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A criterion for H-convergence in elasticity

Björn Gustafsson [a] and Jacqueline Mossino [b]
[a] *Department of Mathematics, Royal Institute of Technology, S-100 44 Stockholm, Sweden*
E-mail: gbjorn@kth.se
[b] *C.M.L.A., Ecole Normale Supérieure de Cachan, 61, Avenue du Président Wilson,*
94235 Cachan cedex, France

Abstract. We give a criterion for H-convergence of elasticity tensors in terms of ordinary weak convergence of the factors in certain quotient representations of the tensors.

Keywords: nonperiodic homogenization, compensated compactness, corrector

1. Introduction

Homogenization of rapidly varying coefficients in elliptic partial differential equations has been studied by mathematicians at least since the 1970s, and by physicists and engineers much longer (see [17, 2,16] and references therein). A typical example is when the conductivity matrix $A^\varepsilon = A^\varepsilon(x)$ in an equation

$$-\mathrm{div}\big(A^\varepsilon \nabla u^\varepsilon\big) = f \tag{1.1}$$

in some (bounded) domain $\Omega \subset \mathbb{R}^N$ oscillates rapidly at length scale $\varepsilon > 0$ and one wants to identify a limit matrix $A = A(x)$ (presumably less oscillating than the A^ε) such that, as $\varepsilon \to 0$, the solutions u^ε converge in some weak sense to the solution u of the corresponding homogenized equation:

$$-\mathrm{div}(A\nabla u) = f.$$

In the 1970s and 1980s, F. Murat and L. Tartar identified the appropriate type of convergence, H-convergence, for the above type of problems, and started developing a general theory for it (see [13–15,18–20]). Earlier work was much concerned with special cases, like strictly periodic structures, and in fact the study of periodic structures has drawn most of the attention in homogenization even in recent time, because of its numerous industrial applications.

The present paper deals with general homogenization in the spirit of the work of F. Murat and L. Tartar. The first part (Section 2), extends the definitions and main properties of H-convergence to the framework of systems, occurring for example in linear elasticity. There are several definitions of H-convergence available in the literature. Some of them are formulated in terms of solutions of the partial differential equations (1.1) (referring now to the diffusion case) while others are formulated directly in terms of the algebraic constitutive equation

$$D^\varepsilon = A^\varepsilon E^\varepsilon. \tag{1.2}$$

The latter relates, in an electrostatic language, the displacement field D^ε, for which one has control of $\operatorname{div} D^\varepsilon$ (by (1.1)), with the electric field E^ε, for which $\operatorname{curl} E^\varepsilon$ is under control (in fact vanishes, because $E^\varepsilon = \nabla u^\varepsilon$). Most definitions involve a localization argument ($\omega \Subset \Omega$). In Section 3 of the paper we give a unified presentation of the theory of H-convergence in linear elasticity, avoiding the localization argument, and we prove the equivalence of three different definitions of H-convergence. In linear elasticity the constitutive relation ("Hooke's law") is still of the form (1.2), but now with A^ε a fourth-order tensor (the elasticity tensor) and D^ε, E^ε the stress and strain tensors respectively (both symmetric second-order tensors).

In Section 4, we state our main result, which is a criterion for H-convergence of elasticity tensors in terms of ordinary weak convergence of the factors in certain quotient representations of the tensors. More precisely, we prove that a sequence of tensors A^ε H-converges to a tensor A if and only if there exist tensors M^ε, M, P^ε, P with entries in L^2 and with M invertible, such that

$$\begin{array}{ll} M^\varepsilon A^\varepsilon = P^\varepsilon, & MA = P, \\ M^\varepsilon \rightharpoonup M \quad \text{weakly in } L^2, & P^\varepsilon \rightharpoonup P \quad \text{weakly in } L^2, \\ \{\operatorname{curl} M^\varepsilon\}_{\varepsilon>0} \quad \text{relatively compact in } H^{-1}, & \{\operatorname{div} P^\varepsilon\}_{\varepsilon>0} \quad \text{relatively compact in } H^{-1}, \end{array}$$

with curl and div defined in Section 2.4. Section 5 is devoted to a "corrector" result, while applications to laminates and periodic homogenization are given in Sections 6 and 7, allowing to recover some well known formulae.

Explicit representations of the type $M^\varepsilon A^\varepsilon = P^\varepsilon$ have been constructed and used for proving H-convergence in a series of papers [3–5,7]. The purpose of this article is to point out that the existence of such quotient representations is a completely general fact in connection with H-convergence. In the diffusion case, the corresponding result was announced in [9]. The proof consists of an adaptation of methods developed by F. Murat and L. Tartar (see, e.g., [13–15,18–20]). Compensated compactness is the crucial technical tool used throughout the paper. Although our results are not entirely new, they give a hopefully fruitful general point of view on linear homogenization, based solely on compensated compactness and quotient representations as above.

2. Preliminaries

2.1. Notation

In this paper we use the Einstein convention of repeated indices. We consider the scalar product $[\cdot,\cdot]$ and the norm $|\cdot|$ defined on $\mathbb{R}^{N\times N}$ by

$$[\xi,\eta] = \xi_{ij}\eta_{ij} \quad \text{and} \quad |\xi|^2 = [\xi,\xi].$$

Besides we use the following product of $N\times N\times N\times N$-tensors: by definition $Q = MP$ is the tensor with entries

$$Q_{ijkl} = M_{ijmn}P_{mnkl}.$$

We denote by A (or A^ε) a $N\times N\times N\times N$ tensor with real coefficients A_{ijkl} for i,j,k,l in $\{1,2,\dots,N\}$, satisfying the symmetry assumptions

$$A_{ijkl} = A_{jikl} = A_{ijlk}, \quad \forall i,j,k,l \in \{1,2,\dots,N\}.$$

We also identify A with the linear map:

$$\xi \in \mathbb{R}^{N\times N} \to \eta = A\xi \in \mathbb{R}^{N\times N} \quad \text{with } \eta_{ij} = A_{ijkl}\xi_{kl}.$$

Since $A_{ijkl} = A_{jikl}$, it maps $\mathbb{R}^{N\times N}$ into $\mathbb{R}_s^{N\times N}$, the set of symmetric matrices with real coefficients. In particular it maps $\mathbb{R}_s^{N\times N}$ into $\mathbb{R}_s^{N\times N}$.

Moreover the above tensor A is supposed to satisfy

$$[A\xi, \xi] \geqslant \alpha|\xi|^2, \quad \forall \xi \in \mathbb{R}_s^{N\times N},$$

for some positive α, and then A is invertible on $\mathbb{R}_s^{N\times N}$.

2.2. *Definition of* $M(\alpha, \beta, \Omega)$

From now on, Ω is a bounded open set in $\mathbb{R}^N$. For given α, β such that $0 < \alpha \leqslant \beta < +\infty$, $M(\alpha, \beta, \Omega)$ denotes the set of $N \times N \times N \times N$ tensors with measurable coefficients $A_{ijkl}(x)$ satisfying

$$\text{a.e. } x \in \Omega,\ \forall i, j, k, l \in \{1, 2, \ldots, N\}, \quad A_{ijkl}(x) = A_{jikl}(x) = A_{ijlk}(x), \tag{2.1}$$

$$\text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}_s^{N\times N}, \quad [A(x)\xi, \xi] \geqslant \alpha|\xi|^2 \tag{2.2}$$

(hence $A(x)$ is invertible on $\mathbb{R}_s^{N\times N}$) and

$$\text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}_s^{N\times N}, \quad [A(x)^{-1}\xi, \xi] \geqslant \beta^{-1}|\xi|^2. \tag{2.3}$$

The latter inequality implies that

$$\text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}_s^{N\times N}, \quad |A(x)\xi| \leqslant \beta|\xi|. \tag{2.4}$$

Because of the symmetries (2.1), the bound (2.4) remains true for nonsymmetric $\xi \in \mathbb{R}^{N\times N}$, as is seen by applying (2.1) to the symmetrized part ξ_s of ξ: $|A(x)\xi| = |A(x)\xi_s| \leqslant \beta|\xi_s| \leqslant \beta|\xi|$.

2.3. *Definition of* div D *and* curl E *for* D, E *in* $\mathcal{D}'(\Omega)^{N\times N}$

We define div D and curl E as div and curl of the line (or row) vectors D_i and E_i of D and E. In other words, div and curl apply to the last index of D_{ij} and E_{ij}:

$$(\operatorname{div} D)_i = \operatorname{div} D_i = \frac{\partial D_{ij}}{\partial x_j},$$

$$(\operatorname{curl} E)_{ijk} = (\operatorname{curl} E_i)_{jk} = \frac{\partial E_{ij}}{\partial x_k} - \frac{\partial E_{ik}}{\partial x_j}.$$

2.4. Definition of div P *and* curl M *for* P, M *in* $\mathcal{D}'(\Omega)^{N\times N\times N\times N}$

Again div and curl apply to the last index: for $P, M \in \mathcal{D}'(\Omega)^{N\times N\times N\times N}$, we set $P_{ijk} = (P_{ijkl})_l$, $M_{ijk} = (M_{ijkl})_l$ and define div $P \in \mathcal{D}'(\Omega)^{N\times N\times N}$ and curl $M \in \mathcal{D}'(\Omega)^{N\times N\times N\times N\times N}$ by

$$(\operatorname{div} P)_{ijk} = \operatorname{div} P_{ijk} = \frac{\partial P_{ijkl}}{\partial x_l},$$
$$(\operatorname{curl} M)_{ijklm} = (\operatorname{curl} M_{ijk})_{lm} = \frac{\partial M_{ijkl}}{\partial x_m} - \frac{\partial M_{ijkm}}{\partial x_l}.$$

2.5. Definition of $e(w)$ *and* ∇w *for* w *in* $\mathcal{D}'(\Omega)^N$

By definition, $e(w)$ is the symmetrized gradient (in distributional sense) of w, that is $e(w)$ is the symmetric $N \times N$ matrix with coefficients in $\mathcal{D}'(\Omega)$ defined by

$$e_{ij}(w) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right).$$

On the contrary, the usual gradient of w is the $N \times N$ matrix ∇w with coefficients

$$\nabla_{ij} w = \frac{\partial w_i}{\partial x_j}.$$

2.6. Definition of H-convergence in elasticity

From now on, ε describes a sequence converging to zero. Let $0 < \alpha \leqslant \beta < +\infty$, $0 < \alpha' \leqslant \beta' < +\infty$. Let $A^\varepsilon \in M(\alpha, \beta, \Omega)$, $A \in M(\alpha', \beta', \Omega)$. We say that A^ε H-converges to A (and we write $A^\varepsilon \xrightarrow{H} A$) when ε tends to zero, if the following holds true:

Whenever $D^\varepsilon, D, E^\varepsilon, E \in L^2(\Omega)^{N\times N}$ satisfy

$$D^\varepsilon = A^\varepsilon E^\varepsilon, \tag{2.5}$$
$$D^\varepsilon \rightharpoonup D \quad \text{and} \quad E^\varepsilon \rightharpoonup E \quad \text{weakly in } L^2(\Omega)^{N\times N}, \tag{2.6}$$
$$\{\operatorname{div} D^\varepsilon\}_{\varepsilon>0} \text{ is relatively compact in } H^{-1}(\Omega)^N, \tag{2.7}$$
$$\{\operatorname{curl} E^\varepsilon\}_{\varepsilon>0} \text{ is relatively compact in } H^{-1}(\Omega)^{N\times N\times N}, \tag{2.8}$$

then

$$D = AE. \tag{2.9}$$

2.7. Definition of H_b-convergence

Let us denote by $L^2(\Omega)_s^{N\times N}$ the set of symmetric matrices of size $N \times N$ with entries in $L^2(\Omega)$. Let $0 < \alpha \leqslant \beta < +\infty$, $0 < \alpha' \leqslant \beta' < +\infty$. Let $A^\varepsilon \in M(\alpha, \beta, \Omega)$, $A \in M(\alpha', \beta', \Omega)$. The subscript “$b$”

refers to “boundary conditions”. We say that A^ε H_b-converges to A (and we write $A^\varepsilon \xrightarrow{H_b} A$) as ε tends to zero, if for any $g \in L^2(\Omega)_s^{N\times N}$ and for any $h \in H^1(\Omega)^N$, the sequence of solutions $u^\varepsilon = u^\varepsilon(g,h)$ of

$$\begin{cases} u^\varepsilon - h \in H_0^1(\Omega)^N \text{ and } \forall w \in H_0^1(\Omega)^N, \\ \displaystyle\int_\Omega \left[A^\varepsilon e\left(u^\varepsilon\right), e(w)\right] \mathrm{d}x = \int_\Omega \left[g, e(w)\right] \mathrm{d}x \end{cases} \tag{2.10}$$

satisfy, as ε tends to zero,

$$\begin{cases} u^\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(\Omega)^N, \\ A^\varepsilon e\left(u^\varepsilon\right) \rightharpoonup Ae(u) \quad \text{weakly in } L^2(\Omega)^{N\times N}, \end{cases} \tag{2.11}$$

where $u = u(g,h)$ is the solution of

$$\begin{cases} u - h \in H_0^1(\Omega)^N \text{ and } \forall w \in H_0^1(\Omega)^N, \\ \displaystyle\int_\Omega \left[Ae(u), e(w)\right] \mathrm{d}x = \int_\Omega \left[g, e(w)\right] \mathrm{d}x. \end{cases} \tag{2.12}$$

Remark 1. From the symmetry properties of A^ε, A and g, it follows that $A^\varepsilon e(u^\varepsilon) = A^\varepsilon \nabla u^\varepsilon$, $Ae(u) = A\nabla u$ and that Eq. (2.10) is equivalent to

$$\begin{cases} u^\varepsilon - h \in H_0^1(\Omega)^N \text{ and } \forall w \in H_0^1(\Omega)^N, \\ \displaystyle\int_\Omega \left[A^\varepsilon \nabla u^\varepsilon, \nabla w\right] \mathrm{d}x = \int_\Omega [g, \nabla w]\, \mathrm{d}x, \end{cases} \tag{2.13}$$

the second line in (2.13) being the weak formulation of the system

$$\frac{\partial}{\partial x_j}\left(A_{ijkl}^\varepsilon \nabla_{kl} u^\varepsilon\right) = \frac{\partial g_{ij}}{\partial x_j} \quad \text{in } \Omega\ (\forall i = 1, \ldots, N). \tag{2.14}$$

For simplicity, we write (2.10) or (2.13) as

$$\begin{cases} \operatorname{div}\left(A^\varepsilon \nabla u^\varepsilon\right) = \operatorname{div} g \quad \text{in } \Omega, \\ u^\varepsilon - h \in H_0^1(\Omega)^N. \end{cases} \tag{2.15}$$

The same remark applies to (2.12).

2.8. *Definition of H_{loc}-convergence*

The following definition of H-convergence is more classical, at least in the diffusion framework (see, e.g., [13,15]). In this paper, we call it H_{loc}-convergence. Let $0 < \alpha \leqslant \beta < +\infty$, $0 < \alpha' \leqslant \beta' < +\infty$. Let $A^\varepsilon \in M(\alpha, \beta, \Omega)$, $A \in M(\alpha', \beta', \Omega)$. We say that A^ε H_{loc}-converges to A (and we write $A^\varepsilon \xrightarrow{H_{\mathrm{loc}}} A$) as ε tends to zero, if for any $\omega \Subset \Omega$ and for any $g \in L^2(\omega)_s^{N\times N}$, the sequence of solutions $u^\varepsilon = u^\varepsilon(\omega, g)$ of

$$\begin{cases} u^\varepsilon \in H_0^1(\omega)^N \text{ and } \forall w \in H_0^1(\omega)^N, \\ \displaystyle\int_\omega \left[A^\varepsilon e\left(u^\varepsilon\right), e(w)\right] \mathrm{d}x = \int_\omega \left[g, e(w)\right] \mathrm{d}x \end{cases} \tag{2.16}$$

is such that, as ε tends to zero,

$$\begin{cases} u^\varepsilon \rightharpoonup u \quad \text{weakly in } H_0^1(\omega)^N, \\ A^\varepsilon e(u^\varepsilon) \rightharpoonup Ae(u) \quad \text{weakly in } L^2(\omega)^{N\times N}, \end{cases} \tag{2.17}$$

where $u = u(\omega, g)$ is the solution of

$$\begin{cases} u \in H_0^1(\omega)^N \text{ and } \forall w \in H_0^1(\omega)^N, \\ \displaystyle\int_\omega [Ae(u), e(w)]\,\mathrm{d}x = \int_\omega [g, e(w)]\,\mathrm{d}x. \end{cases} \tag{2.18}$$

2.9. The div–curl lemma

The "div–curl lemma" of compensated compactness [13,19,6] will be used extensively throughout the paper. We recall that this lemma in general says that if $f^\varepsilon, g^\varepsilon, f, g \in L^2(\Omega)^n$ are vector fields such that $f^\varepsilon \rightharpoonup f$, $g^\varepsilon \rightharpoonup g$ weakly in $L^2(\Omega)^n$ and such that $\operatorname{div} f^\varepsilon$ and the components of $\operatorname{curl} g^\varepsilon$ are all contained in a compact subset of $H^{-1}(\Omega)$, then $[f^\varepsilon, g^\varepsilon] \rightharpoonup [f, g]$ weakly as distributions. (Here $[f, g] = f_i g_i$.)

3. Equivalence of H, H_b and H_{loc}-convergence and preliminary remarks

3.1. Comparison of H and H_b-convergence, uniqueness of the H-limit

Proposition 1. *H and H_b convergences are equivalent. Moreover*

$$A^\varepsilon \xrightarrow{H} A \quad \textit{is equivalent to} \quad {}^tA^\varepsilon \xrightarrow{H} {}^tA$$

with

$$\left({}^tA^\varepsilon\right)_{ijkl} = A^\varepsilon_{klij}, \quad \forall i, j, k, l \in \{1, \dots, N\}.$$

Proof. *Step 1*. In this step we prove that

$$A^\varepsilon \xrightarrow{H} A \quad \text{implies that} \quad A^\varepsilon \xrightarrow{H_b} A.$$

Let $g \in L^2(\Omega)_s^{N\times N}$, $h \in H^1(\Omega)^N$ and let u^ε be the solution of (2.10), $E^\varepsilon = \nabla u^\varepsilon$, $D^\varepsilon = A^\varepsilon \nabla u^\varepsilon$. Similarly, let u be the solution of (2.12), $E = \nabla u$, $D = A\nabla u$. Since $A^\varepsilon \in M(\alpha, \beta, \Omega)$, we get from (2.2), (2.4), (2.10),

$$\alpha \left\| e(u^\varepsilon - h) \right\|_{L^2(\Omega)^{N\times N}} \leqslant \left\| g - A^\varepsilon e(h) \right\|_{L^2(\Omega)^{N\times N}} \leqslant \|g\|_{L^2(\Omega)^{N\times N}} + \beta \|e(h)\|_{L^2(\Omega)^{N\times N}},$$

so that $e(u^\varepsilon - h)$ is bounded in $L^2(\Omega)^{N\times N}$ and, due to the Korn inequality, $u^\varepsilon - h$ is bounded in $H_0^1(\Omega)^N$, u^ε is bounded in $H^1(\Omega)^N$, E^ε and D^ε are bounded in $L^2(\Omega)^{N\times N}$. Hence, for some subsequence of ε

(still denoted by ε), for some $\underline{u} \in H^1(\Omega)^N$ ($\underline{u} - h \in H_0^1(\Omega)^N$) and for some $\underline{D} \in L^2(\Omega)^{N\times N}$,

$$u^\varepsilon \rightharpoonup \underline{u} \quad \text{weakly in } H^1(\Omega)^N,$$
$$E^\varepsilon = \nabla u^\varepsilon \rightharpoonup \underline{E} = \nabla \underline{u} \quad \text{weakly in } L^2(\Omega)^{N\times N},$$
$$D^\varepsilon = A^\varepsilon \nabla u^\varepsilon \rightharpoonup \underline{D} \quad \text{weakly in } L^2(\Omega)^{N\times N}.$$

Finally, from (2.13) it follows that

$$\int_\Omega [\underline{D}, \nabla w]\,\mathrm{d}x = \int_\Omega [g, \nabla w]\,\mathrm{d}x, \quad \forall w \in H_0^1(\Omega)^N.$$

It is enough to prove that $\underline{D} = A\nabla\underline{u}$ because then we have $\underline{u} = u$ and the whole sequences u^ε, E^ε and D^ε converge. But $\underline{D} = A\nabla\underline{u}$ follows from the H-convergence of A^ε to A and from the fact that (see (2.14))

$$\operatorname{div} D_i^\varepsilon = \frac{\partial D_{ij}^\varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left(A_{ijkl}^\varepsilon \nabla_{kl} u^\varepsilon\right) = \frac{\partial g_{ij}}{\partial x_j}$$

is a fixed element in $H^{-1}(\Omega)$ and

$$\operatorname{curl} E_i^\varepsilon = \operatorname{curl} \nabla u_i^\varepsilon = 0.$$

Step 2. Now we prove that

$$A^\varepsilon \xrightarrow{H_b} A \quad \text{is equivalent to} \quad {}^tA^\varepsilon \xrightarrow{H_b} {}^tA.$$

Assume $A^\varepsilon \xrightarrow{H_b} A$. Let $g \in L^2(\Omega)_s^{N\times N}$, $h \in H^1(\Omega)^N$ and let v^ε be the solution of

$$\begin{cases} v^\varepsilon - h \in H_0^1(\Omega)^N \text{ and } \forall w \in H_0^1(\Omega)^N, \\ \displaystyle\int_\Omega \left[{}^tA^\varepsilon \nabla v^\varepsilon, \nabla w\right]\mathrm{d}x = \int_\Omega [g, \nabla w]\,\mathrm{d}x, \end{cases}$$

which we write as before (see (2.15))

$$\begin{cases} \operatorname{div}\left({}^tA^\varepsilon \nabla v^\varepsilon\right) = \operatorname{div} g \quad \text{in } \Omega, \\ v^\varepsilon - h \in H_0^1(\Omega)^N. \end{cases} \tag{3.1}$$

It is clear (as in the beginning of Step 1) that, up to extraction of a subsequence, for some $v \in H^1(\Omega)^N$, $v - h \in H_0^1(\Omega)^N$, and or some $\sigma \in L^2(\Omega)^N$,

$$\begin{cases} v^\varepsilon \rightharpoonup v \quad \text{weakly in } H^1(\Omega)^N, \\ {}^tA^\varepsilon \nabla v^\varepsilon \rightharpoonup \sigma \quad \text{weakly in } L^2(\Omega)^{N\times N}, \\ \displaystyle\int_\Omega [\sigma, \nabla w]\,\mathrm{d}x = \int_\Omega [g, \nabla w]\,\mathrm{d}x, \quad \forall w \in H_0^1(\Omega)^N. \end{cases} \tag{3.2}$$

In order to prove that ${}^tA^\varepsilon \xrightarrow{H_b} {}^tA$, we have to prove that $\sigma = {}^tA\nabla v$.

Let us fix $i, j \in \{1, \dots, N\}$ and define $u_{ij} \in H^1(\Omega)^N$ by

$$u_{ij}(x) = \big(u_{ijk}(x)\big)_k \quad \text{with } u_{ijk}(x) = \delta_{ik}x_j. \tag{3.3}$$

It is clear that

$$\nabla_{kl}u_{ij} = \frac{\partial u_{ijk}}{\partial x_l} = \delta_{ik}\delta_{jl}. \tag{3.4}$$

Now we define u_{ij}^ε as the solution of

$$\begin{cases} \operatorname{div}\big(A^\varepsilon \nabla u_{ij}^\varepsilon\big) = \operatorname{div}(A\nabla u_{ij}) & \text{in } \Omega, \\ u_{ij}^\varepsilon - u_{ij} \in H_0^1(\Omega)^N. \end{cases} \tag{3.5}$$

Since $A^\varepsilon \xrightarrow{H_b} A$, we have

$$\begin{cases} u_{ij}^\varepsilon \rightharpoonup u_{ij} & \text{weakly in } H^1(\Omega)^N, \\ A^\varepsilon\nabla u_{ij}^\varepsilon \rightharpoonup A\nabla u_{ij} & \text{weakly in } L^2(\Omega)^{N\times N}. \end{cases} \tag{3.6}$$

By (3.1), (3.2), (3.5), (3.6) and compensated compactness,

$$\big[{}^t A^\varepsilon \nabla v^\varepsilon, \nabla u_{ij}^\varepsilon\big] \xrightarrow{\mathcal{D}'(\Omega)} [\sigma, \nabla u_{ij}]$$

and

$$\big[{}^t A^\varepsilon \nabla v^\varepsilon, \nabla u_{ij}^\varepsilon\big] = \big[\nabla v^\varepsilon, A^\varepsilon \nabla u_{ij}^\varepsilon\big] \xrightarrow{\mathcal{D}'(\Omega)} [\nabla v, A\nabla u_{ij}] = \big[{}^t A\nabla v, \nabla u_{ij}\big].$$

Hence

$$[\sigma, \nabla u_{ij}] = \big[{}^t A\nabla v, \nabla u_{ij}\big]$$

or, by (3.4),

$$\sigma_{kl}\delta_{ik}\delta_{jl} = \big({}^tA\nabla v\big)_{kl}\delta_{ik}\delta_{jl}$$

or $\sigma_{ij} = ({}^tA\nabla v)_{ij}$. As this is true for all i, j, we get $\sigma = {}^tA\nabla v$ and we conclude that ${}^tA^\varepsilon \xrightarrow{H_b} {}^tA$. Thus $A^\varepsilon \xrightarrow{H_b} A$ implies that ${}^tA^\varepsilon \xrightarrow{H_b} {}^tA$ and since ${}^t({}^tA^\varepsilon) = A^\varepsilon$ (etc.) we get the desired equivalence.

Step 3. Finally we prove that

$${}^tA^\varepsilon \xrightarrow{H_b} {}^tA \quad \text{implies that} \quad A^\varepsilon \xrightarrow{H} A.$$

Let again (for fixed $i, j \in \{1, \ldots, N\}$),

$$u_{ij}(x) = \big(u_{ijk}(x)\big)_k \quad \text{with } u_{ijk}(x) = \delta_{ik} x_j.$$

Now let u_{ij}^ε be the solution of

$$\begin{cases} \operatorname{div}\big({}^t A^\varepsilon\, \nabla u_{ij}^\varepsilon\big) = \operatorname{div}\big({}^t A\, \nabla u_{ij}\big) \quad \text{in } \Omega, \\ u_{ij}^\varepsilon - u_{ij} \in H_0^1(\Omega)^N. \end{cases} \tag{3.7}$$

Since ${}^t A^\varepsilon \xrightarrow{H_b} {}^t A$,

$$\begin{cases} u_{ij}^\varepsilon \rightharpoonup u_{ij} \quad \text{weakly in } H^1(\Omega)^N, \\ {}^t A^\varepsilon\, \nabla u_{ij}^\varepsilon \rightharpoonup {}^t A \nabla u_{ij} \quad \text{weakly in } L^2(\Omega)^{N\times N}. \end{cases} \tag{3.8}$$

Let $(D^\varepsilon, E^\varepsilon)$ be as in the definition of H-convergence. We have to prove that $D = AE$. But

$$[D^\varepsilon, \nabla u_{ij}^\varepsilon] = [A^\varepsilon E^\varepsilon, \nabla u_{ij}^\varepsilon] = [E^\varepsilon, {}^t A^\varepsilon \nabla u_{ij}^\varepsilon]$$

and by compensated compactness

$$[D^\varepsilon, \nabla u_{ij}^\varepsilon] \xrightarrow{\mathcal{D}'(\Omega)} [D, \nabla u_{ij}]$$

while, by (3.7), (3.8) and compensated compactness,

$$[E^\varepsilon, {}^t A^\varepsilon \nabla u_{ij}^\varepsilon] \xrightarrow{\mathcal{D}'(\Omega)} [E, {}^t A \nabla u_{ij}] = [AE, \nabla u_{ij}].$$

Hence $[D, \nabla u_{ij}] = [AE, \nabla u_{ij}]$, for all i, j, giving, as at the end of Step 2, $D = AE$. This finishes Step 3 and hence the entire proof. $\square$

Remark 2. In the same way as we have proved that H-convergence implies H_b-convergence, one can prove that H-convergence allows passing to the limit in many other boundary value problems in linear elasticity (e.g., with mixed Dirichlet and Neumann boundary conditions).

Proposition 2. *The H-limit is unique.*

Proof. By virtue of Proposition 1, this is equivalent to the uniqueness of the H_b-limit. So let us assume that $A^\varepsilon \xrightarrow{H_b} A$. Let $g_{ij} = A \nabla u_{ij}$ with $u_{ijk} = \delta_{ik} x_j$ and let u_{ij}^ε be the solution of (3.5). Then, by (3.6), for any B, H_b-limit of A^ε, we have

$$A^\varepsilon \nabla u_{ij}^\varepsilon \rightharpoonup B \nabla u_{ij} \quad \text{weakly in } L^2(\Omega)^{N\times N}.$$

In particular, $A \nabla u_{ij} = B \nabla u_{ij}$, that is $A_{klij} = B_{klij}$, first for all k, l, and then also for all i, j, that is $A = B$. $\square$

3.2. Comparison of H_b and H_{loc}-convergences and subsequent remarks

Proposition 3. *H_{loc} and H_b-convergences are equivalent.*

Proof. Let us first prove that H_{loc}-convergence ${}^tA^\varepsilon \xrightarrow{H_{\mathrm{loc}}} {}^tA$ implies H_b-convergence $A^\varepsilon \xrightarrow{H_b} A$. Let $g \in L^2(\Omega)_s^{N\times N}$, $h \in H^1(\Omega)^N$, let u^ε be the solution of (2.10). It is clear that, up to a subsequence of ε, for some $u \in H^1(\Omega)^N$, $u - h \in H_0^1(\Omega)^N$ and for some $\xi \in L^2(\Omega)^{N\times N}$, $\operatorname{div}\xi = \operatorname{div} g$,

$$
\begin{aligned}
&u^\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(\Omega)^N,\\
&A^\varepsilon \nabla u^\varepsilon \rightharpoonup \xi \quad \text{weakly in } L^2(\Omega)^{N\times N}.
\end{aligned}
$$

In order to prove H_b-convergence, it is enough to prove that $\xi = A\nabla u$.

Let us consider two subdomains ω_1 and ω of Ω, $\omega_1 \Subset \omega \Subset \Omega$. Let, for fixed $i, j \in \{1, \ldots, N\}$,

$$v_{ij}(x) = \big(v_{ijk}(x)\big)_k \quad \text{with } v_{ijk}(x) = \delta_{ik} x_j \varphi(x),$$

for some given $\varphi \in \mathcal{D}(\omega)$ with $\varphi = 1$ in ω_1. Now let $g_{ij} \in L^2(\omega)_s^{N\times N}$ be defined by $g_{ij} = {}^tA\nabla v_{ij}$ and let v_{ij}^ε be the solution of

$$
\begin{cases}
\operatorname{div}\big({}^tA^\varepsilon\, \nabla v_{ij}^\varepsilon\big) = \operatorname{div} g_{ij} \quad \text{in } \omega,\\
v_{ij}^\varepsilon \in H_0^1(\omega)^N.
\end{cases}
\tag{3.9}
$$

Since, by assumption, ${}^tA^\varepsilon \xrightarrow{H_{\mathrm{loc}}} {}^tA$, it follows that

$$
\begin{cases}
v_{ij}^\varepsilon \rightharpoonup v_{ij} \quad \text{weakly in } H_0^1(\omega)^N,\\
{}^tA^\varepsilon\, \nabla v_{ij}^\varepsilon \rightharpoonup {}^tA\nabla v_{ij} \quad \text{weakly in } L^2(\omega)^{N\times N}.
\end{cases}
\tag{3.10}
$$

By compensated compactness,

$$\big[{}^tA^\varepsilon \nabla v_{ij}^\varepsilon, \nabla u^\varepsilon\big] \xrightarrow{\mathcal{D}'(\omega)} \big[{}^tA\nabla v_{ij}, \nabla u\big] = [\nabla v_{ij}, A\nabla u].$$

Moreover, denoting by $\tilde{v}_{ij}^\varepsilon$ and $\tilde{v}_{ij}$ the extensions of v_{ij}^ε and v_{ij} by zero in $\Omega \setminus \omega$, compensated compactness gives

$$\big[\nabla \tilde{v}_{ij}^\varepsilon, A^\varepsilon\, \nabla u^\varepsilon\big] \xrightarrow{\mathcal{D}'(\Omega)} [\nabla \tilde{v}_{ij}, \xi]$$

and hence

$$\big[{}^tA^\varepsilon\, \nabla v_{ij}^\varepsilon, \nabla u^\varepsilon\big] = \big[\nabla v_{ij}^\varepsilon, A^\varepsilon \nabla u^\varepsilon\big] \xrightarrow{\mathcal{D}'(\omega)} [\nabla v_{ij}, \xi].$$

By comparing the two limits in $\mathcal{D}'(\omega)$, we obtain

$$[\nabla v_{ij}, A\nabla u] = [\nabla v_{ij}, \xi].$$

As this is true for any i, j, we obtain first that $\xi = A\nabla u$ in ω_1. But this is true for any ω_1 and ω, hence $\xi = A\nabla u$ in all Ω.

Similarly one proves that ${}^tA^\varepsilon \xrightarrow{H_b} {}^tA$ implies $A^\varepsilon \xrightarrow{H_{\mathrm{loc}}} A$ (we omit the details for the sake of briefness). In view of Proposition 1, the proof is finished. □

Remark 3. In the diffusion case, it was proved by F. Mura and L. Tartar (see, e.g., [15]) that $M(\alpha, \beta, \Omega)$ is sequentially relatively compact for H_{loc}-convergence, that is, for any sequence $A^\varepsilon \in M(\alpha, \beta, \Omega)$, there exist a subsequence, still denoted ε, and there exists $A \in M(\alpha, \beta, \Omega)$ such that $A^\varepsilon \xrightarrow{H_{\mathrm{loc}}} A$. The proof of [15] can be easily extended to the elasticity case. It follows that, in our definition of H-convergence, we may suppose that $\alpha' = \alpha$ and $\beta' = \beta$, as we shall do in the next section.

4. Main result

Our main result is the following.

Theorem 1. *Let $A^\varepsilon, A \in M(\alpha, \beta, \Omega)$, for some $0 < \alpha \leqslant \beta < \infty$. Then $A^\varepsilon \xrightarrow{H} A$ as $\varepsilon \to 0$ if and only if there exist $N \times N \times N \times N$ tensors M^ε, M, P^ε, P with entries in $L^2(\Omega)$ and with M invertible, such that*

$$M^\varepsilon A^\varepsilon = P^\varepsilon, \tag{4.1}$$

$$MA = P, \tag{4.2}$$

$$M^\varepsilon \rightharpoonup M \quad \textit{weakly in } L^2(\Omega)^{N\times N\times N\times N}, \tag{4.3}$$

$$P^\varepsilon \rightharpoonup P \quad \textit{weakly in } L^2(\Omega)^{N\times N\times N\times N}, \tag{4.4}$$

$$\{\mathrm{curl}\, M^\varepsilon\}_{\varepsilon>0} \textit{ is relatively compact in } H^{-1}(\Omega)^{N\times N\times N\times N\times N}, \tag{4.5}$$

$$\{\mathrm{div}\, P^\varepsilon\}_{\varepsilon>0} \textit{ is relatively compact in } H^{-1}(\Omega)^{N\times N\times N} \tag{4.6}$$

with curl *and* div *defined as in Section* 2.4. *When this is the case, M can be chosen to be the identity tensor ($M_{ijkl} = \delta_{ik}\delta_{jl}$) and M^ε so that* $\mathrm{curl}\, M^\varepsilon = 0$.

Proof. We begin by giving the proof of the "if"-part. Suppose that we have the decompositions (4.1), (4.2) with compactness (4.5), (4.6) and with weak convergences $M^\varepsilon \rightharpoonup M$ and $P^\varepsilon \rightharpoonup P$ as in (4.3), (4.4). We are going to prove that A^ε H-converges to A. We consider $D^\varepsilon, D, E^\varepsilon, E$ satisfying (2.5) to (2.8). Then (4.1) acting on E^ε gives

$$M^\varepsilon D^\varepsilon = P^\varepsilon E^\varepsilon,$$

or in components,

$$M^\varepsilon_{ijkl} D^\varepsilon_{kl} = P^\varepsilon_{ijkl} E^\varepsilon_{kl},$$

that is, for the usual scalar product of vectors,

$$\left(M^\varepsilon_{ijk}, D^\varepsilon_k\right) = \left(P^\varepsilon_{ijk}, E^\varepsilon_k\right). \tag{4.7}$$

We have

curl M^ε_{ijk} relatively compact in $H^{-1}(\Omega)^{N\times N}$,

div D^ε_k relatively compact in $H^{-1}(\Omega)$,

div P^ε_{ijk} relatively compact in $H^{-1}(\Omega)$,

curl E^ε_k relatively compact in $H^{-1}(\Omega)^{N\times N}$.

By the div–curl lemma, we get by passing to the limit in (4.7) in the sense of distribution

$$(M_{ijk}, D_k) = (P_{ijk}, E_k),$$

that is $MD = PE$ which, since M is invertible, is the same as $D = AE$.

Now we prove the "only if" part. More precisely, we prove that we may take $M = I$, i.e., $M_{ijkl} = \delta_{ik}\delta_{jl}$, and $P = A$. Let ${}^tA$ denote the transpose of A. As in the proof of Proposition 1, we set, for fixed $i, j \in \{1, \ldots, N\}$,

$$u_{ij} = (u_{ijk})_k, \qquad u_{ijk}(x) = \delta_{ik}x_j, \qquad g_{ij} = {}^tA\nabla u_{ij}$$

and we consider the solution u^ε_{ij} of

$$\begin{cases} \operatorname{div}\left({}^tA^\varepsilon\nabla u^\varepsilon_{ij}\right) = \operatorname{div} g_{ij} \quad \text{in } \Omega, \\ u^\varepsilon_{ij} - u_{ij} \in H^1_0(\Omega)^N. \end{cases} \tag{4.8}$$

Then let M^ε be defined by

$$M^\varepsilon_{ijkl} = \nabla_{kl}u^\varepsilon_{ij}$$

and let $P^\varepsilon = M^\varepsilon A^\varepsilon$, or in components

$$P^\varepsilon_{ijkl} = M^\varepsilon_{ijmn}A^\varepsilon_{mnkl}.$$

We have

$$\begin{aligned}\left(\operatorname{curl} M^\varepsilon\right)_{ijklm} &= \frac{\partial M^\varepsilon_{ijkl}}{\partial x_m} - \frac{\partial M^\varepsilon_{ijkm}}{\partial x_l} \\ &= \frac{\partial}{\partial x_m}\left(\nabla_{kl}u^\varepsilon_{ij}\right) - \frac{\partial}{\partial x_l}\left(\nabla_{km}u^\varepsilon_{ij}\right) \\ &= \frac{\partial}{\partial x_m}\left(\frac{\partial u^\varepsilon_{ijk}}{\partial x_l}\right) - \frac{\partial}{\partial x_l}\left(\frac{\partial u^\varepsilon_{ijk}}{\partial x_m}\right) = 0,\end{aligned}$$

$$\begin{aligned}(\operatorname{div} P^\varepsilon)_{ijk} &= \frac{\partial P^\varepsilon_{ijkl}}{\partial x_l}\\ &= \frac{\partial}{\partial x_l}\left(M^\varepsilon_{ijmn}A^\varepsilon_{mnkl}\right)\\ &= \frac{\partial}{\partial x_l}\left(\left({}^tA^\varepsilon\right)_{klmn}M^\varepsilon_{ijmn}\right)\\ &= \frac{\partial}{\partial x_l}\left(\left({}^tA^\varepsilon\right)_{klmn}\nabla_{mn}u^\varepsilon_{ij}\right)\\ &= \frac{\partial}{\partial x_l}g_{ijkl}.\end{aligned}$$

Hence the so defined tensors M^ε, P^ε satisfy

$$\operatorname{curl} M^\varepsilon = 0 \quad \text{and} \quad \operatorname{div} P^\varepsilon = \operatorname{div} g.$$

From (4.8) we get for each $i, j \in \{1, \ldots, N\}$ the elliptic estimates

$$\begin{aligned}&\|u^\varepsilon_{ij}\|_{H^1(\Omega)^N} \leqslant C < \infty,\\ &\|{}^tA^\varepsilon e(u^\varepsilon_{ij})\|_{L^2(\Omega)^{N\times N}} = \|{}^tA^\varepsilon \nabla u^\varepsilon_{ij}\|_{L^2(\Omega)^{N\times N}} \leqslant C < \infty.\end{aligned}$$

Thus for some subsequence of $\{\varepsilon\}$ and some limit fields v_{ij} ($v_{ij} - u_{ij} \in H^1_0(\Omega)^N$) and σ_{ij}, we have the convergences

$$u^\varepsilon_{ij} \rightharpoonup v_{ij} \quad \text{weakly in } H^1(\Omega)^N, \tag{4.9}$$

$$\left.\begin{array}{l} e(u^\varepsilon_{ij}) \rightharpoonup e(v_{ij}) \\ \nabla u^\varepsilon_{ij} \rightharpoonup \nabla v_{ij} \end{array}\right\} \quad \text{weakly in } L^2(\Omega)^{N\times N}, \tag{4.10}$$

$$P^\varepsilon_{ij} = {}^tA^\varepsilon \nabla u^\varepsilon_{ij} \rightharpoonup \sigma_{ij} \quad \text{weakly in } L^2(\Omega)^{N\times N}. \tag{4.11}$$

The latter convergence together with (4.8) shows that $\operatorname{div}\sigma_{ij} = \operatorname{div} g_{ij}$. At this point we use the fact mentioned in Proposition 1 that H-convergence carries over to the transposed tensors. Thus ${}^tA^\varepsilon \xrightarrow{H} {}^tA$, and since $\operatorname{curl}\nabla u^\varepsilon_{ij} = 0$ and $\operatorname{div} P^\varepsilon_{ij} = \operatorname{div} g_{ij}$ have components staying in a compact subset of $H^{-1}(\Omega)$, it follows from the definition of H-convergence that $\sigma_{ij} = {}^tA\nabla v_{ij}$. Therefore v_{ij} solves the boundary value problem

$$\begin{cases} \operatorname{div}\left({}^tA\nabla v_{ij}\right) = \operatorname{div} g_{ij},\\ v_{ij} - u_{ij} \in H^1_0(\Omega)^N. \end{cases} \tag{4.12}$$

But this problem has the unique solution u_{ij}. Thus we conclude that $v_{ij} = u_{ij}$ and that $\sigma_{ij} = {}^tA\nabla u_{ij}$. It also follows that in (4.9) to (4.11) we have convergence for the full sequence ε. With this in mind, the convergences (4.10), (4.11) state exactly that

$$M^\varepsilon \rightharpoonup M = I \quad \text{weakly in } L^2(\Omega)^{N\times N\times N\times N},$$
$$P^\varepsilon \rightharpoonup P = A \quad \text{weakly in } L^2(\Omega)^{N\times N\times N\times N}.$$

Actually, (4.10) is equivalent to (for any k, l)

$$M^\varepsilon_{ijkl} = \nabla_{kl} u^\varepsilon_{ij} \rightharpoonup \nabla_{kl} u_{ij} = \delta_{ik}\delta_{jl} = M_{ijkl}$$

and (4.11) is equivalent to (for any k, l)

$$P^\varepsilon_{ijkl} \rightharpoonup \left({}^t A \,\nabla u_{ij}\right)_{kl} = \left({}^t A\right)_{klmn} \nabla_{mn} u_{ij} = A_{mnkl}\delta_{im}\delta_{jn} = A_{ijkl}.$$

Moreover the last two convergences hold true, not only for any k, l, but also for any i, j. This proves the theorem. $\square$

Remark 4. It is clear that the matrices M^ε, M, P^ε, P appearing in the decompositions (4.1), (4.2) are far from being uniquely determined by A^ε, A, even when all the conditions (4.3) to (4.6) are satisfied. For example, none of the conditions (4.1) to (4.6) are affected if M^ε, M, P^ε, P are multiplied from the left by one and the same invertible matrix $Q = Q(x)$ with Lipschitz coefficients. In the case of laminates, Section 6 gives two possible choices of M^ε, M, P^ε and P.

5. Correctors

From the formulation of Theorem 1, one can easily pass to construction of "correctors" (cf. [19,20]).

Theorem 2 (and definition). *Let $A^\varepsilon, A \in M(\alpha, \beta, \Omega)$, for some $0 < \alpha \leqslant \beta < \infty$. Let us assume that $A^\varepsilon \xrightarrow{H} A$. Then, first of all, there exist tensors N^ε and Q^ε with entries in $L^2(\Omega)$ such that*

$$A^\varepsilon N^\varepsilon = Q^\varepsilon,$$
$$N^\varepsilon \rightharpoonup I \quad \text{and} \quad Q^\varepsilon \rightharpoonup A, \quad \text{weakly in } L^2(\Omega)^{N\times N\times N\times N},$$

with

$$\operatorname{curl} {}^t N^\varepsilon \text{ relatively compact in } H^{-1}(\Omega)^{N\times N\times N\times N\times N}$$

and

$$\operatorname{div} {}^t Q^\varepsilon \text{ relatively compact in } H^{-1}(\Omega)^{N\times N\times N}.$$

Secondly, let D^ε, E^ε, E be as in the definition of H-convergence for A^ε. By definition, we say that $N^\varepsilon E$ and $Q^\varepsilon E$ are correctors (*good approximations*) *of E^ε and D^ε respectively, if the following convergences hold true*:

$$\frac{1}{2}\left[\left(E^\varepsilon - N^\varepsilon E\right) + {}^t\left(E^\varepsilon - N^\varepsilon E\right)\right] \to 0 \quad \text{and} \quad D^\varepsilon - Q^\varepsilon E \to 0 \quad \text{strongly in } L^2_{\mathrm{loc}}(\Omega)^{N\times N}. \tag{5.1}$$

Now (5.1) *does hold if at least one of the following conditions is satisfied*:

- *E is in $W^{1,\infty}(\Omega)$,*
- *the sequences $\{N^\varepsilon\}_\varepsilon$ and $\{Q^\varepsilon\}_\varepsilon$ are bounded in $L^\infty(\Omega)^{N\times N\times N\times N}$,*
- *$E \in L^p(\Omega)^{N\times N}$ for some p, $2 < p < \infty$, and the sequences $\{N^\varepsilon\}_\varepsilon$ and $\{Q^\varepsilon\}_\varepsilon$ are bounded in $L^{\frac{2p}{p-2}}(\Omega)^{N\times N\times N\times N}$.*

In the last case we even have the global convergence

$$\frac{1}{2}\big[(E^\varepsilon - N^\varepsilon E) + {}^t(E^\varepsilon - N^\varepsilon E)\big] \to 0 \quad \text{and} \quad D^\varepsilon - Q^\varepsilon E \to 0 \quad \text{strongly in } L^2(\Omega)^{N\times N}. \tag{5.2}$$

Proof. In order to construct N^ε and Q^ε, we apply Theorem 1 to the transposed tensors: if $A^\varepsilon \xrightarrow{H} A$ then ${}^tA^\varepsilon \xrightarrow{H} {}^tA$. Thus there are tensors ${}^tN^\varepsilon$, ${}^tN$, ${}^tQ^\varepsilon$, ${}^tQ$ such that ${}^tN^\varepsilon\, {}^tA^\varepsilon = {}^tQ^\varepsilon$, ${}^tN\, {}^tA = {}^tQ$, ${}^tN^\varepsilon \rightharpoonup {}^tN$, ${}^tQ^\varepsilon \rightharpoonup {}^tQ$ with $\operatorname{div} {}^tQ^\varepsilon$ and $\operatorname{curl} {}^tN^\varepsilon$ relatively compact in $H^{-1}(\Omega)^{N\times N\times N}$ and $H^{-1}(\Omega)^{N\times N\times N\times N\times N}$, respectively. Here we choose the normalization ${}^tN = I$, ${}^tQ = {}^tA$.

Now we consider vector fields D^ε, E^ε as in the definition of H-convergence (for A^ε). To prove the assertion (5.1), we first notice that, due to the symmetry property of A^ε,

$$D^\varepsilon - Q^\varepsilon E = A^\varepsilon(E^\varepsilon - N^\varepsilon E) = A^\varepsilon(E^\varepsilon - N^\varepsilon E)_s$$

with

$$(E^\varepsilon - N^\varepsilon E)_s = \frac{1}{2}\big[(E^\varepsilon - N^\varepsilon E) + {}^t(E^\varepsilon - N^\varepsilon E)\big].$$

Thus,

$$[D^\varepsilon - Q^\varepsilon E, E^\varepsilon - N^\varepsilon E] = [A^\varepsilon(E^\varepsilon - N^\varepsilon E), E^\varepsilon - N^\varepsilon E] = [A^\varepsilon(E^\varepsilon - N^\varepsilon E)_s, (E^\varepsilon - N^\varepsilon E)_s].$$

Therefore, by (2.2) and (2.4), for every $\omega \Subset \Omega$, the $L^2(\omega)^{N\times N}$-norms of both $(E^\varepsilon - N^\varepsilon E)_s$ and $D^\varepsilon - Q^\varepsilon E$ can be estimated from above by $\int_\Omega [D^\varepsilon - Q^\varepsilon E, E^\varepsilon - N^\varepsilon E]\phi \, \mathrm{d}x$, for some $\phi \in \mathcal{D}(\Omega)$, $\phi \geqslant 0$, $\phi = 1$ in ω. But

$$[D^\varepsilon - Q^\varepsilon E, E^\varepsilon - N^\varepsilon E] = [D^\varepsilon, E^\varepsilon] - [D^\varepsilon, N^\varepsilon E] - [Q^\varepsilon E, E^\varepsilon] + [Q^\varepsilon E, N^\varepsilon E]. \tag{5.3}$$

Clearly, by the div–curl lemma, $[D^\varepsilon, E^\varepsilon] \to [D, E]$ in the sense of distributions.

- If E has regularity $W^{1,\infty}(\Omega)$, the relative compactness of $\operatorname{curl} {}^tN^\varepsilon$ and $\operatorname{div} {}^tQ^\varepsilon$ implies that $\operatorname{curl}(N^\varepsilon E)$ and $\operatorname{div}(Q^\varepsilon E)$ are relatively compact in $H^{-1}(\Omega)^{N\times N\times N}$ and $H^{-1}(\Omega)^N$, respectively. Then the div–curl lemma gives that, in the sense of distributions,

$$\begin{aligned}&[D^\varepsilon, E^\varepsilon] - [D^\varepsilon, N^\varepsilon E] - [Q^\varepsilon E, E^\varepsilon] + [Q^\varepsilon E, N^\varepsilon E] \\ &\quad \to [D, E] - [D, NE] - [QE, E] + [QE, NE] = [D - QE, E - NE] = 0.\end{aligned}$$

From this the assertion (5.1) follows.

- Assuming that E is just in $L^2(\Omega)^{N\times N}$, but the sequences $\{N^\varepsilon\}_\varepsilon$ and $\{Q^\varepsilon\}_\varepsilon$ are bounded in $L^\infty(\Omega)^{N\times N\times N\times N}$, we observe that

$$[D^\varepsilon, N^\varepsilon E] = [{}^tN^\varepsilon D^\varepsilon, E].$$

The div–curl lemma gives that ${}^tN^\varepsilon D^\varepsilon$ converges in the sense of distributions to ${}^tND$. But this convergence holds also true in weak-$L^2(\Omega)^{N\times N}$, by boundedness, so that, for $\phi\in\mathcal{D}(\Omega)$,

$$\int_\Omega \left[{}^tN^\varepsilon D^\varepsilon, E\right]\phi\,\mathrm{d}x \to \int_\Omega \left[{}^tND, E\right]\phi\,\mathrm{d}x.$$

In the same way, one can pass to the limit in the term $[Q^\varepsilon E, E^\varepsilon] = [E, {}^tQ^\varepsilon E^\varepsilon]$.

As for the last term in the right-hand side of (5.3),

$$[Q^\varepsilon E, N^\varepsilon E] = \left({}^tQ^\varepsilon\right)_{klij}\left({}^tN^\varepsilon\right)_{stij}E_{kl}E_{st} = \left(\left({}^tQ^\varepsilon\right)_{kli}, \left({}^tN^\varepsilon\right)_{sti}\right)E_{kl}E_{st}.$$

By the div–curl lemma and by boundedness, $(({}^tQ^\varepsilon)_{kli}, ({}^tN^\varepsilon)_{sti}) \rightharpoonup (({}^tQ)_{kli}, ({}^tN)_{sti})$ in weak*-$L^\infty(\Omega)$, so that

$$\int_\Omega [Q^\varepsilon E, N^\varepsilon E]\phi\,\mathrm{d}x \to \int_\Omega [QE, NE]\phi\,\mathrm{d}x.$$

The assertion (5.1) follows again.

• Finally, if E is in $L^p(\Omega)^{N\times N}$ for some $p>2$ and if the sequences $\{N^\varepsilon\}_\varepsilon$ and $\{Q^\varepsilon\}_\varepsilon$ are bounded in $L^{\frac{2p}{p-2}}(\Omega)^{N\times N\times N\times N}$, then ${}^tN^\varepsilon D^\varepsilon$ and ${}^tQ^\varepsilon E^\varepsilon$ are bounded in $L^{p'}(\Omega)^{N\times N\times N\times N}$, with p' the conjugate of p. They converge weakly in this space to ${}^tND$ and ${}^tQE$, respectively. It follows that one can pass to the limit in the second and third terms of the right member of (5.3). Moreover, since $E_{kl}E_{st}$ is in $L^{\frac{p}{2}}(\Omega)$ and $(({}^tQ^\varepsilon)_{kli}, ({}^tN^\varepsilon)_{sti})$ is bounded in $L^{\frac{p}{p-2}}(\Omega)$, one can pass to the limit also in the last term in the right member of (5.3). We get the same corrector result.

We notice that, in the last case above, the integrals over the entire domain Ω of the last three terms in the right-hand side of (5.3) tend to the limit, because of the weak convergences that we have mentioned. The same is also true for the first term $[D^\varepsilon, E^\varepsilon] = [A^\varepsilon E^\varepsilon_s, E^\varepsilon_s]$, which is bounded in $L^{\frac{p}{2}}(\Omega)$, as soon as $\{E^\varepsilon\}_\varepsilon$ is bounded in $L^p(\Omega)^{N\times N}$ for $2<p<\infty$. This implies the stronger corrector result (5.2). □

6. The example of elastic laminates

6.1. H-convergence for elastic laminates: known result

The criterion for H-convergence in Theorem 1 is particularly useful when it is possible to find the tensors M^ε and P^ε *a priori* (without solving any Dirichlet problem, e.g.). In such cases one often gets additional properties of M^ε and P^ε, e.g., L^∞-bounds, so that they converge in weak*-L^∞. The main example for which this occurs is the case of stratified media, i.e., when A^ε depends on only one of the coordinates, say x_N:

$$A^\varepsilon = A^\varepsilon(x_N).$$

Then, following [10], we can write

$$A^\varepsilon = \begin{pmatrix} A^\varepsilon_{i\beta k\delta} & A^\varepsilon_{i\beta kN} \\ A^\varepsilon_{iNk\delta} & A^\varepsilon_{iNkN}\end{pmatrix},$$

where Latin indices run over $\{1,\dots,N\}$ while Greek indices only run over $\{1,\dots,N-1\}$. It follows from (2.2) that the matrix (A^ε_{iNkN}) is invertible. Denoting by (R^ε_{iNkN}) its inverse, we have $A^\varepsilon = (M^\varepsilon)^{-1}P^\varepsilon$ with

$$M^\varepsilon = \begin{pmatrix} \delta_{ik}\delta_{\beta\delta} & -A^\varepsilon_{i\beta sN}R^\varepsilon_{sNkN} \\ 0 & R^\varepsilon_{iNkN} \end{pmatrix},$$
$$P^\varepsilon = \begin{pmatrix} A^\varepsilon_{i\beta k\delta} - A^\varepsilon_{i\beta sN}R^\varepsilon_{sNtN}A^\varepsilon_{tNk\delta} & 0 \\ R^\varepsilon_{iNsN}A^\varepsilon_{sNk\delta} & \delta_{ik} \end{pmatrix}.$$

It is immediate that curl $M^\varepsilon = 0$, div $P^\varepsilon = 0$, and it is proved in [10] that $A^\varepsilon \xrightarrow{H} A$ if and only if $M^\varepsilon \rightharpoonup M$, $P^\varepsilon \rightharpoonup P$ weakly* in $L^\infty(\Omega)^{N\times N\times N\times N}$, with M and P defined from A by the above expressions (without ε). (These conditions of H-convergence for elastic laminates occurred in [12] for the first time, before compensated compactness became a standard technique, while the proof by compensated compactness is given in [10].)

Natural generalizations of the above example can be constructed as in the diffusion case in [7,4], where M^ε and P^ε are supposed to have specific dependence upon coordinates, and in [3], where the general condition, the diffusion analogue of (4.5), (4.6), is introduced. Note that in the diffusion case these generalizations also contain the "isotropic factorizable" case of A. Marino and S. Spagnolo [11].

6.2. Recovery of the formula for laminates from the general theory of Theorem 1

In this subsection, we indicate another way of obtaining the H-limit A of $A^\varepsilon = A^\varepsilon(x_N)$, more in the spirit of Theorem 1.

So we assume that $A^\varepsilon \xrightarrow{H} A$, $A^\varepsilon = A^\varepsilon(x_N)$. For fixed i,j, we set $u_{ijk} = \delta_{ik}x_j$ and we look for $u^\varepsilon_{ij} = (u^\varepsilon_{ijk})$ of the form

$$u^\varepsilon_{ijk}(x) = \delta_{ik}x_j + v^\varepsilon_{ijk}(x_N)$$

solving

$$\operatorname{div}\bigl({}^tA^\varepsilon\nabla u^\varepsilon_{ij}\bigr) = \operatorname{div}\bigl({}^tA\nabla u_{ij}\bigr),$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}x_N}\left(A^\varepsilon_{sNkN}\frac{\mathrm{d}v^\varepsilon_{ijs}}{\mathrm{d}x_N}\right) = \frac{\mathrm{d}}{\mathrm{d}x_N}\bigl(A_{ijkN} - A^\varepsilon_{ijkN}\bigr).$$

As the matrix (A^ε_{sNkN}) is invertible (see [10]), it is possible to solve

$$A^\varepsilon_{sNkN}\frac{\mathrm{d}v^\varepsilon_{ijs}}{\mathrm{d}x_N} = A_{ijkN} - A^\varepsilon_{ijkN},$$

for $\frac{\mathrm{d}v^\varepsilon_{ijs}}{\mathrm{d}x_N}$. This gives, by defining R^ε_{sNkN} as the inverse matrix of A^ε_{sNkN},

$$\frac{\mathrm{d}v^\varepsilon_{ijk}}{\mathrm{d}x_N} = \bigl(A_{ijsN} - A^\varepsilon_{ijsN}\bigr)R^\varepsilon_{sNkN} = \bigl((A-A^\varepsilon)R^\varepsilon\bigr)_{ijkN}$$

and then

$$\frac{\partial u^{\varepsilon}_{ijk}}{\partial x_l} = \delta_{ik}\delta_{jl} + \delta_{Nl}(A - A^{\varepsilon})_{ijsN} R^{\varepsilon}_{sNkN}.$$

Now we define a tensor $\overline{M}^{\varepsilon}$ by

$$\overline{M}^{\varepsilon}_{ijkl} = \nabla_{kl} u^{\varepsilon}_{ij} = \frac{\partial u^{\varepsilon}_{ijk}}{\partial x_l}.$$

We have

$$\overline{M}^{\varepsilon} = \begin{pmatrix} \overline{M}^{\varepsilon}_{i\beta k\delta} & \overline{M}^{\varepsilon}_{i\beta kN} \\ \overline{M}^{\varepsilon}_{iNk\delta} & \overline{M}^{\varepsilon}_{iNkN} \end{pmatrix} = \begin{pmatrix} \delta_{ik}\delta_{\beta\delta} & (A - A^{\varepsilon})_{i\beta sN} R^{\varepsilon}_{sNkN} \\ 0 & A_{iNsN} R^{\varepsilon}_{sNkN} \end{pmatrix}.$$

We also define the tensor $\overline{P}^{\varepsilon}$ by

$$\overline{P}^{\varepsilon} = \overline{M}^{\varepsilon} A^{\varepsilon} = \begin{pmatrix} \overline{P}^{\varepsilon}_{i\beta k\delta} & \overline{P}^{\varepsilon}_{i\beta kN} \\ \overline{P}^{\varepsilon}_{iNk\delta} & \overline{P}^{\varepsilon}_{iNkN} \end{pmatrix} = \begin{pmatrix} A^{\varepsilon}_{i\beta k\delta} + (A - A^{\varepsilon})_{i\beta sN} R^{\varepsilon}_{sNtN} A^{\varepsilon}_{tNk\delta} & A_{i\beta kN} \\ A_{iNsN} R^{\varepsilon}_{sNtN} A^{\varepsilon}_{tNk\delta} & A_{iNkN} \end{pmatrix}.$$

Since the coefficients of A^{ε} and R^{ε} are bounded in $L^{\infty}(\Omega)$ independently of ε (see [10] for R^{ε}), the same is true for the coefficients of $\overline{M}^{\varepsilon}$ and $\overline{P}^{\varepsilon}$, so that, up to a subsequence of ε,

$$\overline{M}^{\varepsilon} \rightharpoonup \overline{M} \quad \text{and} \quad \overline{P}^{\varepsilon} \rightharpoonup \overline{P} \quad \text{weakly}^* \text{ in } L^{\infty}(\Omega)^{N\times N\times N\times N}, \tag{6.1}$$

with

$$\overline{M} = \begin{pmatrix} \delta_{ik}\delta_{\beta\delta} & \overline{M}_{i\beta kN} \\ 0 & \overline{M}_{iNkN} \end{pmatrix}, \qquad \overline{P} = \begin{pmatrix} \overline{P}_{i\beta k\delta} & A_{i\beta kN} \\ \overline{P}_{iNk\delta} & A_{iNkN} \end{pmatrix}, \tag{6.2}$$

for some $\overline{M}_{i\beta kN}$, $\overline{M}_{iNkN}$, $\overline{P}_{i\beta k\delta}$ and $\overline{P}_{iNk\delta}$ in $L^{\infty}(\Omega)$.

We claim that, by using compensated compactness, it is possible to pass to the limit in $\overline{P}^{\varepsilon} = \overline{M}^{\varepsilon} A^{\varepsilon}$ and get $\overline{P} = \overline{M} A$. Indeed, defining $\overline{P}^{\varepsilon}_{ij}$ and $\overline{M}^{\varepsilon}_{ij}$ by $\overline{P}^{\varepsilon}_{ijkl} = (\overline{P}^{\varepsilon}_{ij})_{kl}$ and $\overline{M}^{\varepsilon}_{ijkl} = (\overline{M}^{\varepsilon}_{ij})_{kl}$, the equality $\overline{P}^{\varepsilon} = \overline{M}^{\varepsilon} A^{\varepsilon}$ reads

$$\overline{P}^{\varepsilon}_{ij} = {}^{t}A^{\varepsilon} \nabla u^{\varepsilon}_{ij} = {}^{t}A^{\varepsilon} \overline{M}^{\varepsilon}_{ij},$$

and

$$\operatorname{div} \overline{P}^{\varepsilon}_{ij} = \operatorname{div}\left({}^{t}A \nabla u_{ij}\right)$$

is relatively compact in $H^{-1}(\Omega)^N$, while $\operatorname{curl} \overline{M}^{\varepsilon}_{ij} = \operatorname{curl} \nabla u^{\varepsilon}_{ij} = 0$. As A^{ε} H-converges to A, then ${}^{t}A^{\varepsilon}$ H-converges to ${}^{t}A$ and it follows from the definition of H-convergence that $\overline{P}_{ij} = {}^{t}A\overline{M}_{ij}$, with $(\overline{M}_{ij})_{kl} = \overline{M}_{ijkl}$, $(\overline{P}_{ij})_{kl} = \overline{P}_{ijkl}$: we have proved that $\overline{P} = \overline{M} A$.

With (6.2), this gives the following identities

$$\overline{P}_{i\beta k\delta} = A_{i\beta k\delta} + \overline{M}_{i\beta sN} A_{sNk\delta}, \tag{6.3}$$

$$A_{i\beta kN} = A_{i\beta kN} + \overline{M}_{i\beta sN} A_{sNkN}, \tag{6.4}$$

$$\overline{P}_{iNk\delta} = \overline{M}_{iNsN} A_{sNk\delta}, \tag{6.5}$$

$$A_{iNkN} = \overline{M}_{iNsN} A_{sNkN}. \tag{6.6}$$

As A is coercive, then (A_{iNkN}) is invertible and, denoting its inverse by (R_{iNkN}), we derive by successive use of (6.6), (6.5), (6.4) and (6.3) that $\overline{P} = A$ and that $\overline{M}$ is the identity tensor. Then (6.1) says that

$$(A - A^\varepsilon)_{i\beta sN} R^\varepsilon_{sNkN} \rightharpoonup 0, \tag{6.7}$$

$$A_{iNsN} R^\varepsilon_{sNkN} \rightharpoonup \delta_{ik}, \tag{6.8}$$

$$A^\varepsilon_{i\beta k\delta} + (A - A^\varepsilon)_{i\beta sN} R^\varepsilon_{sNtN} A^\varepsilon_{tNk\delta} \rightharpoonup A_{i\beta k\delta}, \tag{6.9}$$

$$A_{iNsN} R^\varepsilon_{sNtN} A^\varepsilon_{tNk\delta} \rightharpoonup A_{iNk\delta}, \tag{6.10}$$

from which, by using successively (6.8), (6.7), (6.10) and (6.9), we recover the classical formulae of H-convergence:

$$R^\varepsilon_{iNkN} \rightharpoonup R_{iNkN},$$
$$A^\varepsilon_{i\beta sN} R^\varepsilon_{sNkN} \rightharpoonup A_{i\beta sN} R_{sNkN},$$
$$R^\varepsilon_{iNkN} A^\varepsilon_{kNl\delta} \rightharpoonup R_{iNkN} A_{kNl\delta},$$
$$A^\varepsilon_{i\beta k\delta} - A^\varepsilon_{i\beta sN} R^\varepsilon_{sNtN} A^\varepsilon_{tNk\delta} \rightharpoonup A_{i\beta k\delta} - A_{i\beta sN} R_{sNtN} A_{tNk\delta}.$$

Remark 5. With M^ε, P^ε, M, P as in Section 6.1, we have

$$M^{-1} = \begin{pmatrix} \delta_{ik}\delta_{\beta\delta} & A_{i\beta kN} \\ 0 & A_{iNkN} \end{pmatrix}$$
$$\overline{M}^\varepsilon = M^{-1} M^\varepsilon, \qquad \overline{P}^\varepsilon = M^{-1} P^\varepsilon.$$

7. The example of periodic elastic materials

7.1. *The case $A^\varepsilon = A(\frac{x}{\varepsilon})$*

In this section, we use our method to construct the H-limit A^* of a tensor $A^\varepsilon(x) = A(\frac{x}{\varepsilon})$, with A defined in $\mathbb{R}^N$, periodic of period $Y =]0,1[^N$. More precisely, we shall prove that A^ε H-converges to the constant tensor A^* defined by

$$A^* = \int_Y (I + \nabla w) A \, \mathrm{d}y, \tag{7.1}$$

where $I = (I_{ijkl}) = (\delta_{ik}\delta_{jl})$ and

$$(\nabla w)_{ijkl} = \nabla_{kl} w_{ij} = \frac{\partial w_{ijk}}{\partial y_l},$$

$w_{ij} = (w_{ijk})$ being a solution of

$$\begin{cases} \operatorname{div}\left({}^t A \nabla w_{ij}\right) = -\operatorname{div}\left({}^t A e_{ij}\right) & \text{in } Y, \\ w_{ij} \quad Y\text{-periodic}, \end{cases} \tag{7.2}$$

with $e_{ij} = (e_{ijkl}) = \delta_{ik}\delta_{jl}$.

We first notice that the vector function $u^\varepsilon_{ij} = (u^\varepsilon_{ijk})$ defined by

$$u^\varepsilon_{ijk}(x) = \delta_{ik} x_j + \varepsilon w_{ijk}\left(\frac{x}{\varepsilon}\right)$$

solves

$$\operatorname{div}\left({}^t A^\varepsilon \nabla u^\varepsilon_{ij}\right) = \frac{1}{\varepsilon}\frac{\partial}{\partial y_l}\left(A_{ijkl} + A_{stkl}\frac{\partial w_{ijs}}{\partial y_t}\right)\left(\frac{x}{\varepsilon}\right) = 0.$$

Now, setting

$$M^\varepsilon_{ijkl} = \frac{\partial u^\varepsilon_{ijk}}{\partial x_l} = \delta_{ik}\delta_{jl} + \frac{\partial w_{ijk}}{\partial y_l}\left(\frac{x}{\varepsilon}\right),$$
$$P^\varepsilon_{ijkl} = \left(A_{ijkl} + A_{stkl}\frac{\partial w_{ijs}}{\partial y_t}\right)\left(\frac{x}{\varepsilon}\right),$$

we have $P^\varepsilon = M^\varepsilon A^\varepsilon$ and it is clear that

$$M^\varepsilon_{ijkl} \rightharpoonup M_{ijkl} = \delta_{ik}\delta_{jl} + \int_Y \frac{\partial w_{ijk}}{\partial y_l}\,\mathrm{d}y = \delta_{ik}\delta_{jl} = I_{ijkl} \quad \text{weakly in } L^2(\Omega),$$
$$P^\varepsilon_{ijkl} \rightharpoonup P_{ijkl} = \int_Y \left(A_{ijkl} + A_{stkl}\frac{\partial w_{ijs}}{\partial y_t}\right)\mathrm{d}y = A^*_{ijkl} \quad \text{weakly in } L^2(\Omega).$$

Moreover

$$\left(\operatorname{div} P^\varepsilon\right)_{ijk} = \frac{\partial P^\varepsilon_{ijkl}}{\partial x_l} = 0$$

and $M^\varepsilon_{ijkl} = \nabla_{kl} u^\varepsilon_{ij}$, so that $(\operatorname{curl} M^\varepsilon)_{ijklm} = 0$. As $P = MA^*$, it follows from Theorem 1 that A^* is the H-limit of A^ε.

7.2. The case $A^\varepsilon = A(x, \frac{x}{\varepsilon})$

In this subsection, we prove, under some regularity conditions, that formula (7.1) remains true in the case that A depends on both slow and rapid variables: $A^\varepsilon(x) = A(x, \frac{x}{\varepsilon})$, with $A = A(x, y)$ a measurable function, defined almost everywhere in $\Omega \times \mathbb{R}^N$ and periodic of period Y in the second variable, symmetric, coercive and bounded so that $A^\varepsilon \in M(\alpha, \beta, \Omega)$ (see Subsection 2.2) for fixed $0 < \alpha \leqslant \beta < \infty$. This result is known (see, e.g., [2] or [1]), but our proof is new. With $w_{ij}(x, \cdot)$ (as a function of the second variable) denoting a solution of

$$\begin{cases} \operatorname{div}_y\bigl({}^t A \nabla_y w_{ij}\bigr) = -\operatorname{div}_y\bigl({}^t A e_{ij}\bigr) & \text{in } Y, \\ w_{ij} \quad Y\text{-periodic} \end{cases} \tag{7.3}$$

we shall prove that A^ε H-converges to the tensor $A^* = A^*(x)$ defined by

$$A^*(x) = \int_Y \bigl(I + \nabla_y w(x, y)\bigr) A(x, y)\, \mathrm{d}y. \tag{7.4}$$

Let $z_{ij}(x, \cdot)$ solve

$$\begin{cases} \operatorname{div}_y\bigl({}^t A \nabla_y z_{ij}\bigr) = \operatorname{div}_x\bigl({}^t(A^* - A)e_{ij} - {}^t A \nabla_y w_{ij}\bigr) - \operatorname{div}_y\bigl({}^t A \nabla_x w_{ij}\bigr) & \text{in } Y, \\ z_{ij} \quad Y\text{-periodic.} \end{cases} \tag{7.5}$$

Under some additional regularity assumptions on A one can show that, as $\varepsilon \to 0$,

$$\begin{cases} A_{ijkl}\Bigl(\cdot, \dfrac{\cdot}{\varepsilon}\Bigr) \rightharpoonup \displaystyle\int_Y A_{ijkl}(\cdot, y)\, \mathrm{d}y \quad \text{weakly in } L^2(\Omega), \\ \dfrac{\partial w_{ijk}}{\partial y_l}\Bigl(\cdot, \dfrac{\cdot}{\varepsilon}\Bigr) \rightharpoonup \displaystyle\int_Y \dfrac{\partial w_{ijk}}{\partial y_l}(\cdot, y)\, \mathrm{d}y \quad \text{weakly in } L^2(\Omega), \\ A_{stkl}\Bigl(\cdot, \dfrac{\cdot}{\varepsilon}\Bigr) \frac{\partial w_{ijk}}{\partial y_l}\Bigl(\cdot, \dfrac{\cdot}{\varepsilon}\Bigr) \rightharpoonup \displaystyle\int_Y A_{stkl}(\cdot, y) \dfrac{\partial w_{ijk}}{\partial y_l}(\cdot, y)\, \mathrm{d}y \quad \text{weakly in } L^2(\Omega), \end{cases} \tag{7.6}$$

$$\frac{\partial w_{ijk}}{\partial x_l}\Bigl(\cdot, \frac{\cdot}{\varepsilon}\Bigr), \frac{\partial z_{ijk}}{\partial x_l}\Bigl(\cdot, \frac{\cdot}{\varepsilon}\Bigr), \frac{\partial z_{ijk}}{\partial y_l}\Bigl(\cdot, \frac{\cdot}{\varepsilon}\Bigr) \quad \text{are bounded in } L^2(\Omega), \tag{7.7}$$

$$\begin{cases} \operatorname{div}_x\bigl({}^t A \nabla_x w_{ij}\bigr)\Bigl(\cdot, \dfrac{\cdot}{\varepsilon}\Bigr), \operatorname{div}_x\bigl({}^t A \nabla_x z_{ij}\bigr)\Bigl(\cdot, \dfrac{\cdot}{\varepsilon}\Bigr), \operatorname{div}_x\bigl({}^t A \nabla_y z_{ij}\bigr)\Bigl(\cdot, \dfrac{\cdot}{\varepsilon}\Bigr) \\ \text{and } \operatorname{div}_y\bigl({}^t A \nabla_x z_{ij}\bigr)\Bigl(\cdot, \dfrac{\cdot}{\varepsilon}\Bigr) \quad \text{are bounded in } L^2(\Omega). \end{cases} \tag{7.8}$$

Assuming (7.6)–(7.8), set $u^\varepsilon_{ij}(x) = v^\varepsilon_{ij}(x, \frac{x}{\varepsilon})$ with

$$v^\varepsilon_{ijk}(x, y) = \delta_{ik} x_j + \varepsilon w_{ijk}(x, y) + \varepsilon^2 z_{ijk}(x, y).$$

Then

$$\nabla_x v^\varepsilon_{ij} = e_{ij} + \varepsilon \nabla_x w_{ij} + \varepsilon^2 \nabla_x z_{ij},$$

$$\nabla_y v_{ij}^\varepsilon = \varepsilon \nabla_y w_{ij} + \varepsilon^2 \nabla_y z_{ij},$$

$$\operatorname{div}\big({}^t A^\varepsilon \nabla u_{ij}^\varepsilon\big) = \operatorname{div}_x\big({}^t A \nabla_x v_{ij}^\varepsilon\big) + \frac{1}{\varepsilon}\operatorname{div}_x\big({}^t A \nabla_y v_{ij}^\varepsilon\big) + \frac{1}{\varepsilon}\operatorname{div}_y\big({}^t A \nabla_x v_{ij}^\varepsilon\big) + \frac{1}{\varepsilon^2}\operatorname{div}_y\big({}^t A \nabla_y v_{ij}^\varepsilon\big),$$

with the convention that the left-hand side is taken at the point x and the right-hand side is taken at the point $(x, \frac{x}{\varepsilon})$. Hence, by using (7.3), (7.5) and the same convention, it is easy to check that

$$\begin{aligned}\operatorname{div}\big({}^t A^\varepsilon \nabla u_{ij}^\varepsilon\big) = {} & \operatorname{div}_x\big({}^t A^* e_{ij}\big) + \varepsilon \operatorname{div}_x\big({}^t A \nabla_x w_{ij} + {}^t A \nabla_y z_{ij}\big) \\ & + \varepsilon \operatorname{div}_y\big({}^t A \nabla_x z_{ij}\big) + \varepsilon^2 \operatorname{div}_x\big({}^t A \nabla_x z_{ij}\big).\end{aligned} \tag{7.9}$$

We define tensors M^ε and P^ε by

$$\begin{aligned}M_{ijkl}^\varepsilon(x) &= \frac{\partial u_{ijk}^\varepsilon}{\partial x_l}(x) \\ &= \frac{\partial v_{ijk}^\varepsilon}{\partial x_l}\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial v_{ijk}^\varepsilon}{\partial y_l}\left(x, \frac{x}{\varepsilon}\right) \\ &= \delta_{ik}\delta_{jl} + \frac{\partial w_{ijk}}{\partial y_l}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \frac{\partial w_{ijk}}{\partial x_l}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \frac{\partial z_{ijk}}{\partial y_l}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 \frac{\partial z_{ijk}}{\partial x_l}\left(x, \frac{x}{\varepsilon}\right), \\ P_{ijkl}^\varepsilon(x) &= M_{ijst}^\varepsilon(x) A_{stkl}^\varepsilon(x) \\ &= A_{ijkl}\left(x, \frac{x}{\varepsilon}\right) + A_{stkl}\left(x, \frac{x}{\varepsilon}\right)\left(\left(\frac{\partial w_{ijs}}{\partial y_t}\right)\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \frac{\partial w_{ijs}}{\partial x_t}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \frac{\partial z_{ijs}}{\partial y_t}\left(x, \frac{x}{\varepsilon}\right)\right. \\ &\quad \left. + \varepsilon^2 \frac{\partial z_{ijs}}{\partial x_t}\left(x, \frac{x}{\varepsilon}\right)\right).\end{aligned}$$

It follows from (7.6) and (7.7) that

$$M_{ijkl}^\varepsilon \rightharpoonup M_{ijkl} = \delta_{ik}\delta_{jl} + \int_Y \frac{\partial w_{ijk}}{\partial y_l}\,\mathrm{d}y = \delta_{ik}\delta_{jl} = I_{ijkl} \quad \text{weakly in } L^2(\Omega),$$

$$P_{ijkl}^\varepsilon \rightharpoonup P_{ijkl} = \int_Y \left(A_{ijkl} + A_{stkl}\frac{\partial w_{ijs}}{\partial y_t}\right)\mathrm{d}y = A_{ijkl}^* \quad \text{weakly in } L^2(\Omega).$$

Moreover, by using (7.9),

$$\begin{aligned}\big(\operatorname{div} P^\varepsilon\big)_{ijk} = \frac{\partial P_{ijkl}^\varepsilon}{\partial x_l} &= \operatorname{div}_x\big({}^t A^* e_{ij}\big) + \varepsilon \operatorname{div}_x\big({}^t A \nabla_x w_{ij} + {}^t A \nabla_y z_{ij}\big) \\ &\quad + \varepsilon \operatorname{div}_y\big({}^t A \nabla_x z_{ij}\big) + \varepsilon^2 \operatorname{div}_x\big({}^t A \nabla_x z_{ij}\big),\end{aligned}$$

taken again at the point $(x, \frac{x}{\varepsilon})$. By virtue of (7.8), $(\operatorname{div} P^\varepsilon)_{ijk}$ is relatively compact in $H^{-1}(\Omega)$. Moreover $M_{ijkl}^\varepsilon = \nabla_{kl} u_{ij}^\varepsilon$, so that $(\operatorname{curl} M^\varepsilon)_{ijklm} = 0$. As $P = MA^* = A^*$, it follows from Theorem 1 that A^ε H-converges to A^*.

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