Advances in Mathematical Sciences and Applications
Gakkōtosho, Tokyo, Vol. 4, No. 2(1994), pp. 265-284

H-CONVERGENCE FOR STRATIFIED STRUCTURES WITH HIGH CONDUCTIVITY

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(Communicated by M. Chipot; Received March 22, 1993)

Abstract: We are concerned with the homogenization of elliptic problems of the kind $-\mathrm{div}\,(A^\epsilon \nabla u^\epsilon) = f$ in $\Omega \subset \mathbb{R}^N$ (plus boundary conditions) where $\epsilon > 0$ is a parameter ($\epsilon \to 0$) and the conductivity matrices $A^\epsilon = \left(a_{ij}^\epsilon(x)\right)$ are symmetric and depend on only one of the coordinates, say x_1 (stratified medium). We also assume coercivity $\alpha\,|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}^\epsilon(x_1)\xi_i\xi_j \leq \beta^\epsilon\,|\xi|^2$ with $0 < \alpha < \beta^\epsilon < \infty$ (allowing $\beta^\epsilon \to \infty$). If $A = (a_{ij}(x))$ is a matrix satisfying similar conditions then our main result states that we have H-convergence $A^\epsilon \rightharpoonup A$ provided

$$\begin{aligned}
\frac{1}{a_{11}^\epsilon} &\rightharpoonup \frac{1}{a_{11}} && w^* - L^\infty(\Omega) \\
\frac{a_{1j}^\epsilon}{a_{11}^\epsilon} &\rightharpoonup \frac{a_{1j}}{a_{11}} && w - L^2(\Omega) \quad \text{for } j \geq 2, \\
a_{ij}^\epsilon - \frac{a_{i1}^\epsilon\, a_{1j}^\epsilon}{a_{11}^\epsilon} &\rightharpoonup a_{ij} - \frac{a_{i1}\, a_{1j}}{a_{11}} && w^* - M(\overline{\Omega}) \quad \text{for } i,j \geq 2
\end{aligned}$$

($M(\overline{\Omega})$ being the dual space of $C(\overline{\Omega})$). This generalizes previous results of Murat and Tartar.

1. Introduction.

We are concerned with the homogenization of linear elliptic problems of the form

$$(1.1)^\epsilon \qquad \begin{cases} -\,\mathrm{div}\,(A^\epsilon \nabla u^\epsilon) = f & \text{in} \quad \Omega \\ +\ \text{boundary conditions,} \end{cases}$$

where the "conductivity" matrices $A^\epsilon = A^\epsilon(x) = (a_{ij}^\epsilon(x))$ are symmetric and uniformly (in x and ϵ) elliptic, with bounded measurable coefficients. Ω is a bounded domain in

$\mathbf{R}^N (N \geq 2)$ and $\epsilon > 0$ is a parameter which is going to tend to zero. The basic assumption about the A^ϵ will be that they depend on only one variable, let us say that $A^\epsilon = A^\epsilon(x_1)$, where $x = (x_1, \dots, x_N)$. This means that we are regarding Ω as a layered or stratified medium (material). For simplicity we shall let Ω be of cylindrical form

$$\Omega = (0,1) \times \omega \tag{1.2}$$

where ω is a bounded domain in $\mathbf{R}^{N-1}$ (for the $(x_2, \dots, x_N)$-variables).

Under the above circumstances there is a well-known result of Murat and Tartar [9], [12], [13] saying that if, as $\epsilon \to 0$, $A^\epsilon \to A = (a_{ij})(a_{ij} \in L^\infty(\Omega))$ in the sense that

$$\frac{1}{a_{11}^\epsilon} \rightharpoonup \frac{1}{a_{11}} \qquad w^* - L^\infty(\Omega) \tag{1.2}$$

$$\frac{a_{1j}^\epsilon}{a_{11}^\epsilon} \rightharpoonup \frac{a_{1j}}{a_{11}} \qquad w^* - L^\infty(\Omega) \quad \text{for} \quad j \geq 2 \tag{1.3}$$

$$a_{ij}^\epsilon - \frac{a_{i1}^\epsilon a_{1j}^\epsilon}{a_{11}^\epsilon} \rightharpoonup a_{ij} - \frac{a_{i1} a_{1j}}{a_{11}} \qquad w^* - L^\infty(\Omega) \quad \text{for} \quad i,j \geq 2 \tag{1.4}$$

then the problems $(1.1)^\epsilon$ converge to the homogenized limit problem

$$\begin{cases} -\operatorname{div}(A\nabla u) = f & \text{in} \quad \Omega \\ + \text{ boundary conditions,} \end{cases} \tag{1.1}$$

the convergence meaning, among other things, convergence of the solutions, $u^\epsilon \rightharpoonup u$, in $w - H^1(\Omega)$. Murat and Tartar actually allow nonsymmetric matrices, with the transposed condition added to (1.3). The convergence $A^\epsilon \to A$ in the sense (1.2)–(1.4) is an instance of H-convergence (cf. Spagnolo [11] and Murat and Tartar [9], [12]).

Note that (1.3), (1.4) imply that the sequences $\frac{a_{1j}^\epsilon}{a_{11}^\epsilon}$ and $a_{ij}^\epsilon - \frac{a_{i1}^\epsilon a_{1j}^\epsilon}{a_{11}^\epsilon}$ are bounded in L^∞. In many situations this condition is not satisfied but still one has the convergence of $(1.1)^\epsilon$ towards (1.1). See Example in § 2 below. The aim of the present paper is to generalize the Murat-Tartar result to cover such cases. Indeed our main result simply states that the convergence $(1.1)^\epsilon \to (1.1)$ remains to hold if (1.2) is satisfied but (1.3), (1.4) are relaxed to convergence in $w - L^2(\Omega)$ and $w^* - M(\overline{\Omega})$ respectively, $M(\overline{\Omega})$ denoting the space of measures on $\overline{\Omega}$ (the dual space of $C(\overline{\Omega})$). It turns out that the uniform coercivity of the A^ϵ results in a natural precompactness of the quantities appearing in (1.2), (1.3), (1.4) in the spaces $w^* - L^\infty(\Omega), w - L^2(\Omega)$ and $w^* - M(\overline{\Omega})$ respectively, and for this reason it is even enough to assume convergence in $\mathcal{D}'(\Omega)$ (weak distribution sense) in (1.2) and (1.3). In (1.4), however, the $w^* - M(\overline{\Omega})$ convergence cannot be relaxed to convergence in $\mathcal{D}'(\Omega)$. Thus our assumptions on the convergence are virtually the weakest possible in our setting of the problem.

Our method is that of formulating (1.1) and $(1.1)^\epsilon$ as minimization problems, and proving Γ-convergence $F^\epsilon \to F$ of the corresponding functionals involved, namely

$$F^\epsilon(u) = \int_\Omega \left(\sum_{i,j=1}^N a_{ij}^\epsilon \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - 2fu \right) dx \tag{1.6$^\epsilon$}$$

and similarly for F. Murat and Tartar prove their result dealing directly with the differential equation $(1.1)^\epsilon$ and using arguments related to "compensated compactness".

The limit behaviour of quasilinear problems of the form

$$\begin{cases} -\frac{\partial}{\partial x_1} g_1\left(x, a_1^\epsilon \frac{\partial u^\epsilon}{\partial x 1}\right) - \sum\limits_{i=2}^{N} a_i^\epsilon \frac{\partial}{\partial x_i} g_i\left(\frac{\partial u^\epsilon}{\partial x_i}\right) = f \qquad \text{in} \qquad \Omega, \\ + \text{ boundary conditions} \end{cases}$$

is investigated in [7], [6] for stratified structures with high conductivity, and in [8] in the case of both low and high conductivities. Homogenization of periodic (stratified or not) structures with thin inclusions of high conductivity has earlier been studied for linear problems in [2], [4], [10].

Acknowledgements: This work has been partially supported by the Swedish Natural Science Research Council (NFR). We are grateful to Bernard Heron and Francois Murat for helpful discussions and to Siv Sandvik for excellent typing of our manuscript.

2. The main result.

With $\Omega = (0,1) \times \omega$ as in § 1 let $\Gamma = \{0\} \times \omega \subset \partial\Omega$ and set

$$H_L^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}.$$

We shall work with the family of minimization problems

$$(2.1)^\epsilon \qquad \inf\{F^\epsilon(v) : v^\epsilon \in H_L^1(\Omega)\}$$

for $\epsilon > 0$, where

$$(2.2)^\epsilon \qquad F^\epsilon(v) = \int\limits_\Omega \left(\sum_{i,j=1}^{N} a_{ij}^\epsilon \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} - 2 f^\epsilon v \right) dx.$$

Our assumptions on $a_{ij}^\epsilon (1 \le i,j \le N)$ and f^ϵ will be that

$$(2.3)^\epsilon \qquad a_{ij}^\epsilon = a_{ji}^\epsilon,$$

$$(2.4)^\epsilon \qquad a_{ij}^\epsilon = a_{ij}^\epsilon(x_1) \in L^\infty(0,1),$$

$$(2.5)^\epsilon \qquad \sum_{i,j=1}^{N} a_{ij}^\epsilon\,(x_1)\xi_i\xi_j \ge \alpha\,|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^N \text{ and a.e. } x_1 \in (0,1)$$

with $\alpha > 0$ independent of ϵ and x;

$$(2.6)^\epsilon \qquad f^\epsilon \in L^2(\Omega).$$

The condition $(2.5)^\epsilon$ will be referred to as "α-coerciveness" of A^ϵ (where $\alpha > 0$ is a number). When we write (2.5) we mean $(2.5)^\epsilon$ with a_{ij}^ϵ replaced by a_{ij}; similarly for other formulas.

The problems $(2.1)^\epsilon$ are equivalent to the weak formulations of $(1.1)^\epsilon$ (with f^ϵ in place of f) with the boundary conditions

$$\begin{aligned} u^\epsilon &= 0 \quad &\text{on} \quad &\Gamma, \\ (A^\epsilon \nabla u^\epsilon) \cdot n &= 0 \quad &\text{on} \quad &\partial\Omega\setminus\Gamma, \end{aligned}$$

n denoting the exterior unit normal vector on $\partial\Omega$. We have chosen these particular boundary conditions just because they are the ones for which our treatment works most smoothly and because the boundary conditions anyway are not very important (the criteria for convergence are local ones).

Clearly the problems $(2.1)^\epsilon$ have unique solutions u^ϵ. Here ϵ is a small parameter corresponding to the heterogeneity of the medium and we will be concerned with the behaviour of the problems $(2.1)^\epsilon$ as $\epsilon \to 0$. To avoid some trivial and uninteresting complications we shall assume that ϵ tends to zero just through a sequence of values.

In an electrical framework, with A^ϵ interpreted as a conductivity matrix, the vector field $E^\epsilon = \nabla u^\epsilon$ (with components $E_j^\epsilon = \partial u^\epsilon / \partial x_j$) is the electrical field strength and $D^\epsilon = A^\epsilon E^\epsilon$ is the electrical displacement field. As has been remarked by Tartar [13] the coefficients appearing in (1.2)–(1.4) are those which appear when (suppressing the ϵ for a moment) $E_1, D_2, \dots, D_N$ are expressed in terms of $D_1, E_2, \dots, E_N$:

$$\begin{cases} E_1 &= \dfrac{1}{a_{11}} D_1 - \sum\limits_{j\geq 2} \dfrac{a_{1j}}{a_{11}} E_j \quad , \\ D_i &= \dfrac{a_{i1}}{a_{11}} D_1 + \sum\limits_{j\geq 2} \left(a_{ij} - \dfrac{a_{i1}a_{1j}}{a_{11}} \right) E_j \quad , i \geq 2. \end{cases}$$

It is convenient to introduce

$$b_{ij} = a_{ij} - \frac{a_{i1}\, a_{1j}}{a_{11}} \qquad (i, j \geq 2). \tag{2.7}$$

(Similarly for b_{ij}^ϵ.) Then $b_{ij} = b_{ji}$. Note the identities

$$\begin{aligned} \sum_{i,j\geq 1} a_{ij} E_i E_j &= \sum_{i\geq 1} E_i D_i \\ &= \frac{1}{a_{11}} D_1^2 + \sum_{i,j\geq 2} b_{ij} E_i E_j \\ &= a_{11} \left(E_1 + \sum_{j\geq 2} \frac{a_{1j}}{a_{11}} E_j \right)^2 + \sum_{i,j\geq 2} b_{ij} E_i E_j \end{aligned} \tag{2.8}$$

with D, E related as above.

As simple consequences of the coercivity (2.5) we have the following estimates (still suppressing ϵ).

Proposition: *Assume that $A = (a_{ij})$ is α-coercive, i.e. that (2.5) holds, and define (b_{ij}) by (2.7). Then*

$$\frac{1}{a_{11}} \leq \frac{1}{\alpha}; \tag{2.9}$$

(b_{ij}) is also α-coercive:

$$\sum_{i,j\geq 2} b_{ij}\eta_i\eta_j \geq \alpha\,|\eta|^2 \tag{2.10}$$

for all $\eta = (\eta_2, \dots, \eta_N) \in \mathbb{R}^{N-1}$;

$$|b_{ij}| \leq \frac{1}{2}\,(b_{ii} + b_{jj}) \qquad (i,j \geq 2); \tag{2.11}$$

$$\left|\frac{a_{1i}}{a_{11}}\right|^2 \leq \frac{1}{\alpha}\,b_{ii} \qquad (i \geq 2). \tag{2.12}$$

Proof (2.9) is obvious from (2.5). To derive (2.10) one may look at (2.8) and observe that $D_1, E_2, \dots, E_N$ can be varied arbitrarily there. Taking $D_1 = 0$ and combining with (2.5) then gives (2.10). (2.11) is obtained by choosing η with $\eta_i = 1, \eta_j = \pm 1, \eta_r = 0$ for $r \neq i, j$ in (2.10).

To show (2.12), finally, we note that the α-coercivity of A implies that for each $i \geq 2$

$$\det\begin{pmatrix} a_{11} - \alpha & a_{1i} \\ a_{1i} & a_{ii} - \alpha \end{pmatrix} \geq 0.$$

Thus $(a_{1i})^2 \leq (a_{11} - \alpha)\,(a_{ii} - \alpha)$ yielding

$$\begin{aligned} b_{ii} = a_{ii} - \frac{(a_{1i})^2}{a_{11}} &\geq \alpha + (a_{1i})^2 \left(\frac{1}{a_{11} - \alpha} - \frac{1}{a_{11}}\right) \\ &= \alpha + \frac{\alpha a_{11}}{a_{11} - \alpha}\left(\frac{a_{1i}}{a_{11}}\right)^2 \geq \alpha\left(\frac{a_{1i}}{a_{11}}\right)^2, \end{aligned}$$

which is the same as (2.12).

Corollary: *Assume that the $A^\epsilon = \left(a_{ij}^\epsilon(x_1)\right)$ are α-coercive and that the sequences $\{b_{ii}^\epsilon\}\,(i \geq 2)$ are bounded in $L^1(0,1)$, i.e.*

$$\int_0^1 b_{ii}^\epsilon\,dt \leq C < \infty \qquad (i \geq 2). \tag{2.13}$$

Then the sequences $\{1/a_{11}^\epsilon\}, \{a_{1i}^\epsilon/a_{11}^\epsilon\}$ *and* $\{b_{ij}^\epsilon\}\,(i \geq 2, j \geq 2)$ *are bounded in* $L^\infty(0,1)$, $L^2(0,1)$ *and* $L^1(0,1)$ *respectively and hence are precompact in* $w^* - L^\infty(0,1)$, $w - L^2(0,1)$ *and* $w^* - M[0,1]$ *respectively.*

The corollary (which follows directly from the proposition) gives natural topologies for the convergence of the coefficients of A^ϵ. We now state our main result.

Theorem: *Assume* $(2.3)^\epsilon$ *to* $(2.6)^\epsilon$ *and that, for some symmetric matrix* $A = (a_{ij})$ *with*

$$a_{ij} = a_{ij}(x_1) \in L^\infty(0,1)\ ,$$
$$1/a_{11} \in L^\infty(0,1)\ ,$$

$A^\epsilon \rightharpoonup A$ *in the sense that*

$$\text{(2.14)} \qquad \frac{1}{a_{11}^\epsilon} \rightharpoonup \frac{1}{a_{11}} \qquad w^* - L^\infty(0,1),$$

$$\text{(2.15)} \qquad \frac{a_{1j}^\epsilon}{a_{11}^\epsilon} \rightharpoonup \frac{a_{1j}}{a_{11}} \qquad w - L^2(0,1) \qquad (j \geq 2),$$

$$\text{(2.16)} \qquad b_{ij}^\epsilon \rightharpoonup b_{ij} \qquad w^* - M[0,1] \qquad (i, j \geq 2)$$

as $\epsilon \to 0$ *(*b_{ij} *defined by (2.7)). Assume also that, for some* $f \in L^2(\Omega)$,

$$\text{(2.17)} \qquad f^\epsilon \rightharpoonup f \qquad w - L^2(\Omega).$$

Then A *is* α*-coercive, hence there is a unique solution* $u \in H_L^1(\Omega)$ *of*

$$\text{(2.1)} \qquad \inf\{F(v) : v \in H_L^1(\Omega)\},$$

where

$$\text{(2.2)} \qquad F(v) = \int_\Omega \left(\sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} - 2fv \right) dx.$$

As $\epsilon \to 0$

$$\text{(2.18)} \qquad u^\epsilon \rightharpoonup u \qquad w - H^1(\Omega),$$

$$\text{(2.19)} \qquad \frac{1}{a_{11}^\epsilon} \sum_{j=1}^N a_{1j}^\epsilon \frac{\partial u^\epsilon}{\partial x_j} \rightharpoonup \frac{1}{a_{11}} \sum_{j=1}^N a_{1j} \frac{\partial u}{\partial x_j} \qquad w - L^2(\Omega),$$

$$\text{(2.20)} \qquad F^\epsilon(u^\epsilon) \to F(u).$$

The proof of the theorem will be given in § 3–5. The rest of this section is devoted to discussions of the convergence assumptions (2.14)-(2.16).

1. We have preferred to regard the coefficients a_{ij}^{ϵ} as functions on the interval $(0,1)$, since they depend only on x_1. However, the theorem could equally well have been stated with convergence in $w^* - L^{\infty}(\Omega), w - L^2(\Omega)$ and $w^* - M(\overline{\Omega})$ instead, and then one would not have to assume in advance that the limit matrix depends only on x_1, but that would be a (simple) consequence of the convergence. (Notational remark here: to avoid subscripts we will frequently denote x_1 by t, and later on in the paper we will use y for the remaining variables $(x_2, \dots, x_N)$.)

2. Recall that (2.16) means that

$$\int_0^1 b_{ij}^{\epsilon}\varphi\,dt \to \int_0^1 b_{ij}\varphi\,dt \qquad (i,j \geq 2) \tag{2.21}$$

for every $\varphi \in C[0,1]$. Choosing here $\varphi = 1$ and $i = j$ shows that (2.16) implies (2.13). On the other hand, if (2.13) is assumed the corollary yields precompactness of the sequences in (2.14) to (2.16) and therefore that (2.14)-(2.16) automatically hold assuming only convergence in any Hansdorff topology.

Thus, in the presence of (2.13) or (2.16) it is enough to assume convergence in e.g. $\mathcal{D}'(0,1)$ for (2.14) and (2.15). However in (2.16) the convergence can not be replaced by convergence in $\mathcal{D}'(0,1)$ (note that $M[0,1]$ is not a subspace of $\mathcal{D}'(0,1)$), even if (2.13) is assumed separately, as the following example shows.

Take $N = 2$, $a_{11}^{\epsilon} = 1$, $a_{12}^{\epsilon} = a_{21}^{\epsilon} = 0$ and $a_{22}^{\epsilon}(t) = 1 + 1/\epsilon$ for $1 - \epsilon < t < 1$, $a_{22}^{\epsilon}(t) = 1$ for $0 < t < 1 - \epsilon$. Then $b_{22}^{\epsilon} = a_{22}^{\epsilon}$, $b_{22}^{\epsilon} \rightharpoonup 1$ in $\mathcal{D}'(0,1)$ and (2.13) holds true, since $\int_0^1 b_{22}^{\epsilon} = 2$ for all $0 < \epsilon < 1$. But $b_{22}^{\epsilon} \rightharpoonup b_{22} = 1 + \delta \quad w^* - M[0,1]$ where δ denotes the Dirac measure at the point $t = 1$, and one can verify that the "correct" limit problem (which falls outside the scope of this paper) has measurevalued coefficients, namely $a_{11} = 1$, $a_{12} = a_{21} = 0$, $a_{22} = b_{22} = 1 + \delta$. As a minimization problem it can be written $\inf\{G(v) : v \in H(\Omega)\}$ where $H(\Omega) = \{v \in H_L^1(\Omega) : v|_{\Gamma'} \in H^1(\Gamma')\}$, $\Gamma' = \{1\} \times \omega$ and $G(v) = \int_{\Omega} \left(|\nabla v|^2 - 2fv\right) dx_1\,dx_2 + \int_{\Gamma'} \left(\frac{\partial v}{\partial x_2}\right)^2 dx_2$.

3. A further comment on (2.16) is that, due to the coerciveness of b_{ij}^{ϵ}, it implies, and actually is equivalent to, mean convergence on intervals, namely that

$$\int_I b_{ij}^{\epsilon}\,dt \to \int_I b_{ij}\,dt \qquad (i,j \geq 2) \tag{2.22}$$

for every interval $I \subset [0,1]$. To see this first observe that in order to prove (2.22) it is enough to prove that

$$\sum_{i,j\geq 2} \eta_i\eta_j \int_I b_{ij}^{\epsilon}\,dt \to \sum_{i,j\geq 2} \eta_i\eta_j \int_I b_{ij}\,dt \tag{2.23}$$

for every $\eta = (\eta_2, \dots, \eta_N) \in \mathsf{R}^{N-1}$. But $b^\epsilon := \sum \eta_i\eta_j\, b_{ij}^\epsilon \geq 0$, $b := \sum \eta_i\eta_j b_{ij} \geq 0$, and for any $\delta > 0$ there are $\varphi, \psi \in C[0,1]$ with $\varphi \leq \chi_I \leq \psi$ and $\int_0^1 (\psi - \varphi)\, dt < \delta$. Since $\int_0^1 b^\epsilon \varphi\, dt \leq \int_I b^\epsilon\, dt \leq \int_0^1 b^\epsilon \psi$, $\int_0^1 b\varphi\, dt \leq \int_I b\, dt \leq \int_0^1 b\psi\, dt$, $\int_0^1 b\psi\, dt - \int_0^1 b\varphi\, dt \leq \delta\, \|b\|_\infty$, $\int_0^1 b^\epsilon \varphi\, dt \to \int_0^1 b\varphi\, dt$, $\int_0^1 b^\epsilon \psi\, dt \to \int_0^1 b\psi\, dt$ and $\delta > 0$ is arbitrary (2.23) follows.

Conversely (2.22) implies (2.16) because (2.22) guarantees, by (2.11), that the sequences $\{b_{ij}^\epsilon\}$ are bounded in $L^1(0,1)$ and then one just has to use that any function in $C[0,1]$ can be uniformly approximated by step functions. (Since the step functions are dense also in $L^1(0,1)$ and $L^2(0,1)$ all three convergence assumptions (2.14)–(2.16) could actually have been replaced by mean convergence on intervals.)

We thus see that the class $C[0,1]$ of test functions allowed in (2.21) automatically extends to the class of piecewise continuous functions. On the other hand, it does not extend to all $L^\infty(0,1)$. In fact we now give an example in which all hypotheses of the theorem are satisfied without the b_{ij}^ϵ converging weakly in $L^1(0,1)$.

Example: Let, for every $\epsilon > 0$, $J_k^\epsilon \subset [0,1]$, $k = 1, 2, \dots$ be intervals, periodically distributed with period ϵ, of lengths $|J_k^\epsilon| = r^\epsilon \ll \epsilon$ (e.g. $J_1^\epsilon = [0, r^\epsilon]$, $J_2^\epsilon = [\epsilon, \epsilon + r^\epsilon]$ etc.) and let $\lambda^\epsilon \to +\infty$ as $\epsilon \to 0$ in such a way that, for some $0 < \beta < \infty$,

$$\frac{r^\epsilon \lambda^\epsilon}{\epsilon} \longrightarrow \beta \qquad \text{as} \qquad \epsilon \longrightarrow 0.$$

Define

$$a^\epsilon(t) = \begin{cases} 1 + \lambda^\epsilon & \text{if} \quad t \in \bigcup_{k \geq 1} J_k^\epsilon, \\ 1 & \text{otherwise,} \end{cases}$$

$$a(t) = 1 + \beta.$$

Then for every interval $I \subset [0,1]$

$$\int_I a^\epsilon\, dt = |I| + \lambda^\epsilon \frac{|I|}{\epsilon} r^\epsilon + o(\epsilon)$$

showing that

$$\int_I a^\epsilon\, dt \longrightarrow \int_I a\, dt \tag{2.24}$$

as $\epsilon \to 0$. It follows that the sequence a^ϵ is bounded in $L^1(0,1)$ and that $a^\epsilon \rightharpoonup a$ $w^* - M[0,1]$.

However the a^ϵ are not equiintegrable and hence they do not converge to a (or any other function) weakly in $L^1(0,1)$. In fact, taking $E^\epsilon = \bigcup_{k \geq 1} J_k^\epsilon$ we have $|E^\epsilon| = r^\epsilon/\epsilon \to 0$ as $\epsilon \to 0$ while $\int_{E^\epsilon} a^\epsilon\, dt \approx \lambda^\epsilon r^\epsilon/\epsilon \to \beta > 0$, disproving the equiintegrability.

Consider the conductivity matrix

$$A^\epsilon(x_1) = a^\epsilon(x_1)Id$$

corresponding to an isotropic stratified medium with many thin layers of high conductivity. Clearly $\frac{1}{a^\epsilon} \rightharpoonup 1$ in $w^* - L^\infty(0,1)$. This together with (2.24) shows that the A^ϵ converge in our sense (2.14) to (2.16) towards

$$A(x_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1+\beta \end{pmatrix}.$$

Since the a^ϵ do not converge in $w^* - L^\infty(0,1)$, or even in $w - L^1(0,1)$, this convergence $A^\epsilon \rightarrow A$ is not covered by the Murat-Tartar result mentioned in the introduction.

3. Proof of the theorem.

Now, following De Giorgi and others (see e.g. [1], [5]), we shall prove the theorem by proving Γ-convergence of the functionals F^ϵ. The main part of the proof will consist in proving the following two lemmas.

Lemma 1. *For every $v \in H^1_L(\Omega) \cap C^\infty(\mathbb{R}^N)$ there exist $v^\epsilon \in H^1_L(\Omega)$ such that, as $\epsilon \rightarrow 0$,*

$$v^\epsilon \rightharpoonup v \qquad w - H^1_L(\Omega),$$
$$\limsup_{\epsilon\rightarrow 0} F^\epsilon(v^\epsilon) \leq F(v).$$

Corollary (of the proof): *The limit matrix $A = (a_{ij}(x_1))$ is α-coercive.*

Lemma 2. *Let $v^\epsilon, v \in H^1_L(\Omega)(\epsilon > 0)$ and suppose that, as $\epsilon \rightarrow 0$,*

$$v^\epsilon \rightharpoonup v \qquad\qquad w - H^1_L(\Omega),$$
$$\frac{\partial v^\epsilon}{\partial x_1} + \sum_{j\geq 2} \frac{a^\epsilon_{1j}}{a^\epsilon_{11}} \frac{\partial v^\epsilon}{\partial x_j} \rightharpoonup \frac{\partial v}{\partial x_1} + \sum_{j\geq 2} \frac{a_{1j}}{a_{11}} \frac{\partial v}{\partial x_j} \qquad w - L^2(\Omega).$$

Then

$$\liminf_{\epsilon\rightarrow 0} F^\epsilon(v^\epsilon) \geq F(v).$$

Proof of the theorem from lemmas:

Take an arbitrary $v \in H^1_L(\Omega) \cap C^\infty(\mathbb{R}^N)$. Then with v^ϵ as in Lemma 1 (and u^ϵ denoting the solution of $(2.1)^\epsilon$) we have

(3.1) $$\liminf F^\epsilon(u^\epsilon) \leq \limsup F^\epsilon(u^\epsilon) \leq \limsup F^\epsilon(v^\epsilon) \leq F(v) < \infty.$$

By $(2.5)^\epsilon$ this shows that

$$\int_\Omega |\nabla u^\epsilon|^2 \, dx \le C < \infty,$$

$$\int_\Omega \left(\frac{\partial u^\epsilon}{\partial x_1} + \sum_{j \ge 2} \frac{a_{1j}^\epsilon}{a_{11}^\epsilon} \frac{\partial u^\epsilon}{\partial x_j} \right)^2 dx \le C < \infty,$$

where the last estimate is obtained by writing F^ϵ in the form

$$F^\epsilon(v) = \int_\Omega \left[a_{11}^\epsilon \left(\frac{\partial v}{\partial x_1} + \sum_{j \ge 2} \frac{a_{1j}^\epsilon}{a_{11}^\epsilon} \frac{\partial v}{\partial x_j} \right)^2 + \sum_{i,j \ge 2} b_{ij}^\epsilon \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} - 2 f^\epsilon v \right] dx$$

(see (2.8)) and using also (2.10).
Hence it follows, by weak compactness, that for any subsequence $\epsilon \to 0$ there is a subsequence and $u \in H_L^1(\Omega), g \in L^2(\Omega)$ such that as $\epsilon \to 0$ through the latter subsequence

$$u^\epsilon \rightharpoonup u \qquad w - H^1(\Omega), \tag{3.2}$$

$$\frac{\partial u^\epsilon}{\partial x_1} + \sum_{j \ge 2} \frac{a_{1j}^\epsilon}{a_{11}^\epsilon} \frac{\partial u^\epsilon}{\partial x_j} \rightharpoonup g \qquad w - L^2(\Omega). \tag{3.3}$$

By (3.2) $u^\epsilon \to u \ s - L^2(\Omega)$ and together with (2.15) this gives

$$\frac{a_{1j}^\epsilon}{a_{11}^\epsilon} u^\epsilon \rightharpoonup \frac{a_{1j}}{a_{11}} u \qquad w - L^1(\Omega),$$

in particular in the sense of distributions. Hence

$$\frac{a_{1j}^\epsilon}{a_{11}^\epsilon} \frac{\partial u^\epsilon}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{a_{1j}^\epsilon}{a_{11}^\epsilon} u^\epsilon \right) \rightharpoonup \frac{\partial}{\partial x_j} \left(\frac{a_{1j}}{a_{11}} u \right) = \frac{a_{1j}}{a_{11}} \frac{\partial u}{\partial x_j} \qquad \text{in the sense of distributions}$$

for $j \ge 2$ and it follows from (3.3) that

$$g = \frac{\partial u}{\partial x_1} + \sum_{j \ge 2} \frac{a_{1j}}{a_{11}} \frac{\partial u}{\partial x_j}. \tag{3.4}$$

Thus $u^\epsilon \rightharpoonup u$ (for a subsequence) in the sense required in Lemma 2, so that

$$F(u) \le \liminf F^\epsilon(u^\epsilon). \tag{3.5}$$

Combining (3.5) with (3.1) shows that $F(u) \leq F(v)$ for every $v \in H^1_L(\Omega) \cap C^\infty(\mathbf{R}^N)$. Since such v are dense in $s - H^1_L(\Omega)$ and F is continuous in $s - H^1_L(\Omega)$ it follows that $F(u) \leq F(v)$ for all $v \in H^1_L(\Omega)$ and hence that u is a solution of (2.1).

By Corollary of Lemma 1 u then is the unique solution of (2.1) and a standard argument with subsequences shows that

$$u^\epsilon \rightharpoonup u \qquad w - H^1_L(\Omega)$$

with ϵ running through the full sequence. Similarly one gets (3.3), i.e. (2.19), for the full sequence.

Combining again (3.5) with (3.1) gives

$$F(u) \leq \liminf F^\epsilon(u^\epsilon) \leq \limsup F^\epsilon(u^\epsilon) \leq F(v)$$

and letting here $v = v_n \to u \; s - H^1_L(\Omega)$ shows that $\lim F^\epsilon(u^\epsilon)$ exists and equals $F(u)$. Modulo the proofs of the lemmas this finishes the proof of the theorem.

4. **Proof of Lemma 1** with Corollary:

Let $v \in H^1_L(\Omega) \cap C^\infty(\mathbf{R}^N)$ and we shall construct $v^\epsilon \in H^1_L(\Omega)$ such that $v^\epsilon \rightharpoonup v$ $w - H^1(\Omega)$ and $\limsup F^\epsilon(v^\epsilon) \leq F(v)$ (actually such that $F^\epsilon(v^\epsilon) \to F(v)$).

We define v^ϵ as the unique, in $H^1_L(\Omega)$, solution of

$$\sum_{j=1}^{N} a^\epsilon_{1j} \frac{\partial v^\epsilon}{\partial x_j} = \sum_{j=1}^{N} a_{1j} \frac{\partial v}{\partial x_j} \quad \text{in} \quad \Omega \tag{4.1}$$

(explicitly given by (4.7) below). Then

$$\int_\Omega \frac{1}{a^\epsilon_{11}} \left(\sum_{j=1}^{N} a^\epsilon_{1j} \frac{\partial v^\epsilon}{\partial x_j} \right)^2 dx = \int_\Omega \frac{1}{a^\epsilon_{11}} \left(\sum_{j=1}^{N} a_{1j} \frac{\partial v}{\partial x_j} \right)^2 dx \tag{4.2}$$

$$\longrightarrow \int_\Omega \frac{1}{a_{11}} \left(\sum_{j=1}^{N} a_{1j} \frac{\partial v}{\partial x_j} \right)^2 dx$$

as $\epsilon \to 0$ by (2.14).

Write (4.1) in the form

$$\frac{\partial v^\epsilon}{\partial x_1} + \sum_{j \geq 2} \frac{a^\epsilon_{1j}}{a^\epsilon_{11}} \frac{\partial v^\epsilon}{\partial x_j} = \sum_{j \geq 1} \frac{a_{1j}}{a^\epsilon_{11}} \frac{\partial v}{\partial x_j} \tag{4.3}$$

and consider the vector field

$$\frac{\partial}{\partial x_1} + \sum_{j \geq 2} \frac{a^\epsilon_{1j}}{a^\epsilon_{11}} \frac{\partial}{\partial x_j} \qquad \left(\text{or } \left(1, \frac{a^\epsilon_{12}}{a^\epsilon_{11}}, \ldots, \frac{a^\epsilon_{1N}}{a^\epsilon_{11}} \right) \right).$$

It has integral curves

$$\left\{\begin{array}{l} x_1 = t \\ x_j = y_j^\epsilon(t) \qquad (j \geq 2) \end{array}\right. \tag{4.4}$$

where

$$y_j^\epsilon(t) = \int_0^t \frac{a_{1j}^\epsilon(s)}{a_{11}^\epsilon(s)}\, ds + y_j^\epsilon(0). \tag{4.5}$$

The solution of (4.3) can be written

$$v^\epsilon\left(t, y^\epsilon(t)\right) = \int_0^t \sum_{j\geq 1} \frac{a_{1j}(\tau)}{a_{11}^\epsilon(\tau)} \frac{\partial v}{\partial x_j}\left(\tau, y^\epsilon(\tau)\right) dt. \tag{4.6}$$

Inverting (4.4), (4.5) gives

$$\left\{\begin{array}{lll} t & = & x_1 \\ y_j^\epsilon(0) & = & x_j - \int_0^{x_1} \frac{a_{1j}^\epsilon(s)}{a_{11}^\epsilon(s)}\, ds, \end{array}\right.$$

by which (4.6) becomes

$$v^\epsilon(x) = \int_0^{x_1} \sum_{j\geq 1} \frac{a_{1j}(\tau)}{a_{11}^\epsilon(\tau)} \frac{\partial v}{\partial x_j} \left(\tau, x_2 - \int_\tau^{x_1} \frac{a_{12}^\epsilon(s)}{a_{11}^\epsilon(s)} ds, \dots, x_N - \int_\tau^{x_1} \frac{a_{1N}^\epsilon(s)}{a_{11}^\epsilon(s)} ds\right) d\tau. \tag{4.7}$$

Also, for $k \geq 2$,

$$\frac{\partial v^\epsilon}{\partial x_k}(x) = \int_0^{x_1} \sum_{j\geq 1} \frac{a_{1j}(\tau)}{a_{11}^\epsilon(\tau)} \frac{\partial^2 v}{\partial x_j \partial x_k} \left(\tau, x_2 - \int_\tau^{x_1} \frac{a_{12}^\epsilon(s)}{a_{11}^\epsilon(s)} ds, \dots, x_N - \int_\tau^{x_1} \frac{a_{1N}^\epsilon(s)}{a_{11}^\epsilon(s)} ds\right) d\tau,$$

and similarly for higher derivatives in $x_2, \dots, x_N$.

From (4.7), using (2.14), (2.15) we see that v^ϵ is smooth in the variables $x_2, \dots, x_N$ with

$$\|v^\epsilon\|_{L^\infty} \leq C \qquad \text{and} \qquad \|D^\alpha v^\epsilon\|_{L^\infty} \leq C_\alpha \tag{4.8}$$

(with C and C_α independent of ϵ) for any derivative in these variables (i.e. $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multiindex with $\alpha_1 = 0$). By (4.3), (4.8), (2.14), (2.15) we have, for $h > 0$,

$$\begin{aligned} |v^\epsilon(x_1 + h, y) - v^\epsilon(x_1, y)| &\leq \int_{x_1}^{x_1+h} \left|\frac{\partial v^\epsilon}{\partial x_1}(t, y)\right| dt \\ &\leq C\left(h + \sum_{j\geq 2} \int_{x_1}^{x_1+h} \left|\frac{a_{1j}^\epsilon(t)}{a_{11}^\epsilon(t)}\right| dt\right) \leq C\, h^{1/2} \end{aligned}$$

and by (4.8)

$$|v^\epsilon(x_1,y) - v^\epsilon(x_1,y')| \le C\,|y-y'|$$

(C denoting various constants).

It follows that the family $\{v^\epsilon\}$ is equicontinuous on $\overline{\Omega}$. Similarly, $\{\partial v^\epsilon/\partial x_j\}$ is equicontinuous on $\overline{\Omega}$ for any $j \ge 2$. For fixed x and τ

$$\frac{\partial v}{\partial x_j}\left(\tau, x_2 - \int_\tau^{x_1} \frac{a_{12}^\epsilon(s)}{a_{11}^\epsilon(s)}\,ds, \dots, x_N - \int_\tau^{x_1} \frac{a_{1N}^\epsilon(s)}{a_{11}^\epsilon(s)}\,ds\right)$$

$$\longrightarrow \frac{\partial v}{\partial x_j}\left(\tau, x_2 - \int_\tau^{x_1} \frac{a_{12}(s)}{a_{11}(s)}\,ds, \dots, x_N - \int_\tau^{x_1} \frac{a_{1N}(s)}{a_{11}(s)}\,ds\right)$$

as $\epsilon \to 0$ by (2.15). Since these functions are uniformly bounded we also have convergence in the L^1-norm with respect to τ for fixed x. By (4.7) and (2.14) this implies that $v^\epsilon(x) \to v(x)$ pointwise. Similarly, $\partial v^\epsilon/\partial x_j \to \partial v/\partial x_j$ pointwise for $j \ge 2$.Combining this with the uniform equicontinuity and using the Ascoli-Arzela theorem it follows that

$$v^\epsilon \longrightarrow v \tag{4.9}$$

$$\frac{\partial v^\epsilon}{\partial x_j} \longrightarrow \frac{\partial v}{\partial x_j} \qquad (j \ge 2) \tag{4.10}$$

uniformly on $\overline{\Omega}$ as $\epsilon \to 0$. By (4.10) also

$$\frac{\partial v^\epsilon}{\partial x_i}\frac{\partial v^\epsilon}{\partial x_j} \longrightarrow \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} \qquad (i,j \ge 2) \tag{4.11}$$

uniformly on $\overline{\Omega}$ as $\epsilon \to 0$.

Now we can conclude that $F^\epsilon(v^\epsilon) \to F(v)$. Writing (see (2.8))

$$F^\epsilon(v^\epsilon) = \int_\Omega \frac{1}{a_{11}^\epsilon}\left(\sum_{j\ge1} a_{1j}^\epsilon \frac{\partial v^\epsilon}{\partial x_j}\right)^2 dx + \int_\Omega \sum_{i,j\ge2} b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_i}\frac{\partial v^\epsilon}{\partial x_j}\,dx - 2\int_\Omega f^\epsilon v^\epsilon\,dx$$

the three terms converge separately to the corresponding terms in $F(v)$ by (4.2), (4.11) and (2.16), (4.9) and (2.17).

To prove that $v^\epsilon \rightharpoonup v \quad w - H^1(\Omega)$ it just remains to see that $\partial v^\epsilon/\partial x_1 \rightharpoonup \partial v/\partial x_1$ $w - L^2(\Omega)$ and this follows easily from (4.3), (4.10), (2.14) and (2.15). This finishes the proof of Lemma 1.

To prove the corollary it is enough to prove that for every $0 \leq \varphi \in \mathcal{D}(\Omega)$ and every $\xi \in \mathbf{R}^N$ we have

$$\int_\Omega \sum_{i,j\geq 1} a_{ij}\, \xi_i\, \xi_j\, \varphi\, dx \geq \alpha\, |\xi|^2 \int_\Omega \varphi\, dx.$$

So let $0 \leq \varphi \in \mathcal{D}(\Omega)$ and $\xi \in \mathbf{R}^N$ be given. Then we can choose $v \in H^1_L(\Omega) \cap C^\infty(\mathbf{R}^N)$ such that $\nabla v = \xi$ on supp φ, and we shall prove that

$$F_\varphi(v) := \int \sum a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}\, \varphi\, dx \geq \alpha \int |\nabla v|^2\, \varphi\, dx.$$

Define F^ϵ_φ similarly (a^ϵ_{ij} in place of a_{ij}). Clearly the proof of Lemma 1 can be repeated with $F_\varphi, F^\epsilon_\varphi$ in place of F and F^ϵ, and we obtain $v^\epsilon \in H^1_L(\Omega)$ with

$$\begin{aligned} &v^\epsilon \rightharpoonup v \qquad w - H^1_L(\Omega), \\ &F^\epsilon_\varphi(v^\epsilon) \to F_\varphi(v). \end{aligned} \tag{4.12}$$

Since (4.12) implies $\liminf \int_\Omega |\nabla v^\epsilon|^2\, \varphi\, dx \geq \int_\Omega |\nabla v|^2\, \varphi\, dx$ and the $\left(a^\epsilon_{ij}\right)$ are α-coercive we now get

$$\begin{aligned} F_\varphi(v) &= \lim F^\epsilon_\varphi(v^\epsilon) \\ &\geq \liminf \alpha \int_\Omega |\nabla v^\epsilon|^2\, \varphi\, dx \geq \alpha \int_\Omega |\nabla v|^2\, \varphi\, dx \end{aligned}$$

as desired.

5. Proof of Lemma 2: Assume

$$v^\epsilon \rightharpoonup v \qquad w - H^1_L(\Omega), \tag{5.1}$$

$$\frac{\partial v^\epsilon}{\partial x_1} + \sum_{j\geq 2} \frac{a^\epsilon_{1j}}{a^\epsilon_{11}} \frac{\partial v^\epsilon}{\partial x_j} \rightharpoonup \frac{\partial v}{\partial x_1} + \sum_{j\geq 2} \frac{a_{1j}}{a_{11}} \frac{\partial v}{\partial_{x_j}} \qquad w - L^2(\Omega). \tag{5.2}$$

and we shall prove that

$$\liminf F^\epsilon(v^\epsilon) \geq F(v). \tag{5.3}$$

Writing again

$$\begin{aligned} F^\epsilon(v^\epsilon) = &\int_\Omega \frac{1}{a^\epsilon_{11}} \left(\sum_{j\geq 1} a^\epsilon_{1j} \frac{\partial v^\epsilon}{\partial x_j} \right)^2 dx \\ &+ \int_\Omega \sum_{i,j\geq 2} b^\epsilon_{ij} \frac{\partial v^\epsilon}{\partial x_i} \frac{\partial v^\epsilon}{\partial x_j}\, dx - 2 \int_\Omega f^\epsilon v^\epsilon\, dx \end{aligned} \tag{5.4}$$

we shall actually show that (5.3) holds for each term separately.

For the first term in (5.4) a convexity argument combined with (5.2) and (2.14) gives

$$\begin{aligned}\int_\Omega \frac{1}{a_{11}^\epsilon}\left(\sum_{j\geq 1} a_{1j}^\epsilon \frac{\partial v^\epsilon}{\partial x_j}\right)^2 dx \geq \int_\Omega \frac{1}{a_{11}^\epsilon}\left(\sum_{j\geq 1} a_{1j} \frac{\partial v}{\partial x_j}\right)^2 dx \\ + 2\int_\Omega \left(\sum_{j\geq 1} a_{1j}\frac{\partial v}{\partial x_j}\right)\left(\frac{\partial v^\epsilon}{\partial x_1} + \sum_{j\geq 2}\frac{a_{1j}^\epsilon}{a_{11}^\epsilon}\frac{\partial v^\epsilon}{\partial x_j} - \sum_{j\geq 1}\frac{a_{1j}}{a_{11}^\epsilon}\frac{\partial v}{\partial x_j}\right) dx \\ \longrightarrow \int_\Omega \frac{1}{a_{11}}\left(\sum_{j\geq 1} a_{1j}\frac{\partial v}{\partial x_j}\right)^2 dx\end{aligned}$$

as desired. The third term in (5.4) tends to $2\int_\Omega fv\,dx$ by (5.1) and (2.17).

The difficult part of the proof is to handle the second term in (5.4), i.e. to prove that

$$\liminf \int_\Omega \sum b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_i}\frac{\partial v^\epsilon}{\partial x_j}\, dx \geq \int_\Omega \sum b_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx \tag{5.5}$$

(the indices i and j, and later on r and ν, will from now on always range over $2,\ldots,N$). Clearly it is enough to prove (5.5) for every subsequence of ϵ for which

$$\int_\Omega \sum b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_i}\frac{\partial v^\epsilon}{\partial x_j}\, dx \leq C < \infty. \tag{5.6}$$

From now on ϵ denotes any such subsequence.

Utilizing an idea which goes back at least to Chabi [3], (but which has here been considerably developed) we shall approximate the vector fields $(\partial v^\epsilon/\partial x_j)_{j\geq 2}$ by vector fields ξ^ϵ which are step functions with respect to x_1. For any $m = 1, 2, \ldots$ and $1 \leq k \leq m$ let

$$I_k = I_{k,m} = \left[\frac{k-1}{m}, \frac{k}{m}\right].$$

The vector $\xi^\epsilon = \xi^{\epsilon,m}(x) = (\xi_2^\epsilon, \ldots, \xi_N^\epsilon)$ will as a function of x_1 be constant on each interval

I_k. (The index m will often be dropped from the notation.) We have

$$\int_\Omega \sum b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_i}\frac{\partial v^\epsilon}{\partial x_j}\,dx - \int_\Omega \sum b_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx$$

$$= \int_\Omega \sum b_{ij}^\epsilon \left(\frac{\partial v^\epsilon}{\partial x_i}\frac{\partial v^\epsilon}{\partial x_j} - \xi_i^\epsilon \xi_j^\epsilon\right) dx + \int_\Omega \sum \left(b_{ij}^\epsilon - b_{ij}\right)\xi_i^\epsilon\xi_j^\epsilon\,dx$$

$$+ \int_\Omega \sum b_{ij}\left(\xi_i^\epsilon\xi_j^\epsilon - \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\right) dx$$

$$\geq 2\int_\Omega \sum b_{ij}^\epsilon \xi_i^\epsilon \left(\frac{\partial v^\epsilon}{\partial x_j} - \xi_j^\epsilon\right) dx \tag{5.7}$$

$$+ \int_\Omega \sum \left(b_{ij}^\epsilon - b_{ij}\right)\xi_i^\epsilon\xi_j^\epsilon\,dx \tag{5.8}$$

$$+ 2\int_\Omega \sum b_{ij}\frac{\partial v}{\partial x_i}\left(\xi_j^\epsilon - \frac{\partial v}{\partial x_j}\right) dx \tag{5.9}$$

and we will choose $\xi^\epsilon = \xi^{\epsilon,m}$ so that each of (5.7), (5.8), (5.9) tends to zero as $m \to \infty$, $\epsilon \to 0$.

Integrating $(2.10)^\epsilon$ over I_k shows that the matrix $\left(\int_{I_k} b_{ij}^\epsilon\,dt\right)_{i,j\geq 2}$ is positive definite with eigenvalues $\geq \alpha\,|I_k| = \alpha/m$, in particular is invertible. Let $\left(\beta_{ij}^\epsilon\right)_{i,j\geq 2}$ be its inverse. Thus for each fixed ϵ, $m, k, y = (x_2, \ldots, x_N)$ the system

$$\sum_r \xi_r \int_{I_k} b_{ir}^\epsilon\,dt = \sum_j \int_{I_k} b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_j}\,dt \tag{5.10}$$

$(i \geq 2)$ is uniquely solvable for ξ_r:

$$\xi_r = \sum_{i,j} \beta_{ri}^\epsilon \int_{I_k} b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_j}\,dt$$

$(r \geq 2)$. This defines $\xi_r = \xi_r^\epsilon = \xi_r^{\epsilon,m}(y)$ for $x_1 \in I_k$. Hence ξ^ϵ is defined on Ω as a step function in x_1. The system (5.10) is chosen so that the first term (5.7) above vanishes.

As to (5.8) we can , by (2.22) for any m choose $\epsilon(m) > 0$ such that

$$\left|\int_{I_k} \left(b_{ij}^\epsilon - b_{ij}\right) dt\right| < \frac{1}{m^2}$$

whenever $1 \leq k \leq m, 0 < \epsilon < \epsilon(m)$. This gives

$$\left|\int_\Omega \sum_{i,j} (b_{ij}^\epsilon - b_{ij}) \xi_i^\epsilon \xi_j^\epsilon \, dx\right| = \left|\sum_{k=1}^m \sum_{i,j} \int_\omega (\xi_i^\epsilon \xi_j^\epsilon)\Big|_{I_k} \, dy \int_{I_k} (b_{ij}^\epsilon - b_{ij}) \, dt\right|$$

$$\leq \sum_{k=1}^m \sum_{i,j} \left|\int_\omega (\xi_i^\epsilon \xi_j^\epsilon)\Big|_{I_k} \, dy\right| \left|\int_{I_k} (b_{ij}^\epsilon - b_{ij}) \, dt\right|$$

$$\leq \frac{1}{m^2} \sum_{k,i,j} \left|\int_\omega (\xi_i^\epsilon \xi_j^\epsilon)\Big|_{I_k} \, dy\right| \leq \frac{1}{m} \sum_{i,j} \int_\Omega |\xi_i^\epsilon \xi_j^\epsilon| \, dx$$

$$\leq \frac{C}{m} \|\xi^{\epsilon,m}\|^2_{L^2(\Omega)^{N-1}}$$

for $0 < \epsilon < \epsilon(m)$. Thus the term (5.8) will tend to zero as $m \to \infty$, $0 < \epsilon < \epsilon(m)$ if just

(5.11) $$\|\xi^{\epsilon,m}\|_{L^2(\Omega)^{N-1}} \leq C < \infty.$$

This estimate will shortly be proven.

We shall also prove that (after possibly redefining $\epsilon(m)$)

(5.12) $$\xi^{\epsilon,m} \rightharpoonup \left(\frac{\partial v}{\partial x_2}, \dots, \frac{\partial v}{\partial x_N}\right) \quad \text{in the sense of distributions}$$

as $m \to \infty$, $0 < \epsilon < \epsilon(m)$. Combining (5.12) with (5.11) shows that $\xi^{\epsilon,m} \rightharpoonup \left(\frac{\partial v}{\partial x_2}, \dots, \frac{\partial v}{\partial x_N}\right)$ $w - L^2(\Omega)^{N-1}$ and hence that the term (5.9) tends to zero as $m \to \infty$, $0 < \epsilon < \epsilon(m)$. Thus it just remains to prove (5.11) and (5.12).

To prove (5.11) we estimate, in I_k,

(5.13) $$|\xi_r^\epsilon|^2 = \left(\int_{I_k} \sum_{i,j} \beta_{ri}^\epsilon b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_j} \, dt\right)^2$$

$$\leq \int_{I_k} \sum_{i,j} b_{ij}^\epsilon \beta_{ri}^\epsilon \beta_{rj}^\epsilon \, dt \cdot \int_{I_k} \sum_{i,j} b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_i} \frac{\partial v^\epsilon}{\partial x_j} \, dt$$

$$= \sum_j \beta_{rj}^\epsilon \delta_{rj} \cdot \int_{I_k} \sum_{i,j} b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_i} \frac{\partial v^\epsilon}{\partial x_j} \, dt$$

$$= \beta_{rr}^\epsilon \int_{I_k} \sum_{i,j} b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_i} \frac{\partial v^\epsilon}{\partial x_j} \, dt.$$

In the first step above we used the Cauchy-Schwarz inequality in matrix-form with the positive definite matrix (b_{ij}^ϵ) as weight matrix. In the second step we simply used the definition of (β_{ij}^ϵ). Since (β_{ij}^ϵ) is positive definite with eigenvalues $\leq m/\alpha$ (cf. $(2.10)^\epsilon$) we have $\beta_{rr}^\epsilon \leq Cm$. Thus (5.13) gives

$$\begin{aligned}
\|\xi^\epsilon\|^2_{L^2(\Omega)^{N-1}} &= \sum_r \int_\omega dy \sum_{k=1}^m |I_k| \left|\xi_r^\epsilon\Big|_{I_k}\right|^2 \\
&\leq C \int_\omega dy \sum_{k=1}^m \int_{I_k} \sum_{i,j} b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_i} \frac{\partial v^\epsilon}{\partial x_j}\, dt \\
&= C \int_\Omega \sum_{i,j} b_{ij}^\epsilon \frac{\partial v^\epsilon}{\partial x_i} \frac{\partial v^\epsilon}{\partial x_j}\, dx,
\end{aligned}$$

which by (5.6) proves (5.11).

To prove (5.12) let $\psi \in \mathcal{D}(\Omega)$ be a test function. Then for each $r = 2, \dots, N$

$$\begin{aligned}
\left\langle \xi_r^\epsilon - \frac{\partial v^\epsilon}{\partial x_r}, \psi \right\rangle &= \int_\omega \sum_{k=1}^m \int_{I_k} \left(\sum_{i,j} \beta_{ri}^\epsilon \int_{I_k} b_{ij}^\epsilon(\tau) \frac{\partial v^\epsilon(\tau, y)}{\partial x_j}\, d\tau - \frac{\partial v^\epsilon(t,y)}{\partial x_r} \right) \psi(t,y) dt\, dy \\
&= \int_\omega \sum_{k=1}^m \int_{I_k} \sum_{i,j} \beta_{ri}^\epsilon \int_{I_k} b_{ij}^\epsilon(\tau) \frac{\partial}{\partial x_j} \left(v^\epsilon(\tau, y) - v^\epsilon(t,y)\right) d\tau\, \psi(t,y)\, dt\, dy \\
&= \int_\omega \sum_{k=1}^m \int_{I_k} \sum_{i,j} \beta_{ri}^\epsilon \int_{I_k} b_{ij}^\epsilon(\tau) \int_\tau^t \frac{\partial v^\epsilon}{\partial x_1}(s,y)\, ds\, d\tau \frac{\partial \psi}{\partial x_j}(t,y)\, dt\, dy.
\end{aligned}$$

Using the inequality corresponding to (2.11) for β_{ri}^ϵ, $|\beta_{ri}^\epsilon| \leq \sum \beta_{\nu\nu}^\epsilon \leq C_1 m$, we thus get

$$\begin{aligned}
\left|\left\langle \xi_r^\epsilon - \frac{\partial v^\epsilon}{\partial x_r}, \psi \right\rangle\right| &\leq \int_\omega \sum_{k=1}^m \int_{I_k} \int_{I_k} \sum_{i,j} |\beta_{ri}^\epsilon|\, |b_{ij}^\epsilon(\tau)| \left|\frac{\partial \psi}{\partial x_j}(t,y)\right| \int_{I_k} \left|\frac{\partial v^\epsilon}{\partial x_1}(s,y)\right| ds\, d\tau\, dt\, dy \\
&\leq \int_\omega \sum_{k=1}^m \int_{I_k} \int_{I_k} \sum_{i,j} C_1 m \sum_\nu b_{\nu\nu}^\epsilon(\tau) \cdot C_2 \int_{I_k} \left|\frac{\partial v^\epsilon}{\partial x_1}\right| ds\, d\tau\, dt\, dy \\
&\leq C_3 m \cdot \int_\omega \sum_{k=1}^m \int_{I_k} dt \int_{I_k} \sum_\nu b_{\nu\nu}^\epsilon(\tau)\, d\tau \int_{I_k} \left|\frac{\partial v^\epsilon}{\partial x_1}\right| ds\, dy \\
&\leq C_3 \cdot \max_k \sum_\nu \int_{I_k} b_{\nu\nu}^\epsilon(\tau)\, d\tau \cdot \int_\Omega \left|\frac{\partial v^\epsilon}{\partial x_1}\right| dx
\end{aligned}$$

for suitable constants $C_1, \dots, C_3$.

Setting $M = 1 + \sum_{i \geq 2} \|b_{ii}\|_{L^\infty}$ it follows from (2.22) that given any m we can find $\epsilon'(m) > 0\,(0 < \epsilon'(m) < \epsilon(m))$ so small that

$$\sum_i \int_{I_k} b_{ii}^\epsilon(\tau)\,d\tau \leq \frac{1}{m} + \sum_i \int_{I_k} b_{ii}(\tau)\,d\tau \leq \frac{M}{m}$$

for $1 \leq k \leq m$ whenever $0 < \epsilon < \epsilon'(m)$. Since $\int_\Omega \left|\frac{\partial v^\epsilon}{\partial x_1}\right| dx$ is bounded as $\epsilon \to 0$ we therefore get

$$\left|\left\langle \xi_r^\epsilon - \frac{\partial v^\epsilon}{\partial x_r}, \psi \right\rangle\right| \leq \frac{C}{m}$$

whenever $0 < \epsilon < \epsilon'(m)$. This proves (5.12), finishing the proof of Lemma 2.

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