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ON ANNIHILATORS OF HARMONIC VECTOR FIELDS

Abstract. *For $\Omega \subset \mathbb{R}^N$ a smoothly bounded domain we characterize smooth vector fields g on $\partial\Omega$ which annihilate all harmonic vector fields f in Ω continuous up to $\partial\Omega$, with respect to the pairing $\langle f, g\rangle = \int_{\partial\Omega} f \cdot g\, d\sigma$ ($d\sigma$ denotes the hypersurface measure on $\partial\Omega$.) Also, we extend these results to the context of differential forms with harmonic vector fields being replaced by harmonic fields, i.e., forms f satisfying $df = 0$, $\delta f = 0$. Then a smooth form g on $\partial\Omega$ is an annihilator if and only if suitable extensions, u and v, into Ω of its normal and tangential components on $\partial\Omega$ satisfy the generalized Cauchy–Riemann system $du = \delta v$, $\delta u = 0$, $dv = 0$ in Ω. Finally we prove that the smooth annihilators we describe are weak* dense among all annihilators.*

§ 1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geqslant 2$) with smooth boundary $\partial\Omega$. The goal of the present paper is to characterize smooth vector fields g on $\partial\Omega$ which annihilate all harmonic vector fields f in Ω with respect to the pairing

$$\langle f, g\rangle = \int_{\partial\Omega} f \cdot g\, d\sigma. \tag{1.1}$$

Here the "dot" denotes the scalar product in $\mathbb{R}^N$ and $d\sigma$ denotes the hypersurface measure on $\partial\Omega$. "Smooth" means "of class C^∞" for simplicity.

By a harmonic vector field we mean a vector field $f = (f_1, \dots, f_N)$ satisfying

$$\begin{cases} \operatorname{div} f = 0, \\ \operatorname{curl} f = 0, \end{cases} \tag{1.2}$$

i.e., $\sum_{j=1}^N \partial f_j/\partial x_j = 0$ and $\partial f_k/\partial x_j - \partial f_j/\partial x_k = 0$ for all k and j. The set of harmonic vectors fields in Ω continuous up to $\partial\Omega$ will be denoted by

$A(\Omega)$ and will be provided with the uniform norm

$$\|f\| = \sup_{x\in\overline{\Omega}} | f(x) | = \sup_{x\in\overline{\Omega}} \left(\sum_{j=1}^N |f_j(x)|^2\right)^{1/2}.$$

The problem of describing the annihilators of harmonic vector fields came up in connection with generalizing some results concerning "analytic content" from two to higher dimensions. Cf. [7, 5]. In two dimensions our problem is closely related to a celebrated result of F. and M. Riesz [9, 3]. See §2 below.

Besides the harmonical vector fields we also consider the subspace of harmonic gradients

$$B(\Omega) = \{f \in A(\Omega) : f = \nabla u \text{ for some harmonic function } u \text{ in } \Omega\}. \tag{1.4}$$

The annihilator of $B(\Omega)$ turns out to have a simple description in terms of a differential equation on $\partial\Omega$ involving the normal and tangential components of the given vector field g. By adding to this certain conditions on the flux of g through a finite number of surfaces one also gets a description of the annihilator of $A(\Omega)$. The above is stated in Theorem 3.1.

A more elegant description of the annihilator of $A(\Omega)$ however can be obtained by regarding the vector fields as differential forms and using some results from the Hodge theory. Theorem 4.1 obtained along those lines is in fact our main result. Roughly it says that g is an annihilator if and only if suitable extensions into Ω of the normal and tangential components of g satisfy a Cauchy–Riemann-like system of equations. In two dimensions one obtains precisely the Cauchy–Riemann equations.

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§ 2. The two-dimensional case

For $N = 2$ the harmonic vector fields are simply the anti-analytic functions (identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way) and the pairing (1.1) can be written

$$\langle f, g\rangle = \operatorname{Re} \int_{\partial\Omega} \overline{f} g\, |dz|.$$

Since f is anti-analytic if and only if $\overline{f}$ is analytic (and then also if $i\overline{f}$ is analytic) it follows that our problem becomes that of identifying those complex-valued functions g on $\partial\Omega$ for which

$$\int_{\partial\Omega} fg\,|dz| = 0 \tag{2.1}$$

for all analytic f in Ω (continuous up to $\partial\Omega$).

In case Ω is the unit disc (2.1) is equivalent to

$$\int_0^{2\pi} g(e^{i\theta})e^{in\theta}\,d\theta = 0$$

for all $n \geqslant 0$ and the well-known answer then is that (2.1) holds if and only if g belongs to the Hardy space H^1 (with $g(0) = 0$). This is true even if g is assumed *a priori* only to be a complex-valued measure on $\partial\Omega$ (F. and M. Riesz theorem, [9], [3, Theorem 3.8]).

In case of a general two-dimensional domain Ω with smooth boundary (2.1) can be written

$$\int_{\partial\Omega} f(z)g(z)\overline{t(z)}\,dz = 0, \tag{2.2}$$

where $t(z) = dz/|dz|$ denotes the unit tangent vector on $\partial\Omega$, oriented so that Ω lies on the left. It is well-known that (2.2) holds for all analytic f (continuous up to $\partial\Omega$) if and only if $g(z)\overline{t(z)}$ extends as an analytic function into Ω.

Write $ig\bar{t} = u + iv$ where u and v are real-valued. This means that

$$\begin{cases} u = g_n \\ v = g_t \end{cases} \tag{2.3}$$

where $g_n = \operatorname{Re}\left(g\cdot\overline{(-it)}\right)$ denotes the normal component of g regarded as a vector ($n = -it$ is the outward unit normal vector on $\partial\Omega$) and $g_t = \operatorname{Re}(g\cdot\bar{t})$ denotes the tangential component of g. The fact that $g\bar{t}$ (or $ig\bar{t}$) extends analytically into Ω means that u and v can be extended to Ω where they satisfy the Cauchy–Riemann system

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \tag{2.4}$$

in Ω. Thus a function g on $\partial\Omega$ annihilates all analytic f in Ω (2.1) if and only if the normal and tangential components of g can be continued into Ω so that they satisfy the Cauchy–Riemann equations. This result will be generalized to higher dimensions in §4.

Let s denote an arclength parameter along $\partial\Omega$, so that $t = dz/ds$. Then restricted to $\partial\Omega$ the Cauchy–Riemann equations (2.4) can be written

$$\begin{cases} \frac{\partial u}{\partial n} = \frac{\partial v}{\partial s}, \\ \frac{\partial u}{\partial s} = -\frac{\partial v}{\partial n}. \end{cases} \tag{2.5}$$

Conversely (2.5) imply the Cauchy–Riemann system (2.4) in Ω provided u and v are known in advance to be harmonic.

Now let us drop the last equation in (2.5), so that we only require

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s} \quad \text{on} \quad \partial\Omega. \tag{2.6}$$

If Ω is multiply connected then, with u, v harmonic in Ω, (2.6) is strictly weaker than (2.5). Indeed, let w be a harmonic measure, i.e., a harmonic function in Ω with $\partial w/\partial s = 0$ on $\partial\Omega$. Then (2.6) is not affected if v is replaced by $v + w$, while (2.5), if it holds as is, becomes false when v is replaced by $v + w$ (if $w \neq 0$).

Note that (2.6) is independent of the extension of v to Ω. Hence it can be written

$$\frac{\partial u}{\partial n} = \frac{\partial g_t}{\partial s} \quad \text{on} \quad \partial\Omega. \tag{2.7}$$

It turns out that, with u being the harmonic extension of g_n to Ω, (2.7) holds if and only if g annihilates analytic functions f in Ω which are of the form $f = \partial\psi/\partial z$ for some real-valued harmonic function ψ. (This is the same as saying that $\bar{f}$ is a harmonic gradient (1.4).) This class of analytic functions is not closed under multiplication by $i = \sqrt{-1}$. If e.g. Ω is an annulus, say $1 < |z| < 2$, then $f(z) = 1/z = 2\partial(\log|z|)/\partial z$ belongs to the class whereas $i/z = -2\partial(\arg z)/\partial z$ does not ($\arg z$ is not single-valued in Ω).

In §3 we shall generalize the above assertion concerning (2.7) to higher dimensions, and prove it.

§ 3. Annihilators of harmonic gradients

We now turn to the general case, with $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$. Let n denote the outward unit normal vector on $\partial\Omega$. Any vector field g on $\partial\Omega$ can be decomposed uniquely as

$$g = g_t + g_n n,$$

where g_n is a scalar function (the normal component of g) and g_t is a vector function tangent to $\partial\Omega$. In fact, $g_n = g\cdot n$, $g_t = g - (g\cdot n)n$.

If g is any vector field on $\partial\Omega$ which is tangent to $\partial\Omega$ ($g_n = 0$) then its divergence on $\partial\Omega$ can be defined by action on test functions as follows:

$$\int_{\partial\Omega} (\operatorname{div} g)\varphi\, d\sigma = -\int_{\partial\Omega} g \cdot \nabla\varphi\, d\sigma$$

for every smooth function φ defined in a neighbourhood of $\partial\Omega$. Note that $\nabla\varphi$ on the right can be replaced by $(\nabla\varphi)_t$ and that the latter only depends on the values of φ on $\partial\Omega$. It is easy to see that this definition of $\operatorname{div} g$ agrees with the usual one [12] when $\partial\Omega$ is considered as a Riemannian manifold embedded in $\mathbb{R}^N$.

Recall that $A(\Omega)$ and $B(\Omega)$ were defined in §1. Since each component of a harmonic vector field $f \in A(\Omega)$ is a harmonic function, and hence is uniquely determined by its boundary values on $\partial\Omega$, $A(\Omega)$ can be considered as a subspace of $C(\partial\Omega)^N$, $C(\partial\Omega)$ denoting the space of continuous functions on $\partial\Omega$. Note that the norm (1.3) on $A(\Omega)$ agrees with the natural norm on $C(\partial\Omega)^N$. With $M(\partial\Omega)$ being the space of signed Borel measures on $\partial\Omega$, the dual space of $A(\Omega)$ thus is a quotient space of $M(\partial\Omega)^N$, namely $M(\partial\Omega)^N/A(\Omega)^\perp$, where $A(\Omega)^\perp = \{\mu = (\mu_1, \dots, \mu_N) \in M(\partial\Omega)^N : \int_{\partial\Omega} f \cdot d\mu = 0$ for all $f \in A(\Omega)\}$. Similarly for $B(\Omega)$, its dual is $M(\partial\Omega)^N/B(\Omega)^\perp$.

We shall describe here $A(\Omega)^\perp \cap C^\infty(\partial\Omega)^N$ and $B(\Omega)^\perp \cap C^\infty(\partial\Omega)^N$.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geqslant 2$) with smooth boundary $\partial\Omega$, let g be a smooth vector field on $\partial\Omega$ and let u be the harmonic extension of g_n to Ω. Then*

a) *$g \in B(\Omega)^\perp$ if and only if*

$$\operatorname{div} g_t = \frac{\partial u}{\partial n} \quad \text{on} \quad \partial\Omega, \tag{3.1}$$

b) *$g \in A(\Omega)^\perp$ if and only if*

$$\int_{\partial S} g_t \cdot \nu\, ds = \int_S \frac{\partial u}{\partial n}\, d\sigma \tag{3.2}$$

for every smooth oriented hypersurface $S \subset \overline{\Omega}$ with $\partial S \subset \partial\Omega$. Here $\partial u/\partial n$ denotes the normal derivative on S in the direction singled out by the orientation. The orientation of S induces an orientation on ∂S and ν is the oriented normal vector of ∂S regarded as a submanifold of $\partial\Omega$ (hence ν is tangent to $\partial\Omega$).

In both cases a) and b), the smooth annihilators satisfying (3.1) or (3.2) are weak dense in $B(\Omega)^\perp$ and $A(\Omega)^\perp$ respectively.*

Note that (3.2) implies (3.1). Indeed, (3.1) is equivalent to (3.2) holding for every $S \subset \partial\Omega$.

Proof. Now we prove only the statements concerning a). b) will be proved in §4.

With g and u as in the statement, let $f \in B(\Omega)$, $f = \nabla\varphi$. Then φ is harmonic and continuously differentiable up to $\partial\Omega$, and we have, using Green's formula,

$$\begin{aligned}\langle f, g\rangle &= \int_{\partial\Omega} \nabla\varphi \cdot g\, d\sigma = \int_{\partial\Omega} (\nabla\varphi)_t \cdot g_t\, d\sigma + \int_{\partial\Omega} \frac{\partial\varphi}{\partial n} \cdot g_n\, d\sigma \\ &= -\int_{\partial\Omega} \varphi \operatorname{div} g_t\, d\sigma + \int_{\partial\Omega} \varphi \frac{\partial u}{\partial n}\, d\sigma = \int_{\partial\Omega} \varphi\left(\frac{\partial u}{\partial n} - \operatorname{div} g_t\right) d\sigma.\end{aligned}$$

Clearly this vanishes for all φ as above if and only if (3.1) holds, proving statement a).

Now to prove the weak* density of such vector fields in $B(\Omega)^\perp$, it suffices to show that each $f \in C(\partial\Omega)^N$ which is annihilated by all smooth vector fields g satisfying (3.1) does in fact belong to $B(\Omega)$.

So assume that $f \in C(\partial\Omega)^N$ and that

$$\langle f, g\rangle = \int_{\partial\Omega} (f_t \cdot g_t + f_n g_n)\, d\sigma = 0 \tag{3.3}$$

for every smooth g satisfying (3.1). Choosing all g with $g_n = 0$ and $\operatorname{div} g_t = 0$ it easily follows that f_t is "exact", i.e., that there exists $\varphi \in C^1(\partial\Omega)$ with $f_t = (\nabla\varphi)_t$. Then $\int_{\partial\Omega} f_t \cdot g_t\, d\sigma = -\int_{\partial\Omega} \varphi \operatorname{div} g_t\, d\sigma = -\int_{\partial\Omega} \varphi \frac{\partial u}{\partial n}\, d\sigma$ for g satisfying (3.1), so that (3.3) takes the form

$$\int_{\partial\Omega} \varphi \frac{\partial u}{\partial n}\, d\sigma = \int_{\partial\Omega} f_n u\, d\sigma. \tag{3.4}$$

Choosing $u = 1$ (which is allowed) shows that $\int_{\partial\Omega} f_n\, d\sigma = 0$ and hence that the Neumann problem

$$\begin{cases} \Delta\psi = 0 & \text{in} \quad \Omega, \\ \frac{\partial\psi}{\partial n} = f_n & \text{on} \quad \partial\Omega \end{cases}$$

can be solved, with $\psi \in C^\infty(\Omega) \cap C(\overline{\Omega})$. Inserting this into (3.4) and using Green's theorem gives

$$\int_{\partial\Omega} (\varphi - \psi)\frac{\partial u}{\partial n}\, d\sigma = 0. \tag{3.5}$$

This holds for all u harmonic in Ω and smooth up to $\partial\Omega$ for which there exists a g_t on $\partial\Omega$ satisfying div $g_t = \partial u/\partial n$, i.e., for u for which $\int_S \partial u/\partial n \, d\sigma = 0$ for each component S of $\partial\Omega$.

Therefore (3.5) shows that $\varphi - \psi$ =constant on each component of $\partial\Omega$. Hence $(\nabla\psi)_t = (\nabla\varphi)_t = (\nabla f)_t$ and thus $f = \nabla\psi$ on $\partial\Omega$. Also it follows that $\psi \in C^1(\overline{\Omega})$ (e.g. since each component of $\nabla\psi$ must be given by the Poisson integral of the corresponding component of f). This proves that $f \in B(\Omega)$ as desired. □

Remark. If $N = 2$ and Ω is the unit disc then the F. and M. Riesz theorem [3, 9] asserts that all of $A(\Omega)^\perp$ is contained in $L^1(\partial\Omega)^N \subset M(\partial\Omega)^N$. This result easily extends to arbitrary smooth domains in $\mathbb{R}^2$. However, it is clear from Theorem 3.1 (or its proof) that is does not generalize to higher dimensional in this form. There are e.g. plenty of vector measures $g \in M(\partial\Omega)^N \setminus L^1(\partial\Omega)^N$ with div $g_t = 0$ and $g_n = 0$ when $N \geqslant 3$.

As a particular example, take Ω to be simply connected and let γ be a closed oriented curve on $\partial\Omega$. The functional $f \mapsto \int_\gamma f \cdot dx$ on $C^\infty(\partial\Omega)^N$ defines an element in $M(\partial\Omega)^N \setminus L^1(\partial\Omega)^N$ (if $N \geqslant 3$) which annihilates all gradients $f = \nabla\varphi$, in particular all $f \in A(\Omega)$. Note that $A(\Omega) = B(\Omega)$ when Ω is simply connected.

The F. and M. Riesz theorem can, however, be generalized to higher dimensions in another form, namely stating that if a harmonic vector field in Ω has boundary values (in an appropriate sense) in form of measures on $\partial\Omega$ then these in fact have to be absolutely continuous. See [10, Chap. VII §3.2], [11] for a special case. We shall also see in §5 that $A(\Omega)^\perp$ is in some sense larger than $A(\Omega)$ when $N \geqslant 3$.

§ 4. Annihilators of harmonic fields

In §3 we described the annihilators of harmonic gradients in a way which generalizes the two-dimensional formula (2.6) or (2.7). In this section we shall find the higher dimensional counterpart of (2.4) as describing the annihilators of harmonic vector fields. The natural language for this is that of differential forms rather than vector fields.

So we identity vector fields with 1-forms in the usual way, i.e., $f = (f_1, \dots, f_N)$ is identified with $f = f_1 dx_1 + \ldots + f_N dx_N$. Recall [12] that the Hodge's star operator on such f gives the $(N-1)$-form

$$*f = f_1 dx_2 \ldots dx_N - f_2 dx_1 dx_3 \ldots dx_N + \ldots + (-1)^{N-1} f_N dx_1 \ldots dx_{N-1}$$

and that hence $f \wedge *g = \sum_{j=1}^{N} f_j g_j dx_1 \ldots dx_N = (f \cdot g)\omega$, ω denoting the volume form $\omega = dx_1 \ldots dx_N$.

In order to define $\langle f, g\rangle$ $f \wedge *g$ must be turned into a $(N-1)$-form, so that it can be integrated over $\partial\Omega$. Let n denote the unit outward normal vector to $\partial\Omega$ considered as a 1-form. Then any 1-form f on $\partial\Omega$ has a unique decomposition

$$f = f_t + n \wedge f_n \tag{4.1}$$

where f_t is a 1-form free of n, i.e., $n \wedge * f_t = 0$, and f_n is a function (0-form). Indeed, f_n can be defined by $*f_n = n \wedge (*f)$ and then (4.1) determines f_t – cf. [8, 2]. The decomposition (4.1) actually generalizes to p-forms f for any p, $0 \leqslant p \leqslant N$, namely by defining f_n as

$$*f_n = (-1)^{p-1} n \wedge (*f). \tag{4.2}$$

f_t is then a p-form and f_n a $(p-1)$-form.

Example 4.1. Suppose part of $\partial\Omega$ is given by $x_1 = 0$ with Ω lying to the left ($x_1 < 0$). Then $n = dx_1$ on this part and, with $f = f_1 dx_1 + \ldots + f_N dx_N$, $f_t = f_2 dx_2 + \ldots + f_N dx_N$ and $f_n = f_1$. Similarly, if f is a p-form for any p f_t consists of terms of f which do not contain dx_1 and f_n is obtained from the remaining terms by factoring out dx_1 to the left (so that $f = f_t + dx_1 \wedge f_n$). As a particular example we have that $(dx_1 dx_2 \ldots dx_N)_n = dx_2 \ldots dx_N$, i.e., the normal component of the volume form in $\mathbb{R}^N$ is, up to a sign, the $(N-1)$-dimensional "volume" form on $\partial\Omega$.

The right $(N-1)$-form to integrate over $\partial\Omega$, in place of hypersurface measure $d\sigma$, must be $*n$, at least up to a sign. To see that the (plus) sign is correct, consider the Dirichlet problem $\Delta\varphi = 1$ in Ω, $\varphi = 0$ on $\partial\Omega$. The solution φ is negative in Ω, hence on $\partial\Omega$ $d\varphi$ is a positive multiple of the normal 1-form n. Since $\int_{\partial\Omega} *d\varphi = \int_\Omega d * d\varphi = \int_\Omega \Delta\varphi = |\Omega| > 0$ this shows that $\int_{\partial\Omega} *n > 0$, i.e., that $*n$ has the right sign.

According to (4.2) $*n = \omega_n$ where $\omega = dx_1 \ldots dx_N$. If f and g are 1-forms then $f \wedge *g = (f \cdot g)\omega$ where on the right f, g are regarded as vectors. Thus $(f \wedge *g)_n = (f \cdot g) *n$ and it follows that the appropriate definition of $\langle f, g\rangle$ when f, g are 1-forms is

$$\langle f, g\rangle = \int_{\partial\Omega} (f \wedge *g)_n. \tag{4.3}$$

This is still a correct definition when f and g are forms of arbitrary degree $0 \leqslant p \leqslant N$. In particular, we then always have $\langle f, f\rangle \geqslant 0$.

Recall the definition of the coexterior derivative δ:

$$\delta f = (-1)^{N(p+1)+1} * d * f$$

if f is a p-form. A p-form f is called a *harmonic field* if

$$df = 0, \quad \delta f = 0 \tag{4.4}$$

(cf. [4, 8, 2]). If $p = 1$ then (4.4) becomes curl $f = 0$, div $f = 0$ if f is considered as a vector field. Hence the harmonic fields when $p = 1$ can be identified with the harmonic vector fields.

For any fixed p $(0 \leqslant p \leqslant N)$ let $A(\Omega)$ denote the set of harmonic fields of degree p in Ω which extend continuously to $\overline{\Omega}$, provided with the norm (generalizing (1.3))

$$\|f\| = \sup_{\overline{\Omega}} \sqrt{|*(f \wedge *f)|}.$$

(Note that $*(f \wedge *f)$ is a 0-form.)

As in the vector case $A(\Omega)$ can be regarded as a subspace of $C(\partial\Omega)^{\binom{N}{p}}$ and the annihilator $A(\Omega)^{\perp}$ with respect to (4.3) (with g having measures as components) as a subspace of $M(\partial\Omega)^{\binom{N}{p}}$. Here we shall characterize $A(\Omega)^{\perp} \cap C^{\infty}(\partial\Omega)^{\binom{N}{p}}$.

Theorem 4.1. *If g is a smooth p-form on $\partial\Omega$, then $g \in A(\Omega)^{\perp}$ if and only if there exist smooth forms u and v of degrees $p-1$ and $p+1$ respectively on $\overline{\Omega}$ satisfying*

$$u_t = g_n, \quad v_n = g_t \quad on \quad \partial\Omega, \tag{4.5}$$

$$du = \delta v \quad in \quad \Omega, \tag{4.6}$$

$$\delta u = 0, \quad dv = 0 \quad in \quad \Omega. \tag{4.7}$$

Moreover, the smooth fields g satisfying the above conditions generate in the weak $*$ topology the whole annihilator $A(\Omega)^{\perp}$. The last condition (4.7) can be viewed as a normalization and may be omitted.

In short, $A(\Omega)^{\perp}$ is generated in the weak $*$ topology by all smooth fields on $\partial\Omega$ of the from $g = v_n + n \wedge u_t$ with u, v satisfying (4.6) and (deliberate) (4.7).

Example 4.2. If $N = 2$ and $p = 1$, then writting $v = \widetilde{v}\,dx\,dy$ (where $(x, y) = (x_1, x_2)$) (4.6) becomes

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial \widetilde{v}}{\partial y}dx - \frac{\partial \widetilde{v}}{\partial x}dy$$

and (4.7) is automatically satisfied. Thus we recover the Cauchy–Riemann system (2.4).

Example 4.3. If $p = 1$ and $N = 3$, then u is a function and $v = v_1 dx_2 dx_3 + v_2 dx_3 dx_1 + v_3 dx_1 dx_2$ can be identified with the vector field $\widetilde{v} = (v_1, v_2, v_3)$. Then the system (4.6), (4.7) becomes

$$\nabla u = \operatorname{curl} \widetilde{v} \quad \text{in} \quad \Omega, \tag{4.8}$$

$$\operatorname{div} \widetilde{v} = 0 \quad \text{in} \quad \Omega. \tag{4.9}$$

Regarding g as a vector field denoted by $\widetilde{g}$ on $\partial\Omega$ and writing $\widetilde{g} = \widetilde{g}_t + g_n \widetilde{n}$ with $\widetilde{g}_t$ a vector tangent to $\partial\Omega$, g_n a scalar and $\widetilde{n}$ the outward unit normal vector, the first equation in (4.5) simply becomes

$$g_n = u \quad \text{in} \quad \partial\Omega. \tag{4.10}$$

To see how the second equation in (4.5) translates into vector analysis language, suppose that a portion of $\partial\Omega$ is given by $x_1 = 0$ with Ω lying to the left $(x_1 < 0)$. Then $v_n = v_3 dx_2 - v_2 dx_3$ on this portion, so that $g_t = v_n$ means that $g_2 = v_3$, $g_3 = -v_2$, where g_j denotes the components of g (or of $\widetilde{g}$).

Since here $\widetilde{n} = (1, 0, 0)$ these equations can be written

$$\widetilde{g}_t = \widetilde{v} \times \widetilde{n} \quad \text{on} \quad \partial\Omega. \tag{4.11}$$

This is a coordinate free reformulation of the equation $v_n = g_t$. Thus we conclude from Theorem 4.1 that in three dimensions a vector field $\widetilde{g}$ on $\partial\Omega$ annihilates all harmonic vector fields in Ω if and only if there exist a function u and a vector field $\widetilde{v}$ in Ω such that (4.8)–(4.11) hold. Note that u necessarily is the harmonic extension of g_n, $\widetilde{v}$ satisfies curl curl $\widetilde{v} = 0$ and that (4.11) simply means that $\widetilde{g}_t$ and $\widetilde{v}_t$ are related by a 90^0 rotation.

The proof of Theorem 4.1 is based on two lemmas.

Lemma 4.1. *Let f, u, v be smooth forms on $\overline{\Omega}$ of degree p, $p-1$, $p+1$ respectively $(1 \leqslant p \leqslant N-1)$ and let g be the p-form on $\partial\Omega$ defined by $g_t = v_n$, $g_n = u_t$ (i.e., $g = v_n + n \wedge u_t$). Then*

$$\int_{\partial\Omega} (f \wedge *g)_n = \int_{\Omega} f \wedge *(du - \delta v) + \int_{\Omega} df \wedge *v - \int_{\Omega} \delta f \wedge *u. \tag{4.12}$$

In particular, if f is a harmonic field we have, setting $h = du - \delta v$,

$$\int_{\partial\Omega} (f \wedge *g)_n = \int_{\Omega} f \wedge *h. \tag{4.13}$$

Note that h is a harmonic field if and only if $\delta du = 0$ and $d\delta v = 0$.

Remark: In general, a form $f \in A(\Omega)$ does not fulfil the smoothness assumptions in the lemma. However, it is easy to see by an approximation argument that (4.13) still holds for an arbitrary $f \in A(\Omega)$.

Proof (lemma).

$$\int_{\partial\Omega}(f\wedge *g)_n = \int_{\partial\Omega}\big((f_t+n\wedge f_n)\wedge *(v_n+n\wedge u_t)\big)_n$$

$$= \Big[\int_{\partial\Omega}(f_t\wedge *v_n)_n + \int_{\partial\Omega}\big(f_t\wedge *(n\wedge u_t)\big)_n$$

$$+\int_{\partial\Omega}(n\wedge f_n\wedge *v_n)_n + \int_{\partial\Omega}\big(n\wedge f_n\wedge *(n\wedge u_t)\big)_n\Big]$$

$$= \int_{\partial\Omega} f_t\wedge *(n\wedge v_n) + 0 + 0 + \int_{\partial\Omega}\big(n\wedge u_t\wedge *(n\wedge f_n)\big)_n$$

$$= \int_{\partial\Omega} f\wedge *v + \int_{\partial\Omega} u\wedge *f$$

$$= \int_{\Omega} df\wedge *v + (-1)^p\int_{\Omega} f\wedge d*v + \int_{\Omega} du\wedge *f + (-1)^{p-1}\int_{\Omega} u\wedge d*f$$

$$= \int_{\Omega} f\wedge *(du-\delta v) + \int_{\Omega} df\wedge *v - \int_{\Omega}\delta f\wedge *u. \qquad \square$$

Lemma 4.2 (Bo Berntsson). *Given any smooth p-form g on $\partial\Omega$ there exists a $(p-1)$-form u in Ω and a $(p+1)$-form v in Ω, both smooth up to $\partial\Omega$, such that*

$$\begin{cases} \delta u = 0,\ \ \delta du = 0 & \text{in } \Omega, \\ u_t = g_n & \text{on } \partial\Omega; \end{cases} \tag{4.14}$$

$$\begin{cases} dv = 0,\ \ d\delta v = 0 & \text{in } \Omega, \\ v_n = g_t & \text{on } \partial\Omega. \end{cases} \tag{4.15}$$

The forms u and v are uniquely determined up to harmonic fields satisfying, respectively, $u_t = 0$ and $v_n = 0$ on $\partial\Omega$.

The weaker version of the lemma obtained by omitting the requirements $\delta u = 0$ and $dv = 0$ follows from [8, 2]. The present version was shown to us by Bo Berntsson, who kindly consented to including it here, along with his elegant proof.

The proof is based on the following version of the Hodge theorem for Riemannian manifolds with boundary.

Every smooth p-form α on $\overline{\Omega}$ has a unique decomposition

$$\alpha = (d\delta + \delta d)\eta + h \tag{4.16}$$

where η, h are smooth p-forms on $\overline{\Omega}$, $\eta_n = 0$, $(d\eta)_n = 0$, $h_n = 0$ on $\partial\Omega$ and $dh = 0$, $\delta h = 0$ in Ω. Moreover, if $d\alpha = 0$ then $d\eta = 0$; if α is exact then $h = 0$. Thus there is a unique harmonic field with vanishing normal part on $\partial\Omega$ in each de Rham cohomology class of Ω.

This theorem can be proved by imitating the proof of the classical Hodge theorem for closed Riemannian manifolds as presented e.g. in [12, Chap. 6] or by doubling Ω to a closed manifold $\widehat{\Omega}$ [1, 4] and applying the classical Hodge theorem to α extended to $\widehat{\Omega}$ as an "even" form (i.e., satisfying $J^*(\alpha) = \alpha$, where J denotes the natural involution on $\widehat{\Omega}$ and J^* its pull-back map). See [4].

Let us, for later reference, point out that there is another ("dual") version of the Hodge theorem, with the boundary conditions replaced by $\eta_t = 0$, $(\delta\eta)_t = 0$ and $h_t = 0$ on $\partial\Omega$. This corresponds to the Hodge theorem on the double $\widehat{\Omega}$ applied to "odd" forms ($J^*(\alpha) = -\alpha$). It follows from this version of the Hodge theorem that there is a unique harmonic field in Ω with vanishing tangential part of $\partial\Omega$ in each relative cohomology class of Ω (cycles and boundaries taken modulo $\partial\Omega$). See [4, 8, 2] for more explicit statements.

Proof (Lemma 4.2). Choose $\rho \in C^\infty(\mathbb{R}^N)$ such that $\Omega = \{x \in \mathbb{R}^N : \rho(x) < 0\}$ and such that $|\nabla\rho| = 1$ on $\partial\Omega$. Then $d\rho = n$.

Extend g in an arbitrary way to a smooth form in $\mathbb{R}^N$ and set $\xi = d(\rho g)$. Then $d\xi = 0$ and

$$\xi_n = g_t \quad \text{on} \quad \partial\Omega.$$

Now decompose $\alpha = d\delta\xi$ according to (4.16):

$$d\delta\xi = (d\delta + \delta d)\eta + h.$$

Since $d\alpha = 0$ it follows that $d\eta = 0$, hence h is exact. But then $h = 0$. Setting

$$v = \xi - \eta$$

it follows that $dv = 0$, $d\delta v = 0$ in Ω, $v_n = g_t$ on $\partial\Omega$.

This proves (4.6). (4.5) is obtained by applying the same procedure to $*g$ instead, and setting $u = *(\xi - \eta)$ in the final step.

To prove the uniqueness statement for (4.14) means to prove that if (4.14) holds with $g_n = 0$ then $du = 0$. But

$$\int_{\Omega} du\wedge *du = \int_{\partial\Omega} u\wedge *du - (-1)^{p-1}\int_{\Omega} u\wedge d*du$$

$$= \int_{\partial\Omega} u_t\wedge *du + \int_{\Omega} u\wedge *\delta du = 0.$$

The uniqueness statement for (4.15) is proved similarly. □

Proof (theorem). Given a smooth p-form on $\partial\Omega$ g, choose forms u and v on $\overline{\Omega}$ satisfying (4.14) and (4.15). Setting $h = du - \delta v$ h then is a harmonic field and it follows from (4.13) that $\langle f, g\rangle = 0$ for all $f \in A(\Omega)$ if and only if $h = 0$ (for the "only if" part, just choose $f = h$). Note that the conditions $\delta u = 0$ and $dv = 0$ are really not needed, they are just a normalization. From this the first part of the theorem follows.

To prove the second assertion of the theorem means to prove that if for some continuous f on $\partial\Omega$ $\langle f, g\rangle = 0$ for all g satisfying (4.5), (4.6) then f extends to a harmonic field in Ω. Now, given f on $\partial\Omega$ we can always extend f in a unique way to Ω as a harmonic form, i.e., satisfying

$$d\delta f + \delta df = 0 \quad \text{in} \quad \Omega. \tag{4.17}$$

(This just means that each component of f is extended as a harmonic function). But (4.17) means that u and v defined by $u = -\delta f$, $v = df$ satisfy (4.6). If these are smooth on $\partial\Omega$ then $g = v_n + n \wedge u_t$ is one of the smooth annihilators of $A(\Omega)$ so that by (4.12)

$$0 = \langle f, g\rangle = \int_{\partial\Omega} (f \wedge *g)_n = \int_{\Omega} df \wedge *df + \int_{\Omega} \delta f \wedge *\delta f.$$

Thus $df = 0$, $\delta f = 0$ in Ω as desired.

If df and δf are not smooth up to $\partial\Omega$ then the above argument fails and one has to proceed as in the corresponding part of Theorem 3.1. Let us outline a proof along these lines, assuming for the sake of simplicity that Ω is homeomorphic to a ball.

According to the proof of Lemma 4.1 $\langle f, g\rangle = \int_{\partial\Omega} f \wedge *v + \int_{\partial\Omega} u \wedge *f$. Thus $\langle f, g\rangle = 0$ holding for all g satisfying (4.5), (4.6) means that

$$\int_{\partial\Omega} f \wedge *v + \int_{\partial\Omega} u \wedge *f = 0$$

for all smooth forms u and v on $\overline{\Omega}$ satisfying (4.6). Choosing in particular $u = 0$ and $v = \delta\psi$ for ψ a general smooth $(p+2)$-form on $\overline{\Omega}$, and applying the Stokes formula to (4.18) we conclude that

$$df = 0 \quad \text{along} \quad \partial\Omega, \tag{4.19}$$

i.e., $(df)_t = 0$ on $\partial\Omega$, in the sense of distributions.

It follows easily from (4.19) that f_t can be approximated uniformly on $\partial\Omega$ by smooth p-forms φ_ϵ on $\partial\Omega$ ($\epsilon \to 0$) satisfying $d\varphi_\epsilon = 0$. The latter equation shows that φ_ϵ are admissible boundary values for harmonic fields (see [2]), i.e., that there exist harmonic fields f_ϵ in Ω smooth up to $\partial\Omega$ and satisfying $(f_\epsilon)_t = \varphi_\epsilon$ on $\partial\Omega$. Thus

$$(f_\epsilon)_t \to f_t \quad \text{on} \quad \partial\Omega \tag{4.20}$$

uniformly as $\epsilon \to 0$.

For u, v satisfying (4.6) we get

$$\int_{\partial\Omega} f_\epsilon \wedge *v \to \int_{\partial\Omega} f \wedge *v \quad \text{as} \quad \epsilon \to 0,$$

and

$$\int_{\partial\Omega} f_\epsilon \wedge *v = (-1)^p \int_{\Omega} f_\epsilon \wedge d*v = -\int_{\Omega} f_\epsilon \wedge *du$$
$$= -\int_{\Omega} du \wedge *f_\epsilon = -\int_{\partial\Omega} u \wedge *f_\epsilon.$$

Thus, taking (4.18) into account,

$$\lim_{\epsilon\to 0} \int_{\partial\Omega} u \wedge *(f_\epsilon - f) = 0 \tag{4.21}$$

for every smooth u satisfying $\delta du = 0$ in Ω (this is equivalent to saying that v satisfying (4.6) exists).

For such u the smooth tangential boundary value u_t can be prescribed arbitrarily (Lemma 4.2). Therefore (4.21) shows that

$$(f_\epsilon)_n \to f_n$$

in the sense of distributions as $\epsilon \to 0$. Thus using also (4.20),

$$f_\epsilon \to f \quad \text{on} \quad \partial\Omega \tag{4.22}$$

in the sense of distributions. This easily implies that f_ϵ converge uniformly on compact subsets of Ω to a harmonic field which has f as continuous boundary value and hence provides the desired extension of f.

Indeed, extend f to Ω as a harmonic form (4.17). Then the components of $f_\epsilon - f$ are harmonic functions in Ω continuous up to $\partial\Omega$. It follows now from (4.22) and the Poisson integral formula that $f_\epsilon \to f$ uniformly on compact subsets of Ω and hence f is in fact a harmonic field. □

Proof of b) of Theorem 3.1. We first translate the condition (3.2) into the language of forms, regarding g as a 1-form. Choose u and v (4.14), (4.15). Then u is the same as in (3.1), (3.2). The left member of (3.2) can be interpreted as the flux of g_t through ∂S and this can readily be seen to be given by $\int_{\partial S} *(n \wedge g_t)$. (In fact, $*(n \wedge g_t)$ equals $\circledast g_t$ where $\circledast$ denotes the star operator on $\partial\Omega$ regarded as an oriented Riemannian manifold in itself.) Thus (3.2) translates into

$$\int_{\partial S} *(n \wedge g_t) = \int_S *du. \tag{4.23}$$

Now

$$\int_S *du - \int_{\partial S} *(n \wedge g_t) = \int_S *du - \int_{\partial S} *(n \wedge v_n) = \int_S *du - \int_{\partial S} *v$$
$$= \int_S (*du - d * v) = \int_S *(du - \delta v) = \int_S *h,$$

where $h = du - \delta v$, a 1-form. Thus (4.23) says that

$$\int_S *h = 0 \quad \text{for every} \quad S \subset \overline{\Omega} \quad \text{with} \quad \partial S \subset \partial\Omega,$$

whereas Theorem 4.1 says that $g \in A(\Omega)^\perp$ if and only if $h = 0$.

Clearly (4.24) holds if $h = 0$, so we only have to prove, conversely, that (4.24) implies that $h = 0$. But this follows from the dual version of the Hodge theorem (4.16) stated earlier. Indeed, $*h$ is a harmonic field and (4.24) says precisely that it is in the zero relative cohomology class of Ω and that it has vanishing tangential part on $\partial\Omega$. (The latter follows by applying (4.24) for arbitrary $S \subset \partial\Omega$.) Thus $*h = 0$, and hence $h = 0$, as desired.

The fact that (4.24) implies $h = 0$ can also be proved more directly as follows. Since $dh = 0$, h is of the form $h = d\varphi$, locally, where φ is a (harmonic) function. It Ω is not simply connected then φ need not be single-valued. However, Ω can always be made simply connected by cutting it up along a finite number of $(N-1)$-surfaces $S_1, \dots, S_m$ satisfying $S_j \subset \overline{\Omega}$, $\partial S_j \subset \partial\Omega$. (For example, a torus in $\mathbb{R}^3$ can be made simply connected by means of a single cut.) Thus $\Omega' = \Omega \setminus \bigcup_{j=1}^m S_j$ is simply connected.

Each S_j given two contributions to $\partial\Omega'$, S_j^+ and S_j^- say, so that $\partial\Omega' = \partial\Omega + \sum_{j=1}^m S_j^+ - \sum_{j=1}^m S_j^-$. In Ω' φ is single-valued and it is easy to see that its values at corresponding points on S_j^+ and S_j^- differ simply by a constant

(for each j), call it c_j. Using (4.24) and an earlier observation that $(*h)_t = 0$ on $\partial\Omega$ we now obtain

$$\int_\Omega h \wedge *h = \int_{\Omega'} h \wedge *h = \int_{\partial\Omega'} \varphi \wedge *h$$
$$= \int_{\partial\Omega} \varphi \wedge *h + \sum_{j=1}^m \left(\int_{S_j^+} \varphi \wedge *h - \int_{S_j^-} \varphi \wedge *h \right)$$
$$= \int_{\partial\Omega} \varphi \wedge (*h)_t + \sum_{j=1}^m c_j \int_{S_j^+} *h = 0.$$

Thus $h = 0$ as desired. □

§ 5. A REMARK ON THE SYSTEM $du = \delta v$, $\delta u = 0$, $dv = 0$

In §4 we say how solutions of the system $du = \delta v$, $\delta u = 0$, $dv = 0$ generate $A(\Omega)^\perp$. In two dimensions the same system, with a sign change, also describes $A(\Omega)$ itself (§2). In this final section we wish to show how this generalizes to higher dimensions.

Let $e \neq 0$ be a 1-form in Ω with constant coefficients (e.g. $e = dx_1$). Given a p-form f in Ω we define a $(p-1)$-form u and a $(p+1)$-form v by

$$*u = (-1)^{p-1} e \wedge *f, \tag{5.1}$$
$$v = e \wedge f \tag{5.2}$$

(cf. u and v in §4). If e.g. $e = dx_1$ then writting $f = g + dx_1 \wedge h$, where g and h do not contain dx_1, we have $u = h$, $v = dx_1 \wedge g$.

Proposition 5.1. *With e, f, u, and v as above we have*

$$df = 0, \quad \delta f = 0 \tag{5.3}$$

if and only if

$$du + \delta v = 0, \tag{5.4}$$
$$\delta u = 0, \tag{5.5}$$
$$dv = 0. \tag{5.6}$$

Note: If $e = dx_k$ then u and v have, respectively, $\binom{N-1}{p-1}$ and $\binom{N-1}{p}$ nontrivial components (the "trival" ones are those which vanish by definition (5.1), (5.2)) and these are simply the $\binom{N}{p}$ components of f. Written out in components the system (5.4)–(5.6) then is exactly the same as (5.3).

Proof. Let $\tilde{e}$ be the (constant) vector field corresponding to e, let $i(\tilde{e})$ denote the interior multiplication ("contraction") by $\tilde{e}$ and $L_{\tilde{e}}$ the Lie derivative (see [12]). The star operator and the above two operations are related by

$$*(f \wedge e) = i(\tilde{e}) * f, \tag{5.7}$$

$$L_{\tilde{e}} f = di(\tilde{e}) f + i(\tilde{e}) df \tag{5.8}$$

for any form f. Since $\tilde{e}$ is a constant vector field we moreover have

$$L_{\tilde{e}} * f = * L_{\tilde{e}} f. \tag{5.9}$$

Also,

$$de = 0.$$

In view of (5.7) the definitions of u and v can also be written

$$u = i(\tilde{e}) f \tag{5.11}$$

$$*v = (-1)^p i(\tilde{e}) * f. \tag{5.12}$$

It is easy to check that f can be reconstructed from u and v by the formula

$$f = \frac{1}{|e|^2}(i(\tilde{e})v + e \wedge u) \tag{5.13}$$

where $|e|$ denotes the Euclidean length of e (or of $\tilde{e}$), so that $i(\tilde{e})e = |e|^2$.

Using (5.1)–(5.2), (5.7)–(5.12) we now compute

$$\begin{aligned} du + \delta v &= di(\tilde{e}) f + (-1)^{Np+p} * d * v \\ &= di(\tilde{e}) f - (-1)^{p(N-p)} * di(\tilde{e}) * f \\ &= L_{\tilde{e}} f - i(\tilde{e}) df - (-1)^{p(N-p)} * L_{\tilde{e}} * f + (-1)^{p(N-p)} * i(\tilde{e}) d * f \\ &= -i(\tilde{e}) df + (-1)^{p(N-p)} * i(\tilde{e}) d * f, \\ \delta u &= (-1)^{Np+p} * d(e \wedge * f) = (-1)^{Np+p-1} * (e \wedge d * f), \\ dv &= -e \wedge df. \end{aligned}$$

From this we see immediately that (5.3) implies (5.4)–(5.6).

Conversely, assume (5.4)–(5.6). Then by the above computations

$$e \wedge df = 0, \tag{5.14}$$

$$e \wedge d * f = 0, \tag{5.15}$$

$$* i(\tilde{e}) df = i(\tilde{e}) d * f. \tag{5.16}$$

We may assume, without loss of generality, that $e = dx_1$. Then (5.14) says that $df = e \wedge g$ for some p-form g free of e. Thus $*i(\tilde{e})df = *g$. Similarly (5.15) shows that $d*f = e \wedge h$ for some $(N-p)$-form h free of e. Thus $i(\tilde{e})d*f = h$. Now $*g = h$ by (5.16). But since g and h are both free of e this implies $g = 0$, $h = 0$. Hence $df = 0$, $d*f = 0$ as desired. □.

Example 5.1. Take $N = 2$, $p = 1$. Then u is a 0-form and v a 2-form so that (5.5), (5.6) are trivially satisfied. Thus (5.3) is equivalent to (5.4) alone. If $e = dx_1$ then $u = f_1$, $v = f_2 dx_1 dx_2$ and (5.4) is the ordinary anti-Cauchy–Riemann system for f_1 and f_2.

Example 5.2. Take $N \geqslant 3$, $p = 1$. Then (5.5) is still trivially satisfied, but not (5.6). For $N = 3$ e.g., with $v = v_1 dx_2 dx_3 + v_2 dx_3 dx_1 + v_3 dx_1 dx_2$, $\tilde{v} = (v_1, v_2, v_3)$ (as in Example 4.3), (5.4) and (5.6) become, respectively,

$$\begin{aligned} \nabla u + \operatorname{curl} \tilde{v} &= 0, \\ \operatorname{div} \tilde{v} &= 0. \end{aligned}$$

If $e = dx_1$, then $u = f_1$, $\tilde{v} = (0, -f_3, f_2)$.

A particular consequence of (5.4) is that $*du$ is an exact form. If $p = 1$ and $e = dx_k$ we have $u = f_k$. Thus Proposition 5.1 shows that if f is a harmonic vector field then each component f_k is a harmonic function having zero flux through any closed hypersurface S. Indeed,

$$\int_S * df_k = \int_S * du = -\int_S d * v = 0.$$

For a different proof of this fact, see e.g. [5, 6].

The descriptions of $A(\Omega)^\perp$ in Theorem 4.1 and of $A(\Omega)$ in Proposition 5.1 can be summarized in the following way. Let $E_\pm(\Omega)$ denote the set of pairs (u, v) of smooth forms of degree $p-1$ and $p+1$ in Ω, satisfying

$$du = \pm \delta v, \quad \delta u = 0, \quad dv = 0$$

(coupled signs). In the + case we require u, v also to be smooth up to $\partial\Omega$.

Then Theorem 4.1 describes a linear map

$$E_+(\Omega) \to A(\Omega)^\perp,$$

namely $(u, v) \mapsto g = v_n + n \wedge u_t$, with weak* dense range and finite dimensional kernel. Given $e \neq 0$ Proposition 5.1 shows that (5.1), (5.2) define a linear map

$$A(\Omega) \to E_-(\Omega).$$

This is injective, by (5.13), but far from surjective in general. Indeed, the range is readily seen to consist of $(u, v) \in E_-(\Omega) \cap C(\overline{\Omega})$ satisfying $i(\tilde{e})u = 0$, $e \wedge v = 0$.

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ОБ ОДНОМ АНАЛОГЕ ТЕОРЕМЫ РУНГЕ ДЛЯ ГАРМОНИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ

Дифференциальная форма u, заданная в открытом подмножестве Ω пространства $\mathbb{R}^n$, называется гармонической, если

$$du = 0, \quad \delta u = 0 \tag{1}$$

в Ω, где d и δ-операторы внешнего дифференцирования и кодифференцирования. Это понятие обобщает понятие аналитической функции комплексной переменной (при $n = 2$ форма $u = P\,dx + Q\,dy$ удовлетворяет системе (1) тогда и только тогда, когда функция $P - iQ$ аналитична).

В работе [1] (см. также [5]) среди прочего был доказан следующий аналог теоремы Рунге: форма u, гармоническая в окрестности компактного множества $K \subset \mathbb{R}^n$, допускает равномерное на K приближение суммами форм Био-Савара и Кулона (так называются гармонические формы специального вида с особенностями на циклах). В настоящей статье предлагается вариант этой теоремы, который в известном смысле ближе к классической теореме Рунге. Мы вводим некоторые гармонические формы с точечными особенностями (так называемые “рациональные формы”) и доказываем теорему о выводе полюсов аппроксимирующих рациональных форм. Кроме того, мы строим новое интегральное представление (аналогичное формуле Коши) точных и (одновременно) коточных гармонических форм.

§ 1. Формы Био-Савара, порожденные парой циклов

Всюду ниже мы предполагаем, что $n \geqslant 3$. Положим

$$\omega_{c,r} = \frac{1}{\tau_n} \sum_{\alpha \in M_r} \delta_{\alpha,\beta} \left\{ \int_C \frac{d\xi^\beta}{|x-\xi|^{n-2}} \right\} dx^\alpha, \tag{2}$$

где $0 < r < n$, c – некоторый $(n-r)$-цикл в $\mathbb{R}^n$, τ_n – $(n-1)$-мерный объем сферы $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, M_r – множество всех возрастающих r-мерных мультииндексов $\alpha = (i_1, \dots, i_r)$, $i_1 < \dots < i_r$, $1 \leqslant i_s \leqslant n$; β обозначает $(n-r)$-мультииндекс, дополнительный по отношению к