$ in B. But in $B \cap \Omega$, $\Delta u = 1 - F(\mu) \not\equiv 0$ by the definition (2.31) of $\Omega(\mu)$. This is the desired contradiction and finishes the proof. ∎

As to the free and moving boundary problems consider, for example, the moving boundary problem in which one starts with an initial (bounded) domain $\Omega_0 \subset \mathbb{R}^N$ and asks for the (increasing) family of domains $\{\Omega(t): t \ge 0\}$ satisfying

$$\begin{cases} \Omega(0) = \Omega_0, \\ \partial\Omega(t) \qquad \text{moves with velocity } -(\nabla p)|_{\partial\Omega(t)}, \end{cases} \tag{2.49}$$

where, for each $t, p = p(x, t)$ denotes the solution of

$$\begin{cases} -\Delta p = f & \text{in } \Omega(t), \\ p = 0 & \text{on } \partial\Omega(t) \end{cases} \tag{2.50}$$

and $f = f(x, t) \geq 0$ is a given function (which we set equal to zero outside $\Omega(t)$). In two dimensions one interpretation of this problem is that $\Omega(t)$ represents a blob of a viscous incompressible fluid in the narrow gap between two parallel plates, a so-called Hele–Shaw cell (without walls in the present case). Assuming that the surface tension at the free boundary $\partial\Omega(t)$ can be neglected (which is reasonable as long as the curvature of $\partial\Omega(t)$ is not too large) the equations (2.49), (2.50) follow from the "Hele–Shaw equation", with f representing some kind of source, e.g. the injection of more fluid through a hole in one of the surfaces. Instead of having a source, one can think of squeezing the plates together, which is modelled by taking f = constant > 0 (in $\Omega(t)$).

For further information on the Hele–Shaw model, see, for example [3, 23, 24].

In higher dimensions there are other physical interpretations of (2.49)–(2.50), e.g. within porous medium flow, heat conduction with phase change (degenerate Stefan problems) and electrochemical machining.

It is well known that problem (2.49)–(2.50) is well posed and always admits a unique global weak (variational inequality) solution $\{\Omega(t): 0 \leq t < \infty\}$. In terms of our balayage operator this is given by

$$F(\mu(t)) = \chi_{\Omega(t)}, \tag{2.51}$$

where

$$\mu(t) = \chi_{\Omega_0} + \int_0^t f(\cdot, \tau)\, d\tau. \tag{2.52}$$

Thus the fluid domain at time $t > 0$ is obtained simply by balayaging out χ_{Ω_0} plus the accumulated source up to time t. The function $u = U^{\mu(t)} - V^{\mu(t)}$ ($u = u(x, t)$) involved in the definition of F in (2.51) is related to the pressure p by

$$u(x, t) = \int_0^t p(x, \tau)\, d\tau$$

("Baiocchi" transformation).

More generally one has

$$F\left(\chi_{\Omega(\tau)} + \int_\tau^t f\right) = \chi_{\Omega(t)} \tag{2.53}$$

whenever $\tau \leq t$ and $\{\Omega(t)\}$ solves (2.49)–(2.50) (with $f \geq 0$). This makes the operator F interesting also for the ill-posed Hele–Shaw problem in which fluid is extracted instead of injected and the fluid region, hence, shrinks (i.e. (2.49)–(2.50) with $f \leq 0$). In fact, this problem is obtained from the previous well-posed one simply by a time reversal, so in (2.53) (with $f \geq 0$) it corresponds to having, say, $\Omega(0)$ known and looking for domains $\Omega(t)$ for t in some interval $-T \leq t \leq 0$ ($T > 0$) satisfying (2.53) whenever $\tau \leq t$. (Actually it is enough to find $\Omega(-T)$ such that (2.53) holds with $\tau = -T$, $t = 0$; for then the $\Omega(t)$ with $-T < t < 0$ are obtained from (2.53) by choosing $\tau = -T$, and it follows from (2.36) that (2.53) then holds for all $-T \leq \tau \leq t \leq 0$.)

The case of squeezing the plates (f = constant) turns out to be particularly nice. In fact, denoting the distance between the plates by $\rho(t)$, supposed to be a decreasing function of t, one easily finds that (2.53) in this case can be replaced by

$$F_{\rho(t)}(\rho(\tau)\chi_{\Omega(\tau)}) = \rho(t)\chi_{\Omega(t)} \qquad (\tau \leq t),$$

where $F_{\rho(t)}$ is the operator $F_{\rho,R}$ with $\rho = \rho(t)$, $R = \mathbb{R}^N$ ($N = 2$). Thus, we get a quite concrete squeezing interpretation of (some of) our balayage operators.

3. A GEOMETRIC PROPERTY OF $F(\mu)$

This section is entirely devoted to the balayage operator F in example 2.2, i.e. to F_ρ with $R = \mathbb{R}^N$ and $\rho = 1$. As indicated in that example, it is of great interest in several applications to know first of all to what extent $F(\mu)$ is of the form χ_Ω ($\Omega = \Omega(\mu)$) and secondly to have as much information as possible on the shape $\Omega(\mu)$.

Here we prove a theorem on the geometry of $F(\mu)$ outside the convex hull of supp μ when $\mu \geq 0$. The proof uses a reflexion argument which is somewhat similar to symmetry arguments invented by Alexandroff and which have also been used in [7–9], for example. Another related result (and proof) is [10, theorem 4.1]. See also [25, Section III.10].

THEOREM 3.1. Suppose $\mu \in M_c$, $\mu \geq 0$, and that supp $\mu \subset \bar{D}$, where D is an open half-space, say $D = \{x \in \mathbb{R}^N\colon x_N < 0\}$. (We write $x = (x_1, \ldots, x_N) = (x', x_N)$.) Then

$$F(\mu)|_{D^e} = \chi_\Omega, \tag{3.1}$$

$$F(\mu)|_D \geq \chi_{\tilde{\Omega}}, \tag{3.2}$$

where Ω is an open set of the form

$$\Omega = \{x \in \mathbb{R}^N\colon x' \in G,\ 0 < x_N < g(x')\}$$

for some open set $G \subset \mathbb{R}^{N-1}$ and some real analytic function $g\colon G \to \mathbb{R}$ and where $\tilde{\Omega} = \{(x', x_N) \in \mathbb{R}^N\colon (x', -x_N) \in \Omega\}$, the reflexion of Ω in ∂D.

Moreover, the function $u = U^\mu - V^\mu$, which is continuously differentiable in D^e, satisfies

$$u = 0 \qquad \text{in } D^e \backslash \Omega,$$

$$u > 0 \qquad \text{in } \Omega, \tag{3.3}$$

$$\frac{\partial \mu}{\partial x_N} < 0 \qquad \text{in } \Omega. \tag{3.4}$$

Note. In this section the letter Ω (without μ) often denotes just some part of the set $\Omega(\mu)$ (2.31). For example, it follows directly from the statements of the theorem that

$$\Omega = \omega(\mu) \cap D^e = \Omega(\mu) \cap D^e$$

in theorem 3.1. Observe also that (3.2) simply states that $\tilde{\Omega} \subset \Omega(\mu)$.

Proof. It is enough to prove that for every $\varepsilon > 0$ the statements of the theorem hold with $D = \{x \in \mathbb{R}^N\colon x_N < \varepsilon\}$ (with $\tilde{\Omega}$ then defined as the reflexion in $x_N = \varepsilon$). Therefore, we may assume that supp $\mu \subset D$.

In terms of $u = U^{\mu} - V^{\mu}$, we have

$$F(\mu) = \mu + \Delta u$$

and u satisfies, by (2.15)-(2.18),

$$u \geq 0, \tag{3.5}$$

$$\Delta u \leq 1 - \mu, \tag{3.6}$$

$$\Delta u = 1 - \mu \qquad \text{in } \omega(\mu) = \{x \in \mathbb{R}^N: u(x) > 0\}. \tag{3.7}$$

Moreover, u is the smallest function satisfying (3.5), (3.6) alone. Since $\operatorname{supp} \mu \subset D$, u is continuously differentiable in a neighbourhood of D^c.

Let $\tilde{u}$ denote the reflexion of u in the hypersurface $x_N = 0$, i.e.

$$\tilde{u}(x', x_N) = u(x', -x_N)$$

and define

$$v = \begin{cases} \min(u, \tilde{u}) & \text{in } D^e \\ u & \text{on } \bar{D}. \end{cases}$$

Clearly $v \geq 0$ everywhere and $\Delta v \leq 1 - \mu$ in D. Moreover, $\Delta v \leq 1$ in $\mathbb{R}^N$ because $\Delta u \leq 1$, and so $\Delta \tilde{u} \leq 1$. Hence, $\Delta \min(u, \tilde{u}) \leq 1$ and one easily checks that $\Delta v \leq 1$ also near ∂D using standard superharmonicity criteria. Thus, $\Delta v \leq 1 - \mu$. Since u is the smallest function satisfying (3.5), (3.6), it follows that $u \leq v$. Thus,

$$u \leq \tilde{u} \qquad \text{in } D^c, \tag{3.8}$$

which implies that

$$\frac{\partial u}{\partial x_N} \leq 0 \qquad \text{on } \partial D. \tag{3.9}$$

Set

$$\Omega = \omega(\mu) \cap D^e = \{x \in D^e: u(x) > 0\}.$$

Then by (3.5)-(3.7) (or by (2.32))

$$\Delta u = 1 \qquad \text{in } \Omega. \tag{3.10}$$

Moreover, $\Delta u = 0$ a.e. on $D^e \backslash \Omega = \{x \in D^e: u(x) = 0\}$ as a consequence of the regularity of u in D^e ($|\Delta u|$ bounded there). It follows that

$$F(\mu)|_{D^e} = (\Delta u)|_{D^e} = \chi_{\Omega}.$$

Next, by (3.10) $\partial u/\partial x_N$ is harmonic in Ω and by (3.9), the definition of Ω and the regularity of u, $\partial u/\partial x_N$ is continuous up to $\partial\Omega$ with boundary values ≤ 0 there ($=0$ on $(\partial\Omega) \cap D^e$). Thus, by the maximum principle

$$\frac{\partial u}{\partial x_N} \leq 0 \qquad \text{in } \Omega \tag{3.11}$$

(and, hence, in all D^e). Moreover, if $\partial u/\partial x_N$ vanishes at some point in Ω, it vanishes identically in the whole component of Ω containing that point. Since $u = 0$ on $(\partial\Omega) \cap D^e$ it then follows that also u itself vanishes in that component of Ω. But this contradicts the definition of Ω.

Hence, we actually have

$$\frac{\partial u}{\partial x_N} < 0 \qquad \text{in } \Omega.$$

Thus, for each $x' \in \mathbb{R}^{N-1}$ the behaviour of $u(x', x_N)$ as a function of $x_N > 0$ is that either it is identically zero or else there is a number $g(x') > 0$ such that $u(x', x_N) = 0$ for $x_N \geq g(x')$ and $u(x', x_N) > 0$ with $\partial u(x', x_N)/\partial x_N < 0$ for $0 < x_N < g(x')$. This gives that

$$\Omega = \{(x', x_N) \colon x' \in G,\ 0 < x_N < g(x')\} \tag{3.12}$$

with g as above and G the set of $x' \in \mathbb{R}^{N-1}$ for which $u(x', x_N)$ does not vanish identically as a function of $x_N > 0$. Since Ω is open, G must be open in $\mathbb{R}^{N-1}$.

Next we have to prove that g is real analytic. Let $x = (x', x_N) \in (\partial\Omega) \cap D^e$. Then (3.12) shows that the ray $L = \{(x', t) \colon t > x_N\}$ lies entirely in Ω^c. Now, all that we have done up to now can be repeated for other half spaces D', with $\operatorname{supp} \mu \subset D'$, $x \in (D')^e$. Varying D' around our original D then gives that there is a whole open cone of rays starting at x and lying entirely in Ω^c. (Compare corollary 3.3 below, where this is expressed in a more precise form.) Thus, Ω^c is "thick enough" at $x \in \partial\Omega$ for the theory of Caffarelli and others [2, 4, 26–28] to be applied, and the conclusion then is that g is real analytic at x'.

To prove (3.2), finally it is enough, by (2.32) for example, to prove that $u > 0$ in $\tilde{\Omega}$. But this is equivalent to that $\tilde{u} > 0$ in Ω, and this latter inequality follows by combining (3.3) and (3.8) ■

COROLLARY 3.2. Assume $\mu \in M_c$, $\mu \geq 0$ and let K denote the closed convex hull of $\operatorname{supp} \mu$. Then the restriction of $F(\mu)$ to K^c is of the form χ_Ω where $\Omega = \omega(\mu)\backslash K = \Omega(\mu)\backslash K$ is an open set with $(\partial\Omega)\backslash K$ consisting of real analytic hypersurfaces (without singularities). Moreover, $\nabla V^\mu \neq \nabla U^\mu$ in Ω. Finally $(\Omega \cup K)^e$ is connected.

Proof. This follows by applying the theorem to all half-spaces D with $\operatorname{supp} \mu \subset \bar{D}$. The last statement actually follows most easily from the next corollary. ■

Remark. In $K\backslash\operatorname{supp} \mu$, $F(\mu)$ is still of the form χ_Ω but $\partial\Omega$ may have singularities (cf. the remarks after example 3.4 below). To be precise: in two dimensions the regularity question for $\partial\Omega$ outside $\operatorname{supp} \mu$ is completely solved [29]: $(\partial\Omega)\backslash\operatorname{supp} \mu$ is analytic with a few specific types of singularities allowed. This is true for arbitrary $\mu \in M_c$ and also under more general circumstances than above. In higher dimensions the regularity question for $\partial\Omega$ seems not be completely solved. On $\operatorname{supp} \mu$, $F(\mu)$ may be as nasty as μ and not of the form χ_Ω; cf. (2.42).

COROLLARY 3.3. With notations and assumptions as in corollary 3.2, let $x \in (\partial\Omega)\backslash K$ (or just $x \in (\Omega \cup K)^c$). Then the cone

$$K_x^0 = \{z \in \mathbb{R}^N \colon (z - x, y - x) \leq 0 \text{ for all } y \in \operatorname{supp} \mu\}$$

does not intersect $\Omega \cup K$. Observe that K_x^0 is a convex cone with vertex at x and with nonempty interior.

Proof. Let $x \in (\Omega \cup K)^c$, $z \in K_x^0$, and we shall prove that $z \notin \Omega \cup K$. By translation and rotation of the coordinates, we may assume that $x = 0$ and that $z = (0, \ldots, 0, z_N)$ where $z_N > 0$

(the case $z = x$ is trivial). Then the statement $z \in K_x^0$ means that $\operatorname{supp}\mu$, and, hence, K, lies in the half-space $\{y \in \mathbb{R}^N: y_N \leq 0\}$. Thus, $z \notin K$, and if $z \in \Omega$ it easily follows that also $x = 0 \in \Omega$, which is a contradiction. ■

COROLLARY 3.4. With notations and assumptions as in corollary 3.2, let $B = B(x; r)$ be a ball such that $\operatorname{supp}\mu \subset \bar{B}$ (equivalently $K \subset \bar{B}$). Then $\Omega \cup B$ is star shaped with respect to x.

Proof. This follows right away from the theorem or from corollary 3.3. ■

COROLLARY 3.5. With the same assumptions and notations as in corollary 3.2, for any $x \in (\partial\Omega)\backslash K$ the normal of $\partial\Omega$ at x intersects K. Moreover, if $N = 2$ and $\operatorname{supp}\mu$ is connected, the normal in fact intersects $\operatorname{supp}\mu$ itself (cf. Figs 1 and 2).

Proof. Suppose that the normal N_x of $\partial\Omega$ at $x \in (\partial\Omega)\backslash K$ did not intersect K. Then there would be a hyperplane $H_x \supset N_x$ which also did not meet K, and K would be contained in one of the two components of $\mathbb{R}^N\backslash H_x$, call it D_x. Observe that D_x is an open half-space. Therefore, the cone K_x^0 in corollary 3.3 would contain in its interior the outward normal of $\partial D_x = H_x$ at x. But this normal is perpendicular to N_x, hence, is tangent to $\partial\Omega$ at x. Thus, using corollary 3.3, we reach the contradiction that a conic neighbourhood of (half of) a tangent line of $\partial\Omega$ at x does not intersect Ω. ■

COROLLARY 3.6. Let μ and K be as in corollary 3.2 and let $B = B(x; r)$ be a ball such that $\operatorname{supp}\mu \subset \bar{B}$. Then:

(a) if $F(\mu) \geq \chi_K$ then (2.43) holds with $\Omega = \Omega(\mu)$ solid;
(b) if $F(\mu) \geq \chi_B$ then (2.43) holds with $\Omega = \Omega(\mu)$ star shaped with respect to x.

Proof. (a) Since $F(\mu) \leq 1$ the assumptions imply that $F(\mu) = 1$ on K. Thus, with Ω_1 the open set obtained in corollary 3.2, $F(\mu) = \chi_{K\cup\Omega_1}$. Since K is convex ∂K has measure zero. It

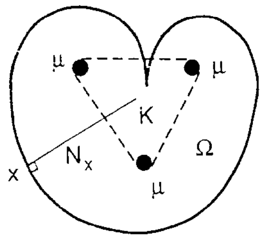

Fig. 1. $\mu \geq 0$, $F(\mu) = \chi_\Omega$, $K = \operatorname{conv}\operatorname{supp}\mu$.

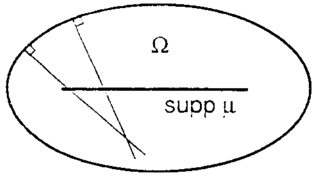

Fig. 2. $\mu \geq 0$, $F(\mu) = \chi_\Omega$, $K = \operatorname{conv}\operatorname{supp}\mu$.

follows that the open set $(\operatorname{int} K) \cup \Omega_1$ has the same measure as $K \cup \Omega_1$. Set $\Omega = \operatorname{int}(K \cup \Omega_1)$. Then $(\operatorname{int} K) \cup \Omega_1 \subset \Omega \subset K \cup \Omega_1$, $(K \cup \Omega_1)\backslash\Omega$ has measure zero (actually Ω is the largest open set with this property) and (hence)

$$F(\mu) = \chi_\Omega .$$

To prove that Ω is solid, it is enough to prove the following: for every $x \in \Omega^c$ there is a sequence $\{x_n\} \subset \Omega^c$ such that $x_n \to x$ as $n \to \infty$ and such that x_n is the vertex of a nonempty open cone contained in Ω^c. In fact, if this is the case then every $x \in \Omega^c$ is in the closure of $\Omega^e = \operatorname{int}(\Omega^c)$, hence, $\partial\Omega = \partial(\Omega^e)$, and every $x \in \Omega^e$ can be connected with infinity via a cone (with vertex close to x).

So let $x \in \Omega^c$, i.e. $x \notin \operatorname{int}(K \cup \Omega_1)$. By definition this means that there is a sequence $x_n \to x$ with $x_n \notin K \cup \Omega_1$ and corollary 3.3 then shows that $\{x_n\}$ has the required properties.

(b) This now follows easily, using corollary 3.4.

The condition $F(\mu) \geq \chi_K$ in corollary 3.6 is sometimes easy to verify. Some examples of this follow.

Example 3.1. Suppose $\mu \geq 0$ has support in a hyperplane. Then $\chi_K = 0$ (a.e.) in $\mathbb{R}^N$ so that $F(\mu) \geq \chi_K$ is trivially true.

Example 3.2. Let $\mu = \sum_{j=1}^m a_j\delta_{x_j}$, where $a_j > 0$ and suppose that

$$\operatorname{conv}\{x_1, \ldots, x_m\} \subset \bigcup_{j=1}^m \bar{B}_j,$$

where B_j is the open ball with center x_j and volume a_j. Then $F(a_j\delta_{x_j}) = \chi_{B_j}$, and using (2.36) we get

$$F(\mu) = F\left(\sum_{j=1}^m a_j\delta_{x_j}\right) = F\left(\sum_{j=1}^m F(a_j\delta_{x_j})\right) = F\left(\sum_{j=1}^m \chi_{B_j}\right).$$

By assumption $\sum_{j=1}^m \chi_{B_j} \geq \chi_K$ ($K = \operatorname{conv}\{x_1, \ldots, x_m\} = \operatorname{conv}\operatorname{supp}\mu$) and it follows from (2.37) that $F(\mu) = F(\sum_{j=1}^m \chi_{B_j}) \geq \chi_K$.

More generally, the same conclusion holds with $\mu = \sum_{j=1}^m a_j\delta_{x_j} + \nu$ with x_j as before and $\nu \geq 0$, $\operatorname{supp}\nu \subset \operatorname{conv}\{x_1, \ldots, x_m\}$.

Similarly, the stronger condition $F(\mu) \geq \chi_B$ in (b) can also be verified in many cases.

Example 3.3. Suppose $\mu \geq 0$, $\operatorname{supp}\mu \subset \bar{B}$ ($B = B(x; r)$) and that $\mu(\bar{B}) \geq 6^N m(B)$. Then one can prove that $F(\mu) \geq \chi_{B_1}$ where $B_1 = B(x; 3r)$. (See the proof of theorem 2.4 (v) in [13].) Thus, it follows that $F(\mu) = \chi_\Omega$, with Ω star shaped with respect to x and, moreover, $B_1 \subset \Omega$.

Example 3.4. Suppose $\mu = a\delta_x + \nu$, where $\nu \geq 0$, $\operatorname{supp}\nu \subset \bar{B}$ and B is the ball with center x and volume a. Then (as in example 3.2) $F(\mu) \geq \chi_B$ holds.

In (a) of corollary 3.6 it does not follow that Ω is connected or (if $N \geq 3$) simply connected (for example). This is clear from example 3.1: by taking suitable $\mu \geq 0$ with support in a hyperplane one can produce $\Omega(\mu)$ which approximates any type of configuration in that hyperplane.

Nor does it follow that $\partial\Omega$ is real analytic everywhere. As an example, let K be a closed equilateral triangle ($N = 2$) and let $\mu = a\chi_K$ for some $a > 1$. Then $F(\mu) \geq \chi_K$ by (2.37) but for a close to 1 $\partial\Omega$ will meet K at the corners and have an angle there (thus, $\partial\Omega$ will not even be of class C^1 and the singularities will be nonanalytic) (cf. [30; 17, corollary 13.3; 31, Section 1]). Another example is $\mu = a(\delta_{x_1} + \delta_{x_2})$, where $x_j \in \mathbb{R}^N$ and $a > 0$ is the volume of the ball with radius $r = \frac{1}{2}|x_1 - x_2|$. Then K is the line segment joining x_1 and x_2, $\Omega = B(x_1; r) \cup B(x_2; r)$ and $\partial\Omega$ has an analytic singularity (a double point) at the midpoint of K, thus in $K\setminus\operatorname{supp}\mu$.

As another application of corollary 3.6, we have the uniqueness question for quadrature domains: if $\mu \geq 0$, $F(\mu) \geq \chi_K$ so that $\Omega(\mu)$ is solid, then it follows easily from proposition 2.4 that $\Omega \in Q(\mu, HL^1)$ implies $[\Omega] = \Omega(\mu)$ (cf. [17, corollary 4.10; 13, corollary 3.6]).

Finally, let us interpret some of the results in this section in terms of the moving boundary problem (2.49)–(2.50) for Hele-Shaw flows. We shall assume for simplicity that $\operatorname{supp} f(\cdot, t) \subset \bar{\Omega}_0$ for all $t \geq 0$. Then $\operatorname{supp}\mu(t) = \bar{\Omega}_0$ in (2.51)–(2.52). Thus, corollary 3.2 says that $\partial\Omega(t)$ is analytic outside the convex hull K of $\bar{\Omega}_0$ and that there are no holes in $\Omega(t)$ outside K. (Inside K there may very well be holes.) If Ω_0 itself is convex the above information is of course particularly precise. (But it does not follow that $\Omega(t)$ is convex; suppose, for example, that $\partial\Omega_0$ contains a straight line segment and choose f to be a point source (in Ω_0) very close to that line segment. This f will cause a "bubble" on $\partial\Omega(t)$ near $\operatorname{supp} f$ when $t > 0$ is small, making $\Omega(t)$ nonconvex.)

Corollary 3.5 shows that for every $x \in \partial\Omega(t)$ the normal of $\partial\Omega(t)$ at x intersects K. Clearly this gives interesting geometric information about $\partial\Omega(t)$ which is particularly precise when $\partial\Omega(t)$ is far away from Ω_0.

If Ω_0 is a ball $B(x; r)$, then corollary 3.6 shows that $\Omega(t)$ is star shaped with respect to x (independently of the form of f) (cf. [10, theorem 4.1]).

For the backward (or suction) problem, i.e. for solutions $\{\Omega(t): -T \leq t \leq 0\}$ of (2.53) with $\Omega(0)$ given, we have the following, assuming that $N = 2$ and $\operatorname{supp} f(\cdot, t) \subset \bar{\Omega}(-T)$ for all t: for any $x \in \partial\Omega(0)$ the normal of $\partial\Omega(0)$ at x will intersect all the $\Omega(t)$ $(-T \leq t \leq 0)$ as long as these are connected. Thus, for example, if $\Omega(0)$ contains a "finger" with parallel sides, this cannot be completely emptied by means of sinks outside it (cf. Fig. 3).

Similarly, if D is a half-plane containing the sinks, then it follows from (3.2) that the part $\Omega(0)\setminus\bar{D}$ of the initial fluid region $\Omega(0)$ cannot be completely emptied unless its reflexion in ∂D is contained in $\Omega(0)$. Neither can $\Omega(0)\setminus\bar{D}$ be completely emptied if it contains holes (or more generally if $\partial\Omega(0)\setminus\bar{D}$ is not the graph of a function) (Fig. 3).

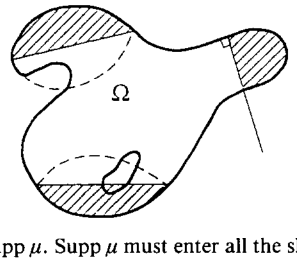

Fig. 3. $\mu \geq 0$, $F(\mu) = \chi_\Omega$, $K = \operatorname{conv}\operatorname{supp}\mu$. Supp μ must enter all the shaded areas. The same is true, in the Hele-Shaw model, for $\Omega(s)$ for $s < t$ if $\Omega = \Omega(t)$.

4. A GENERALIZATION

In this section we generalize part of theorem 3.1 to the case of an arbitrary $F = F_{\rho,R}$ (with the limitation $\rho \geq 0$, though) and with D an arbitrary subdomain of R.

THEOREM 4.1. Let R and ρ satisfy (2.1)–(2.4) and, moreover, $\rho \geq 0$, and let $F = F_{\rho,R}$. Furthermore, let D be an open set with $D \subset R$ such that ∂D has a barrier at each point. Then for any $\mu \in M_c$ with $\mu \geq 0$ and $\operatorname{supp}\mu \subset \bar{D}$, there exists a $\nu \in M_c$ with $\operatorname{supp}\nu \subset \partial D$ and $\nu \geq \mu|_{\partial D}$ (in particular $\nu \geq 0$) such that

$$F(\nu)|_{D^c} = F(\mu)|_{D^c}, \tag{4.1}$$

$$F(\nu) \leq F(\mu). \tag{4.2}$$

Remark. It is important that $\nu \geq 0$. Taking $R = \mathbb{R}^N$, $\rho = 1$, and a half-space D, one easily derives theorem 3.1 as in the second half of the proof of that theorem.

Proof. Let $F = F_{\rho,R}$ as in the statement of the theorem and let $G = F_{\rho,D}$. The proof goes by successive balayage in an infinite number of steps as follows. Define measures σ_n, τ_n, λ_n, $\nu_n \in M_c$ for $n = 0, 1, 2, \ldots$ inductively by

$$\sigma_0 = \tau_0 = 0, \tag{4.3}$$

$$\lambda_0 = G(\mu)|_D, \tag{4.4}$$

$$\nu_0 = G(\mu)_{D^c} \tag{4.5}$$

and

$$\sigma_{n+1} = F(\sigma_n + \tau_n + \nu_n)|_D, \tag{4.6}$$

$$\tau_{n+1} = F(\sigma_n + \tau_n + \nu_n)|_{D^c}, \tag{4.7}$$

$$\lambda_{n+1} = G(\lambda_n + (\sigma_{n+1} - \sigma_n))|_D, \tag{4.8}$$

$$\nu_{n+1} = G(\lambda_n + (\sigma_{n+1} - \sigma_n))|_{D^c}. \tag{4.9}$$

(It is always understood that when we take the restriction of a measure to some set the restricted measure is extended by zero to all $\mathbb{R}^N$.) Equations (4.6)–(4.9) can also be written

$$F(\sigma_n + \tau_n + \nu_n) = \sigma_{n+1} + \tau_{n+1}, \tag{4.10}$$

$$G(\lambda_n + (\sigma_{n+1} - \sigma_n)) = \lambda_{n+1} + \nu_{n+1}, \tag{4.11}$$

where the right members are the decomposition of the left members into sums of two measures concentrated on D and D^c, respectively. Note also that since $\mu = \sigma_n = \lambda_n = 0$ in D^e, $\nu_n = 0$ in D^e by (2.39). Hence,

$$\operatorname{supp}\nu_n \subset \partial D. \tag{4.12}$$

The following properties of σ_n, τ_n, λ_n, ν_n will be established for all n.

$$0 = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_n; \tag{4.13}$$

$$0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n; \tag{4.14}$$

$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n; \tag{4.15}$$

$$\nu_n \geq 0; \tag{4.16}$$

$$\sigma_n \leq \lambda_n; \tag{4.17}$$

$$F(\nu_0 + \nu_1 + \cdots + \nu_n) = \sigma_{n+1} + \tau_{n+1}; \tag{4.18}$$

$$F(\lambda_n + \nu_n + \tau_n) = F(\mu); \tag{4.19}$$

$$\lambda_n + \tau_n \leq F(\mu); \tag{4.20}$$

$$\sigma_n + \tau_n \leq F(\mu). \tag{4.21}$$

Assume that (4.13)–(4.21) have been proved and define

$$\nu = \nu_0 + \nu_1 + \nu_2 + \cdots;$$

the sum converges to a measure $\nu \in M_c$ because $\nu_n \geq 0$ by (4.16) and because it follows from (4.18), (4.21) that

$$F(\nu_0 + \nu_1 + \cdots + \nu_n) \leq F(\mu)$$

and, hence, by (2.34) that

$$\langle \nu_0 + \nu_1 + \cdots + \nu_n, 1\rangle \leq \langle \mu, 1\rangle < \infty. \tag{4.22}$$

(4.22) shows in particular that

$$\langle \nu_n, 1\rangle \to 0 \qquad \text{as } n \to \infty. \tag{4.23}$$

From (4.18) and (vi) of theorem 2.2 we obtain

$$\sigma_n + \tau_n \nearrow F(\nu) \qquad (n \to \infty). \tag{4.24}$$

Moreover, using (4.14), (4.15), (4.20) and that, by (2.34), (4.19), (4.23), $\langle F(\mu) - \lambda_n - \tau_n, 1\rangle = \langle \nu_n, 1\rangle \to 0$, we see that

$$\lambda_n + \tau_n \nearrow F(\mu) \qquad (n \to \infty). \tag{4.25}$$

Now (4.2) follows from (4.21), (4.24) and (4.1) follows from (4.24), (4.25), observing that $\sigma_n = \lambda_n = 0$ on D^c. That supp $\nu \subset \partial D$ follows from (4.12), and $\nu|_{\partial D} \geq \mu|_{\partial D}$ follows from (4.5), which by (2.37) shows that already $\nu_0|_{\partial D} \geq \mu|_{\partial D}$.

Thus, it just remains to verify (4.13)–(4.21).

Equations (4.13)–(4.16): these hold by definition and (2.37) for $n = 0$. Assuming the validity of (4.13)–(4.16) for n one immediately deduces their validity for $n + 1$ using just (4.6)–(4.9) and (2.38). Thus (4.13)–(4.16) follow by induction.

Equation (4.17): this also follows by induction, using (4.8), (2.38) and that $\sigma_n = \lambda_n = 0$ on D^c.

Equation (4.18): this follows by induction, using (4.3), (4.16), (4.10) and (2.36).

Equation (4.19): for $n = 0$, we have $F(\lambda_0 + \nu_0 + \tau_0) = FG(\mu) = F(\mu)$ by (4.3)–(4.5) and (2.36). For arbitrary n, we have, using (4.10)–(4.11), (4.14), (4.17) and (2.36),

$$\begin{aligned} F(\lambda_{n+1} + \nu_{n+1} + \tau_{n+1}) &= F(G(\lambda_n + \sigma_{n+1} - \sigma_n) + \tau_{n+1}) \\ &= F(\lambda_n + \sigma_{n+1} - \sigma_n + \tau_{n+1}) \\ &= F(F(\sigma_n + \tau_n + \nu_n) + \lambda_n - \sigma_n) \\ &= F(\sigma_n + \tau_n + \nu_n + \lambda_n - \sigma_n) \\ &= F(\lambda_n + \nu_n + \tau_n). \end{aligned}$$

Thus, (4.19) follows by induction.

Equation (4.20): since by definition (4.7), (4.8) $\lambda_n + \tau_n \leq \rho$ in R, (4.16), (2.38) and (4.19) show that $\lambda_n + \tau_n = F(\lambda_n + \tau_n) \leq F(\lambda_n + \tau_n + \nu_n) = F(\mu)$ as required.

Equation (4.21): this follows from (4.17) and (4.20).

This finishes the proof of theorem 4.1. ■

There is also another proof of theorem 4.1, more in the spirit of the proof of theorem 3.1. It goes as follows (outline, assuming some additional smoothness of μ and ∂D).

Write

$$F(\mu) = \mu + \Delta u,$$

where $u (= U^\mu - V^\mu)$ satisfies (2.15)–(2.18), and is the smallest function satisfying (2.15)–(2.16). Set

$$\psi = \begin{cases} 0 & \text{in } D \\ u & \text{on } D^c \end{cases}$$

and let v be the smallest function (or distribution) satisfying

$$v \geq \psi \qquad \text{in } \mathbb{R}^N, \tag{4.26}$$

$$\Delta v \leq \rho \qquad \text{in } R. \tag{4.27}$$

Observe that u satisfies (4.26)–(4.27) so the set of competing functions is not empty. It also follows that

$$v = u \qquad \text{on } D^c, \tag{4.28}$$

$$v \leq u \qquad \text{in } D. \tag{4.29}$$

By (4.27) Δv is a (signed) measure in R. Define a measure ν on ∂D by

$$\nu = (\rho - \Delta v)|_{\partial D} = (-\Delta v)|_{\partial D} \tag{4.30}$$

and extend ν by $\nu = 0$ outside ∂D. This gives immediately

$$v \geq 0 \qquad \text{in } \mathbb{R}^N,$$

$$\nu + \Delta v \leq \rho \qquad \text{in } R,$$

$$v = 0 \qquad \text{on } R^c.$$

If we can prove that

$$\nu + \Delta v = \rho \qquad \text{in } \omega = \{x \in R\colon v(x) > 0\} \tag{4.31}$$

then it will follow that

$$F(\nu) = \nu + \Delta v, \tag{4.32}$$

at least if ν has finite energy. (If μ has finite energy that ν does have finite energy, but we do not go through the verification of this.)

Now (4.31) certainly holds in $\omega \cap \partial D$ by (4.30). In D^e, $v = u$, $\mu = \nu = 0$, hence, (4.31) holds in $\omega \cap D^e$ because of (2.18). Finally, that $\Delta v = \rho$ in $\omega \cap D$ follows from the minimality of v (as in the proof of (d) of theorem 2.1). This proves (4.31).

From (4.32), (4.28) we see that $F(\nu) = F(\mu)$ in D^e, proving (4.1). As to (4.2), i.e. $\nu + \Delta v \leq \mu + \Delta u$, this holds in ω because $\mu + \Delta u = \rho$ there by (4.29), (2.18). In $D\backslash\omega$, $v = 0$, hence,

$\nu + \Delta v = 0$ (a.e.) there (by the regularity of v in D). Since $F(\mu) \geq 0$ this shows that (4.2) holds also in $D\backslash\omega$. Since ∂D has measure zero and $F(\mu) = F(\nu)$ in D^e by (4.1), (4.2) now follows.

It remains to check that $\nu \geq \mu|_{\partial D}$. But (4.28), (4.29) give that $-\Delta u \leq -\Delta v$ on ∂D, hence, $\mu \leq \rho - \Delta u \leq \rho - \Delta v = \nu$ on ∂D as desired. ■

Equation (4.1) says that, as to the effect outside D, ν is equivalent to μ in a certain sense. Since this certainly reminds of classical balayage, let us indicate the relation between the two operations. Let τ be the classical balayage measure of μ (onto ∂D). Thus, by example 2.1,

$$\tau = H(\mu),$$

where $H = F_{0,D}$, assuming that D is bounded.

The relation between ν and τ can be expressed by the formula

$$\tau - \nu = H(F(\mu) - F(\nu)), \tag{4.33}$$

i.e. their difference equals the classical balayage of the difference in (4.2). Observe that $F(\mu) - F(\nu)$ is a positive measure with support in $\bar{D}$. In particular

$$\nu \leq \tau$$

and equality holds if and only if $F(\nu) = F(\mu)$.

To prove (4.33) it is enough to prove that the potentials of both members agree in D^e (since both members are measures on ∂D). But $U^{H(F(\mu)-F(\nu))} = U^{F(\mu)-F(\nu)} = U^{F(\mu)} - U^{F(\nu)} = V^\mu - V^\nu$ in D^e and $U^{\tau-\nu} = U^\tau - U^\nu = U^\mu - U^\nu$ in D^e, hence, the desired conclusion follows, for example, from (4.28) (observing that $U^\nu - V^\nu = v$ by (4.32)).

The equality $F(\nu) = F(\mu)$, and, hence, $\nu = \tau$, does hold in some cases, e.g. if $\rho = 0$ in D or more generally if ρ is small enough in D compared to the size of μ. However, in the applications we have in mind, ν usually differs from τ. If, for example, D is unbounded as in theorem 3.1, then ν never equals τ (unless $\mu = 0$).

Example 4.1. Take $N = 2$, $R = \mathbb{R}^2$, $\rho = 1$ and let $D = B(0; 1)$. Let $t \geq 0$ be a parameter and consider the measures

$$\mu = \mu(t) = \pi t^2 \delta.$$

Then $F(\mu(t)) = \chi_{B(0;t)}$. For each t in the interval $1 < t < \sqrt{e}$, there is a unique $s = s(t)$ in the interval $0 < s < 1$ satisfying

$$\int_s^t r \log r \, \mathrm{d}r = 0 \tag{4.34}$$

Now, denoting by $\mathrm{d}\theta$ the arc length measure of $\partial B(0; 1)$, the measures $\tau = \tau(t)$ and $\nu = \nu(t)$ considered above are given by

$$\mathrm{d}\tau(t) = \frac{t^2}{2}\,\mathrm{d}\theta \qquad \text{for all } t \geq 0,$$

$$\mathrm{d}\nu(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ ((t^2 - s(t)^2)/2)\,\mathrm{d}\theta & \text{if } 1 < t < \sqrt{e}, \\ (t^2/2)\,\mathrm{d}\theta & \text{if } t \leq \sqrt{e}. \end{cases}$$

For $1 < t < \sqrt{e}$, $F(\nu(t)) = \chi_{\Omega(t)}$ where $\Omega(t) = \{x \in \mathbb{R}^2: s(t) < |x| < t\}$ and for $t \geq \sqrt{e}$, $F(\nu(t)) = \chi_{B(0;t)}$. The equation (4.34) for $s(t)$ is the "quadrature identity" (cf. (2.33) or (2.48)) $\int_{\Omega(t)} \varphi \, dm = \int \varphi \, d\nu(t)$ with $\varphi(x) = \log|x|$.

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