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Stratified materials allowing asymptotically prescribed equipotentials

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1. Introduction

Let us consider the sequence of minimization problems:

$$P(a_n): \inf\left\{\frac{1}{p}\int_\Omega a_n^{p-1}|\nabla v|^p\,dx - \int_\Omega fv\,dx\,;\, v\in W_L^{1,p}(\Omega)\right\},$$

where Ω is a bounded domain in $\mathbf{R}^N$, $1<p<\infty$, L stands for suitable Dirichlet boundary conditions and, for each $n\in\mathbf{N}$, $0<a_n\in L^\infty(\Omega)$. In the applications we have in mind Ω is a conductor, the a_n represent rapidly oscillating (thermic or electric) conductivity coefficients and we are interested in the possible convergence, as $n\to\infty$, of the problems $P(a_n)$ to some "homogenized" limit problem when a_n converges to some $a\in L^\infty(\Omega)$ in a suitable sense.

In the case $N=1$ it is well known (cf. [12] for $p=2$) that if $1/a_n, 1/a$ and a are in $L^\infty(\Omega)$ and

$$(H) \qquad \frac{1}{a_n}\to\frac{1}{a} \quad \text{in } w^*\text{-}L^\infty(\Omega),$$

then the solution u_n of $P(a_n)$ converges in w-$W^{1,p}(\Omega)$ to the solution u of $P(a)$ and

$$\int_\Omega a_n^{p-1}|u_n'|^p\,dx \to \int_\Omega a^{p-1}|u'|^p\,dx,$$

which is to say that $P(a_n)$ converges to $P(a)$.

In the case $N>1$ the situation is more complicated and the hypothesis (H) by no means implies that $P(a_n)$ converges to $P(a)$. In general not very much can be said, as far as we know, but if the a_n happen to depend on only one variable, say x_1, then it is known (cf. [11], [13] for $p=2$ and [9] for more general situations) that if (H) holds and moreover $a_n^{p-1}\to(a^*)^{p-1}$ in w^*-$L^\infty(\Omega)$ for some $a^*\in L^\infty(\Omega)$, then $P(a_n)$ (with $|\nabla v|^p=\sum_{i=1}^N|\partial v/\partial x_i|^p$) converges to the problem

$$\inf\left\{\frac{1}{p}\int_\Omega a^{p-1}\left|\frac{\partial v}{\partial x_1}\right|^p dx+\frac{1}{p}\int_\Omega (a^*)^{p-1}\sum_{i=2}^N\left|\frac{\partial v}{\partial x_i}\right|^p dx-\int_\Omega fv\,dx\,;v\in W_L^{1,p}(\Omega)\right\}.$$

The present paper is a natural sequel to [7], [8] and is concerned with a kind of singular version of the above, namely corresponding to the case $a^*=+\infty$. Let ϕ be a given smooth function on $\bar\Omega$ and assume that the a_n depend only on $t=\phi(x)$, so that $a_n(x)=\mathbf{a}_n(t)$ say. Assume also that (H) holds. In [7], [8] we proved that if, in addition to the above, Ω contains an increasing (as $n\to\infty$) number of leaves of perfect conductors which are uniformly distributed level surfaces of ϕ (this corresponds to having the additional constraint "$v=$ constant on each leaf" in $P(a_n)$) then $P(a_n)$ converges to a limit problem P whose admissible functions are constant on each level surface of ϕ. In practice P then is a one-dimensional problem.

In this paper we obtain the same conclusion under more relaxed conditions, namely with the leaves of perfect conductors replaced by the assumption that a_n is very large along many of the level surfaces of ϕ. Precisely, the right condition on a_n turns out to be that

$$(H')\qquad \int_I \mathbf{a}_n^{p-1}(t)\,dt\to+\infty\quad\text{as } n\to\infty$$

for every interval I of positive length. Thus, if (H) and (H') hold, then $P(a_n)$ converges to the same homogenized limit problem P as before, the solution of which is constant on all the level surfaces of the prescribed function ϕ.

This is our main result. It contains as special cases earlier results in e.g. [4] concerning periodical reinforced structures. A typical example is when $a_n=a$ (independent of n) except for an increasing number of thin layers of very high conductivity. If there are n uniformly distributed layers of thickness $\varepsilon=\varepsilon_n$ and conductivity $\lambda=\lambda_n$ then $(H),(H')$ hold if

$$n\varepsilon\to0 \text{ and } n\varepsilon\lambda^{p-1}\to\infty\quad\text{as } n\to\infty.$$

In the body of the paper we actually work with more general problems than $P(a_n)$, namely

$$(P_n)\qquad \inf\left\{\int_\Omega\frac{1}{a_n}G_n(x,a_n\nabla v)\,dx-\int_\Omega f_n v\,dx\,;v\in W_L^{1,p}(\Omega)\right\},$$

where the functions $G_n(x,z)$ satisfy certain natural conditions, e.g. $G_n(x,z)=|z|^p/p$, where $|\cdot|$ is either the euclidean norm ($|z|^p=(\sum_{i=1}^N |z_i|^2)^{p/2}$) or the l^p-norm ($|z|^p=\sum_{i=1}^N |z_i|^p$). Note that problem P_n is equivalent to the weak formulation of the quasilinear boundary value problem

$$\begin{cases} -\operatorname{div} g_n(x, a_n \nabla u_n) = f_n & \text{in } \Omega, \\ u_n \in W_L^{1,p}(\Omega), \end{cases}$$

where g_n is the gradient of G_n.

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2. Statement of the main result

We shall work with domains Ω of annulus (or shell) type (cf. however §4). Let $\Omega=\Omega_0\setminus\bar{\Omega}_1$ where Ω_0 and Ω_1 are bounded domains in $\mathbf{R}^N$, $(N\geq 2)$ with smooth boundaries and satisfying $\Omega_0\supset\bar{\Omega}_1$. Let $\phi\in C^1(\bar{\Omega},\mathbf{R})$ satisfy $\phi=0$ on $\partial\Omega_0$, $\phi=1$ on $\partial\Omega_1$ and $\nabla\phi\neq 0$ on $\bar{\Omega}$. It then follows that $0<\phi<1$ in Ω; the condition $\nabla\phi\neq 0$ also imposes topological restrictions on Ω. The geometry we think of is that with Ω_0 and Ω_1 homeomorphic to balls, but the above assumptions also allow Ω_0 and Ω_1 to be e.g. nested tori.

Let us consider the following sequence of minimization problems

$$(P_n) \qquad \inf\left\{\int_\Omega \frac{1}{a_n} G_n(x, a_n\nabla v)\,dx - \int_\Omega f_n v\,dx\,;\, v\in W_L^{1,p}(\Omega)\right\}$$

where

- $a_n\in L^\infty(\Omega)$, $a_n(x)\geq c>0$ for every $n\in\mathbf{N}$ and a.e. $x\in\Omega$,
- $W_L^{1,p}(\Omega)=\{v\in W^{1,p}(\Omega); v=0 \text{ on } \partial\Omega_0, v=1 \text{ on } \partial\Omega_1\}$, $(1<p<\infty)$,
- $f_n\in L^{p'}(\Omega)$, $1/p+1/p'=1$, $f_n\to f$ in w-$L^{p'}(\Omega)$,
- G_n are standard functions in the calculus of variations, that is:
  - $G_n\colon(x,z)\in\Omega\times\mathbf{R}^N\to G_n(x,z)\in\mathbf{R}$ is a Carathéodory function (that is, measurable with respect to x, continuous with respect to z)
  - for every $n\in\mathbf{N}$, for almost every $x\in\Omega$, $G_n(x,\cdot)$ is a strictly convex function which admits a gradient denoted by $g_n(x,\cdot)$,
  - there exist constants $c_1,c_2,c_4>0$ and $c_3\in L^1(\Omega)$ such that, for every $n\in\mathbf{N}$, for almost every $x\in\Omega$ and for every $z\in\mathbf{R}^N$,

$$(1) \qquad c_1|z|^p\leq G_n(x,z)\leq c_2|z|^p+c_3(x);$$

$$|g_n(x,z)| \le c_4(1+|z|^{p-1}). \tag{2}$$

- There exists G satisfying the same properties as G_n, such that for almost every $x\in\Omega$ and for every $z\in\mathbf{R}^N$,

$$G_n(x,z)\to G(x,z) \quad \text{as } n\to\infty, \tag{3}$$

$$g_n(x,z)\to g(x,z) \quad \text{as } n\to\infty. \tag{4}$$

Clearly (cf. [10]), problem (P_n) admits a unique solution u_n, and u_n is also the unique weak solution of

$$\begin{cases} -\operatorname{div}\, g_n(x,a_n\nabla u_n)=f_n & \text{in } \Omega,\\ u_n\in W_L^{1,p}(\Omega). \end{cases}$$

Theorem. *We assume that* (a_n) *satisfies the following hypothesis*:

$$a_n=\mathbf{a}_n\circ\phi \text{ with } \mathbf{a}_n\in L^\infty(0,1) \text{ and } \exists c>0:\forall n\in\mathbf{N},\ a.e.\ t\in]0,1[,\ c\le\mathbf{a}_n(t), \tag{5}$$

$$\exists \mathbf{a}\in L^\infty(0,1)\colon \frac{1}{\mathbf{a}_n}\to\frac{1}{\mathbf{a}} \text{ weakly}^* \text{ in } L^\infty(0,1) \text{ as } n\to\infty, \tag{6}$$

$$\text{for every non degenerate interval } I\subset[0,1],\ \int_I \mathbf{a}_n^{p-1}(t)\,dt\to+\infty. \tag{7}$$

Then, as $n\to\infty$, *the solution* u_n *of* (P_n) *converges weakly in* $W^{1,p}(\Omega)$ *to the solution* u *of*

$$\inf\left\{\int_\Omega \frac{1}{a}G(x,a\nabla v)\,dx-\int_\Omega fv\,dx\,;\,v=\mathbf{v}\circ\phi,\mathbf{v}\in W_L^{1,p}(0,1)\right\}, \tag{P}$$

where $a=\mathbf{a}\circ\phi$ *and* $W_L^{1,p}(0,1)=\{\mathbf{v}\in W^{1,p}(0,1);\mathbf{v}(0)=0,\mathbf{v}(1)=1\}$. *Moreover*

$$\int_\Omega \frac{1}{a_n}G_n(x,a_n\nabla u_n)\,dx\to\int_\Omega\frac{1}{a}G(x,a\nabla u)\,dx, \tag{8}$$

which is to say that the infimum of (P_n) *converges to the infimum of* (P).

Remark 1. The assumptions in the theorem are actually slightly excessive. In (5) we could allow $c>0$ to depend on n. This would still guarantee that $1/\mathbf{a}_n\in$

$L^\infty(0,1)$ and then it would follow from (6) and the uniform boundedness principle that c actually could be taken independent of n.

Conversely, with (5) as it is, (6) could be replaced by the weaker condition that

$$\int_I \frac{dt}{\mathbf{a}_n(t)} \to \int_I \frac{dt}{\mathbf{a}(t)}$$

for every interval $I\subset[0,1]$ (making (6) more similar to (7)). In fact, by (5) the sequence $(1/\mathbf{a}_n)$ is bounded in $L^\infty(0,1)$ and then it is enough to have the convergence

$$\int_0^1 \frac{1}{\mathbf{a}_n(t)}\psi(t)\,dt \to \int_0^1 \frac{1}{\mathbf{a}(t)}\psi(t)\,dt$$

for a dense set of functions $\psi\in L^1(0,1)$, e.g. for all step functions.

Remark 2. The limit problem (P) of (P_n) is the same as that obtained for a foliated material with leaves of a perfect conductor in [8] and by Lemma 2.2 of [8], (P) can also be formulated

(P)

$$\inf\left\{\int_\Omega \frac{1}{a}G(x,a\nabla v)\,dx-\int_\Omega fv\,dx\,;v\in W_L^{1,p}(\Omega),\forall t\in]0,1[,v=\text{ constant on }\Gamma_t\right\},$$

where Γ_t is the level surface $\{\phi=t\}$.

Actually (P) is a one dimensional problem (cf. [8], §3.b, where the coarea formula of [6] is used). More precisely, let

- $\mathbf{G}(t,z)=\displaystyle\int_{\Gamma_t}\frac{G(x,z\nabla\phi)}{|\nabla\phi|}\,d\gamma,$
- $f(t)=\displaystyle\int_{\Gamma_t}\frac{f}{|\nabla\phi|}\,d\gamma,$
- $(\mathbf{P}):\inf\left\{\displaystyle\int_0^1\frac{1}{\mathbf{a}}\mathbf{G}(t,\mathbf{a}\mathbf{v}')\,dt-\int_0^1\mathbf{f}\mathbf{v}\,dt\,;\mathbf{v}\in W^{1,p}L(0,1)\right\},$
- $\mathbf{u}$ the solution of $(\mathbf{P})$.

Then $u=\mathbf{u}\circ\phi$ and

$$\int_\Omega \frac{1}{a}G(x,a\nabla u)\,dx-\int_\Omega fu\,dx=\int_0^1\frac{1}{\mathbf{a}}\mathbf{G}(t,\mathbf{a}\mathbf{u}')\,dt-\int_0^1\mathbf{f}\mathbf{u}\,dt.$$

Remark 3. In [9] we investigate a case when $\int_I \mathbf{a}_n^{p-1}(t)dt$ is bounded; more precisely we determine the limit problem of (P_n) assuming $\int_I \mathbf{a}_n^{p-1}(t)dt\to\int_I a^{*p-1}(t)dt$ where $a^*\in L^\infty(0,1)$ instead of hypothesis (7).

Example. Stratified annulus containing numerous thin layers of very high conductivity which are uniformly distributed in Ω.

For each $n\in\mathbf{N}$, let $T_n=\{t_{i,n};0\le i\le n\}$ where $(t_{i,n})_i$ is a sequence of points in $[0,1]$ such that $0=t_{0,n}<t_{1,n}<...<t_{n,n}=1$. Let $\varepsilon=\varepsilon_n$ such that

$$0<\varepsilon<\tfrac{1}{2}\min\{t_{i,n}-t_{i-1,n}\,;1\le i\le n\}$$

and let $\Sigma_{i,n,\varepsilon}$ be the layer located between the two level surfaces of ϕ of values $t_{i,n}-\varepsilon$ and $t_{i,n}+\varepsilon$, that is

$$\Sigma_{i,n,\varepsilon}=\{x\in\Omega\,;t_{i,n}-\varepsilon<\phi(x)<t_{i,n}+\varepsilon\},\ 1\le i\le n-1 \quad \text{and} \quad \text{set } \Sigma_{n,\varepsilon}=\bigcup\Sigma_{i,n,\varepsilon}.$$

Let us suppose that this stratified annulus Ω (which contains the $n-1$ thin layers $\Sigma_{i,n,\varepsilon}$) has a conductivity coefficient a_n such that

$$a_n=\begin{cases} b_n & \text{in } \Omega\backslash\Sigma_{n,\varepsilon},\\ \lambda_n & \text{in } \Sigma_{n,\varepsilon},\end{cases}$$

where $b_n=\mathbf{b}_n\circ\phi$, $\mathbf{b}_n\in L^\infty(0,1)$, $\lambda_n=\boldsymbol{\lambda}_n\circ\phi$, $\boldsymbol{\lambda}_n\in L^\infty(0,1)$, $\boldsymbol{\lambda}_n(t)\ge\Lambda_n>0$ and $\Lambda_n\to\infty$ as $n\to\infty$.

The problem (P_n) can be written

$$\inf\left\{\int_{\Omega\backslash\Sigma_{n,\varepsilon}}\frac{1}{b_n}G_n(x,b_n\nabla v)\,dx+\int_{\Sigma_{n,\varepsilon}}\frac{1}{\lambda_n}G_n(x,\lambda_n\nabla v)\,dx-\int_\Omega f_n v\,dx\,;v\in W_L^{1,p}(\Omega)\right\}.$$

Corollary. *Let us assume that*

- $\exists c>0{:}\forall n\in\mathbf{N}$, *a.e.* $t\in]0,1[$, $c\le\mathbf{b}_n(t)$,
- $\exists\mathbf{b}\in L^\infty(0,1){:}1/\mathbf{b}_n\to1/\mathbf{b}$ *weakly* in* $L^\infty(0,1)$ *as* $n\to\infty$,
- $\exists\beta>0{:}\forall n\in\mathbf{N}$, $\forall 1\le i\le n$, $t_{i,n}-t_{i-1,n}\le\beta/n$,
- $n\varepsilon\to0$ *and* $n\varepsilon\Lambda_n^{p-1}\to\infty$ *as* $n\to\infty$.

Then, the solution u_n *of* (P_n) *converges weakly in* $W^{1,p}(\Omega)$ *to the solution* u *of*

$$\inf\left\{\int_\Omega\frac{1}{b}G(x,b\nabla v)\,dx-\int_\Omega fv\,dx\,;v=\mathbf{v}\circ\phi,\mathbf{v}\in W_L^{1,p}(0,1)\right\}$$

where $b=\mathbf{b}\circ\phi$. *Moreover,*

$$\int_{\Omega\backslash\Sigma_{n,\varepsilon}}\frac{1}{b_n}G_n(x,b_n\nabla u_n)\,dx+\int_{\Sigma_{n,\varepsilon}}\frac{1}{\lambda_n}G_n(x,\lambda_n\nabla u_n)\,dx\to\int_\Omega\frac{1}{b}G(x,b\nabla u)\,dx.$$

Proof. We have to prove that the sequence (a_n) has the properties (5), (6), and (7) of the theorem. It is clear that (5) holds. As to property (6), we have $1/\mathbf{a}_n \to 1/\mathbf{b}$ weakly* in $L^\infty(0,1)$, since $1/\mathbf{b}_n \to 1/\mathbf{b}$ weakly* in $L^\infty(0,1)$, $n\varepsilon \to 0$ and $\Lambda_n \to \infty$.

To verify (7) finally, let I be a subinterval of $[0,1]$ and denote by k the number of intervals $[t_{i-1,n}, t_{i,n}]$ which meet I. We have $|I| \le k\beta/n$. The number of intervals $[t_{i,n}-\varepsilon, t_{i,n}+\varepsilon]$ contained in I is at least $k-3$. Hence we get

$$\int_I \mathbf{a}_n^{p-1}(t)\,dt \ge (k-3)2\varepsilon\Lambda_n^{p-1} \ge 2\left(\frac{1}{\beta}n|I|-3\right)\varepsilon\Lambda_n^{p-1} \to +\infty.$$

Remark 4. Periodical reinforced structures have been studied in [2], [3] and [4]. In [4], $p=2$, $a_n=1$ in $\Omega\setminus\Sigma_{n,\varepsilon}$, $a_n=\lambda$ in $\Sigma_{n,\varepsilon}$, Γ_t are hyperplanes, G is "less general" and the limit behavior of (P_n) was obtained if $n\varepsilon\lambda \to k \in [0,+\infty]$. The previous example extends the case $n\varepsilon\lambda \to +\infty$, that is the case of "very high" conductivity. The case $n\varepsilon\lambda \to k \in [0,+\infty[$ of "high" conductivity is a particular case of the results of [9].

3. Proof of the theorem

Since the convergence of minimization problems is related to the Γ-convergence of the functionals we want to minimize (cf. [5] and also [1]), the theorem will be easily deduced from the following three lemmas:

Lemma 1. *Under conditions* (5) *and* (6), *for every* $v=\mathbf{v}\circ\phi$ *with* $\mathbf{v}\in W_L^{1,p}(0,1)$ *there exists a sequence* $v_n \in W_L^{1,p}(\Omega)$ *such that* v_n *converges to* v *in* w-$W^{1,p}(\Omega)$ *and*

$$\limsup \int_\Omega \frac{1}{a_n} G_n(x, a_n\nabla v_n)\,dx \le \int_\Omega \frac{1}{a} G(x, a\nabla v)\,dx.$$

Lemma 2. *Under conditions* (5) *and* (6), *if* v_n *converges to* v *in* w-$W^{1,p}(\Omega)$, *then*

$$\liminf \int_\Omega \frac{1}{a_n} G_n(x, a_n\nabla v_n)\,dx \ge \int_\Omega \frac{1}{a} G(x, a\nabla v)\,dx.$$

Lemma 3. *Under conditions* (5) *and* (7), *if* $v_n \in W_L^{1,p}(\Omega)$ *and converges to* v *in* $w-W^{1,p}(\Omega)$ *and if* $\int_\Omega a_n^{p-1}|\nabla v_n|^p dx$ *is bounded, then there exists* $\mathbf{v}\in W_L^{1,p}(0,1)$ *such that* $v=\mathbf{v}\circ\phi$.

Before proving these lemmas we establish the theorem:

Proof of the theorem. Let u_n be the unique solution of (P_n). Let $v=\mathbf{v}\circ\phi$ with $\mathbf{v}\in W_L^{1,p}(0,1)$. By Lemma 1, there exist $v_n\in W_L^{1,p}(\Omega)$ such that v_n converges to v in w-$W^{1,p}(\Omega)$ (therefore in $L^p(\Omega)$) and

(9)
$$\begin{aligned}\int_\Omega \frac{1}{a}G(x,a\nabla v)\,dx-\int_\Omega fv\,dx &\geq \limsup\biggl(\int_\Omega \frac{1}{a_n}G_n(x,a_n\nabla v_n)\,dx-\int_\Omega f_nv_n\,dx\biggr)\\ &\geq \limsup\biggl(\int_\Omega \frac{1}{a_n}G_n(x,a_n\nabla u_n)\,dx-\int_\Omega f_nu_n\,dx\biggr)\\ &\geq \limsup\biggl(\int_\Omega c_1c^{p-1}|\nabla u_n|^p\,dx-\int_\Omega f_nu_n\,dx\biggr).\end{aligned}$$

Using Poincaré's inequality, we deduce that (u_n) is bounded in $W^{1,p}(\Omega)$ and that $\int_\Omega a_n^{-1}G_n(x,a_n\nabla u_n)dx$ is bounded. Hence a subsequence of u_n, say u_n again, converges to some u in w-$W_L^{1,p}(\Omega)$ and in $L^p(\Omega)$ and due to hypothesis (1), $\int_\Omega a_n^{p-1}|\nabla u_n|^p dx$ is bounded. By Lemma 3, there exists $\mathbf{u}\in W_L^{1,p}(0,1)$ such that $u=\mathbf{u}\circ\phi$ and, by Lemma 2,

$$\int_\Omega \frac{1}{a}G(x,a\nabla u)\,dx-\int_\Omega fu\,dx\leq \liminf\biggl(\int_\Omega \frac{1}{a_n}G_n(x,a_n\nabla u_n)\,dx-\int_\Omega f_nu_n\,dx\biggr).$$

Consequently, by (9), for all $v=\mathbf{v}\circ\phi$ with $\mathbf{v}\in W_L^{1,p}(0,1)$, we have

$$\int_\Omega \frac{1}{a}G(x,a\nabla u)\,dx-\int_\Omega fu\,dx\leq \int_\Omega \frac{1}{a}G(x,a\nabla v)\,dx-\int_\Omega fv\,dx.$$

Therefore, u is the unique solution of (P), the whole sequence (u_n) converges to u in w-$W_L^{1,p}(\Omega)$ and in $L^p(\Omega)$ and

$$\int_\Omega \frac{1}{a_n}G_n(x,a_n\nabla u_n)\,dx\to\int_\Omega \frac{1}{a}G(x,a\nabla u)\,dx.$$

Proof of Lemma 1. Let $v=\mathbf{v}\circ\phi$ with $\mathbf{v}\in W_L^{1,p}(0,1)$. Let v_n be defined by

$$\mathbf{v}_n(t)=\frac{1}{\delta_n}\int_0^t \frac{\mathbf{a}}{\mathbf{a}_n}\mathbf{v}'\,ds,\quad \text{where}\quad \delta_n=\int_0^1 \frac{\mathbf{a}}{\mathbf{a}_n}\mathbf{v}'\,ds,$$

and let $v_n=\mathbf{v}_n\circ\phi$. Then $v_n\in W_L^{1,p}(\Omega)$, $v_n\to v$ in w-$W^{1,p}(\Omega)$ and in $L^p(\Omega)$ (cf. [8], Lemma 2.4 where the same functions were used), and

$$G_n(x,a_n\nabla v_n)=G_n\biggl(x,\frac{1}{\delta_n}a\nabla v\biggr).$$

Let us write

$$G_n(x,a_n\nabla v_n)-G(x,a\nabla v) = G_n\left(x,\frac{1}{\delta_n}a\nabla v\right)-G_n(x,a\nabla v)+G_n(x,a\nabla v)-G(x,a\nabla v).$$

Using hypotheses (3), (1) and Lebesgue's theorem, we get $G_n(x,a\nabla v)\to G(x,a\nabla v)$ in $L^1(\Omega)$. Moreover,

$$G_n\left(x,\frac{1}{\delta_n}a\nabla v\right)-G_n(x,a\nabla v)=\int_1^{1/\delta_n} a\nabla v\cdot g_n(x,ta\nabla v)\,dt;$$

since $\delta_n\to 1$ and using (2), (1) and Lebesgue's theorem, we deduce that

$$G_n\left(x,\frac{1}{\delta_n}a\nabla v\right)-G_n(x,a\nabla v)\to 0 \quad \text{in } L^1(\Omega).$$

Consequently,

$$G_n(x,a_n\nabla v_n)\to G(x,a\nabla v) \quad \text{in } L^1(\Omega).$$

Since, by hypothesis (6) and Lemma 2.1 of [8], we have $1/a_n\to 1/a$ in w^*-$L^\infty(\Omega)$, it follows that

$$\int_\Omega \frac{1}{a_n}G_n(x,a_n\nabla v_n)\,dx\to\int_\Omega\frac{1}{a}G(x,a\nabla v)\,dx.$$

Proof of Lemma 2. Let $v_n\to v$ in w-$W^{1,p}(\Omega)$. Since $G_n(x,\cdot)$ is convex,

$$\int_\Omega \frac{1}{a_n}G_n(x,a_n\nabla v_n)\,dx \geq \int_\Omega\frac{1}{a_n}G_n(x,a\nabla v)\,dx+\int_\Omega\frac{1}{a_n}g_n(x,a\nabla v)\cdot(a_n\nabla v_n-a\nabla v)\,dx.$$

We have

$$\int_\Omega\frac{1}{a_n}G_n(x,a\nabla v)\,dx\to\int_\Omega\frac{1}{a}G(x,a\nabla v)\,dx$$

since $G_n(x,a\nabla v)\to G(x,a,\nabla v)$ in $L^1(\Omega)$ and $1/a_n\to 1/a$ in w^*-$L^\infty(\Omega)$. Moreover,

$$\int_\Omega\frac{1}{a_n}g_n(x,a\nabla v)\cdot(a_n\nabla v_n-a\nabla v)\,dx=\int_\Omega g_n(x,a\nabla v)\cdot\left(\nabla v_n-\frac{a}{a_n}\nabla v\right)dx\to 0$$

since $g_n(x,a\nabla v)\to g(x,a\nabla v)$ in s-$L^{p'}(\Omega)$ (using hypotheses (4), (2) and Lebesgue's theorem), $\nabla v_n\to\nabla v$ in w-$L^p(\Omega)$ and $(a/a_n)\nabla v\to\nabla v$ in w-$L^p(\Omega)$. Therefore,

$$\liminf\int_\Omega\frac{1}{a_n}G_n(x,a_n\nabla v_n)\,dx\geq\int_\Omega\frac{1}{a}G(x,a\nabla v)\,dx.$$

Proof of Lemma 3. Let $v_n \in W_L^{1,p}(\Omega)$ and $v_n \to v$ in w-$W_L^{1,p}(\Omega)$. Suppose that

$$\int_\Omega a_n^{p-1} |\nabla v_n|^p \, dx \leq C.$$

In order to simplify the computations, we switch to "cylindrical" coordinates on Ω: it is easy to see that $\bar{\Omega}$ is C^1-diffeomorphic with (e.g.) $[0,1] \times \Gamma_0$ by

$$D = (\phi, \psi) \colon x \in \bar{\Omega} \to (t, y) \in [0,1] \times \Gamma_0,$$

where $t = \phi(x)$ and $y = \psi(x)$ e.g., can be defined to be the point of Γ_0 which lies on the orthogonal trajectory to the level surface $\Gamma_t = \{\phi(x) = t\}$ which passes through x (cf. [8], Appendix).

Let $V_n = v_n \circ D^{-1}$ and $V = v \circ D^{-1}$. We have $V_n \to V$ in $w = W^{1,p}(\,]0,1[\times \Gamma_0)$. We will prove that $\nabla_y V = 0$ a.e.; therefore $V(t,y) = V(t)$ for a.e. $t \in [0,1]$ and then $v = \mathbf{v} \circ \phi$ with $\mathbf{v} \in W_L^{1,p}(0,1)$ and $\mathbf{v} = V$.

For that purpose, let us approximate the functions $V_n(t,y)$ by the functions $W_{m,n}(t,y)$ (which are step functions with respect to t) defined as follows: given $m \in \mathbf{N}$, let $I_k = [(k-1)/m, k/m]$, for $k = 1, \ldots, m$ and let

$$W_{m,n}(t,y) = \int_{I_k} V_n(s,y) \frac{\mathbf{a}_n(s)^{p-1}}{\int_{I_k} \mathbf{a}_n^{p-1}} \, ds, \quad \text{for } t \in I_k \text{ and } y \in \Gamma_0,$$

that is, for $t \in [0,1]$ and $y \in \Gamma_0$

$$W_{m,n}(t,y) = \sum_{k=1}^{m} X_{I_k}(t) \int_{I_k} V_n(s,y) \, d\mu_{n,k}(s),$$

where $X_{I_k} = 1$ on I_k and $X_{I_k} = 0$ elsewhere and $d\mu_{n,k}(s) = (\mathbf{a}_n(s)^{p-1}) / (\int_{I_k} \mathbf{a}_n^{p-1}) ds$ on I_k. Observe that $\mu_{n,k}$ is a probability measure on I_k.

We have

$$\begin{aligned}
\int_0^1 & |W_{m,n}(t,y)-V_n(t,y)|^p\,dt \\
&= \sum_{k=1}^m \int_{I_k} |W_{m,n}(t,y)-V_n(t,y)|^p\,dt \\
&= \sum_{k=1}^m \int_{I_k} \left| \int_{I_k} (V_n(s,y)-V_n(t,y))\,d\mu_{n,k}(s)\right|^p dt \\
&\le \sum_{k=1}^m \int_{I_k} \int_{I_k} |V_n(s,y)-V_n(t,y)|^p\,d\mu_{n,k}(s)\,dt \\
&= \sum_{k=1}^m \int_{I_k} \int_{I_k} \left| \int_s^t \frac{\partial V_n}{\partial \tau}(\tau,y)\,d\tau\right|^p d\mu_{n,k}(s)\,dt \\
&\le \sum_{k=1}^m \int_{I_k} \int_{I_k} \left(\int_{I_k} \left| \frac{\partial V_n}{\partial \tau}(\tau,y)\right| d\tau\right)^p d\mu_{n,k}(s)\,dt \\
&\le \sum_{k=1}^m \int_{I_k} \int_{I_k} |I_k|^{p-1} \int_{I_k} \left| \frac{\partial V_n}{\partial \tau}(\tau,y)\right|^p d\tau\,d\mu_{n,k}(s)\,dt \\
&= \sum_{k=1}^m |I_k|\,|I_k|^{p-1} \int_{I_k} \left| \frac{\partial V_n}{\partial \tau}(\tau,y)\right|^p d\tau \\
&= \frac{1}{m^p} \int_0^1 \left| \frac{\partial V_n}{\partial t}(t,y)\right|^p dt \\
&\le \frac{1}{m^p} \int_0^1 |\nabla V_n(t,y)|^p\,dt.
\end{aligned}$$

Thus, integrating with respect to $y\in\Gamma_0$,

$$\int_0^1 \int_{\Gamma_0} |W_{m,n}(t,y)-V_n(t,y)|^p\,dt\,d\gamma(y) \le \frac{1}{m^p} \int_0^1 \int_{\Gamma_0} |\nabla V_n(t,y)|^p\,dt\,d\gamma(y) \le \frac{C}{m^p},$$

since V_n is bounded in $W^{1,p}(]0,1[\times\Gamma_0)$. Consequently $W_{m,n}\in L^p(]0,1[\times\Gamma_0)$ and

$$\|W_{m,n}-V_n\|_{L^p(]0,1[\times\Gamma_0)} \le \frac{C}{m} \tag{10}$$

with C independent of n and m.

Next, for $t\in[0,1]$ and $y\in\Gamma_0$

$$\nabla_y W_{m,n}(t,y) = \sum_{k=1}^m X_{I_k}(t) \int_{I_k} \nabla_y V_n(s,y)\,d\mu_{n,k}(s),$$

$$\begin{aligned}\int_0^1 |\nabla_y W_{m,n}(t,y)|^p\,dt &= \sum_{k=1}^m \int_{I_k} \left| \int_{I_k} \nabla_y V_n(s,y)\,d\mu_{n,k}(s)\right|^p dt \\ &\le \sum_{k=1}^m \int_{I_k}\int_{I_k} |\nabla_y V_n(s,y)|^p\,d\mu_{n,k}(s)\,dt \\ &= \sum_{k=1}^m |I_k| \int_{I_k} |\nabla_y V_n(s,y)|^p\,d\mu_{n,k}(s) \\ &= \sum_{k=1}^m |I_k| \int_{I_k} \frac{\mathbf{a}_n^{p-1}(s)}{\int_{I_k}\mathbf{a}_n^{p-1}} |\nabla_y V_n(s,y)|^p\,ds \\ &\le \frac{1}{m}\frac{1}{\min_k \int_{I_k}\mathbf{a}_n^{p-1}} \int_0^1 \mathbf{a}_n^{p-1}(s)|\nabla V_n(s,y)|^p\,ds.\end{aligned}$$

Thus, integrating with respect to $y\in\Gamma_0$, we deduce that

$$\begin{aligned}&\int_0^1\int_{\Gamma_0} |\nabla_y W_{m,n}(t,y)|^p\,dt\,d\gamma(y) \\ &\qquad \le \frac{1}{m}\frac{1}{\min_k \int_{I_k}\mathbf{a}_n^{p-1}} \int_0^1\int_{\Gamma_0} \mathbf{a}_n^{p-1}(t)|\nabla V_n(t,y)|^p\,dt\,d\gamma(y) \\ &\qquad \le \frac{C}{m}\frac{1}{\min_k \int_{I_k}\mathbf{a}_n^{p-1}},\end{aligned}$$

since $\int_\Omega a_n^{p-1}|\nabla v_n|^p\,dx$ is bounded.

Now, given any m, we can choose $M=M(m)$ so large that $\min_k \int_{I_k}\mathbf{a}_n^{p-1}\ge 1$ (e.g.) whenever $n\ge M(m)$ (this is by assumption (7) in the theorem). Thus

$$\int_0^1\int_{\Gamma_0} |\nabla_y W_{m,n}(t,y)|^p\,dt\,d\gamma(y) \le \frac{C}{m} \quad \text{whenever } n\ge M(m). \tag{11}$$

For each m, we choose an n such that $n\ge m$, $n\ge M(m)$. Then, it follows from (10) and (11) that $W_{m,n}\to V$ in s-$L^p(\,]0,1[\times\Gamma_0)$ as $m\to\infty$ and that $\nabla_y W_{m,n}\to 0$ in s-$L^p(\,]0,1[\times\Gamma_0)$ as $m\to\infty$. Thus we have $\nabla_y V=0$ *a.e.* as desired.

The proof of the theorem is now complete.

Correctors. The convergence of u_n to u in w-$W^{1,p}(\Omega)$ can be made more precise, introducing correctors. Let r_n be defined by $\nabla u_n=\delta_n^{-1}aa_n^{-1}\nabla u+r_n$. Assume that the operators g_n are uniformly strongly monotone, that is there exists $\alpha>0$ such that for every $n\in\mathbf{N}$, $x\in\Omega$, $z_1,z_2\in\mathbf{R}^N$,

$$\alpha|z_1-z_2|^p \le (g_n(x,z_1)-g_n(x,z_2))\cdot(z_1-z_2).$$

Assume also that either G_n is positively homogeneous of degree p or $G_n = G$. Then $r_n \to 0$ in s-$L^p(\Omega)$.

Proof. Let $\mathbf{v}_n(t) = \delta_n^{-1} \int_0^t \mathbf{a}\mathbf{a}_n^{-1}\mathbf{u}' ds$ where $\delta_n = \int_0^1 \mathbf{a}\mathbf{a}_n^{-1}\mathbf{u}' ds$ and let $v_n = \mathbf{v}_n \circ \phi$. Then $\nabla v_n = \delta_n^{-1} a a_n^{-1} \nabla u$. Since the operators g_n are strongly monotone, we get

$$\alpha c^{p-1} \int_\Omega |\nabla u_n - \nabla v_n|^p \, dx \le \int_\Omega (g_n(x, a_n \nabla u_n) - g_n(x, a_n \nabla v_n)) \cdot (\nabla u_n - \nabla v_n) \, dx$$
$$\le \int_\Omega f_n(u_n - v_n) \, dx - \int_\Omega g_n\left(x, \frac{1}{\delta_n} a \nabla u\right) \cdot (\nabla u_n - \nabla v_n) \, dx.$$

Since $u_n - v_n \to 0$ in w-$W^{1,p}(\Omega)$ and in s-$L^p(\Omega)$, it follows that

$$\int_\Omega |\nabla u_n - \nabla v_n|^p \, dx \to 0 \quad \text{as } n \to \infty.$$

Hence $\nabla u_n - \delta_n^{-1} a a_n^{-1} \nabla u = r_n \to 0$ in s-$L^p(\Omega)$.

4. Some generalizations

Other geometric settings can be considered with practically no change in the proof. In fact, we never used the assumption that $\Gamma = \Gamma_0$ (or Γ_1) was the boundary of a domain Ω_0 (Ω_1 respectively). Therefore Γ could as well be any bounded smooth hypersurface (with or without boundary) in $\mathbf{R}^N$ and Ω could be any domain for which we have, as in the proof of Lemma 3, a diffeomorphism $D = (\phi, \psi)\colon x \in \bar{\Omega} \to (t, y) \in [0,1] \times \Gamma$. In this case $\Gamma_t \subset \bar{\Omega}$ $(0 \le t \le 1)$ is to be the inverse image under D of $\{t\} \times \Gamma$ and Γ_0 and Γ_1 now just make up part of the boundary $\partial\Omega$ of Ω (in general). Thus e.g. Ω could be any kind of deformed rectilinear box with Γ_0 and Γ_1 being two opposite faces.

The proof goes through as in the case $\Omega = \Omega_0 \setminus \bar{\Omega}_1$ with $W_L^{1,p}(\Omega)$ now defined as $\{v \in W^{1,p}(\Omega); v = 0 \text{ on } \Gamma_0, v = 1 \text{ on } \Gamma_1\}$. The minimization problem (P_n) will be equivalent to (the weak formulation of):

$$\begin{cases} -\operatorname{div} g_n(x, a_n \nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma_0, \\ u_n = 1 & \text{on } \Gamma_1, \\ g_n(x, a_n \nabla u_n) \cdot \nu = 0 & \text{on } \partial\Omega \setminus (\Gamma_0 \cup \Gamma_1), \end{cases}$$

where ν denotes the outward normal vector of $\partial\Omega$.

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