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# TOPICS IN GEOMETRIC FUNCTION THEORY AND RELATED QUESTIONS OF HYDRODYNAMICS

by

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Papers summarized in this dissertation

[A] Quadrature identities and the Schottky double.
TRITA-MAT-1977-3

[B] On the motion of a vortex in two-dimensional flow of an ideal fluid in simply and multiply connected domains.
TRITA-MAT-1979-7

[C] Applications of variational inequalities to a moving boundary problem for Hele Shaw flows.
TRITA-MAT-1981-9

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This dissertation summarizes the papers [A],[B] and [C].

Paper [A] deals with domains in the complex plane called "quadrature domains". $\Omega \subset \mathbb{C}$ is a quadrature domain if a "quadrature identity" of the kind

$$\int_\Omega f\,dx\,dy = \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} a_{kj} f^{(j)}(z_k), \qquad (1)$$

where $a_{kj} \in \mathbb{C}$, $z_j \in \Omega$ and $m$, $n_j$ are integers $\geq 1$, holds for all $f$ in some specified test class of integrable, holomorphic functions in $\Omega$. Usually this test class is $L^p_a(\Omega)$ (consisting of functions $f$ holomorphic in $\Omega$ with $\int |f|^p\,dx\,dy < +\infty$, $1 \leq p < \infty$) or $L^p_{as}(\Omega)$ ($f \in L^p_a(\Omega)$ and moreover having a single-valued integral in $\Omega$). Quadrature domains arose for D.Aharonov and H.S.Shapiro in studying certain extremal problems for univalent functions. The same authors then began investigating quadrature domains for their own sake in [1]. Among the results proved in [1] are the following.

1) $\Omega$ is a quadrature domain if and only if there exists a meromorphic function $h(z)$ in $\Omega$, continuously extendible to $\overline{\Omega}$, such that

$$h(z) = \overline{z} \quad \text{on} \quad \partial\Omega \qquad (2)$$

(Lemma 2.3 in [1]; (2) is to hold in a certain technical sense also at $z = \infty$ if $\Omega$ is unbounded).

2) A simply connected domain $\Omega$ is a quadrature domain if and only if $\Omega$ is the conformal image of $\mathbb{D}$ (the open unit disc) under a rational function (with no poles on $\overline{\mathbb{D}}$). In particular there exist plenty of simply connected quadrature domains. (Thm. 1 in [1].)

3) If $\Omega$ is a quadrature domain, $\partial\Omega$ is part of an algebraic curve (Theorem 3 in [1]).

In all three cases the apriori assumption about $\Omega$ is that

$$\int_{\Omega} \frac{dxdy}{|z|} < \infty \qquad (z = x + iy), \tag{3}$$

and the test class of functions is $L_a^1(\Omega)$.

Among the questions left open in [1] are:

4) Do there exist multiply connected quadrature domains and, if so, how can one produce them? (Equivalently, find an analogue to 2) for multiply connected domains.)

5) Can two different quadrature domains have the same quadrature identity?

Paper [A] grew out of the attempts to answer questions 4) and 5). The main results obtained are the following.

a) We give a positive answer to question 4) by giving a method, analogous to 2), for producing multiply connected quadrature domains. (See 2´) below.)

b) We develop point 3) above by showing that the boundary of a quadrature domain must be a whole algebraic curve (Theorem 3.4 in [A]) and by finding explicit relations between the coefficients of the polynomial function of that curve on the one hand and the data $a_{kj}$, $z_j$, m, $n_j$ of the quadrature identity on the other (Theorem 5.1).

c) We give a partial answer to 5): it turns out that, in the multiply connected case, there in general exist continuous families of domains admitting the same quadrature identity. The number of real parameters in such families depend on the connectivity of the domains and on the test class of functions. In Section 7 of [A] we compute what these numbers are in certain cases (Theorem 7.1, 7.2, Suggestion 7.3). In Section 6 a specific example is worked out, yielding a one-real-parameter family of doubly connected quadrature domains all having the same quadrature identity for the test class $L_a^2(\Omega)$.

Part of question 5) is however still unsolved: can two different simply connected quadrature domains have the same quadrature identity for the test class $L_a^2(\Omega)$ ? Almost certainly the answer is yes but we have so far no specific example of this. The reason for expecting the answer to be yes is that there seems to be no reason for the answer to be no, in particular in view of a recent example of Sakai ([16]) showing that, in a very closely related problem, the corresponding answer is yes.

d) We prove, in Section 4, a couple of results on non-existence of multiply connected quadrature domain admitting quadrature identities of certain low orders. (The order of a quadrature identity like (1) is by definition $\sum_{k=1}^{m} n_k$, assuming that all $z_j$ are distinct and that $a_{k,n_k-1} \neq 0$.) For example it is proved (Corollary 4.5) that a multiply connected domain never admits a quadrature identity of order $\leq 2$ for the test class $L_a^2(\Omega)$. (Here the domains considered are subject to our apriori assumptions, (i) and (ii) below.) The same result has been obtained also by Avci (Theorem 6 in [3]), by using different methods, and special cases of it are proved by Aharonov-Shapiro (Theorem 4 in [1]) and C.Ullemar (Theorem 2 in [18].)

The general idea, underlying most of the results in [A], is that of completing a plane domain $\Omega$ with a "backside" $\tilde{\Omega}$, so that a compact Riemann surface

$$\hat{\Omega} = \Omega \cup \partial\Omega \cup \tilde{\Omega},$$

the Schottky double of $\Omega$, is obtained ([8], p. 47 f).

In order for this to be possible (in a certain technical sense) $\Omega$ must be conformally equivalent to a domain bounded by finitely many analytic Jordan curves. Therefore all domains in [A] are assumed apriori to be

(i) finitely connected, with no boundary component consisting of a single point.

Moreover, the test class $L_a^2(\Omega)$ suits us better than $L_a^1(\Omega)$. Since we want to have $1 \in L_a^2(\Omega)$ (which also ensures that $L_a^2(\Omega) \subset L_a^1(\Omega)$) we only consider domains that have

(ii) finite area.

These apriori assumptions are stronger than that one, (3), used in [1].

From the Schottky double point of view equation (2) simply means that the pair $(h(z),z)$ defines a meromorphic function on $\hat{\Omega}$, namely that function which equals $h(z)$ on $\Omega$, equals $\bar{z}$ on $\tilde{\Omega}$ and extends continously over $\partial\Omega$ by (2). By means of this interpretation of (2) we are able to generalize property 2) to the multiply connected case as follows.

2´) Let $W$ be a standard domain bounded by analytic Jordan curves, representing a certain conformal type. Then all quadrature domains conformally equivalent to $W$ are obtained as conformal images of $W$ under functions meromorphic on the Schottky double $\hat{W}$ of $W$ (Theorem 3.1). (Note that, in 2), the rational functions are just the meromorphic functions on $\hat{\mathbb{D}} \cong$ the Riemann sphere).

Although the classical theory of compact Riemann surfaces does guarantee a good supply of meromorphic functions on $\hat{W}$ one must have meromorphic functions which moreover are univalent (and pole-free) on $W$ in order to produce quadrature domains from 2´). The existence of such functions is proved by approximating some explicit function, defined and univalent in some neighbourhood of $W \cup \partial W$ in $\hat{W}$, with functions meromorphic on $\hat{W}$. This explicit function can e.g. be chosen to be $z$ (the identity function) on $W$. This function extends holomorphically over $\partial W$ in $\hat{W}$ because $\partial W$ was assumed to be analytic. The possibility of approximating such a function by functions meromorphic on $\hat{W}$ follows from Runge-like approximation theorems as proved e.g. in [12] or [9]. The approximation is to be uniform on a neighbourhood of $W \cup \partial W$ and it is easy to see that the approximating meromorphic functions then will be univalent on (a neighborhood of) $W \cup \partial W$ whenever the approximation is good enough. This is the way existence of multiply connected quadrature domains is proved.

Part of the results in [A] concern quadrature identities for the test class $L^2_{as}(\Omega)$ instead of $L^2_a(\Omega)$. In that case equation (2) for property 1) has to be replaced by

$$h(z) = \bar{z} + \text{constant} \quad \text{on each component of } \partial\Omega, \qquad (4)$$

with (in general) different constants on different boundary components. In terms of the Schottky double $\hat{\Omega}$ of $\Omega$ (4) means that the pair $(h'(z)dz\ ,\ dz)$ defines a meromorphic differential on $\hat{\Omega}$ (on observing that (4) can be written

$$h'(z)dz = d\bar{z} \qquad \text{along } \partial\Omega\ ). \tag{5}$$

In [B] we study the motion of a vortex in two-dimensional flow of an ideal fluid in simply and multiply connected domains. This study was initiated by questions posed by Prof. Bengt Joel Andersson (Dept. of Hydromechanics, KTH, Stockholm), concerning existence and uniqueness of "equilibrium points" for such a vortex, i.e. points where a free vortex is at rest ( in general a free vortex moves).

The hydrodynamical setting is this. Let $\Omega$ be a finitely connected, possibly unbounded, domain regarded as a subdomain of the complex plane. In $\Omega$ we have an incompressible, (locally) irro - tational, time-dependent flow with a vortex of constant strength at a moving point $z_0 = z_0(t)$.

Suppose first that the vortex is kept fixed (in some way). Then the surrounding fluid will exert a force $F_\beta$ on it (the parameter $\beta$ will be explained below). Since this force depends on the position of the vortex it may be regarded as a vector field in $\Omega$ . This vector field turns out to be a potential field, i.e.

$$F_\beta = \text{grad } u_\beta \tag{6}$$

for some (real) function $u_\beta$ in $\Omega$ .

If on the other hand the vortex can move freely (which is perhaps the most natural situation), then its velocity will be

$$\frac{dz_0}{dt} = i \cdot (\text{real constant}) \cdot F_\beta(z_0) \tag{7}$$

($i = \sqrt{-1}$). Thus its velocity is always perpendicular to $F_\beta$ and it follows that it moves along a level line of $u_\beta$ . (The fact that a free vortex moves along the level line of a function has, in the simply connected case, earlier been observed by H. Villat ([19]) and B.J.Andersson ([2]).)

The domain functions $F_\beta$ and $u_\beta$ are in the center of interest in [B] . They are expressible in terms of certain "modified" Green's functions $g_\beta(z,\zeta)$ for the domain. These differ from the ordinary Green's function $g(z,\zeta)$ in that, instead of being constantly equal to zero on the boundary, they are free to take arbitrary constant values on the individual boundary components, and are determined by having their conjugate periods prescribed (together with a normalization condition). The parameter $\beta$ in $g_\beta(z,\zeta)$ (and in $u_\beta$ and $F_\beta$) just is the list of these conjugate periods ($\beta = (\beta_1,\ldots,\beta_m)$ ). The presence of this parameter reflects the fact that for a flow in a multiply connected domain one can prescribe the circulations around the "holes" of the domain. When the domain is simply connected the two kinds of Green's functions coincide.

Expanding the analytic completion (with respect to z) $G_\beta(z,\zeta)$ of $g_\beta(z,\zeta)$ in a power series about $z = \zeta$ ,

$$G_\beta(z,\zeta) = -\log(z-\zeta) + c_{\beta 0}(\zeta) + c_{\beta 1}(\zeta)\cdot(z-\zeta) + c_{\beta 2}(\zeta)\cdot(z-\zeta)^2 + \ldots, \quad (8)$$

we find that

$$F_\beta(\zeta) = -\text{(positive constant)}\cdot\overline{c}_{\beta 1}(\zeta) \quad (9)$$

$$u_\beta(\zeta) = -\text{(positive constant)}\cdot c_{\beta 0}(\zeta) \quad (10)$$

($c_{\beta 0}(\zeta)$ is chosen real in (8) ).

The functions $c_{\beta 0}(\zeta)$, $c_{\beta 1}(\zeta)$,.., as well as the corresponding functions $c_0(\zeta)$, $c_1(\zeta)$,... defined in terms of the ordinary Green's function, are studied in [B] with regard to boundary behaviour, transformation properties under conformal mappings etc. Although these investigations give few or no results which are new from a purely mathematical standpoint they give some results which are interesting in their hydrodynamical context.

For example we find that

$$u_\beta(\zeta) \to +\infty \quad \text{as} \quad \zeta \to \partial\Omega \quad (11)$$

if $\Omega$ is bounded. Thus $u_\beta$ must have a point where it attains its minimum in $\Omega$ , and such a point is a point where a free vortex is at rest. This proves the existence of equilibrium points for bounded domains.

It is also shown that $u_\beta$ is subharmonic, more precisely that

$$\Delta u_\beta = (\text{positive constant})\cdot K_s(\zeta,\zeta), \tag{12}$$

where $K_s(z,\zeta)$ is the reduced Bergman kernel for $\Omega$ (thus $K_s(\zeta,\zeta) > 0$).

If $\Omega$ is simply connected $u_\beta$ satisfies a remarkable differential equation , namely

$$\Delta u_\beta = A\, e^{Bu_\beta}, \tag{13}$$

where $A$ and $B$ are positive constants. This equation, which has no obvious physical interpretation and does not hold if $\Omega$ is multiply connected, has an interesting history, being studied by Liouville (it sometimes appears under the name Liouville's equation), H.A.Schwarz, Picard, Poincaré, Bieberbach and others for example in early attempts to prove the so-called uniformization theorem. See [4]. Recently, and in our context, it has been taken up by S.Richardson (Dept. of Mathematics, Univ. of Edinburgh) ([15]).

If $\Omega$ moreover is convex, and not an infinite strip, then we find that $u_\beta$ can have at most one local extremum (equivalently, $F_\beta$ has at most one zero). Thus there is at most one equilibrum point for a vortex in a convex domain (other that an infinite strip). (This result was conjectured by B.J.Andersson, and was originally thought to be the main result of [B]. However, as we found this result to be already known in a function theoretic context (Sats 4 in [10]) the emphasis of [B] changed towards a general study of vortex motion in finitely connected domains.)

The condition of convexity for the above property to hold cannot be relaxed to starlikeness, as we show by giving an explicit example of a starlike domain with three zeroes for $F_\beta$.

In paper [C] we use the technique of variational inequalities to prove existence and uniqueness for a kind of weak solution to a moving boundary problem arising in the theory of Hele Shaw flow. This moving boundary problem also has some relevance to quadrature identities (paper [A]). See p. 11 f below.

The moving boundary problem we study in [C] was introduced by S.Richardson in [14]. The hydrodynamical background for it can be briefly described as follows.
Let two large plane surfaces be lying parallel to each other, separated by a narrow gap. In one of the surfaces there is a hole for injection of fluid. At time zero a prescribed region (covering the hole of injection) in the space between the two surfaces is already occupied by fluid. From then on fluid is injected through the hole at a moderate constant rate so that the region of fluid grows. It is the growth of this region (regarded as a two-dimensional set) we study.

The assumptions on the fluid are that it shall be Newtonian and incompressible, but it might well have high viscosity. Then, under some mild further assumptions, the flow will be what is known as a Hele Shaw flow ([13], p. 581 f), more precisely a two-dimensional Hele Shaw flow with free boundaries and with a source point.

Mathematically, the problem of describing the growing region of fluid is a kind of moving boundary problem. Identifying the plane of the two surfaces with the complex plane and letting the point of injection be the origin the Hele Shaw flow moving boundary problem turns out to be the following:

Given an initial domain $D_0 \subset \mathbb{C}$ with $0 \in D_0$ find a family of domains, $t \to D_t$ for $t \geq 0$ $(0 \in D_t \subset \mathbb{C})$, such that, at each instant $t$, the normal velocity of $\partial D_t$ at $z \in \partial D_t$ equals $-\operatorname{grad} g_{D_t}(z)$, where $g_{D_t}$ is the Green's function for $D_t$ with respect to the origin ($g_{D_t}(z) = -\log|z| +$ harmonic in $D_t$, $g_{D_t}(z) = 0$ on $\partial D_t$).

For this moving boundary problem we formulate a concept of weak solution, which is as follows:

$$[0,\infty) \ni t \to D_t \,, \tag{14}$$

where $D_t \subset \mathbb{C}$ are open sets with $0 \in D_t$, is a weak solution if there exist (for each $t$) distributions $u_t$ with compact support in $\mathbb{C}$ such that

$$\chi_{D_t} - \chi_{D_0} = \Delta u_t + 2\pi t\cdot\delta_0 \tag{15}$$

$$u_t \geq 0 \quad \text{and} \tag{16}$$

$$D_t = D_0 \cup \{z \in \mathbb{C} : u_t(z) > 0\} \ . \tag{17}$$

Here $\chi_D$ is the characteristic function of $D$, $\delta_0$ the Dirac measure at the origin and the statement $u_t(z) > 0$ in (17) is meaningful because $u_t$, by (15), can be represented by a function continuous outside the origin. If $D_0$ is connected the term $D_0$ can be deleted from (17) (for $t > 0$).

(Within the body of the paper we actually have to work with a slightly more technical variant of the concept of weak solution (called (C)) than the one described).

To see how the concept of weak solution is related to the first description of the moving boundary problem one can formulate the latter within the language of distribution theory by the equation

$$\frac{\partial}{\partial t}\chi_{D_t} = \Delta g_{D_t} + 2\pi\delta_0 \ , \tag{18}$$

in which $g_{D_t}$ now is extended to all $\mathbb{C}$ by setting it equal to zero outside $D_t$. (Both members of (18) will be distributions supported by $\partial D_t$.) Integrating (18) with respect to $t$ from $0$ to $t$ yields (15) for

$$u_t = \int_0^t g_{D_\tau}\, d\tau \ . \tag{19}$$

This $u_t$ also satisfies (16) and (17) because $g_{D_t} \geq 0$, with strict inequality (only) in $D_t$, and because $D_t$ must be increasing with $t$ (since the right member of (18) is a non-negative distribution). Thus a solution in the first sense, a "classical solution", is also a weak solution.

Now our main result, Theorem 13, states that, given $D_0 \subset \mathbb{C}$, bounded and with $0 \in D_0$, a weak solution always exists and is unique. It also states that a (weak) solution has a property we call the "moment inequality", namely that

$$\int_{D_t} \varphi \geq \int_{D_0} \varphi + 2\pi t \cdot \varphi(0) \tag{20}$$

for every function $\varphi$, subharmonic in $D_t$. Actually, this moment property is equivalent to the property of being a weak solution (Theorem 10).

Choosing $\varphi(z) = \pm \operatorname{Re} z^n$ and $\pm \operatorname{Im} z^n$ in (20) gives

$$|D_t| = |D_0| + 2\pi t \quad (n = 0) \tag{21}$$

($|..|$ denotes area) and

$$\int_{D_t} z^n = \int_{D_0} z^n \qquad (n = 1,2,\dots). \tag{22}$$

This nice property, of preserving the "complex moments" $\int_{D_t} z^n$ $(n \geq 1)$ for the domains $D_t$, of a solution of the Hele Shaw problem was discovered by S.Richardson ([14]).

The proof of existence and uniqueness of weak solutions goes via the theory of variational inequalities. Let $\Omega$ be a sufficiently large disc, centered at the origin, and let $\psi_t$ be the function in $\Omega$ defined by

$$\begin{aligned} \Delta\psi_t &= \chi_{D_0} - 1 + 2\pi t \cdot \delta_0 && \text{in } \Omega \\ \psi_t &= 0 && \text{on } \partial\Omega \end{aligned} \tag{23}$$

Then, if $u_t$ satisfies (15) to (17),

$$\begin{aligned} &\Delta u_t + \Delta\psi_t \leq 0 \\ &u_t \geq 0 \\ &u_t \cdot (\Delta u_t + \Delta\psi_t) = 0 . \end{aligned} \tag{24}$$

This is a so-called linear complementarity problem for $u_t$, and it is equivalent to the following variational inequality.

Find $u_t$ , defined in $\Omega$ and $= 0$ on $\partial\Omega$, such that

$$\Delta u_t + \Delta\psi_t \leq 0$$

and

$$\int\nabla(u-u_t)\cdot\nabla u_t \geq 0 \quad \text{for all } u \tag{25}$$

satisfying $u = 0$ on $\partial\Omega$ and
$\Delta u + \Delta\psi_t \leq 0$ (in $\Omega$ ).

Such a variational inequality always has a unique solution (in an appropriate Sobolev space), and if $u_t$ is this solution $D_t$ , defined by (17), are shown to satisfy (15) to (17) for $t$ small enough compared with $\Omega$. Thus we get a weak solution of the moving boundary problem by solving a variational inequality for each $t$.

The variational inequality (25) can be transformed to a more familiar form by the substitution

$$v_t = u_t + \psi_t \; . \tag{26}$$

Then (25) will be equivalent to:

Find $v_t$ , defined in $\Omega$ and $= 0$ on $\partial\Omega$ , such that

$$v_t \geq \psi_t$$

and

$$\int\nabla(v - v_t)\cdot\nabla v_t \geq 0 \quad \text{for all } v \tag{27}$$

satisfying $v = 0$ on $\partial\Omega$ and
$v \geq \psi_t$ (in $\Omega$).

This is the kind of variational inequality occurring for example in the famous "obstacle problem" ([11]).

The existence of weak solutions to the Hele Shaw problem leads to the existence of certain kinds of quadrature identities. For example the following is proved (part of Corollary 16.1):

[11] Kinderlehrer,D.-Stampacchia,G.: An introduction to variational inequalities and their applications, Academic Press 1980.

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[3] Avci,Y.: Quadrature identities and the Schwarz function, Doctoral dissertation, Stanford University 1977.

[4] Bieberbach,L.: $\Delta u = e^u$ und die Automorphen Funktionen, Math.Ann. 77 (1916), 173-212.

[5] Duvaut,G.: The solution of a two-phase Stefan problem by a variational inequality, pp 173-181 in "Moving boundary problems in heat flow and diffusion", edited by Ockendon and Hodgkins (Clarendon press, Oxford 1975).

[6] Elliott,C.M.: On a variational inequality formulation of an electrochemical macining moving boundary problem and its approximation by the finite element method, J.Inst.Maths Applics 25 (1980), 121-131.

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[9] Gustafsson,B: The Runge approximation theorem on compact Riemann surfaces, TRITA-MAT-1977-2, Mahtematics, Royal Institute of Technology, Stockholm.

[10] Haegi,H.R.: Extremalprobleme und Ungleichungen konformer Gebietsgrössen, Compositio Mathematica 8 (1951), 81-111.

Given $z_1,\ldots,z_m \in \mathbb{C}$ , $a_1,\ldots,a_m > 0$ there exists an open set D such that

$$\int_D f(z)dxdy = \sum_{j=1}^{m} a_j f(z_j) \quad \text{for all} \quad f \in L_a^1(D) \tag{28}$$

The theory of variational inequalities has already been used by others to handle moving boundary problems of various kinds. Works closely connected with ours are [5] by G.Duvaut on the classical two-phase Stefan problem, [6] by C.M.Elliott on a problem arising in electro-chemistry and [7] by C.M.Elliott-V.Janovsky on the Hele-Shaw problem.

Recently M.Sakai, in a paper ([17]) primarily devoted to quadrature domains (in a more general sense than ours), has obtained results on the Hele Shaw problem similar to ours, but by using quite different methods. Sakai works with a concept of weak solution which is essentially the same as our moment inequality (20).

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