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ASYMPTOTIC ANALYSIS FOR MONOTONE QUASILINEAR PROBLEMS IN THIN MULTIDOMAINS

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Abstract. In this paper we perform the asymptotic analysis of a class of monotone quasilinear Neumann problems, with exponent $p \in]1, +\infty[$ and nonstandard transmission condition, originating by change of variables from the quasilinear Neumann problem in a thin multidomain. This completes the Γ-convergence approach previously considered by the same authors. In particular, the corrector type result which is given here is more general.

0. Introduction and motivation

Let $N \geq 2$, let $\omega \subset \mathbf{R}^{N-1}$ be a bounded open connected set with C^1 boundary such that the origin in $\mathbf{R}^{N-1}$, denoted by $0'$, belongs to ω. Let $\Omega^{(1)} = \omega \times (0,1)$, $\Omega^{(2)} = \omega \times (-1,0)$ and let $\{r_n\}_{n\in\mathbf{N}}$, $\{h_n\}_{n\in\mathbf{N}}$ be two sequences of positive numbers converging to 0. We consider two vertical cylinders with small volumes: $\Omega_n^{(1)} = r_n\omega \times (0,1)$ with small cross section

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$r_n\omega$ and constant height, $\Omega_n^{(2)} = \omega \times (-h_n, 0)$ with small height h_n and constant cross section. Their interface is $S_n = r_n\omega \times \{0\}$ (see figure).

Then we consider the thin multidomain $\Omega_n = \Omega_n^{(1)} \cup \Omega_n^{(2)} \cup S_n$ and the following model Neumann problem in Ω_n:

$$\begin{cases} -\Delta_p U_n + |U_n|^{p-2}U_n = F & \text{in } \Omega_n, \\ \frac{\partial U_n}{\partial \nu} = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{0.1}$$

where $p \in]1, +\infty[$, $\Delta_p U_n = div(|\nabla U_n|^{p-2}\nabla U_n)$, $|\cdot|$ is the euclidean norm in $\mathbf{R}^N$ or the absolute value in $\mathbf{R}$, $F \in L^{\frac{p}{p-1}}(\omega \times (-1, 1))$. Assuming that, when $n \to +\infty$, the sequence $\left\{\frac{h_n}{r_n^{N-1}}\right\}_{n\in\mathbf{N}}$ converges to some finite positive number, the asymptotic behaviour of U_n can be obtained from a general theorem of [11], where one considers a large class of minimization problems, including the one corresponding to (0.1), with "energy"

$$\frac{1}{p}\int_{\Omega_n}(|\nabla V|^p + |V|^p)dX - \int_{\Omega_n} FV\,dX.$$

In this paper our approach is more direct, as we study the asymptotic behaviour of a wider class of monotone Neumann problems, not necessarily connected to an energy functional, but still having (0.1) as model equation, written in its variational form : $U_n \in W^{1,p}(\Omega_n)$ and, for every V in $W^{1,p}(\Omega_n)$,

$$\int_{\Omega_n} [|\nabla U_n|^{p-2}(\nabla U_n, \nabla V) + (|U_n|^{p-2}U_n - F)V]dX = 0,$$

$(\cdot,\cdot)$ denoting the scalar product in $\mathbf{R}^N$. To study the asymptotic behaviour of $\{U_n\}_{n\in\mathbf{N}}$ as $n \to +\infty$, we introduce the scallings $T_n^{(i)} : \Omega^{(i)} \to \Omega_n^{(i)}$ $(i = 1, 2)$, defined by $x = (x_1, \ldots, x_{N-1}, x_N) = (x', x_N) \to T_n^{(i)}(x) = X =$

$(X_1, \dots X_{N-1}, X_N) = (X', X_N)$,

$$T_n^{(1)}(x', x_N) = (X', X_N) = (r_n x', x_N),$$

$$T_n^{(2)}(x', x_N) = (X', X_N) = (x', h_n x_N).$$

By these homeomorphisms, each function $v^{(i)} : \Omega^{(i)} \to \mathbf{R}$ is connected with $V^{(i)} : \Omega_n^{(i)} \to \mathbf{R}$, by

$$v^{(i)}(x) = V^{(i)}(X) = V^{(i)}(T_n^{(i)}(x)), \quad (i = 1, 2)$$

and one can define $\nabla_n^{(i)}$ $(i = 1, 2)$, by

$$\begin{aligned}\nabla_n^{(1)} v^{(1)}(x) &= \Big(\frac{1}{r_n}\nabla' v^{(1)}(x), \frac{\partial v^{(1)}}{\partial x_N}(x)\Big)\\ &= \nabla V^{(1)}(X) = \Big(\nabla' V^{(1)}(X), \frac{\partial V^{(1)}}{\partial X_N}(X)\Big),\\ \nabla_n^{(2)} v^{(2)}(x) &= \Big(\nabla' v^{(2)}(x), \frac{1}{h_n}\frac{\partial v^{(2)}}{\partial x_N}(x)\Big)\\ &= \nabla V^{(2)}(X) = \Big(\nabla' V^{(2)}(X), \frac{\partial V^{(2)}}{\partial X_N}(X)\Big),\end{aligned}$$

where $\nabla' v = (\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_{N-1}})$, $\nabla' V = (\frac{\partial V}{\partial X_1}, \dots, \frac{\partial V}{\partial X_{N-1}})$. This allows to transform the above problem, posed in the thin multidomain Ω_n depending on n, into a problem which is posed in the fixed domain $\Omega = \omega \times (-1, 1)$ (compare for instance [5, 6, 7, 12, 13, 14, and 16]). Defining $U_n^{(i)}$ and $F_n^{(i)}$ by $U_n^{(i)} = U_n|_{\Omega_n^{(i)}}$ and $F_n^{(i)} = F|_{\Omega_n^{(i)}}$, then defining $u_n^{(i)}$ and $f_n^{(i)}$ by $u_n^{(i)}(x) = U_n^{(i)}(X)$ and $f_n^{(i)}(x) = F_n^{(i)}(X)$ as above, it is easy to see that for every $n \in \mathbf{N}$, $u_n = (u_n^{(1)}, u_n^{(2)})$ is the unique solution of the variational problem:

$$\begin{aligned}u_n \in K_n = \Big\{ v = (v^{(1)}, v^{(2)}) \in W^{1,p}(\Omega^{(1)}) \times W^{1,p}(\Omega^{(2)});\\ v^{(1)}(x', 0) = v^{(2)}(r_n x', 0),\ x' \text{ a.e. in } \omega \Big\}\end{aligned} \tag{0.2}$$

and for every $v = (v^{(1)}, v^{(2)}) \in K_n$,

$$\int_{\Omega^{(1)}} \Big[|\nabla_n^{(1)} u_n^{(1)}|^{p-2} (\nabla_n^{(1)} u_n^{(1)}, \nabla_n^{(1)} v^{(1)}) + \big(|u_n^{(1)}|^{p-2} u_n^{(1)} - f_n^{(1)}\big) v^{(1)} \Big] dx +$$
$$\frac{h_n}{r_n^{N-1}} \int_{\Omega^{(2)}} \Big[|\nabla_n^{(2)} u_n^{(2)}|^{p-2} (\nabla_n^{(2)} u_n^{(2)}, \nabla_n^{(2)} v^{(2)}) + \big(|u_n^{(2)}|^{p-2} u_n^{(2)} - f_n^{(2)}\big) v^{(2)} \Big] dx = 0.$$

More generally, the aim of this paper is to study the asymptotic behaviour, as $n \to +\infty$, of the following variational problem: $u_n = (u_n^{(1)}, u_n^{(2)}) \in K_n$ and for every $v = (v^{(1)}, v^{(2)}) \in K_n$,

$$\int_{\Omega^{(1)}} \left[\left(a(x, B_n^{(1)} u_n^{(1)}), B_n^{(1)} v^{(1)} \right) - f_n v^{(1)} \right] dx \tag{0.3}$$

$$+ \frac{h_n}{r_n^{N-1}} \int_{\Omega^{(2)}} \left[\left(a(x, B_n^{(2)} u_n^{(2)}), B_n^{(2)} v^{(2)} \right) - f_n v^{(2)} \right] dx = 0,$$

where $(.,.)$ is now the scalar product in $\mathbf{R}^{N+1}$, $B_n^{(i)}$ $(i = 1, 2)$ is the linear operator from $W^{1,p}(\Omega^{(i)})$ to $\left(L^p(\Omega^{(i)})\right)^{N+1}$ given by $B_n^{(i)} v^{(i)} = (v^{(i)}, \nabla_n^{(i)} v^{(i)})$, $f_n = f_n(x)$ is defined in Ω, $a = a(x, s, \xi)$ is a Carathéodory function, satisfying suitable monotonicity, coercivity and growth conditions (see the precise assumptions in the next section).

We assume that the volumes of the two cylinders $\Omega_n^{(1)}$ and $\Omega_n^{(2)}$ tend to zero with same rate, as n tends to infinity, that is

$$r_n \to 0, \quad h_n \to 0, \quad q_n = \frac{h_n}{r_n^{N-1}} \to q \in (0, \infty) \tag{0.4}$$

and we assume also that

$$f_n \rightharpoonup f \quad \text{weakly in} \quad L^{\frac{p}{p-1}}(\Omega). \tag{0.5}$$

(For instance (0.5) holds true, up to a subsequence, for the model problem, if $F \in L^\infty(\Omega)$.)

It is proved in this paper that for any sequence of solutions $\{u_n\}_{n \in \mathbf{N}}$ of (0.3), there exists a subsequence, still denoted $\{u_n\}_{n \in \mathbf{N}}$ for simplicity, such that $u_n^{(i)} \rightharpoonup u^{(i)}$ weakly in $W^{1,p}(\Omega^{(i)})$, $(i = 1, 2)$,

$$\frac{1}{r_n} \nabla' u_n^{(1)} \rightharpoonup \nabla' y^{(1)} \quad \text{weakly in} \quad \left(L^p(\Omega^{(1)})\right)^{N-1},$$

$$\frac{1}{h_n} \frac{\partial u_n^{(2)}}{\partial x_N} \rightharpoonup \frac{\partial y^{(2)}}{\partial x_N} \quad \text{weakly in} \quad L^p(\Omega^{(2)}),$$

and $((u^{(1)}, u^{(2)}), (y^{(1)}, y^{(2)}))$ is a solution of a variational problem involving $a(x, (u^{(1)}, \nabla' y^{(1)}, \frac{du^{(1)}}{dx_N}))$ and $a(x, (u^{(2)}, \nabla' u^{(2)}, \frac{\partial y^{(2)}}{\partial x_N}))$.

In the case $p \leq N - 1$, the limit problem does not depend on q and splits into two independent variational problems posed in the respective domains $\Omega^{(1)}$ and $\Omega^{(2)}$. In the case $p > N - 1$, the limit problem is coupled and depends on q. This fact was already observed in [11].

If a is strongly monotone, the problem (0.3) and its limit have unique solutions and, in such a case, the above convergences occur for the whole sequence and they are proved to be strong, a result which was not obtained in such a generality in [11].

Finally, let us mention some other references. A preliminary version of [11], concerning the model problem, but including oscillating coefficients, was published in [10] with sketch of proofs. We recall that [2] and [3] deal with the case of oscillating coefficients having measure limits, but with $\Omega_n^{(2)} = \Omega^{(2)}$ independent of n and with a simpler (purely algebraic) transmission condition. The homogenization of a monotone quasilinear problem in a domain having oscillating boundary was considered in [9] (see also [17]). For general background on homogenization of thin structures, the reader is referred to [8]. The present paper, as well as [11], is inspired by [16].

1. Main results

We impose the following conditions on the nonlinear term a:

- Denoting $V = (s, \xi) = (s, \xi', \xi_N) \in \mathbf{R} \times \mathbf{R}^{N-1} \times \mathbf{R}$, the function $a = a(x, V) = a(x, s, \xi) = a(x, s, \xi', \xi_N)$, with values in $\mathbf{R}^{N+1}$, is a Carathéodory function, that is, it is measurable with respect to $x \in \Omega = \omega \times (-1, 1)$ and continuous with respect to the other variables.
- It satisfies the monotonicity condition:

$$(a(x, V) - a(x, W), V - W) \geq 0, \text{ a.e. } x \in \Omega, \text{ for all } V, W \in \mathbf{R}^{N+1}. \quad (1.1)$$

- It is coercive, that is there exist $p \in (1, \infty)$, $\alpha > 0$ and $g \in L^1(\Omega)$, such that

$$(a(x, V), V) \geq \alpha |V|^p - g(x), \text{ a.e. } x \in \Omega, \text{ for all } V \in \mathbf{R}^{N+1}. \quad (1.2)$$

- It satisfies the following growth condition: there exist $\beta > 0$ and $h \in L^{p'}(\Omega)$ with $p' = \dfrac{p}{p-1}$, such that

$$|a(x, V)| \leq \beta |V|^{p-1} + h(x), \text{ a.e. } x \in \Omega, \text{ for all } V \in \mathbf{R}^{N+1}. \quad (1.3)$$

It is well known that this growth condition implies that the mapping $V \to a(., V(.))$ is continuous from $(L^p(\Omega))^{N+1}$ to $(L^{p'}(\Omega))^{N+1}$.

Under the above hypotheses, for every f_n in $L^{p'}(\Omega)$, the problem (0.3) admits at least one solution $u_n = (u_n^{(1)}, u_n^{(2)})$ in the subspace K_n defined in (0.2) (see also beginning of Section 2).

When n tends to infinity, we assume (0.4) and (0.5).

The asymptotic behaviour of u_n in such a case is our main result, given in Theorem 1.1, where we use the following notations:

$$W_m^{1,p}(\omega) = \Big\{ v \in W^{1,p}(\omega), \int_\omega v dx' = 0 \Big\},$$

$$W_m^{1,p}((-1,0)) = \Big\{ v \in W^{1,p}((-1,0)), \int_{-1}^0 v dx_N = 0 \Big\},$$

$$K = W^{1,p}((0,1)) \times W^{1,p}(\omega) \text{ if } 1 < p \leq N-1,$$

$$K = \big\{ v = (v^{(1)}, v^{(2)}) \in W^{1,p}((0,1)) \times W^{1,p}(\omega); v^{(1)}(0) = v^{(2)}(0') \big\} \text{ if } p > N-1,$$

identified in the canonical way to a subspace of $W^{1,p}(\Omega^{(1)}) \times W^{1,p}(\Omega^{(2)})$,

$$\begin{aligned} a(x,V) &= a(x,s,\xi) = a(x,s,\xi',\xi_N) \\ &= \big(a_0(x,s,\xi',\xi_N), a'(x,s,\xi',\xi_N), a_N(x,\xi',\xi_N)\big) \in \mathbf{R} \times \mathbf{R}^{N-1} \times \mathbf{R}. \end{aligned}$$

Theorem 1.1. *Let $W_m^{1,p}(\omega)$, $W_m^{1,p}((-1,0))$, K and $a = (a_0, a', a_N)$ be as above, with a verifying* (1.1), (1.2), (1.3). *Let $u_n = (u_n^{(1)}, u_n^{(2)})$ be solutions of* (0.3). *Assuming* (0.4) *and* (0.5) *when $n \to +\infty$, there exists $u = (u^{(1)}, u^{(2)}) \in K$ and $y = (y^{(1)}, y^{(2)}) \in L^p(0,1; W_m^{1,p}(\omega)) \times L^p(\omega; W_m^{1,p}((-1,0)))$, such that for a subsequence, the following convergences hold*

$$u_n^{(i)} \rightharpoonup u^{(i)} \text{ weakly in } W^{1,p}(\Omega^{(i)}), \quad (i = 1,2); \tag{1.4}$$

$$\frac{1}{r_n}\Big(u_n^{(1)} - \frac{1}{|\omega|}\int_\omega u_n^{(1)} dx'\Big) \rightharpoonup y^{(1)} \quad \text{weakly in } L^p(0,1; W_m^{1,p}(\omega)), \tag{1.5}$$

$$\frac{1}{h_n}\Big(u_n^{(2)} - \int_{-1}^0 u_n^{(2)} dx_N\Big) \rightharpoonup y^{(2)} \quad \text{weakly in } L^p(\omega; W_m^{1,p}((-1,0))), \tag{1.6}$$

hence

$$\frac{1}{r_n}\nabla' u_n^{(1)} \rightharpoonup \nabla' y^{(1)} \text{ weakly in } \big(L^p(\Omega^{(1)})\big)^{N-1}, \tag{1.7}$$

$$\frac{1}{h_n}\frac{\partial u_n^{(2)}}{\partial x_N} \rightharpoonup \frac{\partial y^{(2)}}{\partial x_N} \text{ weakly in } L^p(\Omega^{(2)});$$

$$a(., B_n^{(1)} u_n^{(1)}) \rightharpoonup a(., u^{(1)}, \nabla' y^{(1)}, \frac{du^{(1)}}{dx_N}) \text{ weakly in } \big(L^{p'}(\Omega^{(1)})\big)^{N+1}, \tag{1.8}$$

$$a(., B_n^{(2)} u_n^{(2)}) \rightharpoonup a(., u^{(2)}, \nabla' u^{(2)}, \frac{\partial y^{(2)}}{\partial x_N}) \text{ weakly in } \big(L^{p'}(\Omega^{(2)})\big)^{N+1}. \tag{1.9}$$

The limit $(u,y)=((u^{(1)},u^{(2)}),(y^{(1)},y^{(2)}))$ *satisfies the following variational conditions, for every* $(v,z)=((v^{(1)},v^{(2)}),(z^{(1)},z^{(2)}))$ *verifying the same constraints as* (u,y),

$$\int_{\Omega^{(1)}}\Big(a'\Big(x,u^{(1)},\nabla' y^{(1)},\frac{du^{(1)}}{dx_N}\Big),\nabla' z^{(1)}\Big)dx=0, \tag{1.10}$$

$$\int_{\Omega^{(2)}}a_N\Big(x,u^{(2)},\nabla' u^{(2)},\frac{\partial y^{(2)}}{\partial x_N}\Big)\frac{\partial z^{(2)}}{\partial x_N}dx=0, \tag{1.11}$$

$$\begin{aligned}&\int_{\Omega^{(1)}}\Big[a_N\Big(x,u^{(1)},\nabla' y^{(1)},\frac{du^{(1)}}{dx_N}\Big)\frac{dv^{(1)}}{dx_N}\\&\quad+\Big(a_0\Big(x,u^{(1)},\nabla' y^{(1)},\frac{du^{(1)}}{dx_N}\Big)-f\Big)v^{(1)}\Big]dx\\&+q\int_{\Omega^{(2)}}\Big[\Big(a'\Big(x,u^{(2)},\nabla' u^{(2)},\frac{\partial y^{(2)}}{\partial x_N}\Big),\nabla' v^{(2)}\Big)\\&\quad+\Big(a_0\Big(x,u^{(2)},\nabla' u^{(2)},\frac{\partial y^{(2)}}{\partial x_N}\Big)-f\Big)v^{(2)}\Big]dx=0.\end{aligned} \tag{1.12}$$

Remark 1.1. Clearly, the equations (1.10) and (1.11) automatically extend to every $z^{(1)}\in L^p(0,1;W^{1,p}(\omega))$, $z^{(2)}\in L^p(\omega;W^{1,p}(]-1,0[))$.

Remark 1.2. As in [11] we notice that the limit problem is coupled and its solutions depend on q if $p>N-1$; otherwise it is uncoupled, its solutions do not depend on q and (1.12) splits into

$$\int_{\Omega^{(1)}}\Big[a_N\Big(x,u^{(1)},\nabla' y^{(1)},\frac{du^{(1)}}{dx_N}\Big)\frac{dv^{(1)}}{dx_N}+\Big(a_0\Big(x,u^{(1)},\nabla' y^{(1)},\frac{du^{(1)}}{dx_N}\Big)-f\Big)v^{(1)}\Big]dx=0,$$

$$\int_{\Omega^{(2)}}\Big[\Big(a'\Big(x,u^{(2)},\nabla' u^{(2)},\frac{\partial y^{(2)}}{\partial x_N}\Big),\nabla' v^{(2)}\Big)+\Big(a_0\Big(x,u^{(2)},\nabla' u^{(2)},\frac{\partial y^{(2)}}{\partial x_N}\Big)-f\Big)v^{(2)}\Big]dx=0.$$

Next we assert strong convergences, assuming strong monotonicity.

Theorem 1.2. *If we suppose further that* a *is strongly monotone, that is, with* $\gamma>0$,

$$(a(x,V)-a(x,W),V-W)\geq\gamma|V-W|^p,$$

in the case $p\geq 2$, *or*

$$(a(x,V)-a(x,W),V-W)\geq\gamma\frac{|V-W|^2}{(|V|+|W|)^{2-p}},$$

in the case $1 < p < 2$, *then* (0.3) *and* (1.10) *to* (1.12) *have unique respective solutions* $u_n = (u_n^{(1)}, u_n^{(2)})$ *and* $(u, y) = ((u^{(1)}, u^{(2)}), (y^{(1)}, y^{(2)}))$. *Moreover we have strong convergences in* (1.4) *to* (1.9), *for the whole sequence* u_n.

Remark 1.3. The strong convergences in (1.5), (1.6) mean that $y^{(1)}$, $y^{(2)}$ are correctors for the limit problem in the same sense as in [16].

Remark 1.4. For the model problem,

$$a_0(x, s, \xi', \xi_N) = |s|^{p-2}s, \quad a'(x, s, \xi', \xi_N) = (|\xi'|^2 + \xi_N^2)^{\frac{p-2}{2}}\xi',$$

$$a_N(x, s, \xi', \xi_N) = (|\xi'|^2 + \xi_N^2)^{\frac{p-2}{2}}\xi_N,$$

so that (1.10), (1.11) reduce to $y^{(1)} = 0$, $y^{(2)} = 0$ and the limit problem reads $(u^{(1)}, u^{(2)}) \in K$ and for any $(v^{(1)}, v^{(2)}) \in K$,

$$|\omega| \int_0^1 \Big[\Big|\frac{du^{(1)}}{dx_N}\Big|^{p-2} \frac{du^{(1)}}{dx_N} \frac{dv^{(1)}}{dx_N} + \Big(|u^{(1)}|^{p-2} u^{(1)} - \frac{1}{|\omega|} \int_\omega f dx' \Big) v^{(1)} \Big] dx \quad (1.13)$$

$$+ q \int_\omega \Big[|\nabla' u^{(2)}|^{p-2} \Big(\nabla' u^{(2)}, \nabla' v^{(2)} \Big) + \Big(|u^{(2)}|^{p-2} u^{(2)} - \int_{-1}^0 f dx_N \Big) v^{(2)} \Big] dx' = 0.$$

We notice that (1.13) has a unique solution $(u^{(1)}, u^{(2)}) \in K$ and it follows from Theorem 1.2 that, for the whole sequence n,

$$u_n^{(i)} \to u^{(i)} \text{ strongly in } W^{1,p}(\Omega^{(i)}) \ (i = 1, 2),$$

$$\frac{1}{r_n} \nabla' u_n^{(1)} \to 0 \text{ strongly in } \big(L^p(\Omega^{(1)})\big)^{N-1},$$

$$\frac{1}{h_n} \frac{\partial u_n^{(2)}}{\partial x_N} \to 0 \text{ strongly in } L^p(\Omega^{(2)}).$$

It is interesting to remark that the limit problem obtained for (0.1) is the same as for the model problem considered in [11], whose energy is

$$\frac{1}{p} \int_{\Omega_n} \Big(|\nabla' V|^p + \Big| \frac{\partial V}{\partial X_N} \Big|^p + |V|^p \Big) dX - \int_{\Omega_n} FV \, dX,$$

instead of

$$\frac{1}{p} \int_{\Omega_n} (|\nabla V|^p + |V|^p) \, dX - \int_{\Omega_n} FV \, dX.$$

2. Proof of Theorem 1.1

Before proving the main result, we must observe that (0.3) admits at least one solution. Indeed K_n, defined in (0.2), is a closed subspace of $W = W^{1,p}(\Omega^{(1)}) \times W^{1,p}(\Omega^{(2)})$ and the operator $A_n : W \to W'$ given by

$$< A_n(u), v > := \int_{\Omega^{(1)}} \left(a(x, B_n^{(1)} u^{(1)}), B_n^{(1)} v^{(1)}\right) dx + q_n \int_{\Omega^{(2)}} \left(a(x, B_n^{(2)} u^{(2)}), B_n^{(2)} v^{(2)}\right) dx$$

is monotone, bounded, continuous from strong-W to weak-W' and coercive, under the hypotheses we are considering. Then, defining $A'_n : K_n \to K'_n$ by $A'_n = A_{n|K_n}$, it is clear that A'_n enjoys the same properties as A_n. This implies (see [1], [4] or [15]) that $A'_n(K_n) = K'_n$. In particular, for any $f \in L^{p'}(\Omega)$, one has $(f_{|\Omega^{(1)}}, f_{|\Omega^{(2)}}) \in L^{p'}(\Omega^{(1)}) \times L^{p'}(\Omega^{(2)}) \subset W' \subset K'_n$ and there exists $u_n \in K_n$ such that, for any $v \in K_n$,

$$< A_n(u_n), v >= \int_{\Omega^{(1)}} f v^{(1)}\, dx + \int_{\Omega^{(2)}} f v^{(2)}\, dx.$$

Now we turn to the proof of Theorem 1.1. As usual, we begin with **a priori estimates**.

Lemma 2.1. *For any sequence of solutions u_n of (0.3), for $i = 1, 2$, $\{u_n^{(i)}\}_n$ is bounded in $L^p(\Omega^{(i)})$ and $\{\nabla_n^{(i)}(u_n^{(i)})\}_n$ is bounded in $\left(L^p(\Omega^{(i)})\right)^N$.*

Proof. In the sequel, C will denote any positive constant, independent of n. Consider $v = u_n$, as test function in (0.3). We obtain

$$\begin{aligned} < A_n(u_n), u_n > :=& \int_{\Omega^{(1)}} \left(a(x, B_n^{(1)} u_n^{(1)}), B_n^{(1)} u_n^{(1)}\right) dx \\ &+ q_n \int_{\Omega^{(2)}} \left(a(x, B_n^{(2)} u_n^{(2)}), B_n^{(2)} u_n^{(2)}\right) dx \qquad (2.1)\\ =& \int_{\Omega^{(1)}} f_n u_n^{(1)} dx + q_n \int_{\Omega^{(2)}} f_n u_n^{(2)} dx. \end{aligned}$$

As $\{q_n\}_n$ is bounded away from ∞ and $\{f_n\}$ is bounded in $L^{p'}(\Omega)$, the right member may be estimated from above,

$$\int_{\Omega^{(1)}} f_n u_n^{(1)} dx + q_n \int_{\Omega^{(2)}} f_n u_n^{(2)} dx \leq C(\|u_n^{(1)}\|_{L^p(\Omega^{(1)})} + \|u_n^{(2)}\|_{L^p(\Omega^{(2)})}). \quad (2.2)$$

As $\{q_n\}_n$ is bounded away from 0 and by using the coercivity condition on a, the left member in (2.1) is estimated from below:

$$< A_n(u_n), u_n > \geq C\Big(\int_{\Omega^{(1)}} \Big(|\nabla_n^{(1)} u_n^{(1)}|^p + |u_n^{(1)}|^p\Big) dx + \int_{\Omega^{(2)}} \Big(|\nabla_n^{(2)} u_n^{(2)}|^p + |u_n^{(2)}|^p\Big) dx\Big) - C. \qquad (2.3)$$

From (2.1), (2.2) and (2.3) one easily obtains the claims of the lemma, by using the fact that the second members in (2.2) and (2.3) have different homogeneities.

The following proposition was proved in [11] (see Proposition 2.1 in [11]):

Proposition 2.1. *Let h_n, r_n satisfy (0.4). Let $\{v_n^{(2)}\}_n$ be a bounded sequence in $L^p(\Omega^{(2)})$, such that $\{\nabla_n^{(2)} v_n^{(2)}\}_n$ is bounded in $\left(L^p(\Omega^{(2)})\right)^N$. Then, up to a subsequence,*

$$v_n^{(2)} \rightharpoonup v^{(2)} \quad \textit{weakly in} \quad W^{1,p}(\Omega^{(2)}),$$

for some $v^{(2)}$ which does not depend on x_N. Moreover, if $p > N-1$,

$$\lim_n \int_\omega v_n^{(2)}(r_n x', 0) dx' = |\omega| v^{(2)}(0').$$

As in [11], the following general **compactness** result follows from Proposition 2.1 and Poincaré-Wirtinger inequality.

Proposition 2.2. *Let h_n, r_n satisfy (0.4). Let $v_n = (v_n^{(1)}, v_n^{(2)})$ be elements in K_n such that, for $i = 1, 2$, $\{v_n^{(i)}\}_n$ is bounded in $L^p(\Omega^{(i)})$ and $\{\nabla_n^{(i)} v_n^{(i)}\}_n$ is bounded in $\left(L^p(\Omega^{(i)})\right)^N$. Then, up to a subsequence,*

$$v_n^{(i)} \rightharpoonup v^{(i)} \ \textit{weakly in} \ W^{1,p}(\Omega^{(i)}), \quad (i = 1, 2),$$

for some $v^{(1)}$ independent of x' and $v^{(2)}$ independent of x_N, and for some $z^{(1)}, z^{(2)}$,

$$\frac{1}{r_n}\Big(v_n^{(1)} - \frac{1}{|\omega|}\int_\omega v_n^{(1)} dx'\Big) \rightharpoonup z^{(1)} \qquad \textit{weakly in } L^p(0,1; W_m^{1,p}(\omega)),$$

$$\frac{1}{h_n}\Big(v_n^{(2)} - \int_{-1}^0 v_n^{(2)} dx_N\Big) \rightharpoonup z^{(2)} \qquad \textit{weakly in } L^p(\omega; W_m^{1,p}((-1,0))).$$

Moreover, if $p > N-1$,

$$v^{(1)}(0) = v^{(2)}(0').$$

Applying Proposition 2.2 to the solutions of (0.3), we obtain part of Theorem 1.1, namely we get $(u^{(1)}, u^{(2)})$ in K, $y^{(1)}$ in $L^p((0,1); W_m^{1,p}(\omega))$, $y^{(2)}$ in

$L^p(\omega; W_m^{1,p}((-1,0)))$ and the convergences (1.4) to (1.7). It remains to prove (1.8) and (1.9) and to prove that (u, y) solves the limit problem (1.10) to (1.12). This part of the proof of Theorem 1.1 proceeds with a suitable choice of **test functions** and with **monotonicity**. It differs from [11], except that it uses the same **density argument**.

As a consequence of Lemma 2.1 and the growth condition on a, the sequences $\{a(x, B_n^{(i)} u_n^{(i)})\}_n$ are bounded in $(L^{p'}(\Omega^{(i)}))^{N+1}$, so we may suppose that, for some η,

$$a(x, B_n^{(i)} u_n^{(i)}) \rightharpoonup \eta^{(i)} \text{ weakly in } \left(L^{p'}(\Omega^{(i)})\right)^{N+1}. \tag{2.4}$$

Let

$$\tilde{K} = \left\{ (v^{(1)}, v^{(2)}) \in W^{1,\infty}((0,1)) \times W^{1,\infty}(\omega),\ v^{(1)}(0) = v^{(2)}(0') \right\},$$

$$\tilde{Z} = C^1(\overline{\Omega}^{(1)}) \times C^1(\overline{\Omega}^{(2)}).$$

It was proved in [11] (see Proposition 3.1) that $\tilde{K}$ is dense in K in $W^{1,p}$-norm and $\tilde{Z}$ is dense in $Z = L^p(0,1; W^{1,p}(\omega)) \times L^p(\omega; W^{1,p}((-1,0)))$. So it is enough to prove that (1.10) to (1.12) are satisfied for any $(v, z) \in \tilde{K} \times \tilde{Z}$ (cf. Remark 1.1).

In order to do this, we consider in (0.3) two types of test functions.

The **first test function** is the following one. Let $v = (v^{(1)}, v^{(2)}) \in \tilde{K}$ and define $v_n = (v_n^{(1)}, v_n^{(2)}) \in K_n$ by

$$v_n^{(2)} = v^{(2)},$$

$$v_n^{(1)} = \begin{cases} v^{(1)}(x_N), & \text{if } x_N > \alpha_n, \\ v^{(1)}(\alpha_n)\frac{x_N}{\alpha_n} + v^{(2)}(r_n x')\frac{\alpha_n - x_N}{\alpha_n}, & \text{if } 0 < x_N < \alpha_n, \end{cases}$$

where $\alpha_n \in (0,1)$ will be chosen later on. We obtain

$$\begin{aligned} &\int_{x_N > \alpha_n} (a(x, B_n^{(1)} u_n^{(1)}), \{v^{(1)}, 0, \frac{\partial v^{(1)}}{\partial x_N}\}) dx + \int_{0 < x_N < \alpha_n} (a(x, B_n^{(1)} u_n^{(1)}), B_n^{(1)} v_n^{(1)}) dx \\ &+ q_n \int_{\Omega^{(2)}} \left(a(x, B_n^{(2)} u_n^{(2)}), \{v^{(2)}, \nabla' v^{(2)}, 0\}\right) dx \\ &= \int_{x_N > \alpha_n} f_n v^{(1)} dx + \int_{0 < x_N < \alpha_n} f_n v_n^{(1)} dx + q_n \int_{\Omega^{(2)}} f_n v^{(2)} dx. \end{aligned} \tag{2.5}$$

Now we compute and take the limits of each term in the left side of (2.5). From (2.4),

$$\begin{aligned}
&\int_{x_N>\alpha_n} (a(x, B_n^{(1)}u_n^{(1)}), \{v^{(1)}, 0, \frac{\partial v^{(1)}}{\partial x_N}\})\,dx \\
&= \int_{x_N>\alpha_n} a_0(x, B_n^{(1)}u_n^{(1)})v^{(1)}dx + \int_{x_N>\alpha_n} a_N(x, B_n^{(1)}u_n^{(1)})\frac{\partial v^{(1)}}{\partial x_N}dx \qquad (2.6)\\
&\to \int_{\Omega^{(1)}} (\eta_0^{(1)}v^{(1)} + \eta_N^{(1)}\frac{\partial v^{(1)}}{\partial x_N})\,dx,
\end{aligned}$$

as soon as α_n tends to zero. For the first integral on $0 < x_N < \alpha_n$, (2.4) and Hölder's inequality give

$$\Big|\int_{0<x_N<\alpha_n} (a(x, B_n^{(1)}u_n^{(1)}), B_n^{(1)}v_n^{(1)})\,dx\Big| \le C\|B_n^{(1)}v_n^{(1)}\|_{L^p(0<x_N<\alpha_n)}.$$

Computing $B_n^{(1)}v_n^{(1)}$ on $0 < x_N < \alpha_n$ gives

$$B_n^{(1)}v_n^{(1)} = \begin{pmatrix} v_n^{(1)} \\ \frac{\alpha_n - x_N}{\alpha_n}\nabla' v^{(2)}(r_n x') \\ \frac{1}{\alpha_n}(v^{(1)}(\alpha_n) - v^{(2)}(r_n x')) \end{pmatrix}.$$

We can now estimate

$$\begin{aligned}
&|B_n^{(1)}v_n^{(1)}| \le C + C|\nabla' v^{(2)}(r_n x')| + \frac{C}{\alpha_n}|v^{(1)}(\alpha_n) - v^{(2)}(r_n x')| \\
&\le C + \frac{C}{\alpha_n}\Big|v^{(1)}(0) + \int_0^{\alpha_N} \frac{\partial v^{(1)}}{\partial x_N}(x_N)dx_N - v^{(2)}(0') - \int_0^{r_n} \nabla' v^{(2)}(tx')\cdot x'dt\Big| \\
&= C + \frac{C}{\alpha_n}\Big|\int_0^{\alpha_N} \frac{\partial v^{(1)}}{\partial x_N}(x_N)dx_N - \int_0^{r_n} \nabla' v^{(2)}(tx')\cdot x'dt\Big| \\
&\le C + \frac{C}{\alpha_n}(\alpha_n + r_n) \le C,
\end{aligned}$$

as soon as $\{r_n/\alpha_n\}_n$ is bounded.

So if we choose α_n with this property, we obtain that

$$\int_{0<x_N<\alpha_n} \Big(a(x, B_n^{(1)}u_n^{(1)}), B_n^{(1)}v_n^{(1)}\Big)\,dx \to 0. \qquad (2.7)$$

For the last integral in the left side of (2.5), we have from (2.4)

$$\int_{\Omega^{(2)}} \Big(a(x, B_n^{(2)}u_n^{(2)}), \Big\{v^{(2)}, \nabla' v^{(2)}, 0\Big\}\Big)\,dx \qquad (2.8)$$

$$= \int_{\Omega^{(2)}} a_0(x, B_n^{(2)} u_n^{(2)}) v^{(2)} dx + \int_{\Omega^{(2)}} \left(a'(x, B_n^{(2)} u_n^{(2)}), \nabla' v^{(2)} \right) dx$$
$$\to \int_{\Omega^{(2)}} \eta_0^{(2)} v^{(2)} dx + \int_{\Omega^{(2)}} \left(\eta^{(2)\prime}, \nabla' v^{(2)} \right) dx.$$

The right member of (2.5) can be written

$$\int_{\Omega^{(1)}} f_n v^{(1)} dx + q_n \int_{\Omega^{(2)}} f_n v^{(2)} dx + \int_{0<x_N<\alpha_n} f_n (v_n^{(1)} - v^{(1)}) dx;$$

the last integral tends to zero and we have from (0.4) and (0.5)

$$\int_{\Omega^{(1)}} f_n v^{(1)} dx + q_n \int_{\Omega^{(2)}} f_n v^{(2)} dx \to \int_{\Omega^{(1)}} f v^{(1)} dx + q \int_{\Omega^{(2)}} f v^{(2)} dx. \quad (2.9)$$

By passing to the limit in (2.5) and using (2.6) to (2.9), we obtain

$$\int_{\Omega^{(1)}} (\eta_0^{(1)} v^{(1)} dx + \eta_N^{(1)} \frac{\partial v^{(1)}}{\partial x_N}) dx + q \int_{\Omega^{(2)}} \left[\eta_0^{(2)} v^{(2)} + \left(\eta^{(2)\prime}, \nabla' v^{(2)} \right) \right] dx$$
$$= \int_{\Omega^{(1)}} f v^{(1)} dx + q \int_{\Omega^{(2)}} f v^{(2)} dx, \quad (2.10)$$

which is true for all $v \in \tilde{K}$ and then, by density and continuity, for all $v \in K$, in particular for u.

Next we take a **second test function**. Given $z = (z^{(1)}, z^{(2)}) \in \tilde{Z}$, we define $v_n \in K_n$ by the following relations:

$$v_n^{(2)} = h_n z^{(2)},$$
$$v_n^{(1)} = \begin{cases} r_n z^{(1)}, & \text{if } x_N > \alpha_n, \\ r_n z^{(1)}(x', \alpha_n) \frac{x_N}{\alpha_n} + h_n z^{(2)}(r_n x', 0) \frac{\alpha_n - x_N}{\alpha_n}, & \text{if } 0 < x_N < \alpha_n. \end{cases}$$

Using v_n as test function in (0.3), we obtain

$$\int_{x_N>\alpha_n} (a(x, B_n^{(1)} u_n^{(1)}), B_n^{(1)} v_n^{(1)}) dx + \int_{0<x_N<\alpha_n} (a(x, B_n^{(1)} u_n^{(1)}), B_n^{(1)} v_n^{(1)}) dx$$
$$+ q_n \int_{\Omega^{(2)}} \left(a(x, B_n^{(2)} u_n^{(2)}), \left\{ h_n z^{(2)}, h_n \nabla' z^{(2)}, \frac{\partial z^{(2)}}{\partial x_N} \right\} \right) dx$$
$$= \int_{\Omega^{(1)}} f_n v_n^{(1)} dx + q_n \int_{\Omega^{(2)}} f_n v_n^{(2)} dx. \quad (2.11)$$

We remark that $v_n^{(i)} \to 0$ in $L^\infty(\Omega^{(i)})$. It follows that the right side of (2.11) tends to zero. It remains to pass to the limit in the integrals involving a.

On $x_N > \alpha_n$, we have from (2.4),

$$\int_{x_N>\alpha_n} \left(a(x, B_n^{(1)}u_n^{(1)}), B_n^{(1)}v_n^{(1)}\right) dx = r_n \int_{x_N>\alpha_n} a_0(x, B_n^{(1)}u_n^{(1)}) z^{(1)} dx$$
$$+ \int_{x_N>\alpha_n} \left(a'(x, B_n^{(1)}u_n^{(1)}), \nabla' z^{(1)}\right) dx + r_n \int_{x_N>\alpha_n} a_N(x, B_n^{(1)}u_n^{(1)}) \frac{\partial z^{(1)}}{\partial x_N} dx$$
$$\to \int_{\Omega^{(1)}} \left(\eta^{(1)'}, \nabla' z^{(1)}\right) dx. \tag{2.12}$$

Let us estimate the integral containing a on $0 < x_N < \alpha_n$. We have

$$B_n^{(1)}v_n^{(1)} = \begin{pmatrix} v_n^{(1)} \\ \nabla' z^{(1)}(x', \alpha_n)\frac{x_N}{\alpha_n} + h_n \nabla' z^{(2)}(r_n x', 0)\frac{\alpha_n - x_N}{\alpha_n} \\ \frac{1}{\alpha_n}\left[r_n z^{(1)}(x', \alpha_n) - h_n z^{(2)}(r_n x', 0)\right] \end{pmatrix}.$$

Now it is easy to see that, if we choose α_n such that $\{r_n/\alpha_n\}_n$ and $\{h_n/\alpha_n\}_n$ are bounded, then $|B_n^{(1)}v_n^{(1)}| \le C$ and from (2.4), if α_n tends to zero,

$$\left| \int_{0<x_N<\alpha_n} \left(a(x, B_n^{(1)}u_n^{(1)}), B_n^{(1)}v_n^{(1)}\right) dx \right| \le C \|B_n^{(1)}v_n^{(1)}\|_{L^p(0<x_N<\alpha_n)} \to 0. \tag{2.13}$$

For the integral involving a on $\Omega^{(2)}$, we have, using (2.4) again,

$$\int_{\Omega^{(2)}} \left(a(x, B_n^{(2)}u_n^{(2)}), \left\{h_n z^{(2)}, h_n \nabla' z^{(2)}, \frac{\partial z^{(2)}}{\partial x_N}\right\}\right) dx \tag{2.14}$$
$$= h_n \int_{\Omega^{(2)}} a_0(x, B_n^{(2)}u_n^{(2)}) z^{(2)} dx + h_n \int_{\Omega^{(2)}} \left(a'(x, B_n^{(2)}u_n^{(2)}), \nabla' z^{(2)}\right) dx$$
$$+ \int_{\Omega^{(2)}} a_N(x, B_n^{(2)}u_n^{(2)}) \frac{\partial z^{(2)}}{\partial x_N} dx \to \int_{\Omega^{(2)}} \eta_N^{(2)} \frac{\partial z^{(2)}}{\partial x_N} dx.$$

From (2.11) and (2.12) to (2.14), we obtain

$$\int_{\Omega^{(1)}} \left(\eta^{(1)'}, \nabla' z^{(1)}\right) dx + q \int_{\Omega^{(2)}} \eta_N^{(2)} \frac{\partial z^{(2)}}{\partial x_N} dx = 0, \tag{2.15}$$

which holds for any $z \in \tilde{Z}$ and, by density and continuity, for any $z \in Z$, in particular for y.

In the remainder of the proof, we identify the quantities $\eta^{(i)}$ which, together with (2.10) and (2.15), will give the desired result. For this we use the **monotonicity** condition.

Let $V^{(i)} = (v^{(i)}, \tau^{(i)}) = (v^{(i)}, \tau^{(i)\prime}, \tau_N^{(i)}) \in \left(L^p(\Omega^{(i)})\right)^{N+1}$. We have

$$\begin{aligned}
0 \leq & \int_{\Omega^{(1)}} \left(a(x, B_n^{(1)}u_n^{(1)}) - a(x, V^{(1)}), B_n^{(1)}u_n^{(1)} - V^{(1)}\right) dx \qquad (2.16)\\
& + q_n \int_{\Omega^{(2)}} \left(a(x, B_n^{(2)}u_n^{(2)}) - a(x, V^{(2)}), B_n^{(2)}u_n^{(2)} - V^{(2)}\right) dx\\
& = \int_{\Omega^{(1)}} \left(a(x, B_n^{(1)}u_n^{(1)}), B_n^{(1)}u_n^{(1)}\right) dx + q_n \int_{\Omega^{(2)}} \left(a(x, B_n^{(2)}u_n^{(2)}), B_n^{(2)}u_n^{(2)}\right) dx\\
& - \int_{\Omega^{(1)}} \left(a(x, B_n^{(1)}u_n^{(1)}), V^{(1)}\right) dx + \int_{\Omega^{(1)}} \left(a(x, V^{(1)}), V^{(1)} - B_n^{(1)}u_n^{(1)}\right) dx\\
& - q_n \int_{\Omega^{(2)}} \left(a(x, B_n^{(2)}u_n^{(2)}), V^{(2)}\right) dx + q_n \int_{\Omega^{(2)}} \left(a(x, V^{(2)}), V^{(2)} - B_n^{(2)}u_n^{(2)}\right) dx.
\end{aligned}$$

Using u_n as test function in (0.3), we obtain, by means of (2.10),

$$\begin{aligned}
& \int_{\Omega^{(1)}} \left(a(x, B_n^{(1)}u_n^{(1)}), B_n^{(1)}u_n^{(1)}\right) dx + q_n \int_{\Omega^{(2)}} \left(a(x, B_n^{(2)}u_n^{(2)}), B_n^{(2)}u_n^{(2)}\right) dx\\
& = \int_{\Omega^{(1)}} f_n u_n^{(1)} dx + q_n \int_{\Omega^{(2)}} f_n u_n^{(2)} dx \to \int_{\Omega^{(1)}} f u^{(1)} dx + q \int_{\Omega^{(2)}} f u^{(2)} dx \qquad (2.17)\\
& = \int_{\Omega^{(1)}} \left(\eta_0^{(1)} u^{(1)} + \eta_N^{(1)} \frac{\partial u^{(1)}}{\partial x_N}\right) dx + q \int_{\Omega^{(2)}} \left[\eta_0^{(2)} u^{(2)} + \left(\eta^{(2)\prime}, \nabla' u^{(2)}\right)\right] dx.
\end{aligned}$$

Now, with (2.17), using the convergences (1.4) to (1.7) as well as (2.4), we pass to the limit in (2.16) and, regrouping the terms, we obtain:

$$\begin{aligned}
0 \leq & \int_{\Omega^{(1)}} \left[\eta_0^{(1)}(u^{(1)} - v^{(1)}) + \eta_N^{(1)}\left(\frac{du^{(1)}}{dx_N} - \tau_N^{(1)}\right)\right] dx \qquad (2.18)\\
& + q \int_{\Omega^{(2)}} \left[\eta_0^{(2)}(u^{(2)} - v^{(2)}) + \left(\eta^{(2)\prime}, \nabla' u^{(2)} - \tau^{(2)\prime}\right)\right] dx\\
& + \int_{\Omega^{(1)}} \left(a(x, V^{(1)}), \left\{v^{(1)} - u^{(1)}, \tau^{(1)\prime} - \nabla' y^{(1)}, \tau_N^{(1)} - \frac{du^{(1)}}{dx_N}\right\}\right) dx\\
& + q \int_{\Omega^{(2)}} \left(a(x, V^{(2)}), \left\{v^{(2)} - u^{(2)}, \tau^{(2)\prime} - \nabla' u^{(2)}, \tau_N^{(2)} - \frac{\partial y^{(2)}}{\partial x_N}\right\}\right) dx\\
& - \int_{\Omega^{(1)}} \left(\eta^{(1)\prime}, \tau^{(1)\prime}\right) dx - q \int_{\Omega^{(2)}} \eta_N^{(2)} \tau_N^{(2)} dx.
\end{aligned}$$

By taking $v = u$, $\tau^{(1)'} = \nabla' y^{(1)}$, $\tau_N^{(2)} = \frac{\partial y^{(2)}}{\partial x_N}$ in (2.18) and by using (2.15), we get

$$0 \leq \int_{\Omega^{(1)}} \left(\eta_N^{(1)} - a_N(x, u^{(1)}, \nabla' y^{(1)}, \tau_N^{(1)})\right) \left(\frac{du^{(1)}}{dx_N} - \tau_N^{(1)}\right) dx \tag{2.19}$$
$$+ q \int_{\Omega^{(2)}} \left(\eta^{(2)'} - a'(x, u^{(2)}, \tau^{(2)'}, \frac{\partial y^{(2)}}{\partial x_N}), \nabla' u^{(2)} - \tau^{(2)'}\right) dx.$$

In (2.19) we choose $\tau_N^{(1)} = \frac{du^{(1)}}{dx_N} - t\phi$, $\tau^{(2)'} = \nabla' u^{(2)} - t\psi$ with t positive and $\phi \in L^p(\Omega^{(1)})$, $\psi \in \left(L^p(\Omega^{(2)})\right)^{N-1}$, we divide by t and we pass to the limit for $t \to 0$. It follows that

$$0 \leq \int_{\Omega^{(1)}} \left(\eta_N^{(1)} - a_N(x, u^{(1)}, \nabla' y^{(1)}, \frac{du^{(1)}}{dx_N})\right) \phi \, dx \tag{2.20}$$
$$+ q \int_{\Omega^{(2)}} \left(\eta^{(2)'} - a'(x, u^{(2)}, \nabla' u^{(2)}, \frac{\partial y^{(2)}}{\partial x_N})\right) \psi \, dx.$$

As ϕ and ψ may be arbitrarily chosen, we find that

$$\eta_N^{(1)} = a_N(x, u^{(1)}, \nabla' y^{(1)}, \frac{du^{(1)}}{dx_N}), \quad \eta^{(2)'} = a'(x, u^{(2)}, \nabla' u^{(2)}, \frac{\partial y^{(2)}}{\partial x_N}). \tag{2.21}$$

In the same way, in (2.18) we take $v = u$, $\tau^{(1)'} = \nabla' y^{(1)} - t\phi$, with ϕ in $\left(L^p(\Omega^{(1)})\right)^{N-1}$, $\tau_N^{(1)} = \frac{du^{(1)}}{dx_N}$, $\tau^{(2)'} = \nabla' u^{(2)}$, $\tau_N^{(2)} = \frac{\partial y^{(2)}}{\partial x_N} - t\psi$, with ψ in $L^p(\Omega^{(2)})$, we divide by t and we pass to the limit for $t \to 0$, using (2.15) again, and obtain

$$0 \leq -\int_{\Omega^{(1)}} \left(a'(x, u^{(1)}, \nabla' y^{(1)}, \frac{du^{(1)}}{dx_N}) - \eta^{(1)'}, \phi\right) dx$$
$$- q \int_{\Omega^{(2)}} \left(a_N(x, u^{(2)}, \nabla' u^{(2)}, \frac{\partial y^{(2)}}{\partial x_N}) - \eta_N^{(2)}\right) \psi dx,$$

which gives

$$\eta^{(1)'} = a'(x, u^{(1)}, \nabla' y^{(1)}, \frac{du^{(1)}}{dx_N}), \quad \eta_N^{(2)} = a_N(x, u^{(2)}, \nabla' u^{(2)}, \frac{\partial y^{(2)}}{\partial x_N}). \tag{2.22}$$

The last identification we need is obtained by taking, in (2.18), $\tau^{(1)'} = \nabla' y^{(1)}, \tau_N^{(1)} = \frac{\partial u^{(1)}}{\partial x_N}, \tau^{(2)'} = \nabla' u^{(2)}, \tau_N^{(2)} = \frac{\partial y^{(2)}}{\partial x_N}$, giving

$$\eta_0^{(1)} = a_0(x, u^{(1)}, \nabla' y^{(1)}, \frac{du^{(1)}}{dx_N}), \quad \eta_0^{(2)} = a_0(x, u^{(2)}, \nabla' u^{(2)}, \frac{\partial y^{(2)}}{\partial x_N}). \tag{2.23}$$

From (2.4), (2.10), (2.15), (2.21) to (2.23), we finish the proof of the theorem, that is we obtain (1.8), (1.9) and (1.10), (1.11), (1.12).

3. Proof of Theorem 1.2

Assuming that a is strongly monotone, we first prove strong convergences in (1.4) and (1.7). Then the strong convergences in (1.8) and (1.9) follow immediately, as well as the ones in (1.5) and (1.6) since, by Poincaré-Wirtinger inequality,

$$\left\| \frac{1}{r_n}\Big(u_n^{(1)} - \frac{1}{|\omega|}\int_\omega u_n^{(1)}dx'\Big) - y^{(1)} \right\|_{L^p(\Omega^{(1)})} \leq \left\| \frac{1}{r_n}\nabla' u_n^{(1)} - \nabla' y^{(1)} \right\|_{(L^p(\Omega^{(1)}))^{N-1}},$$

$$\left\| \frac{1}{h_n}\Big(u_n^{(2)} - \int_{-1}^0 u_n^{(2)}dx_N\Big) - y^{(2)} \right\|_{L^p(\Omega^{(2)})} \leq \left\| \frac{1}{h_n}\frac{\partial u_n^{(2)}}{\partial x_N} - \frac{\partial y^{(2)}}{\partial x_N} \right\|_{L^p(\Omega^{(2)})}.$$

Let us begin with $p \geq 2$. Using the strong monotonicity condition in (2.16), we obtain, after passing to the limit, that

$$\begin{aligned} 0 \leq C \limsup \Big(&\|\nabla_n^{(1)} u_n^{(1)} - \tau^{(1)}\|_{L^p}^p + q\|\nabla_n^{(2)} u_n^{(2)} - \tau^{(2)}\|_{L^p}^p \\ &+ \|u_n^{(1)} - v^{(1)}\|_{L^p}^p + q\|u_n^{(2)} - v^{(2)}\|_{L^p}^p \Big) \leq \text{ second member of (2.18).} \end{aligned} \tag{3.1}$$

Now choosing $u = v$, $\tau_N^{(1)} = \frac{du^{(1)}}{dx_N}$, $\tau^{(2)\prime} = \nabla' u^{(2)}$, $\tau^{(1)\prime} = \nabla' y^{(1)}$, $\tau_N^{(2)} = \frac{dy^{(2)}}{dx_N}$, and taking (2.15) into account, we immediately obtain that the convergences (1.4) and (1.7) are strong.

The case $p < 2$ is very similar, except that, in the first line of (3.1), each term of the form $\|A_n - B\|_{L^p}^p$ is replaced by

$$\int \frac{|A_n - B|^2}{(|A_n| + |B|)^{2-p}}\, dx,$$

tending to zero as before. But, from Hölder inequality,

$$\int |A_n - B|^p\, dx \leq \Big(\int \frac{|A_n - B|^2}{(|A_n| + |B|)^{2-p}}\, dx \Big)^{\frac{p}{2}} \Big(\int (|A_n| + |B|)^p\, dx \Big)^{\frac{2-p}{2}},$$

where the last integral is bounded independently of n. Hence $\|A_n - B\|_{L^p}^p$ tends to zero again.

REFERENCES

[1] V. Barbu, "Convexity and Optimization in Banach Spaces," 2nd edition, Kluwer Academic Publishers, (1986).

[2] M. Boutkrida, Thesis, E.N.S.-Cachan, (1999).

[3] M. Boutkrida, J. Mossino, *Un problème limite dégénéré pour un multidomaine localement mince*, C.R. Acad. Sci. Paris, 330, Série I (2000), 55-60.

[4] H. Brezis, "Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert," North-Holland Publishing Co., (1973).

[5] P.G. Ciarlet, "Plates and Junctions in Elastic Multi-Structures: An Asymptotic Analysis," Masson, Paris, (1990).

[6] P.G. Ciarlet, "Mathematical Elasticity, vol. II: Theory of Plates," Nort-Holland, Amsterdam, (1997).

[7] P.G. Ciarlet and P. Destuynder, *A Justification of the Two-Dimensional Linear Plate Model*, J. Mécanique, 18 (1979), 315-344.

[8] D. Cioranescu and J. Saint Jean Paulin, "Homogenization of Reticulated Structures", Appl. Math. Sc., 139, Springer-Verlag, New-York, (1999).

[9] D. Blanchard, L. Carbone, and A. Gaudiello, *Homogenization of a monotone problem in a domain with oscillating boundary*, M2AN, 33 (1999), 1057-1070.

[10] A. Gaudiello, B. Gustafsson, C. Lefter, and J. Mossino, *Coupled and Uncoupled Limits for a N-Dimensional Multidomain Neumann Problem*, C.R. Acad. Sci. Paris, 330, Série I, (2000), 985-990.

[11] A. Gaudiello, B. Gustafsson, C. Lefter, and J. Mossino, *Asymtotic analysis of a class of minimization problems in a thin multidomain*, Calculus of Variations and Partial Diff. Eq., to appear.

[12] H. Le Dret, "Problèmes Variationnels dans les Multi-Domaines, modélisation des jonctions et applications" Masson, Paris, (1991).

[13] H. Le Dret and A. Raoult, *The Nonlinear Membrane Model as Variational Limit of Nonlinear Three-Dimensional Elasticity*, J. Math. Pures Appl. **74**, (1995), 549-578.

[14] H. Le Dret and A. Raoult, *The Membrane Shell Model in Nonlinear Elasticity, a Variational Asymptotic Derivation*, J. Nonlinear Sc., 6 (1996), 58-84.

[15] J.L. Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires", Dunod, Paris, (1969).

[16] F. Murat, A. Sili, *Problèmes monotones dans des cylindres de faible diamètre formés de matériaux hétérogènes*, C.R. Acad. Sci. Paris, 320, Série I (1995), 1199-1204.

[17] L. Tartar, Cours Peccot, Collège de France (1977), partially written in F. Murat, *H-convergence*, Séminaire d'analyse fonctionelle et numérique de l'Université d'Alger (1977-78), english translation in "Mathematical Modelling of Composite Materials", A. Cherkaev and R.V. Kohn editors, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser-Verlag (1997), 21-44.