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Γ-convergence of stratified media with measure-valued limits

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Abstract. We consider energy functionals, or Dirichlet forms,

$$J^\varepsilon_\Omega(u) = \int_\Omega (A^\varepsilon \nabla u, \nabla u)\,\mathrm{d}x = \sum_{i,j=1}^N \int_\Omega a^\varepsilon_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,\mathrm{d}x,$$

for a class $\mathcal{G}$ of bounded domains $\Omega \subset \mathbb{R}^N$, with $\varepsilon > 0$ a fine structure parameter and with symmetric conductivity matrices $A^\varepsilon = (a^\varepsilon_{ij}) \in L^\infty_{\mathrm{loc}}(\mathbb{R})^{N\times N}$ which are functions only of the first coordinate x_1 and which are locally uniformly elliptic for each fixed $\varepsilon > 0$. We show that if the functions (of x_1) $b^\varepsilon_{11} = 1/a^\varepsilon_{11}$, $b^\varepsilon_{1j} = a^\varepsilon_{1j}/a^\varepsilon_{11}$ ($j \geqslant 2$), $b^\varepsilon_{ij} = a^\varepsilon_{ij} - a^\varepsilon_{i1}a^\varepsilon_{1j}/a^\varepsilon_{11}$ ($i,j \geqslant 2$) converge weakly* as measures towards corresponding limit measures b_{ij} as $\varepsilon \to 0$, if the $(1,1)$-coefficient m^ε_{11} of $(A^\varepsilon)^{-1}$ is bounded in $L^1_{\mathrm{loc}}(\mathbb{R})$ and if none of its weak* cluster measures has atoms in common with b_{ii}, $i \geqslant 2$, then the family $J^\varepsilon = \{J^\varepsilon_\Omega\}_{\Omega\in\mathcal{G}}$ Γ-converges in a local sense towards a naturally defined limit family $J = \{J_\Omega\}_{\Omega\in\mathcal{G}}$ as $\varepsilon \to 0$. An alternative way of formulating the conclusion is to say that the energy densities $(A^\varepsilon\nabla u, \nabla u)$ Γ-converge in a distributional sense towards the corresponding limit density.

Writing J^ε_Ω in terms of $B^\varepsilon = (b^\varepsilon_{ij})$ it becomes

$$J^\varepsilon_\Omega(u) = \int_\Omega \left(\frac{\partial u}{\partial x_1} + \sum_{j=2}^N b^\varepsilon_{1j}\frac{\partial u}{\partial x_j}\right)^2 \frac{1}{b^\varepsilon_{11}}\,\mathrm{d}x + \sum_{i,j=2}^N \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} b^\varepsilon_{ij}\,\mathrm{d}x,$$

and the definition of J_Ω and the limit density $(A\nabla u, \nabla u)$ is obtained by properly replacing the $b^\varepsilon_{ij} \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ by the limit measures b_{ij} and making sense to everything for u in a certain linear subspace of $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

1. Introduction

This paper is a natural follow-up of a sequence of papers [7–9] devoted to investigations of Γ-convergence, or H-convergence, of stratified media with singular and/or degenerate material characteristica.

Basically we are interested in linear elliptic problems of the type

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega, \\ + \text{ boundary conditions} \end{cases} \tag{1.1}$$

($\Omega \subset \mathbb{R}^N$ bounded, $N \geqslant 2$), where the "conductivity" matrices $A^\varepsilon = A^\varepsilon(x) = (a^\varepsilon_{ij}(x))$ are assumed symmetric, positive definite and to depend on only one of the coordinates ("stratified medium"), say $A^\varepsilon = A^\varepsilon(x_1)$. Here $\varepsilon > 0$ is a small parameter indicating the length scale of the fine structure oscillations of $A^\varepsilon(x_1)$. However, no real coupling between ε and x_1 is made, in particular no periodicity assumption is imposed.

As $\varepsilon \to 0$ the matrices A^ε are assumed to converge to a limit matrix A in a sense which is supposed to ensure convergence of the solutions u^ε towards the solution u of a corresponding limit problem determined by A. It is well-known from the work of F. Murat and L. Tartar (see, e.g., [12,14]), that the appropriate quantities in this respect are the functions (of x_1)

$$b^\varepsilon_{11} = \frac{1}{a^\varepsilon_{11}},$$

$$b^\varepsilon_{1j} = \frac{a^\varepsilon_{1j}}{a^\varepsilon_{11}} \quad (j \geqslant 2),$$

$$b^\varepsilon_{ij} = a^\varepsilon_{ij} - \frac{a^\varepsilon_{i1} a^\varepsilon_{1j}}{a^\varepsilon_{11}} \quad (i, j \geqslant 2)$$

and that these should converge weakly (or weakly*) in suitable spaces.

Our convergence assumptions indeed are that the $\{b^\varepsilon_{ij}\}$ $(i, j \geqslant 1)$ converge weakly* as measures towards some limit measures b_{ij}. In addition to this we have to require that the (1, 1)-coefficient of the inverse matrix of A^ε is locally bounded in L^1 and that none of its weak* cluster measures has atoms in common with b_{ii} for $i \geqslant 2$. See Assumptions 1–4 in Section 2 for details.

As to problem (1.1) we shall only be interested in local or interior properties of solutions. Moreover, it is well-known that the source term f plays no essential role for the convergence of the solutions u^ε. Therefore everything can be formulated as a question of convergence of the energy functionals, or Dirichlet forms, associated with the matrices A^ε, namely

$$J^\varepsilon_\Omega(u) = \int_\Omega (A^\varepsilon \nabla u, \nabla u)\, dx = \sum_{i,j=1}^N \int_\Omega a^\varepsilon_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx.$$

For these the relevant convergence concept, in our context, is Γ-convergence (see [1,4,10,11]). In the general case, when $a^\varepsilon_{12}, \ldots, a^\varepsilon_{1N}$ do not all vanish, technical difficulties prevent us from working in one fixed domain Ω. Instead, we work simultaneously in a whole class $\mathcal{G}$ of domains and with a local version of Γ-convergence formulated for the entire family $J^\varepsilon = \{J^\varepsilon_\Omega\}_{\Omega \in \mathcal{G}}$. See Conditions $\Gamma 1$ and $\Gamma 2$ in Section 2 for the exact definition. Our main result then states that the family $J^\varepsilon = \{J^\varepsilon_\Omega\}_{\Omega \in \mathcal{G}}$ Γ-converges in this local sense towards a certain limit family $J = \{J_\Omega\}_{\Omega \in \mathcal{G}}$ which has a natural, but somewhat complicated, definition in terms of the measures b_{ij}.

In a preliminary form, our results were stated and proofs indicated in [7]. In that paper we actually only treated the case of diagonal matrices A^ε. On the other hand we allowed nonlinear versions of the problem, with the energy forms of the type

$$J^\varepsilon(u) = \sum_{i=1}^N \int_\Omega a^\varepsilon_i(x_1) \left| \frac{\partial u}{\partial x_i} \right|^p dx$$

for some $1 < p < \infty$ and we worked directly with problems of the form (1.1).

After [7] was written the nonlinear diagonal case has been carefully treated in [2]. We therefore feel no need to pursue this matter any further here. Instead we concentrate on the linear full matrix case, but then also take advantage of some technical developments made in [2]. Specifically we find the idea of working in a BV-setting and using a convergence lemma such as Lemma 3.1 in [2] useful. The corresponding lemma which is needed for our purposes is formulated as Lemma 6.2 in Appendix (Section 6).

The organization of the paper is as follows. In Section 2 we give the necessary definitions and assumptions in precise forms and also state the main result of Γ-convergence. Section 3 contains an approximation lemma and some estimates needed for the main part of the proof.

The definition of Γ-convergence consists of two limit assertions. We call these Condition $\Gamma 1$ and Condition $\Gamma 2$. We thus have to prove that Conditions $\Gamma 1$ and $\Gamma 2$ are satisfied in our case, and this is done in Sections 4 and 5 respectively.

Some of the main steps in the proofs in Sections 3–5 consist of applying a couple of general results related to weak* convergence of measures (or BV-functions). For convenience we have placed these general results in an Appendix (Section 6). They are probably not very new but we have not found them in the literature, except for variants of them occuring in the recent papers [7] (Lemma 4 there) and [2] (Lemma 3.1).

Some notation used

For vectors $x \in \mathbb{R}^N$ we write $x = (x_1, x_2, \dots, x_N) = (x_1, x')$ with $x' = (x_2, \dots, x_N) \in \mathbb{R}^{N-1}$. We never use primes for derivatives in this paper.

If μ is a positive (=nonnegative) measure on an open (or just measurable) set $\Omega \subset \mathbb{R}^N$, $L^p(\mu; \Omega)$ denotes the usual Banach space of equivalence classes of functions with norm

$$\|u\|_{L^p(\mu;\Omega)} = \left(\int_\Omega |u|^p \,\mathrm{d}\mu\right)^{1/p}$$

for $1 \leqslant p < \infty$, and the usual supremum norm for $p = \infty$. When $p = 2$ the definition of $L^p(\mu; \Omega)$ can in a natural way be extended to the case that μ is a positive semi-definite matrix-valued measure and with the elements of the space being equivalence classes of vector-valued functions; see Appendix.

$L^p_{\mathrm{loc}}(\mu; \Omega)$ denotes the Frechet space determined by the seminorms $\|u\|_{L^p(\mu;K)}$ for all compact $K \subset \Omega$. Thus $u^\varepsilon \to u$ in $L^p_{\mathrm{loc}}(\mu; \Omega)$ means that $\|u^\varepsilon - u\|_{L^p(\mu;K)} \to 0$ for each compact $K \subset \Omega$ ($\varepsilon \to 0$).

In case μ is Lebesgue measure it is usually deleted from notation, and also Ω may be deleted if it is clear from the context.

$W^{1,p}(\Omega)$ etc. denote the usual Sobolev spaces.

$M(\Omega)$ denotes the set of signed Radon measures on $\Omega \subset \mathbb{R}^N$, i.e., signed regular Borel measures on Ω which are finite on compact sets. Weak* convergence $\mu^\varepsilon \rightharpoonup \mu$ in $M(\Omega)$ means that $\int \varphi \,\mathrm{d}\mu^\varepsilon \to \int \varphi \,\mathrm{d}\mu$ for each $\varphi \in C_0(\Omega)$, where $C_0(\Omega)$ denotes the set of continuous functions in Ω with compact support. Also, $C_0^\infty(\Omega)$ denotes the set of smooth functions with compact support in Ω.

For measures in integrals we use two different notations: sometimes we simply write (e.g.) $\int \varphi \,\mathrm{d}\mu$, or $\int \varphi(x) \,\mathrm{d}\mu(x)$ while many times it fits better to write $\int \varphi(x)\mu(\mathrm{d}x)$. The latter notation is particularly useful when (as is often the case) working with L^∞-functions b^ε (say) converging towards a measure b. Then we usually write $b^\varepsilon(x)\,\mathrm{d}x$, or even $b^\varepsilon(\mathrm{d}x)$, for b^ε considered as a measure (namely the one with density function b^ε with respect to Lebesgue measure) and it is then more natural to write, for the limit measure, $b(\mathrm{d}x)$ than to write $\mathrm{d}b(x)$.

If $b \in M(\mathbb{R})$, $\mathbb{R}$ representing the first coordinate axis in $\mathbb{R}^N$, we often identify b with its product with Lebesgue measure in the remaining variables, e.g., we have $b(\mathrm{d}x) = b(\mathrm{d}x_1)\,\mathrm{d}x'$.

2. Assumptions, definitions and statement of the main result

We shall discuss convergence of conductivity matrices $A^\varepsilon(x) = (a^\varepsilon_{ij}(x))^N_{i,j=1}$ towards a limit matrix $A(x) = (a_{ij}(x))^N_{i,j=1}$ in terms of Γ-convergence of corresponding energy functionals

$$J^\varepsilon_\Omega(u) = \int_\Omega (A^\varepsilon \nabla u, \nabla u)\,\mathrm{d}x = \sum_{i,j=1}^N \int_\Omega a^\varepsilon_{ij}(x)\,\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\,\mathrm{d}x,$$

$$J_\Omega(u) = \int_\Omega (A\nabla u, \nabla u)\,\mathrm{d}x.$$

A general assumption will be that the A^ε, A represent material characteristica, e.g., electric conductivities, for stratified or layered media. This means mathematically that they are constant on planes perpendicular to a certain given direction, which we take to be the e_1-direction. Thus A^ε, A are taken to depend only on the coordinate x_1.

A second main idea is that we allow the limit material, represented by A, to be as general as possible, e.g., to be singular or degenerate in the sense of having infinite or zero conductivities in certain directions. Mathematically this will mean that a certain transformed matrix $B = \Phi(A)$, appearing when writing J_Ω in a different form, has measure-valued entries.

Working in this generality causes considerable technical difficulties, and we were not able to incorporate, e.g., boundary conditions in our treatment, or even to work in one fixed domain. Therefore we shall in this paper work only in a local setting, having the data defined in all $\mathbb{R}^N$ but discussing convergence questions locally, on bounded domains $\Omega \subset \mathbb{R}^N$. Put in another way, what we are discussing is convergence, in a distributional sense, of the energy densities $(A^\varepsilon \nabla u, \nabla u)$. The general arena will be the space $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

Below we list our assumptions, *Assumptions 1–4*, in precise forms. We work in $N \geqslant 2$ dimensions and with a suitable class $\mathcal{G}$ of bounded domains $\Omega \subset \mathbb{R}^N$ (see Definition 2.3). The parameter $\varepsilon > 0$ is restricted to take values only in a *sequence* of numbers tending to zero, e.g., $\{1/n\colon\ n = 1, 2, \ldots\}$.

For the matrices A^ε, $\varepsilon > 0$, we assume first of all uniform ellipticity for fixed ε, as follows.

Assumption 1. For each fixed $\varepsilon > 0$,

$$A^\varepsilon = A^\varepsilon(x_1) = \left(a^\varepsilon_{ij}(x_1)\right)^N_{i,j=1} \in L^\infty_{\mathrm{loc}}(\mathbb{R})^{N\times N}$$

is symmetric and locally uniformly (in x_1) elliptic. In other words, A^ε is symmetric and there exist positive functions α^ε and β^ε on $\mathbb{R}$ with $1/\alpha^\varepsilon, \beta^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ such that

$$\alpha^\varepsilon(x_1)|\xi|^2 \leqslant (A^\varepsilon(x_1)\xi, \xi) \leqslant \beta^\varepsilon(x_1)|\xi|^2 \tag{2.1}$$

for all $\xi \in \mathbb{R}^N$ and $x_1 \in \mathbb{R}$.

The appropriate energy functionals, or Dirichlet forms, corresponding to A^ε are defined, for any bounded domain $\Omega \subset \mathbb{R}^N$, by

$$J^\varepsilon_\Omega(u) = \int_\Omega (A^\varepsilon(x_1)\nabla u, \nabla u)\,\mathrm{d}x = \sum_{i,j=1}^N \int_\Omega a^\varepsilon_{ij}(x_1)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,\mathrm{d}x = \|\nabla u\|^2_{L^2(A^\varepsilon;\Omega)} \tag{2.2}$$

if $u \in W^{1,2}(\Omega)$, $J^\varepsilon_\Omega(u) = +\infty$ if $u \in L^2(\Omega) \setminus W^{1,2}(\Omega)$. Then J^ε_Ω is a convex functional $L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$. Composing J^ε_Ω with the restriction map $L^2_{\mathrm{loc}}(\mathbb{R}^N) \to L^2(\Omega)$ it can be viewed also as a map

$$J^\varepsilon_\Omega : L^2_{\mathrm{loc}}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}.$$

We shall study Γ-convergence of the family $J^\varepsilon = \{J^\varepsilon_\Omega\}_{\Omega\in\mathcal{G}}$ towards a corresponding limit family $J = \{J_\Omega\}_{\Omega\in\mathcal{G}}$. The space

$$W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N) = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^N):\ u|_\Omega \in W^{1,2}(\Omega) \text{ for all } \Omega \in \mathcal{G}\}$$

can be interpreted as the space of *potentials of locally finite energy* for A^ε, $\varepsilon > 0$. In fact, $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N) = V^\varepsilon$ for all $\varepsilon > 0$ where, generally speaking,

$$V^\varepsilon = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^N):\ J^\varepsilon_\Omega(u) < +\infty \text{ for all } \Omega \in \mathcal{G}\}.$$

We shall introduce also for the limit case a subspace $V \subset L^2_{\mathrm{loc}}(\mathbb{R}^N)$ of potentials of locally finite energy. It is related to J by

$$V = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^N):\ J_\Omega(u) < +\infty \text{ for all } \Omega \in \mathcal{G}\}.$$

The exact definitions of J and V are complicated and will occupy most of the remaining parts of this section. Assuming for a moment that J and V have already been defined we may formulate the appropriate notion of Γ-convergence $J^\varepsilon \to J$ as $\varepsilon \to 0$ as follows (cf. [1,4,10,11]).

Condition Γ1. Whenever $u^\varepsilon, u \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $u^\varepsilon \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ we have, for each $\Omega \in \mathcal{G}$,

$$\varliminf_{\varepsilon\to 0} J^\varepsilon_\Omega(u^\varepsilon) \geqslant J_\Omega(u). \tag{2.3}$$

Condition Γ2. For every $u \in V$ there exist $u^\varepsilon \in V^\varepsilon$ such that $u^\varepsilon \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and such that, for each $\Omega \in \mathcal{G}$,

$$\varlimsup_{\varepsilon\to 0} J^\varepsilon_\Omega(u^\varepsilon) \leqslant J_\Omega(u). \tag{2.4}$$

To emphasize the distributional character of this Γ-convergence we also formulate it in terms of test functions instead of integration over domains. Define, for nonnegative $\varphi \in C^\infty_0(\mathbb{R}^N)$,

$$J^\varepsilon_\varphi(u) = \int_{\mathbb{R}^N} (A^\varepsilon\nabla u, \nabla u)\varphi\,\mathrm{d}x.$$

With a corresponding definition of the limit functional J_φ we get the following two conditions for distributional Γ-convergence.

Condition Γ1′. Whenever $u^\varepsilon, u \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $u^\varepsilon \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ we have, for each nonnegative $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\varliminf_{\varepsilon\to 0} J^\varepsilon_\varphi(u^\varepsilon) \geqslant J_\varphi(u). \tag{2.5}$$

Condition Γ2′. For every $u \in V$ there exist $u^\varepsilon \in V^\varepsilon$ such that $u^\varepsilon \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and such that, for each nonnegative $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\varlimsup_{\varepsilon\to 0} J^\varepsilon_\varphi(u^\varepsilon) \leqslant J_\varphi(u). \tag{2.6}$$

The limit functionals J_Ω and J_φ can in general not be defined directly in terms of a matrix $A(x_1)$ corresponding to the $A^\varepsilon(x_1)$, because this A may be too singular or degenerate. In the electric conductivity interpretations the conductivity matrix A^ε relates the electric displacement field D^ε to the electric field E^ε by

$$D^\varepsilon = A^\varepsilon E^\varepsilon, \tag{2.7}$$

and E^ε is related to the electric potential u^ε by

$$E^\varepsilon = \nabla u^\varepsilon.$$

As we allow A^ε to be rapidly oscillating as a function of x_1 some of the components of E^ε and D^ε may also be rapidly oscillating. But there are some components which are better behaved than the others. These are D^ε_1, the first component of the displacement field (good behaviour because of the continuity equation) and $E^{\varepsilon\prime} = (E^\varepsilon_2, \ldots, E^\varepsilon_N)$, the tangential (to the sheets $x_1 = \mathrm{const}$) part of the electric field (good behaviour because they are gradients of $u^\varepsilon|_{x_1=\mathrm{const}}$).

One main idea lying behind much of the work on stratified media (see [12,14], e.g.) is to express the "bad" (rapidly varying) quantities in terms of the "good" (slowly varying) ones. Thus we rewrite (2.7) as, suppressing $\varepsilon > 0$, and also the dependence on x, for a moment,

$$E_1 = b_{11}D_1 - \sum_{j\geqslant 2} b_{1j}E_j,$$

$$D_i = b_{i1}D_1 + \sum_{j\geqslant 2} b_{ij}E_j \quad (i \geqslant 2),$$

where

$$b_{11} = \frac{1}{a_{11}},$$

$$b_{1j} = b_{j1} = \frac{a_{1j}}{a_{11}} \quad \text{for } j \geqslant 2,$$

$$b_{ij} = a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}} \quad \text{for } i, j \geqslant 2.$$

Note that $a_{11} = (Ae_1, e_1) > 0$ by virtue of the ellipticity of A (see (2.1)).

Setting $B = (b_{ij})_{i,j=1}^N$ the above formulas define a transformation Φ taking any symmetric matrix A with $a_{11} \neq 0$ to a symmetric matrix $B = \Phi(A)$ with $b_{11} \neq 0$. This transformation is investigated in more detail in [9, §2]. In particular, we recall the following [9, Lemma 2.5]. Let $M = (m_{ij})$ denote the inverse matrix of A and $\lambda > 0$ the smallest eigenvalue of A (i.e., the largest constant λ in the ellipticity estimate $(A\xi, \xi) \geqslant \lambda|\xi|^2$ ($\xi \in \mathbb{R}^N$)). Then

$$0 < b_{11} \leqslant m_{11} \leqslant \frac{1}{\lambda}, \tag{2.8}$$

$$|b_{1j}|^2 \leqslant m_{11} b_{jj} \quad (j \geqslant 2), \tag{2.9}$$

$$2|b_{ij}| \leqslant b_{ii} + b_{jj} \quad (i, j \geqslant 2). \tag{2.10}$$

We also note the identities, when $D = AE$,

$$\sum_{i,j\geqslant 1} a_{ij} E_i E_j = (AE, E) = (D, E) = b_{11} D_1^2 + \sum_{i,j\geqslant 2} b_{ij} E_i E_j = b_{11} D_1^2 + (B'E', E'), \tag{2.11}$$

where we have set $B' = (b_{ij})_{i,j\geqslant 2}$, $E' = (E_i)_{i\geqslant 2}$. As a further notation, to be used later on, we set

$$b' = (b_{1j})_{j\geqslant 2}.$$

It is known that the matrix B' is elliptic with the same ellipticity constant λ as A (see [9, Lemma 2.7]).

To continue our list of assumptions, let $B^\varepsilon(x_1) = \Phi(A^\varepsilon(x_1))$. Then our second assumption on A^ε is that the coefficients $b_{ij}^\varepsilon(x_1)$ of $B^\varepsilon(x_1)$ converge weakly* as measures towards some measures $b_{ij} \in M(\mathbb{R})$ as $\varepsilon \to 0$:

Assumption 2. There exists $B \in M(\mathbb{R})^{N\times N}$ such that

$$B^\varepsilon \rightharpoonup B \quad \text{weakly* in } M(\mathbb{R})^{N\times N} \tag{2.12}$$

as $\varepsilon \to 0$.

By definition, the weak* convergence (2.12) means that $\int \varphi(x_1) b_{ij}^\varepsilon(x_1)\,\mathrm{d}x_1 \to \int \varphi(x_1) b_{ij}(\mathrm{d}x_1)$ for every $\varphi \in C_0(\mathbb{R})$ and every $1 \leqslant i, j \leqslant N$.

Assumption 2 implies in particular that for each bounded interval I there exists a constant $C_I < \infty$ such that

$$\int_I |b_{ij}^\varepsilon|\,\mathrm{d}t \leqslant C_I \tag{2.13}$$

for all $1 \leqslant i, j \leqslant N$ and all $\varepsilon > 0$. Our third assumption says that such an estimate holds also for the $(1,1)$-component m_{11}^ε of $M^\varepsilon = (A^\varepsilon)^{-1}$, the inverse matrix of A^ε:

Assumption 3. For every bounded interval I there exists $C_I < \infty$ such that

$$\int_I m_{11}^\varepsilon(t)\,\mathrm{d}t \leqslant C_I \tag{2.14}$$

for all $\varepsilon > 0$.

In the special case that $a^\varepsilon_{12} = \cdots = a^\varepsilon_{1N} = 0$ we have $m^\varepsilon_{11} = b^\varepsilon_{11}$, and then (2.14) is a consequence of (2.12), but in general (2.14) is independent of (2.12) (see Example after Remark 3.3 in [9]).

By (2.14) the sequence of positive functions $\{m^\varepsilon_{11}\}_{\varepsilon>0}$ has various weak* cluster points in $M(\mathbb{R})$. The final assumption is that atoms of these cluster points do not coincide with atoms of b_{ii} for $i \geqslant 2$. Precisely:

Assumption 4. Let μ denote any weak* cluster point of $\{m^\varepsilon_{11}\}_{\varepsilon>0}$ in $M(\mathbb{R})$. Then, for $i \geqslant 2$, μ and b_{ii} have no common atom (point mass), i.e.,

$$\mu(\{t\})b_{ii}(\{t\}) = 0 \tag{2.15}$$

for every $t \in \mathbb{R}$.

Spelling out (2.15) directly in terms of the sequences $\{m^\varepsilon_{11}\}$, $\{b^\varepsilon_{ii}\}$ it is found (cf. the proof of Lemma 6.2, in Appendix) to be equivalent to the following statement.

Assumption 4′. For every bounded interval I and every $\eta > 0$ there exists $\delta > 0$ such that

$$\int_I \int_I \chi_{|s-t|<\delta}\, m^\varepsilon_{11}(t)\, b^\varepsilon_{ii}(s)\, \mathrm{d}s\, \mathrm{d}t < \eta \tag{2.16}$$

for $\varepsilon > 0$ small enough, and $i \geqslant 2$.

Here $\chi_{|s-t|<\delta} = \chi_{|s-t|<\delta}(s,t)$ denotes the characteristic function of the set $\{(s,t) \in \mathbb{R}^2\colon\ |s-t| < \delta\}$.

The above finishes the list of assumptions. We immediately deduce from Assumptions 1–4 the following.

Lemma 2.1.

(i) *Let ν_{1j} denote any weak* cluster point in $M(\mathbb{R})$ of $\{|b^\varepsilon_{1j}|\}_{\varepsilon>0}$, $j \geqslant 2$. Then ν_{1j} has no atom. Equivalently, for every $t \in \mathbb{R}$ and $\eta > 0$ there exists $\delta > 0$ such that*

$$\int_{t-\delta}^{t+\delta} \left|b^\varepsilon_{1j}(s)\right| \mathrm{d}s < \eta \tag{2.17}$$

for $\varepsilon > 0$ small enough, and $j \geqslant 2$.

(ii) *The statement in Assumption 4′ holds with b^ε_{ii} replaced by $|b^\varepsilon_{ij}|$ for any $i, j \geqslant 2$.*

Corollary 2.2.

(i) *The entries b_{1j}, $j \geqslant 2$, of the limit matrix B have no atoms. Moreover, for every bounded interval $I \subset \mathbb{R}$ we have*

$$b^\varepsilon_{1j}(I) \to b_{1j}(I) \tag{2.18}$$

as $\varepsilon \to 0$ $(j \geqslant 2)$.

(ii) *For every bounded interval $I \subset \mathbb{R}$ we have*

$$b^{\varepsilon}_{ij}(I) \to b_{ij}(I) \tag{2.19}$$

also when $i = j = 1$ or $i, j \geqslant 2$, provided $b_{ii}(\partial I) = b_{jj}(\partial I) = 0$.

Proof of Lemma 2.1. (i) Keeping $t \in \mathbb{R}$ fixed we first notice that (2.17) holds for arbitrarily given $\eta > 0$ when $\delta > 0$, $\varepsilon > 0$ are small enough if and only if $\nu_{1j}(\{t\}) = 0$ for every weak* cluster point ν_{1j} of $\{|b^{\varepsilon}_{1j}|\}_{\varepsilon>0}$. This is elementary to verify, using that $\nu_{1j}(\{t\}) = \lim_{\delta\to 0}\int_{t-\delta}^{t+\delta} \mathrm{d}\nu_{1j}$ and approximating the characteristic function $\chi_{(t-\delta,\, t+\delta)}$ by smooth functions. Thus, for (i) it remains to prove (2.17).

By (2.9) and the Cauchy–Schwarz inequality we have

$$\int_{t-\delta}^{t+\delta} |b^{\varepsilon}_{1j}(s)|\,\mathrm{d}s \leqslant \left(\int_{t-\delta}^{t+\delta} m^{\varepsilon}_{11}(s)\,\mathrm{d}s\right)^{1/2}\left(\int_{t-\delta}^{t+\delta} b^{\varepsilon}_{jj}(s)\,\mathrm{d}s\right)^{1/2}.$$

Here each factor in the right member is bounded from above by (2.13) and (2.14), and at least one of them tends to zero as $\delta \to 0$, $\varepsilon \to 0$ by Assumption 4. Thus the left member tends to zero as $\delta \to 0$, $\varepsilon \to 0$, showing that (2.17) holds.

Statement (ii) in the lemma is an immediate consequence of (2.10).

Part (i) of the corollary can either be seen as a direct consequence of statement (i) in the lemma, or else be obtained by combining it with Lemma 6.2 in Appendix (choose $f^{\varepsilon} = 1$ there). For part (ii) one may use (2.10) together with Lemma 6.2, applied with $f^{\varepsilon} = 1$ and g^{ε} a primitive function of b^{ε}_{ij}. Basically, the corollary should be seen as a consequence of the general principle that weak* convergence $\mu^{\varepsilon} \to \mu$ of positive Radon measures implies convergence $\mu^{\varepsilon}(S) \to \mu(S)$ for every Borel set S with $\mu(\partial S) = 0$ ([6, Theorem 1.9.1]). □

We introduce below two classes of domains. The first one, $\mathcal{G}$, is the class for which the Γ-convergence Conditions $\Gamma 1$ and $\Gamma 2$ are stated, and the second, $\mathcal{F}$, is a countable subclass which generates $\mathcal{G}$ and for which the technical details of our proofs work well.

Definition 2.3.

(i) Let $\mathcal{G}$ denote the class of all bounded domains $\Omega \subset \mathbb{R}^N$ such that $\partial\Omega$ has Lebesgue measure zero and, for all $1 \leqslant i \leqslant N$, also b_{ii} measure zero.

(ii) Choose a countable dense set $S \subset \mathbb{R}$ satisfying

$$b_{ii}(S) = 0 \tag{2.20}$$

for all $1 \leqslant i \leqslant N$ and let $\mathcal{F}$ be the class of all domains of the form

$$\Omega = I_1 \times I_2 \times \cdots \times I_N,$$

where $I_1, I_2, \ldots, I_N$ are bounded open intervals with endpoints in S.

We note the following.

(i) $\mathcal{F} \subset \mathcal{G}$.
(ii) $\mathcal{F}$ is countable.
(iii) $\mathcal{F}$ is a basis for the topology of $\mathbb{R}^N$.
(iv) $\mathcal{F}$ is an exhaustion of $\mathbb{R}^N$ in the sense that every compact set is contained in some member of $\mathcal{F}$.

Since the first coordinate x_1 plays a distinguished role we shall usually write domains Ω in $\mathcal{F}$ in the form

$$\Omega = I \times \Omega', \tag{2.21}$$

where, referring to the previous notation, $I = I_1$ (for the x_1-variable) and $\Omega' = I_2 \times \cdots \times I_N \subset \mathbb{R}^{N-1}$ (for the $x' = (x_2, \ldots, x_N)$-variables).

For later purposes it is convenient to assume that S is chosen together with a "filtration" consisting of discrete subsets $S_n \subset S$ such that

$$S_1 \subset S_2 \subset \cdots \subset S, \qquad \bigcup_{n=1}^{\infty} S_n = S. \tag{2.22}$$

One may, e.g., think of $S_n = \{k2^{-n}\colon k \in \mathbb{Z}\}$ in the case that S consists of those rational numbers which have only powers of two in the denominator. The general case may be thought of as a perturbation of this.

Returning now to the energy functionals we may use (2.11) to express $J_\Omega^\varepsilon = J_\Omega^\varepsilon(u^\varepsilon)$ in terms of the slowly varying components of $E^\varepsilon = \nabla u^\varepsilon$ and $D^\varepsilon = A^\varepsilon E^\varepsilon$ as

$$J_\Omega^\varepsilon(u^\varepsilon) = \|D_1^\varepsilon\|^2_{L^2(b_{11}^\varepsilon;\Omega)} + \|E^{\varepsilon\prime}\|^2_{L^2(B^{\varepsilon\prime};\Omega)}, \tag{2.23}$$

for $u^\varepsilon \in W^{1,2}(\Omega)$, where D_1^ε, $E_i^\varepsilon \in L^2(\Omega)$ are given in terms of u^ε by

$$D_1^\varepsilon = \sum_{j\geqslant 1} a_{1j}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j}, \tag{2.24}$$

$$E^{\varepsilon\prime} = (E_i^\varepsilon)_{i\geqslant 2} = \left(\frac{\partial u^\varepsilon}{\partial x_i}\right)_{i\geqslant 2}. \tag{2.25}$$

Here (2.24) can also be written as

$$D_1^\varepsilon b_{11}^\varepsilon = \frac{\partial u^\varepsilon}{\partial x_1} + \sum_{j\geqslant 2} \frac{\partial u^\varepsilon}{\partial x_j} b_{1j}^\varepsilon. \tag{2.26}$$

The above is the clue for defining the limit functionals J_Ω. Intuitively, looking at (2.23), they should be

$$J_\Omega(u) = \|D_1\|^2_{L^2(b_{11};\Omega)} + \|E'\|^2_{L^2(B';\Omega)} \tag{2.27}$$

provided there exist $D_1 \in L^2(b_{11};\Omega)$, $E' \in L^2(B';\Omega)$ related to u by

$$D_1 b_{11} = \frac{\partial u}{\partial x_1} + \sum_{j\geqslant 2} \frac{\partial u}{\partial x_j} b_{1j}, \tag{2.28}$$

$$E' = \left(\frac{\partial u}{\partial x_j}\right)_{j\geqslant 2}, \tag{2.29}$$

$J_\Omega(u) = +\infty$ otherwise. The fields D_1 and E' should be uniquely determined by u when they exist.

In order to make such a definition of $J_\Omega(u)$ rigorous we have to make precise sense to (2.28), (2.29). Since these are local statements we may then assume that $\Omega \in \mathcal{F}$, so that Ω is of the form (2.21). By multiplying with test functions and integrating by parts, using that $\partial b_{ij}/\partial x_j = 0$ for $j \geqslant 2$, we replace, as a first step, (2.28) by its weak formulation

$$\int_\Omega \varphi D_1 b_{11}(\mathrm{d}x) + \int_\Omega u \frac{\partial \varphi}{\partial x_1}\,\mathrm{d}x + \sum_{j\geqslant 2}\int_\Omega u \frac{\partial \varphi}{\partial x_j} b_{1j}(\mathrm{d}x) = 0$$

or, more accurately,

$$(\varphi, D_1)_{L^2(b_{11};\Omega)} + \int_\Omega u \frac{\partial \varphi}{\partial x_1}\,\mathrm{d}x + \sum_{j\geqslant 2}\int_I \left(u(x_1,\cdot), \frac{\partial \varphi}{\partial x_j}(x_1,\cdot)\right)_{L^2(\Omega')} b_{1j}(\mathrm{d}x_1) = 0 \tag{2.30}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Here the first two terms make unquestionable sense when $u \in L^2(\Omega)$, $D_1 \in L^2(b_{11};\Omega)$. For the last term to make sense it is necessary that the function

$$f_j(x_1) = \left(u(x_1,\cdot), \frac{\partial \varphi}{\partial x_j}(x_1,\cdot)\right)_{L^2(\Omega')}, \tag{2.31}$$

which apriori is just an $L^2(I)$-function, is integrable with respect to the total variation measure $|b_{1j}|$. Therefore, u must be in some more restricted space than $L^2(\Omega)$. With b_{11} a measure and D_1 square-integrable, in particular integrable, with respect to b_{11} the term $D_1 b_{11}$ in (2.28) will be at worst a (signed) measure. Therefore, by (2.28) also $\partial u/\partial x_1$ may be expected to be a measure, i.e., u as a function of x_1 should be of bounded variation.

By definition, a function $f \in L^1(I)$ is of bounded variation, $f \in BV(I)$, if an estimate

$$\left|\int_I f(x_1)\frac{\mathrm{d}\varphi(x_1)}{\mathrm{d}x_1}\,\mathrm{d}x_1\right| \leqslant C\|\varphi\|_{L^\infty(I)} \tag{2.32}$$

holds for all $\varphi \in C_0^\infty(I)$. The total variation $TV(f;I)$ of f on I can then be defined as the smallest possible constant C in (2.32).

An equivalent way of expressing that a function $f \in L^1(I)$ is of bounded variation is saying that, after having extended f by zero outside I, its distributional derivative μ on $\mathbb{R}$ is a signed Radon measure

($\mu \in M(\mathbb{R})$). Then $\operatorname{supp}\mu \subset \bar{I}$ and f agrees almost everywhere with the cumulative distribution function [13,5] of μ, i.e., with

$$\tilde{f}(x_1) = \mu((-\infty, x_1]).$$

This function $\tilde{f}$ is a canonical representative of f in the sense that it is the unique function which agrees with f almost everywhere and is continuous from the right ($\tilde{f}(x_1) = \lim_{h\searrow 0} \tilde{f}(x_1+h)$).

As a convention, whenever a pointwise definition of a function $f \in BV(I)$ is required it will be understood that the right continuous version shall be used. See Appendix (Section 6) for a few more details concerning functions of bounded variation.

For a function $u \in L^2(\Omega)$, $\Omega = I \times \Omega' \in \mathcal{F}$, a suitable way of expressing that it is of bounded variation with respect to x_1 is to say that it satisfies an estimate

$$\left|\int_\Omega u \frac{\partial\varphi}{\partial x_1}\,\mathrm{d}x\right| \leqslant C\|\varphi\|_{L^\infty(I;L^2(\Omega'))} \tag{2.33}$$

for some constant C and all $\varphi \in C_0^\infty(\Omega)$. Here

$$\|\varphi\|_{L^\infty(I;L^2(\Omega'))} = \sup_{x_1\in I}\left(\int_{\Omega'} |\varphi(x_1,x')|^2\,\mathrm{d}x'\right)^{1/2}.$$

We shall denote the space of functions of bounded variation in this sense by $BV(I;L^2(\Omega'))$ and we equip it with its natural seminorm. Thus

$$BV(I;L^2(\Omega')) = \{u \in L^2(\Omega)\text{: an estimate (2.33) holds for all } \varphi \in C_0^\infty(\Omega)\},$$

with seminorm

$$\|u\|_{BV} = \text{the smallest possible } C \text{ in (2.33)}.$$

This space $BV(I;L^2(\Omega'))$ is actually identical with the space of those functions $I \to L^2(\Omega')$ which, after possible modification on a nullsubset of I, have finite total variation measured in terms of the norm of $L^2(\Omega')$. See [3], Appendice, in particular Proposition A.5. However, we shall not need much of the theory of $L^2(\Omega')$-valued functions of bounded variation, only the following simple lemma, which in particular shows that functions in $BV(I;L^2(\Omega'))$ are good enough for (2.30) to make sense.

Lemma 2.4. *Let $u \in BV(I;L^2(\Omega'))$.*

(i) *For every $\psi \in L^2(\Omega')$ the function*

$$f(x_1) = \big(u(x_1,\cdot),\psi\big)_{L^2(\Omega')} \tag{2.34}$$

is of bounded variation on I with

$$TV(f;I) \leqslant \|u\|_{BV}\,\|\psi\|_{L^2(\Omega')}, \tag{2.35}$$

$$|\tilde{f}(x_1)| \leqslant (|I|^{-1/2}\|u\|_{L^2(\Omega)} + \|u\|_{BV})\|\psi\|_{L^2(\Omega')} \tag{2.36}$$

for $x_1 \in I$ and $\tilde{f}$ denoting the right continuous version of f.

(ii) *For every $\psi \in C_0^\infty(\Omega)$ the function $f(x_1) = (u(x_1,\cdot), \psi(x_1,\cdot))_{L^2(\Omega')}$ is of bounded variation on I.*

(iii) *For every $t \in I$, u has a limit-from-the-right trace $T_t u = u|_{x_1=t} \in L^2(\Omega')$, defined by the identity*

$$(T_t u, \psi)_{L^2(\Omega')} = \lim_{h \searrow 0} (u(t+h,\cdot), \psi)_{L^2(\Omega')} \tag{2.37}$$

for $\psi \in L^2(\Omega')$. Moreover,

$$\|T_t u\|_{L^2(\Omega')} \leqslant |I|^{-1/2}\|u\|_{L^2(\Omega)} + \|u\|_{BV}.$$

Proof. (i) For $\varphi \in C_0^\infty(I)$, $\psi \in C_0^\infty(\Omega')$ we have, by definition of the space BV,

$$\left|\int_\Omega \frac{\mathrm{d}\varphi(x_1)}{\mathrm{d}x_1}\psi(x')u(x)\,\mathrm{d}x\right| \leqslant \|u\|_{BV}\,\|\varphi\|_\infty\,\|\psi\|_{L^2(\Omega')}.$$

This inequality extends by continuity to all $\psi \in L^2(\Omega')$. Thus, with f as in (2.34),

$$\left|\int_I \frac{\mathrm{d}\varphi(x_1)}{\mathrm{d}x_1}f(x_1)\,\mathrm{d}x_1\right| \leqslant \|u\|_{BV}\,\|\varphi\|_\infty\,\|\psi\|_{L^2(\Omega')},$$

from which we conclude that $f \in BV(I)$ with

$$TV(f;I) \leqslant \|u\|_{BV}\,\|\psi\|_{L^2(\Omega')}.$$

For any $x_1, t \in I$ we have

$$|\tilde{f}(x_1) - \tilde{f}(t)| \leqslant TV(f;I).$$

Integrating with respect to t gives

$$|\tilde{f}(x_1)| \leqslant |I|^{-1}\|f\|_{L^1(I)} + TV(f;I).$$

Since

$$\|f\|_{L^1(I)} \leqslant \|u\|_{L^1(I;L^2(\Omega'))}\|\psi\|_{L^2(\Omega')} \leqslant |I|^{1/2}\|u\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega')}$$

the last estimate in (i) follows.

(ii) For $\psi \in C_0^\infty(\Omega)$ and any $\varphi \in C_0^\infty(I)$ we have, by (2.33),

$$\left|\int_I \frac{\mathrm{d}\varphi(x_1)}{\mathrm{d}x_1}f(x_1)\,\mathrm{d}x_1\right| = \left|\int_\Omega \frac{\mathrm{d}\varphi(x_1)}{\mathrm{d}x_1}\psi(x)u(x)\,\mathrm{d}x\right|$$

$$= \left| \int_\Omega \frac{\partial}{\partial x_1}(\varphi(x_1)\psi(x))u(x)\,\mathrm{d}x - \int_\Omega \varphi(x_1)\frac{\partial \psi(x)}{\partial x_1}u(x)\,\mathrm{d}x \right|$$
$$\leqslant \|u\|_{BV}\|\varphi\psi\|_{L^\infty(I;L^2(\Omega'))} + \|\varphi\|_\infty \left\| \frac{\partial \psi}{\partial x_1} u \right\|_{L^1(\Omega)} \leqslant C\|\varphi\|_\infty$$

(C depending on ψ and u). Thus $f \in BV(I)$.

(iii) For fixed $\psi \in L^2(\Omega')$ the limit on the right-hand side of (2.37) exists since the dependence on h is of bounded variation, by (i). Clearly this limit is a linear function of ψ, and by (2.36) its modulus is bounded by $(\|u\|_{BV} + |I|^{-1/2}\|u\|_{L^2(\Omega)})\,\|\psi\|_{L^2(\Omega')}$. Therefore the right-hand side of (2.37) defines a bounded linear functional on $L^2(\Omega')$ with norm $\leqslant \|u\|_{BV} + |I|^{-1/2}\|u\|_{L^2(\Omega)}$. This completes the proof of the lemma. □

Now, assuming $u \in BV(I; L^2(\Omega'))$ the last term in (2.30) makes good sense, the $BV(I)$-functions (2.31) occurring there being interpreted pointwise as their right continuous representatives. These are certainly integrable with respect to the b_{1j}, being bounded and defined everywhere.

Equation (2.29) is naturally regarded as an identity in $L^2(B'; \Omega)$, for weak derivatives of u. Using again that $\partial b_{ij}/\partial x_j = 0$ for $j \geqslant 2$ we may therefore express it as

$$\sum_{i,j\geqslant 2} \int_\Omega \left(\varphi_i E_j + u\,\frac{\partial \varphi_i}{\partial x_j}\right) b_{ij}(\mathrm{d}x) = 0$$

or, more accurately, as

$$(\varphi', E')_{L^2(B';\Omega)} + \sum_{i,j\geqslant 2} \int_I \left(u(x_1,\cdot), \frac{\partial \varphi_i}{\partial x_j}(x_1,\cdot)\right)_{L^2(\Omega')} b_{ij}(\mathrm{d}x_1) = 0, \tag{2.38}$$

to hold for all $\varphi' = (\varphi_i)_{i\geqslant 2} \in C_0^\infty(\Omega)^{N-1}$. By similar remarks as above this makes sense when $u \in BV(I; L^2(\Omega'))$.

Clearly D_1 is uniquely determined as an element in $L^2(b_{11}; \Omega)$ by Eq. (2.30) when u is given. Indeed, D_1 exists if and only if the functional $\Lambda : C_0^\infty(\Omega) \to \mathbb{R}$ (depending on u) given by

$$\Lambda(\varphi) = -\int_\Omega u\,\frac{\partial \varphi}{\partial x_1}\,\mathrm{d}x - \sum_{j\geqslant 2} \int_I \left(u(x_1,\cdot), \frac{\partial \varphi}{\partial x_j}(x_1,\cdot)\right)_{L^2(\Omega')} b_{1j}(\mathrm{d}x_1)$$

is continuous in the $L^2(b_{11}; \Omega)$-norm. When this is the case Λ has a unique extension to $L^2(b_{11}; \Omega)$, $D_1 \in L^2(b_{11}; \Omega)$ is its Riesz representative and $\|D_1\|_{L^2(b_{11};\Omega)}$ equals the norm of the functional, namely $\|\Lambda\|_{L^2(b_{11};\Omega)^*} = \sup\{\Lambda(\varphi)\colon \varphi \in C_0^\infty(\Omega), \|\varphi\|_{L^2(b_{11};\Omega)} \leqslant 1\}$. In this way one could avoid mentioning D_1 and write its presence in $J_\Omega(u)$ as $\|\Lambda\|^2_{L^2(b_{11};\Omega)^*}$, or even as

$$\left\| \frac{\partial u}{\partial x_1} + \sum_{j\geqslant 2} \frac{\partial u}{\partial x_j}\, b_{1j} \right\|^2_{L^2(b_{11};\Omega)^*}.$$

Similar remarks apply to E': it exists if and only if the functional $\Lambda' : C_0^\infty(\Omega)^{N-1} \to \mathbb{R}$ defined by

$$\Lambda'(\varphi') = -\sum_{i,j\geqslant 2} \int_I \left(u(x_1, \cdot), \frac{\partial \varphi_i}{\partial x_j}(x_1, \cdot) \right)_{L^2(\Omega')} b_{ij}(\mathrm{d}x_1)$$

is continuous in the $L^2(B'; \Omega)$-norm. When this is the case E' is defined to be the Riesz representative of Λ' in $L^2(B'; \Omega)$ and is uniquely determined as an element of $L^2(B'; \Omega)$. Moreover, $\|E'\|_{L^2(B';\Omega)}$ equals the norm of the functional, namely $\|\Lambda'\|_{L^2(B';\Omega)^*} = \sup\{\Lambda'(\varphi') \colon \varphi' \in C_0^\infty(\Omega)^{N-1}, \|\varphi'\|_{L^2(B';\Omega)} \leqslant 1\}$. Without mentioning D_1 and E' the expression (2.27) for $J_\Omega(u)$ can thus be written

$$J_\Omega(u) = \left\| \frac{\partial u}{\partial x_1} + \sum_{j\geqslant 2} \frac{\partial u}{\partial x_j} b_{1j} \right\|^2_{L^2(b_{11};\Omega)^*} + \|\nabla' u\|^2_{L^2(B';\Omega)}.$$

To sum up, we are now able to define V_Ω and J_Ω for domains $\Omega \in \mathcal{F}$, as follows.

$$V_\Omega = \{u \in BV(I; L^2(\Omega')) \colon \text{ there exist } D_1 \in L^2(b_{11}; \Omega) \text{ and}$$
$$E' \in L^2(B'; \Omega) \text{ satisfying (2.30) and (2.38), respectively}\}.$$

If $u \in V_\Omega$, then D_1 and E' are uniquely determined by u and we define

$$J_\Omega(u) = \|D_1\|^2_{L^2(b_{11};\Omega)} + \|E'\|^2_{L^2(B';\Omega)}. \tag{2.39}$$

For $u \in L^2(\Omega) \setminus V_\Omega$ we simply set $J_\Omega(u) = +\infty$. For $u \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$, $u \in V_\Omega$ naturally means $u|_\Omega \in V_\Omega$, and $J_\Omega(u)$ means $J_\Omega(u|_\Omega)$.

Next we want to extend the above definitions to arbitrary domains $\Omega \subset \mathbb{R}^N$. The particular form (2.21) of domains in $\mathcal{F}$ was useful essentially only to express the relationship between u, D_1, E' as in (2.30), (2.38). It is easy to verify that if $u \in V_{\Omega_1} \cap V_{\Omega_2}$, with $\Omega_1, \Omega_2 \in \mathcal{F}$ and $\Omega_1 \cap \Omega_2 \neq \emptyset$, then the fields D_1 and E' corresponding to u agree on $\Omega_1 \cap \Omega_2$. Therefore, given $u \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and any domain $\Omega \subset \mathbb{R}^N$ we can unambigously define V_Ω and J_Ω as follows. Write Ω as a union of elements in $\mathcal{F}$, say $\Omega = \bigcup \Omega_j$, $\Omega_j \in \mathcal{F}$ and j ranging over some index set. If $u \in V_{\Omega_j}$ for all j then we get uniquely defined fields D_1 and E' in Ω. In case $D_1 \in L^2(b_{11}; \Omega)$, $E' \in L^2(B'; \Omega)$ we say that $u \in V_\Omega$ and we define $J_\Omega(u)$ again by (2.39). Otherwise, or if $u \notin V_{\Omega_j}$ for some j, we say that $u \notin V_\Omega$ and put $J_\Omega(u) = +\infty$.

The space of potentials of locally finite energy is defined as

$$V = \bigcap_{\Omega\in\mathcal{F}} V_\Omega = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^N) \colon u_\Omega \in V_\Omega \text{ for all } \Omega \in \mathcal{F}\}$$
$$= \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^N) \colon J_\Omega(u) < +\infty \text{ for all } \Omega \in \mathcal{F}\}.$$

Here, $\mathcal{F}$ can be replaced by $\mathcal{G}$, or any other exhaustion of $\mathbb{R}^N$ with bounded domains.

When $u \in V$ we have global fields $D_1 \in L^2_{\mathrm{loc}}(b_{11}; \mathbb{R}^N)$ and $E' \in L^2_{\mathrm{loc}}(B'; \mathbb{R}^N)$, which can be defined directly by global versions of (2.30) and (2.38) to $\mathbb{R}^N$. These are

$$(\varphi_1, D_1)_{L^2(b_{11};\mathbb{R}^N)} + \int_{\mathbb{R}^N} u \frac{\partial \varphi_1}{\partial x_1} \mathrm{d}x + \sum_{j\geqslant 2} \int_{\mathbb{R}} \left(u(x_1, \cdot), \frac{\partial \varphi_1}{\partial x_j}(x_1, \cdot) \right)_{L^2(\mathbb{R}^{N-1})} b_{1j}(\mathrm{d}x_1) = 0 \tag{2.40}$$

and

$$(\varphi', E')_{L^2(B';\mathbb{R}^N)} + \sum_{i,j\geqslant 2} \int_{\mathbb{R}} \left(u(x_1,\cdot), \frac{\partial \varphi_i}{\partial x_j}(x_1,\cdot)\right)_{L^2(\mathbb{R}^{N-1})} b_{ij}(\mathrm{d}x_1) = 0 \tag{2.41}$$

respectively, to hold for all $\varphi = (\varphi_1, \varphi') = (\varphi_i)_{i\geqslant 1} \in C_0^\infty(\mathbb{R}^N)^N$. Setting

$$BV_{\mathrm{loc}} = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^N)\colon\ u|_\Omega \in BV(I; L^2(\Omega')) \text{ for all } \Omega \in \mathcal{F}\}$$

we can therefore summarize the global versions of the definitions as follows.

Definition 2.5. The space of potentials of locally finite energy is

$$\begin{aligned} V = \{&u \in BV_{\mathrm{loc}}\colon \text{ there exist } D_1 \in L^2_{\mathrm{loc}}(b_{11};\mathbb{R}^N) \text{ and} \\ &E' \in L^2_{\mathrm{loc}}(B';\mathbb{R}^N) \text{ such that (2.40) and (2.41) hold}\}. \end{aligned}$$

For $u \in V$ and an arbitrary domain $\Omega \subset \mathbb{R}^N$ the energy $J_\Omega(u)$ is given by (2.39), i.e.,

$$J_\Omega(u) = \|D_1\|^2_{L^2(b_{11};\Omega)} + \|E'\|^2_{L^2(B';\Omega)}.$$

Finally, we define J_φ for nonnegative functions $\varphi \in C_0^\infty(\mathbb{R}^N)$. Given such a φ, setting $\Omega = \{x \in \mathbb{R}^N\colon \varphi(x) > 0\}$ and assuming that $u \in V_\omega$ for every $\omega \in \mathcal{F}$ compactly contained in Ω we define

$$J_\varphi(u) = \sup \sum_j c_j J_{\Omega_j}(u), \tag{2.42}$$

where the supremum is taken over all step functions $0 \leqslant \sum_j c_j \chi_{\Omega_j} \leqslant \varphi$ (finite sum) with $\Omega_j \in \mathcal{F}$ compactly contained in Ω. Here the supremum, and hence $J_\varphi(u)$, may be equal to $+\infty$. In the case that $u \notin V_\omega$ for some ω as above we also set $J_\varphi(u) = +\infty$.

Now we can state our main result.

Theorem 2.6. *Under Assumptions* 1–4, J^ε *Γ-converges, as* $\varepsilon \to 0$, *towards* J *in the sense that Conditions* $\Gamma 1$ *and* $\Gamma 2$, *as well as* $\Gamma 1'$ *and* $\Gamma 2'$, *hold.*

The direct proofs of Conditions $\Gamma 1$ and $\Gamma 2$ will be given in Sections 4 and 5, respectively. The next section is devoted to some auxiliary results needed for these proofs.

3. Some auxiliary results

In this section we prepare for the proofs of Conditions $\Gamma 1$ and $\Gamma 2$ by showing that every $u \in V$ can be approximated by functions which are smooth in the x'-directions and by establishing a representation formula for these latter (Lemma 3.1). Moreover, a useful estimate is given (Lemma 3.2).

Lemma 3.1. *For every* $u \in V$ *there exist* $u^\eta \in V$ $(\eta > 0)$ *which are smooth in the* x'*-variables and which satisfy* (i)–(iii) *below*:

(i) *As* $\eta \to 0$,

$$u^\eta \to u \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^N),$$
$$D_1^\eta \to D_1 \quad \text{in } L^2_{\mathrm{loc}}(b_{11};\mathbb{R}^N),$$
$$E'^\eta \to E' \quad \text{in } L^2_{\mathrm{loc}}(B';\mathbb{R}^N),$$

with D_1^η *and* E'^η *related to* u^η *as in* (2.40), (2.41).

(ii) *For each fixed* $\Omega \in \mathcal{F}$, $\eta > 0$ *and* $i, j \geqslant 2$,

$$u^\eta, \frac{\partial u^\eta}{\partial x_i}, \frac{\partial^2 u^\eta}{\partial x_i \partial x_j} \in L^\infty(\Omega), \tag{3.1}$$

$$\int_I \sup_{x' \in \Omega'} \left|D_1^\eta(x_1, x')\right|^2 b_{11}(\mathrm{d}x_1) < \infty, \tag{3.2}$$

$$\int_I \sup_{x' \in \Omega'} \left|\frac{\partial D_1^\eta}{\partial x_i}(x_1, x')\right|^2 b_{11}(\mathrm{d}x_1) < \infty, \tag{3.3}$$

$$\int_I \sup_{x' \in \Omega'} \left|\frac{\partial^2 D_1^\eta}{\partial x_i \partial x_j}(x_1, x')\right|^2 b_{11}(\mathrm{d}x_1) < \infty. \tag{3.4}$$

(iii) *The following representation formulas express* u^η *in terms of* D_1^η *and the trace of* u^η *at* $x_1 = s$, *for any* $s \in \mathbb{R}$:

$$u^\eta(x) = u^\eta(s, x' - b'(s, x_1]) + \int_{(s,x_1]} D_1^\eta(t, x' - b'(t, x_1])b_{11}(\mathrm{d}t) \tag{3.5}$$

for $x_1 \geqslant s$, *and*

$$u^\eta(x) = u^\eta(s, x' + b'(x_1, s]) - \int_{(x_1,s]} D_1^\eta(t, x' + b'(x_1, t])b_{11}(\mathrm{d}t) \tag{3.6}$$

for $x_1 < s$ $(x = (x_1, x') \in \mathbb{R}^N)$.

Remark. The representation formulas in (iii) of the lemma will be used in the proof that Conditions $\Gamma 2$ and $\Gamma 2'$ hold (Lemma 5.1). Note that for $x \in \Omega$ the right members of (3.5), (3.6) may invoke values of u^η and D_1^η at points far outside Ω (in the x'-directions). Thus in order to have a representation formula in Ω it is necessary to start with $u \in V_{\Omega_1}$ for a larger domain Ω_1, or simply with $u \in V$. This is the explanation that we do not work in one fixed domain throughout the paper and that, in Condition $\Gamma 2$ for example, we assume $u \in V$ even to deduce (2.4) for one particular domain.

Proof. To prove (i), take $u \in V$ and let $D_1 \in L^2_{\mathrm{loc}}(b_{11};\mathbb{R}^N))$ and $E' \in L^2_{\mathrm{loc}}(B';\mathbb{R}^N)$ be the functions occurring in Definition 2.5. We shall approximate u by convolving with standard mollifiers in the x'-directions.

Take $h = h_{N-1} \in C_0^\infty(\mathbb{R}^{N-1})$ a radially symmetric mollifier satisfying $h \geqslant 0$, $\int_{\mathbb{R}^{N-1}} h(x')\,\mathrm{d}x' = 1$, $\operatorname{supp} h \subset B(0,1)$ and set, for $\eta > 0$,

$$h^\eta(y') = \frac{1}{\eta^{N-1}} h\left(\frac{y'}{\eta}\right).$$

Define u^η by

$$u^\eta(x_1, x') = \left(u(x_1, \cdot) * h^\eta\right)(x') = \int_{\mathbb{R}^{N-1}} u(x_1, x' - y')h^\eta(y')\,\mathrm{d}y',$$

the convolution of u with h^η with respect to the x'-variables, and similarly

$$D_1^\eta(x_1, x') = \left(D_1(x_1, \cdot) * h^\eta\right)(x') = \int_{\mathbb{R}^{N-1}} D_1(x_1, x' - y')h^\eta(y')\,\mathrm{d}y',$$

$$E'^\eta(x_1, x') = \left(E'(x_1, \cdot) * h^\eta\right)(x') = \int_{\mathbb{R}^{N-1}} E'(x_1, x' - y')h^\eta(y')\,\mathrm{d}y'.$$

Then

$$u^\eta \longrightarrow u \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N), \tag{3.7}$$

$$D_1^\eta \longrightarrow D_1 \quad \text{in } L^2_{\text{loc}}(b_{11}; \mathbb{R}^N), \tag{3.8}$$

$$E'^\eta \longrightarrow E' \quad \text{in } L^2_{\text{loc}}(B'; \mathbb{R}^N) \tag{3.9}$$

as $\eta \longrightarrow 0$. This all follows by standard arguments. Considering, e.g., D_1 and given a small number $\delta > 0$ we can first approximate D_1 on any $\Omega \in \mathcal{F}$ by a function $f \in C_0^\infty(\mathbb{R}^N)$ so that

$$\|D_1 - f\|_{L^2(b_{11};\Omega)} < \delta.$$

Since f is uniformly continuous the function $f^\eta = f * h^\eta$ (defined by $f^\eta(x_1, x') = (f(x_1, \cdot) * h^\eta)(x')$) tends to f uniformly in Ω, hence also in the $L^2(b_{11}; \Omega)$-norm, as $\eta \longrightarrow 0$. As convolution with h^η is a meanvalue process it decreases L^2-norms, so that

$$\|D_1^\eta - f^\eta\|_{L^2(b_{11};\Omega)} \leqslant \|D_1 - f\|_{L^2(b_{11};\Omega)}.$$

Combining these estimates gives (3.8), and (3.7), (3.9) are proved similarly.

Thus

$$\|u^\eta - u\|_{L^2(\Omega)} \longrightarrow 0,$$

$$\|D_1^\eta - D_1\|^2_{L^2(b_{11};\Omega)} + \|E'^\eta - E'\|^2_{L^2(B';\Omega)} \longrightarrow 0$$

for any $\Omega \in \mathcal{F}$, as $\eta \longrightarrow 0$. It remains to be checked that the second left member above equals $J_\Omega(u^\eta - u)$, namely that D_1^η and E'^η are related to u^η as in (2.40), (2.41).

To see that this is the case, choose any test functions $\psi_i \in C_0^\infty(\mathbb{R}^N)$ and apply (2.40), (2.41) with the test functions $\varphi_i \in C_0^\infty(\mathbb{R}^N)$ given by

$$\varphi_i(x) = \left(\psi_i(x_1, \cdot) * h^\eta\right)(x') = \int_{\mathbb{R}^{N-1}} \psi_i(x_1, y')h^\eta(y' - x')\,\mathrm{d}y'.$$

Then the resulting equations, after having moved the convolutions from φ to u, D_1 and E', simply say that u^η, D_1^η and E'^η are related the right way.

The statements in part (ii) of the lemma follow easily from the immediate estimate

$$\left|(f * g)(x)\right| \leqslant \|g\|_\infty \int_{x' - \eta\,\mathrm{supp}\,h} \left|f(x_1, y')\right| \mathrm{d}y'$$

used with $f = u$ or D_1, $g = h^\eta$ or one of its derivatives and taking into account the facts that, for any $\Omega \in \mathcal{F}$, $u \in L^\infty(I; L^2(\Omega'))$ (by Lemma 2.4) and $D_1 \in L^2(b_{11}; \Omega)$.

To prove the representation formula (3.5) in part (iii) of the lemma, another kind of test functions will be used in (2.40). It is most convenient then to start from the original u and D_1 because the convolution with h^η will be built into the test functions we shall use.

Ideally, we would like to take, in place of φ_1 in (2.40), the test function

$$\varphi(t, y') = \chi_{(s,x_1]}(t)h^\eta\left(y' - x' + b'(t, x_1]\right)$$

because, as a formal computation shows, this would give (3.5). However, that φ is not in $C_0^\infty(\mathbb{R}^N)$ so we have to approximate it.

We shall take a doubly indexed approximation family $\varphi_{mn} \in C_0^\infty(\mathbb{R}^N)$ given by

$$\varphi_{mn}(t, y') = f_m(t)h^\eta\left(y' - x' + b'_n(t, x_1]\right),$$

where f_m are smooth versions of $\chi_{(s,x_1]}$ and b'_n smooth versions of b'. Letting $h_1^\eta \in C_0^\infty(\mathbb{R})$ denote one-dimensional mollifiers we may take

$$f_m(t) = \left(\chi_{(s,x_1]} * h_1^{1/m}\right)(t - 1/m),$$

which approximate the characteristic function $\chi_{(s,x_1]}$ from the right, and

$$b'_n = b' * h_1^{1/n},$$

i.e.,

$$b'_n(\mathrm{d}t) = \mathrm{d}t \int_{\mathbb{R}} h_1^{1/n}(t - \tau)b'(\mathrm{d}\tau).$$

We regard b'_n as vector-valued measures and hence write, e.g., $b'_n(t, x_1]$ for its value on the interval $(t, x_1]$. The dependence on t and x_1 is of course smooth. We have

$$b'_n \rightharpoonup b' \quad \text{weakly* in } M(\mathbb{R}) \tag{3.10}$$

as $n \to \infty$.

Let $b' = b'_+ - b'_-$ be the decomposition of b' into its positive and negative parts. Then we even have

$$b'_+ * h_1^{1/n} \rightharpoonup b'_+,$$
$$b'_- * h_1^{1/n} \rightharpoonup b'_-$$

weakly* in $M(\mathbb{R})$. Since b' has no atom (Corollary 2.2) none of b'_+ and b'_- has atoms, and it follows that

$$(b'_\pm * h_1^{1/n})(t, x_1] \to b'_\pm(t, x_1]$$

for each of the signs and for all t ([6, Theorem 1.9.1]). Thus $b'_n(t, x_1]$ tends to $b'(t, x_1]$ for all t. Since

$$|b'_n| = |b'_+ * h_1^{1/n} - b'_- * h_1^{1/n}| \leqslant b'_+ * h_1^{1/n} + b'_- * h_1^{1/n}$$

we also conclude that for every $\tau \in \mathbb{R}$, $\rho > 0$ there exists $\delta > 0$ such that

$$|b'_n|(\tau - \delta, \tau + \delta) < \rho \tag{3.11}$$

for n large enough.

The above φ_{mn} are in $C_0^\infty(\mathbb{R}^N)$, and inserting them into (2.40) gives

$$\begin{aligned}
0 &= \iint \varphi_{mn} D_1 b_{11}(\mathrm{d}t)\,\mathrm{d}y' + \iint u\,\frac{\partial \varphi_{mn}}{\partial t}\,\mathrm{d}t\,\mathrm{d}y' + \iint u \nabla' \varphi_{mn} \cdot b'(\mathrm{d}t)\,\mathrm{d}y' \\
&= \iint f_m(t) h^\eta(y' - x' + b'_n(t, x_1]) D_1(t, y') b_{11}(\mathrm{d}t)\,\mathrm{d}y' \\
&\quad + \iint u(t, y')\,\frac{\mathrm{d}f_m(t)}{\mathrm{d}t}\,h^\eta(y' - x' + b'_n(t, x_1])\,\mathrm{d}t\,\mathrm{d}y' \\
&\quad + \iint u(t, y') f_m(t) \nabla' h^\eta(y' - x' + b'_n(t, x_1]) \cdot \frac{\mathrm{d}b'_n(t, x_1]}{\mathrm{d}t}\,\mathrm{d}t\,\mathrm{d}y' \\
&\quad + \iint u(t, y') f_m(t) \nabla' h^\eta(y' - x' + b'_n(t, x_1]) \cdot b'(\mathrm{d}t)\,\mathrm{d}y',
\end{aligned} \tag{3.12}$$

all double integrals taken over $\mathbb{R}^N$.

Given η, the functions f_m, h^η, $\nabla' h^\eta$ above are uniformly bounded (independent of m and n). We can therefore easily (by dominated convergence, e.g.) pass to the limit with the first term in the last member of (3.12), letting $m, n \to \infty$ in an arbitrary fashion:

$$\begin{aligned}
&\iint f_m(t) h^\eta(y' - x' + b'_n(t, x_1]) D_1(t, y') b_{11}(\mathrm{d}t)\,\mathrm{d}y' \\
&\quad \to \iint \chi_{(s, x_1]}(t) h^\eta(y' - x' + b'(t, x_1]) D_1(t, y') b_{11}(\mathrm{d}t)\,\mathrm{d}y' \\
&\quad = \int_{(s, x_1]} D_1^\eta(t, x' - b'(t, x_1]) b_{11}(\mathrm{d}t).
\end{aligned}$$

For the second term in (3.12) we may pass to the limit first in m, giving

$$\begin{aligned}&\iint u(t,y')\,\frac{\mathrm{d}f_m(t)}{\mathrm{d}t}\,h^\eta\big(y'-x'+b_n'(t,x_1]\big)\,\mathrm{d}t\,\mathrm{d}y'\\&\quad\to\int u(s,y')h^\eta\big(y'-x'+b_n'(s,x_1]\big)\,\mathrm{d}y'-\int u(x_1,y')h^\eta(y'-x')\,\mathrm{d}y'\\&\quad=u^\eta\big(s,x'-b_n'(s,x_1]\big)-u^\eta(x_1,x').\end{aligned}$$

Recall that we always work with right-continuous versions of functions of bounded variation, and it is easy to see from the way $f_m(t)$ was defined that the factor $\mathrm{d}f_m(t)/\mathrm{d}t$ in the integral above picks up the limit-from-the-right traces of the functions at $t=s$ and $t=x_1$. Letting then $n\to\infty$ we find that the last expression tends to

$$u^\eta\big(s,x'-b'(s,x_1]\big)-u^\eta(x_1,x').$$

The last two terms in (3.12) can, taken together, be written

$$\iint u(t,y')f_m(t)\nabla'h^\eta\big(y'-x'+b_n'(t)\big)\,\mathrm{d}y'\,\mathrm{d}\big(b_n'(t,x_1]-b'(t,x_1]\big). \tag{3.13}$$

We would like to show that this tends to zero by using Lemma 6.2 in Appendix. The measure $b_n'-b'$ tends to zero weak* by (3.10), as $n\to\infty$, and (6.15) in Lemma 6.2 is satisfied in view of (3.11) (and b' having no atom). Hence to apply Lemma 6.2 we would like to show that the remaining part of (3.13) is bounded in $BV(I)$, uniformly in m and n, where I is taken to be a fixed interval which contains the domain of integration with respect to t. Then (6.16) will also follow from (3.11).

Thus we look for an estimate

$$\left|\int_I\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}\int_{\Omega'}u(t,y')f_m(t)\nabla'h^\eta\big(y'-x'+b_n'(t,x_1]\big)\,\mathrm{d}y'\,\mathrm{d}t\right|\leqslant C\|\psi\|_\infty, \tag{3.14}$$

to hold for all $\psi\in C_0^\infty(I)$ with C independent of m and n. Choosing one component $\partial h^\eta/\partial x_j$ of $\nabla'h^\eta$ we have

$$\begin{aligned}&\int_I\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}\int_{\Omega'}u(t,y')f_m(t)\,\frac{\partial h^\eta}{\partial x_j}\,\big(y'-x'+b_n'(t,x_1]\big)\,\mathrm{d}y'\,\mathrm{d}t\\&\quad=\int_I\int_{\Omega'}u(t,y')\,\frac{\partial}{\partial t}\left(\psi(t)f_m(t)\,\frac{\partial h^\eta}{\partial x_j}\,\big(y'-x'+b_n'(t,x_1]\big)\right)\mathrm{d}y'\,\mathrm{d}t\\&\qquad-\int_I\int_{\Omega'}u(t,y')\psi(t)\,\frac{\mathrm{d}f_m(t)}{\mathrm{d}t}\,\frac{\partial h^\eta}{\partial x_j}\,\big(y'-x'+b_n'(t,x_1]\big)\,\mathrm{d}y'\,\mathrm{d}t\\&\qquad-\int_I\int_{\Omega'}u(t,y')\psi(t)f_m(t)\nabla'\frac{\partial h^\eta}{\partial x_j}\big(y'-x'+b_n'(t,x_1]\big)\cdot\frac{\mathrm{d}b_n'(t,x_1]}{\mathrm{d}t}\,\mathrm{d}t\,\mathrm{d}y'.\end{aligned}$$

Here the first term can be estimated by

$$C\left\|\psi f_m\,\frac{\partial h^\eta}{\partial x_j}\right\|_\infty\leqslant C\|\psi\|_\infty$$

because $u \in BV(I; L^2(\Omega'))$, the second term can be estimated by

$$C\|u\|_{L^\infty(I;L^2(\Omega'))}\|\psi\|_\infty \left\|\frac{\partial h^\eta}{\partial x_j}\right\|_\infty \int_I \left|\frac{\mathrm{d}f_m}{\mathrm{d}t}\right| \mathrm{d}t \leqslant C\|\psi\|_\infty$$

and the third similarly (using (3.10) by

$$C\|u\|_{L^\infty(I;L^2(\Omega'))}\|\psi\|_\infty \, \|f_m\|_\infty \left\|\nabla' \frac{\partial h^\eta}{\partial x_j}\right\|_\infty |b'_n|(I) \leqslant C\|\psi\|_\infty.$$

Thus we get the desired estimate (3.14) and Lemma 6.2 shows that (3.13) tends to zero as $n \to \infty$, independent of the behaviour of m.

Returning now to (3.12) and putting the pieces together we find that as $m \to \infty$ first and $n \to \infty$ afterwards we get

$$0 = \int_{(s,x_1]} D_1^\eta(t, x' - b'(t, x_1])b_{11}(\mathrm{d}t) + u^\eta(s, x' - b'(s, x_1]) - u^\eta(x_1, x'),$$

which is (3.5). Formula (3.6) is proved similarly. This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let $\Omega \in \mathcal{F}$. For any $u^\varepsilon \in W^{1,2}(\Omega)$ we have*

$$\left\|\frac{\partial u^\varepsilon}{\partial x_1}\right\|^2_{L^1(I;L^2(\Omega'))} \leqslant J^\varepsilon_\Omega(u^\varepsilon) \int_I m^\varepsilon_{11}(x_1)\,\mathrm{d}x_1.$$

Proof. Let $e_1 = {}^t(1, 0, \ldots, 0)$ and set $\xi^\varepsilon = (A^\varepsilon)^{-1}e_1$. Then $(A^\varepsilon\xi^\varepsilon, \xi^\varepsilon) = m^\varepsilon_{11}$. Using the Cauchy–Schwarz inequality for the positive definite form determined by A^ε we get (pointwise a.e.)

$$\left|\frac{\partial u^\varepsilon}{\partial x_1}\right|^2 = \left|(e_1, \nabla u^\varepsilon)\right|^2 = \left|(A^\varepsilon\xi^\varepsilon, \nabla u^\varepsilon)\right|^2 \leqslant (A^\varepsilon\xi^\varepsilon, \xi^\varepsilon)\,(A^\varepsilon\nabla u^\varepsilon, \nabla u^\varepsilon) = m^\varepsilon_{11}(A^\varepsilon\nabla u^\varepsilon, \nabla u^\varepsilon).$$

Thus, using the Cauchy–Schwarz inequality once more,

$$\begin{aligned}
\left\|\frac{\partial u^\varepsilon}{\partial x_1}\right\|^2_{L^1(I;L^2(\Omega'))} &= \left(\int_I \left\|\frac{\partial u^\varepsilon}{\partial x_1}(x_1, \cdot)\right\|_{L^2(\Omega')} \mathrm{d}x_1\right)^2 = \left(\int_I \left(\int_{\Omega'} \left|\frac{\partial u^\varepsilon}{\partial x_1}(x_1, x')\right|^2 \mathrm{d}x'\right)^{1/2} \mathrm{d}x_1\right)^2 \\
&\leqslant \left(\int_I \left(\int_{\Omega'} m^\varepsilon_{11}(x_1)\,(A^\varepsilon\nabla u^\varepsilon, \nabla u^\varepsilon)\,\mathrm{d}x'\right)^{1/2} \mathrm{d}x_1\right)^2 \\
&= \left(\int_I m^\varepsilon_{11}(x_1)^{1/2}\left(\int_{\Omega'} (A^\varepsilon\nabla u^\varepsilon, \nabla u^\varepsilon)\,\mathrm{d}x'\right)^{1/2} \mathrm{d}x_1\right)^2 \\
&\leqslant \int_I m^\varepsilon_{11}(x_1)\,\mathrm{d}x_1 \int_I\int_{\Omega'} (A^\varepsilon\nabla u^\varepsilon, \nabla u^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}x_1 \\
&= \int_I m^\varepsilon_{11}(x_1)\,\mathrm{d}x_1 \cdot J^\varepsilon_\Omega(u^\varepsilon),
\end{aligned}$$

as desired.

4. Proof of Γ-convergence: Condition $\Gamma 1$

In this section we prove that Conditions $\Gamma 1$ and $\Gamma 1'$ hold under Assumptions 1–4. In fact, Condition $\Gamma 1$ follows immediately from the lemma below. Condition $\Gamma 1'$ is obtained by applying the lemma to domains in $\mathcal{F}$, approximating φ from below by step functions based on $\mathcal{F}$ and using the definition (2.42) of J_φ. The easy details of this are omitted.

Lemma 4.1. *Let $\Omega \in \mathcal{G}$, suppose u^ε, $u \in L^2(\Omega)$ and*

$$u^\varepsilon \to u \quad \text{in } L^2(\Omega). \tag{4.1}$$

Then

$$\underline{\lim}_{\varepsilon\to 0} \; J^\varepsilon_\Omega(u^\varepsilon) \geqslant J_\Omega(u). \tag{4.2}$$

Proof. By a standard argument with subsequences we may assume that

$$J^\varepsilon_\Omega(u^\varepsilon) \leqslant C < \infty, \tag{4.3}$$

and we only need to prove that (4.2) holds for some subsequence of $\{\varepsilon\}$.

From Assumption 1, (2.2), (2.23), (4.3) we see that $u^\varepsilon \in W^{1,2}(\Omega)$ and

$$\|D^\varepsilon_1\|_{L^2(b^\varepsilon_{11};\Omega)} \leqslant C < \infty,$$
$$\|E^{\varepsilon'}\|_{L^2(B^{\varepsilon'};\Omega)} \leqslant C < \infty.$$

From Assumption 2 it follows that $B^\varepsilon \rightharpoonup B$ weakly* in $M(\overline{\Omega})^{N\times N}$, $\overline{\Omega}$ denoting the closure of Ω, hence that $b^\varepsilon_{11} \rightharpoonup b_{11}$ weakly* in $M(\overline{\Omega})$ and $B^{\varepsilon'} \rightharpoonup B'$ weakly* in $M(\overline{\Omega})^{(N-1)\times(N-1)}$. We may therefore apply Lemma 6.1 (Appendix) to conclude that there exist $D_1 \in L^2(b_{11};\overline{\Omega})$, $E' \in L^2(B';\overline{\Omega})$ such that, for a subsequence,

$$D^\varepsilon_1 b^\varepsilon_{11} \rightharpoonup D_1 b_{11} \quad \text{weakly* in } M(\overline{\Omega}), \tag{4.4}$$
$$B^{\varepsilon'} E^{\varepsilon'} \rightharpoonup B'E' \quad \text{weakly* in } M(\overline{\Omega})^{(N-1)\times(N-1)}, \tag{4.5}$$
$$\|D_1\|_{L^2(b_{11};\overline{\Omega})} \leqslant \underline{\lim}_{\varepsilon\to 0} \|D^\varepsilon_1\|_{L^2(b^\varepsilon_{11};\overline{\Omega})}, \tag{4.6}$$
$$\|E'\|_{L^2(B';\overline{\Omega})} \leqslant \underline{\lim}_{\varepsilon\to 0} \|E^{\varepsilon'}\|_{L^2(B^{\varepsilon'};\overline{\Omega})}. \tag{4.7}$$

Since $\partial\Omega$ has Lebesgue measure zero (and hence b^ε_{ii} measure zero) we can in (4.6) and (4.7) replace $\overline{\Omega}$ by Ω.

In view of the expression (2.23) for $J^\varepsilon_\Omega(u^\varepsilon)$, (4.2) thus follows once we know that $\|D_1\|^2_{L^2(b_{11};\Omega)} + \|E'\|^2_{L^2(B';\Omega)}$ equals $J_\Omega(u)$, i.e., once we know that D_1 and E' obtained in the limits (4.4), (4.5) are related to u in the right way.

This is a purely local question, hence it is enough to prove it for every subdomain belonging to the class $\mathcal{F}$. With a change of notation we may as well assume that Ω itself is in $\mathcal{F}$. Thus Ω is of the form

(2.21) and what we then have to show is that $u \in BV(I; L^2(\Omega'))$ and that D_1 and E' are related to u by (2.30), (2.38).

First, using (4.3) and Lemma 3.2, we have

$$\left|\int_\Omega u^\varepsilon \frac{\partial \varphi}{\partial x_1}\,dx\right| = \left|\int_\Omega \frac{\partial u^\varepsilon}{\partial x_1}\varphi\,dx\right| \leqslant \left\|\frac{\partial u^\varepsilon}{\partial x_1}\right\|_{L^1(I;L^2(\Omega'))}\|\varphi\|_{L^\infty(I;L^2(\Omega'))} \leqslant C\|\varphi\|_{L^\infty(I;L^2(\Omega'))}$$

for $\varphi \in C_0^\infty(\Omega)$. Since $u^\varepsilon \to u$ in $L^2(\Omega)$ this shows that u satisfies the same estimate, i.e., u satisfies (2.33). Thus we conclude that $u \in BV(I; L^2(\Omega'))$.

Next, the definitions (2.24), (2.25) of D_1^ε and $E^{\varepsilon\prime}$ can clearly be written in the same form as (2.30), (2.38), namely

$$(\varphi_1, D_1^\varepsilon)_{L^2(b_{11}^\varepsilon;\Omega)} + \int_\Omega u^\varepsilon \frac{\partial \varphi_1}{\partial x_1}\,dx + \sum_{j\geqslant 2}\int_I \left(u^\varepsilon(x_1,\cdot), \frac{\partial \varphi_1}{\partial x_j}(x_1,\cdot)\right)_{L^2(\Omega')} b_{1j}^\varepsilon(x_1)\,dx_1 = 0, \tag{4.8}$$

$$(\varphi', E^{\varepsilon\prime})_{L^2(B^{\varepsilon\prime};\Omega)} + \sum_{i,j\geqslant 2}\int_I \left(u^\varepsilon(x_1,\cdot), \frac{\partial \varphi_i}{\partial x_j}(x_1,\cdot)\right)_{L^2(\Omega')} b_{ij}^\varepsilon(x_1)\,dx_1 = 0 \tag{4.9}$$

for $\varphi = (\varphi_1, \varphi') \in C_0^\infty(\Omega)^N$. We claim that, as $\varepsilon \to 0$, each term in (4.8), (4.9) converges to the corresponding term in (2.30), (2.38).

It follows immediately from (4.1), (4.4), (4.5) that

$$(\varphi_1, D_1^\varepsilon)_{L^2(b_{11}^\varepsilon;\Omega)} \to (\varphi_1, D_1)_{L^2(b_{11};\Omega)},$$

$$(\varphi', E^{\varepsilon\prime})_{L^2(B^{\varepsilon\prime};\Omega)} \to (\varphi', E')_{L^2(B';\Omega)},$$

$$\int_\Omega u^\varepsilon \frac{\partial \varphi_1}{\partial x_1}\,dx \to \int_\Omega u \frac{\partial \varphi_1}{\partial x_1}\,dx.$$

Hence we are left with proving that

$$\int_I (u^\varepsilon(x_1,\cdot), \psi(x_1,\cdot))_{L^2(\Omega')} b_{ij}^\varepsilon(x_1)\,dx_1 \to \int_I (u(x_1,\cdot), \psi(x_1,\cdot))_{L^2(\Omega')} b_{ij}(dx_1), \tag{4.10}$$

for any $i \geqslant 1$, $j \geqslant 2$, $\psi \in C_0^\infty(\Omega)$.

Set

$$f^\varepsilon(x_1) = (u^\varepsilon(x_1,\cdot), \psi(x_1,\cdot))_{L^2(\Omega')}, \qquad f(x_1) = (u(x_1,\cdot), \psi(x_1,\cdot))_{L^2(\Omega')} \tag{4.11}$$

so that assertion (4.10) becomes

$$\int_I f^\varepsilon(x_1) b_{ij}^\varepsilon(x_1)\,dx_1 \to \int_I f(x_1) b_{ij}(dx_1). \tag{4.12}$$

Here we know from Lemma 2.4 that $f \in BV(I)$. We shall use Lemma 6.2 in Appendix to prove (4.12).

For a.e. $x_1 \in I$ we have (recall that $u^\varepsilon \in W^{1,2}(\Omega)$ by (4.3),

$$\begin{aligned}\left|\frac{\mathrm{d}f^\varepsilon}{\mathrm{d}x_1}(x_1)\right| &= \left|\int_{\Omega'} \frac{\partial u^\varepsilon}{\partial x_1}(x_1,x')\,\psi(x_1,x')\,\mathrm{d}x' + \int_{\Omega'} u^\varepsilon(x_1,x')\,\frac{\partial \psi}{\partial x_1}(x_1,x')\,\mathrm{d}x'\right| \\ &\leqslant C\int_{\Omega'}\left|\frac{\partial u^\varepsilon}{\partial x_1}(x_1,x')\right|\mathrm{d}x' + C\int_{\Omega'}|u^\varepsilon(x_1,x')|\,\mathrm{d}x'.\end{aligned}$$

Thus, for any subinterval $(s,t)\subset I$,

$$\begin{aligned}\int_s^t\left|\frac{\mathrm{d}f^\varepsilon}{\mathrm{d}x_1}\right|\mathrm{d}x_1 &\leqslant C\left\|\frac{\partial u^\varepsilon}{\partial x_1}\right\|_{L^1((s,t);L^1(\Omega'))} + C\|u^\varepsilon\|_{L^1((s,t);L^1(\Omega'))} \\ &\leqslant C\left\|\frac{\partial u^\varepsilon}{\partial x_1}\right\|_{L^1((s,t);L^2(\Omega'))} + C|t-s|^{1/2}\|u^\varepsilon\|_{L^2((s,t)\times\Omega')}.\end{aligned}$$

Together with Lemma 3.2 and (4.1), (4.3) this implies that

$$\left(\int_s^t\left|\frac{\mathrm{d}f^\varepsilon}{\mathrm{d}x_1}\right|\mathrm{d}x_1\right)^2 \leqslant C\int_s^t m_{11}^\varepsilon(x_1)\,\mathrm{d}x_1 + C|t-s| \tag{4.13}$$

for $(s,t)\subset I$.

By (4.11) and the Cauchy–Schwarz inequality

$$\|f^\varepsilon\|_{L^1(I)} \leqslant \|u^\varepsilon\|_{L^2(\Omega)}\,\|\psi\|_{L^2(\Omega)} \leqslant C, \tag{4.14}$$

$$\|f^\varepsilon - f\|_{L^1(I)} \leqslant \|u^\varepsilon - u\|_{L^2(\Omega)}\,\|\psi\|_{L^2(\Omega)} \leqslant C\|u-u^\varepsilon\|_{L^2(\Omega)}. \tag{4.15}$$

It follows from (4.14), (4.13) and Assumption 3 that $\{f^\varepsilon\}$ is bounded in $W^{1,1}(I)$, hence in $BV(I)$. Using (4.15) and (4.1) it also follows that $f^\varepsilon \to f$ in $L^1(I)$. Thus we conclude that

$$f^\varepsilon \rightharpoonup f \quad \text{weakly* in } BV(I), \tag{4.16}$$

for a subsequence.

Using (2.13) and (4.13) we next find that

$$\begin{aligned}&\left(\int_I\int_I \chi_{|x_1-s|<\delta}\left|\frac{\mathrm{d}f^\varepsilon}{\mathrm{d}x_1}\right|\mathrm{d}x_1\,|b_{ij}^\varepsilon(s)|\,\mathrm{d}s\right)^2 \\ &\quad\leqslant \int_I\left(\int_I \chi_{|x_1-s|<\delta}\left|\frac{\mathrm{d}f^\varepsilon}{\mathrm{d}x_1}\right|\mathrm{d}x_1\right)^2|b_{ij}^\varepsilon(s)|\,\mathrm{d}s\cdot\int_I|b_{ij}^\varepsilon(s)|\mathrm{d}s \\ &\quad\leqslant C\int_I\int_I \chi_{|x_1-s|<\delta}m_{11}^\varepsilon(x_1)\,\mathrm{d}x_1\,|b_{ij}^\varepsilon(s)|\mathrm{d}s + C\delta.\end{aligned}$$

This is smaller than any given number $\eta > 0$ when $i,j \geqslant 2$ and $\delta > 0$, $\varepsilon > 0$ are small enough, by Assumption 4′ and (2.10). When $i = 1$ and $j \geqslant 2$, we write, using (2.9) and Assumption 3,

$$\int_I \chi_{|x_1-s|<\delta}|b_{1j}^\varepsilon(s)|\mathrm{d}s \leqslant \int_I \chi_{|x_1-s|<\delta}m_{11}^\varepsilon(s)^{1/2}b_{jj}^\varepsilon(s)^{1/2}\,\mathrm{d}s \leqslant C\left(\int_I \chi_{|x_1-s|<\delta}b_{jj}^\varepsilon(s)\,\mathrm{d}s\right)^{1/2}$$

so that, using again Assumption 3,

$$\int_I \int_I \chi_{|x_1-s|<\delta} |b^\varepsilon_{1j}(s)| m^\varepsilon_{11}(x_1)\, dx_1\, ds \leqslant C \int_I m^\varepsilon_{11}(x_1) \left(\int_I \chi_{|x_1-s|<\delta} b^\varepsilon_{jj}(s)\, ds \right)^{1/2} dx_1$$

$$\leqslant C \left(\int_I m^\varepsilon_{11}(x_1) \int_I \chi_{|x_1-s|<\delta} b^\varepsilon_{jj}(s)\, ds\, dx_1 \right)^{1/2},$$

which gives the same conclusion as for $i, j \geqslant 2$.

Using the ε-version of (2.10), (2.17) and (2.20) it also follows that

$$\int_{t-\delta}^{t+\delta} |b^\varepsilon_{ij}(s)|\, ds < \eta \tag{4.17}$$

for $t \in \partial I$ and $\delta > 0$, $\varepsilon > 0$ small enough.

By combining the above with (4.16) and Assumption 2, and using Lemma 6.2 (in Appendix) (4.12) now follows. This finishes the proof of Lemma 4.1 □

5. Proof of Γ-convergence: Condition Γ2

Here we prove that Conditions $\Gamma 2$ and $\Gamma 2'$ hold under Assumptions 1–4. This will follow from

Lemma 5.1. *For every $u \in V$ there exist $u^\varepsilon \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N)$ such that, for each $\Omega \in \mathcal{F}$,*

$$u^\varepsilon \to u \quad \text{in } L^2(\Omega),$$

$$\overline{\lim_{\varepsilon \to 0}}\, J^\varepsilon_\Omega(u^\varepsilon) \leqslant J_\Omega(u).$$

Before proving the lemma we deduce Conditions $\Gamma 2$ and $\Gamma 2'$ from it. As for Condition $\Gamma 2$, let $u \in V$ and $\Omega \in \mathcal{G}$ be given. Referring to the sets S_n in (2.22) we may approximate Ω from outside by, for example,

$$\Omega_n = \text{the union of those components of } (\mathbb{R} \setminus S_n)^N \text{ which meet } \Omega.$$

Then $\Omega \subset \Omega_n$ except for a set of b_{ii} measure zero and Lebesgue measure zero, and $\bigcap_{n=1}^\infty \Omega_n \subset \overline{\Omega}$. Since $\partial\Omega$ has b_{ii} measure zero we conclude that

$$J_{\Omega_n}(u) \searrow J_\Omega(u)$$

as $n \to \infty$. Since the lemma clearly holds for each Ω_n, Ω_n being a finite disjoint union of elements in $\mathcal{F}$, we conclude that it holds for Ω. Condition $\Gamma 2$ now follows.

Condition $\Gamma 2'$ is obtained in a similar manner, by approximating a given nonnegative $\varphi \in C_0^\infty(\mathbb{R}^N)$ from above by step functions based on $\mathcal{F}$. The simple details are omitted.

Proof of Lemma 5.1. For clarity we split this long and technical proof into subsections, a) to h).

a) Definition of u^ε

Let $u \in V$. By Lemma 3.1 there exists a sequence u^η of elements of V, u^η smooth in the x' variables, $u^\eta \to u$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, $D_1^\eta \to D_1$ in $L^2_{\text{loc}}(b_{11};\mathbb{R}^N)$, $E'^\eta \to E'$ in $L^2_{\text{loc}}(B';\mathbb{R}^N)$, when $\eta \to 0$. Moreover u^η is given by the following representation formula:

$$u^\eta(x) = u^\eta(0, x' - b'(0, x_1]) + \int_{(0,x_1]} D_1^\eta(t, x' - b'(t, x_1])b_{11}(\mathrm{d}t) \tag{5.1}$$

for $x \in \mathbb{R}^N$ with $x_1 \geqslant 0$, and

$$u^\eta(x) = u^\eta(0, x' + b'(x_1, 0]) - \int_{(x_1,0]} D_1^\eta(t, x' + b'(x_1, t])b_{11}(\mathrm{d}t) \tag{5.2}$$

when $x_1 < 0$.

The sequence u^ε to be constructed in this proof will be defined from a sequence $u^{\varepsilon,n,\eta}$ depending on three parameters, ε, n, η. Here $\varepsilon > 0$ is the same parameter as in J^ε, $\eta > 0$ is the smoothing parameter above (and in Lemma 3.1) and n is an integer parameter corresponding to subdivisions of $\mathbb{R}$ into intervals, in fact the same parameter as in (2.22). The final sequence u^ε will be obtained by specifying n and η as functions of ε. Specifically, it will be defined as

$$u^\varepsilon = u^{\varepsilon,\overline{n}(p),1/p}, \tag{5.3}$$

where the integer parameter p is defined in terms of ε by $\overline{\varepsilon}(p+1) < \varepsilon \leqslant \overline{\varepsilon}(p)$ and where $\overline{\varepsilon}(p)$ and $\overline{n}(p)$ are conveniently selected subsequences of ε and n.

Note. We will temporarily mix two notations with superscripts. In writing u^ε, the superscript ε is the final homogenization parameter, while in writing u^η, or sometimes $u^{1/p}$ where $\eta = 1/p$, the superscript η is the smoothing parameter in the x'-directions (Lemma 3.1). Similarly for D_1^ε, D_1^η and $D_1^{1/p}$. We shall only use the letters indicated above (ε, η, $1/p$) and therefore hope that no confusion will arise.

Now to the details. We use the decompositions of $\mathbb{R}$ into intervals defined by the sets S_n in (2.22). Thus we write

$$\mathbb{R} \setminus S_n = \bigcup_{k \in \mathbb{Z}} I_k,$$

where $I_k = I_k^n$ are open intervals (namely the components of $\mathbb{R} \setminus S_n$), indexed by $\mathbb{Z}$ for convenience (we assume that the S_n are infinite). We may further assume that the S_n are defined so that $|I_k^n| < 1/n$ for all k and n. Since $b_{ii}(S_n) = 0$ by (2.20), (2.22) we have

$$b_{11}(\partial I_k^n) = 0 \tag{5.4}$$

for all k and n.

We shall define $u^{\varepsilon,n,\eta}$ by first constructing its field $D_1^{n,\eta}$ with respect to A^ε (or J^ε), and this will only depend on n and η. In fact, we set

$$D_1^{n,\eta}(x) = \sum_{k\in\mathbb{Z}} \chi_{I_k}(x_1)\psi_k(x'),$$

where the functions $\psi_k = \psi_k^{n,\eta}$ are given by

$$\psi_k(x') = \frac{1}{b_{11}(I_k)} \int_{I_k} D_1^\eta(t,x') b_{11}(\mathrm{d}t) \quad \text{if } b_{11}(I_k) > 0,$$
$$\psi_k(x') = 0 \quad \text{if } b_{11}(I_k) = 0.$$

Thus $D_1^{n,\eta}$ is a step function with respect to x_1, obtained by b_{11}-weighted meanvalues. Next we define $u^{\varepsilon,n,\eta}$ by

$$\begin{aligned} u^{\varepsilon,n,\eta}(x) &= u^\eta\big(0, x' - (b^\varepsilon)'\,(0,x_1]\big) + \int_{(0,x_1]} D_1^{n,\eta}\big(t, x' - (b^\varepsilon)'\,(t,x_1]\big) b_{11}^\varepsilon(\mathrm{d}t) \\ &= u^\eta\big(0, x' - (b^\varepsilon)'\,(0,x_1]\big) + \sum_{k\in\mathbb{Z}} \int_{I_k\cap(0,x_1]} \psi_k\big(x' - (b^\varepsilon)'\,(t,x_1]\big) b_{11}^\varepsilon(\mathrm{d}t) \end{aligned} \tag{5.5}$$

for $x \in \mathbb{R}^N$ with $x_1 \geqslant 0$, and similarly for $x_1 < 0$ (see (3.6), (5.2)). Note that only finitely many terms are nonzero in the last summation.

The function $u^{\varepsilon,n,\eta}$ is smooth in the x'-directions and $D_1^{n,\eta}$ is a bounded function in each fixed $\Omega \in \mathcal{F}$. Thus we may differentiate (5.5) to obtain

$$\begin{aligned} \frac{\partial u^{\varepsilon,n,\eta}}{\partial x_1}(x) &= -(b^\varepsilon)'\,(x_1) \cdot \nabla' u^\eta\big(0, x' - (b^\varepsilon)'\,(0,x_1]\big) + b_{11}^\varepsilon(x_1) D_1^{n,\eta}(x_1, x') \\ &\quad - (b^\varepsilon)'\,(x_1) \cdot \int_{(0,x_1]} \nabla' D_1^{n,\eta}\big(t, x' - (b^\varepsilon)'\,(t,x_1]\big) b_{11}^\varepsilon(\mathrm{d}t), \end{aligned}$$

$$\nabla' u^{\varepsilon,n,\eta}(x) = \nabla' u^\eta\big(0, x' - (b^\varepsilon)'\,(0,x_1]\big) + \int_{(0,x_1]} \nabla' D_1^{n,\eta}\big(t, x' - (b^\varepsilon)'\,(t,x_1]\big) b_{11}^\varepsilon(\mathrm{d}t).$$

Multiplying the latter relation with $(b^\varepsilon)'(x_1)$ and adding to the first one gives

$$\frac{\partial u^{\varepsilon,n,\eta}}{\partial x_1}(x) + (b^\varepsilon)'\,(x_1) \cdot \nabla' u^{\varepsilon,n,\eta}(x) = b_{11}^\varepsilon(x_1) D_1^{n,\eta}(x). \tag{5.6}$$

Relation (5.6) shows that $D_1^{n,\eta}$ is the field related to $u^{\varepsilon,n,\eta}$ via B^ε as in (2.26). This means that the B^ε-version of (2.40) holds for $D_1^{n,\eta}$ and $u^{\varepsilon,n,\eta}$. With notations as in (5.3) the field D_1^ε coupled to u^ε will thus finally be

$$D_1^\varepsilon = D_1^{\overline{n}(p),1/p}.$$

Since $u^{\varepsilon,n,\eta}$ is differentiable with respect to the x'-variables it is also immediate that $\nabla' u^{\varepsilon,n,\eta}$ is in $L^2(B^{\varepsilon\prime})$ and represents the field E' there, i.e., that (2.41) holds with $\nabla' u^{\varepsilon,n,\eta}$ in place of E', $u^{\varepsilon,n,\eta}$ in place of u and b_{ij}^ε in place of b_{ij}. With u^ε as in (5.3), the corresponding E'-field will thus be $\nabla' u^\varepsilon$.

b) General outline of proof

Defining u^ε and D_1^ε as above we shall prove that

$$u^\varepsilon \to u \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0, \tag{5.7}$$

$$\overline{\lim_{\varepsilon\to 0}} \|D_1^\varepsilon\|_{L^2(b_{11}^\varepsilon;\Omega)} \leqslant \|D_1\|_{L^2(b_{11};\Omega)}, \tag{5.8}$$

$$\lim_{\varepsilon\to 0} \|\nabla' u^\varepsilon\|_{L^2(B^{\varepsilon\prime};\Omega)} = \|E'\|_{L^2(B';\Omega)}, \tag{5.9}$$

for each $\Omega \in \mathcal{F}$. By the above remarks and (2.23), (2.27) this will imply the lemma.

c) Proof of (5.8)

We first prove (5.8). Let $\Omega = I \times \Omega' \in \mathcal{F}$. By (2.22) and the definition of $\mathcal{F}$, the endpoints of I are in S_n for n large enough, say for $n \geqslant \overline{n}$. Thus, for fixed $n \geqslant \overline{n}$, I is the union of finitely many of the I_k plus finitely many points (with no b_{ii} mass).

Applying the Cauchy–Schwarz inequality to the definition of ψ_k and $D_1^{n,\eta}$ we have

$$\int_{I_k} \left|D_1^{n,\eta}(x_1,x')\right|^2 b_{11}(\mathrm{d}x_1) \leqslant \int_{I_k} \left|D_1^{\eta}(x_1,x')\right|^2 b_{11}(\mathrm{d}x_1) \tag{5.10}$$

for each $x' \in \mathbb{R}^{N-1}$ and k. This gives

$$\int_I \left|D_1^{n,\eta}\right|^2 b_{11}(\mathrm{d}x_1) \leqslant \int_I \left|D_1^{\eta}\right|^2 b_{11}(\mathrm{d}x_1),$$

and hence

$$\int_\Omega \left|D_1^{n,\eta}\right|^2 b_{11}(\mathrm{d}x) \leqslant \int_\Omega \left|D_1^{\eta}\right|^2 b_{11}(\mathrm{d}x). \tag{5.11}$$

Since $\int_{\Omega'} |D_1^{n,\eta}(x_1,x')|^2\,\mathrm{d}x'$ is piecewise constant as a function of x_1 with jumps only at the partition points t_k, and since (2.19), (5.4) together with Corollary 2.2 implies that

$$b_{11}^\varepsilon(I_k) \to b_{11}(I_k) \tag{5.12}$$

for each k we have, for fixed n and η,

$$\int_\Omega \left|D_1^{n,\eta}(x)\right|^2 b_{11}^\varepsilon(\mathrm{d}x) \to \int_\Omega \left|D_1^{n,\eta}(x)\right|^2 b_{11}(\mathrm{d}x)$$

as $\varepsilon \to 0$. Together with (5.11) this shows that

$$\int_\Omega \left|D_1^{n,\eta}(x)\right|^2 b_{11}^\varepsilon(\mathrm{d}x) \leqslant \int_\Omega \left|D_1^{\eta}(x)\right|^2 b_{11}(\mathrm{d}x) + \eta \tag{5.13}$$

for $\varepsilon \leqslant \overline{\varepsilon}(n,\eta)$ (say) and $n \geqslant \overline{n}$. Later on we will have to redefine $\overline{n}$ and let it depend also on η, $\overline{n} = \overline{n}(\eta)$. We take $\eta = 1/p$ where p is an integer parameter, we write $\overline{\varepsilon}(p)$ instead of $\overline{\varepsilon}(\overline{n}(1/p), 1/p)$

and $\overline{n}(p)$ instead of $\overline{n}(1/p)$ and we can assume $\overline{\varepsilon}(p)$ monotone decreasing. Defining $D_1^\varepsilon = D_1^{\overline{n}(p),1/p}$ for $\overline{\varepsilon}(p+1) < \varepsilon \leqslant \overline{\varepsilon}(p)$, we have by (5.13)

$$\int_\Omega \left|D_1^\varepsilon(x)\right|^2 b_{11}^\varepsilon(\mathrm{d}x) = \int_\Omega \left|D_1^{\overline{n}(p),1/p}\right|^2 b_{11}^\varepsilon(\mathrm{d}x) \leqslant \int_\Omega \left|D_1^{1/p}(x)\right|^2 b_{11}(\mathrm{d}x) + \frac{1}{p}.$$

Since $p \longrightarrow \infty$ as $\varepsilon \longrightarrow 0$ this gives, by Lemma 3.1,

$$\overline{\lim_{\varepsilon\to 0}} \int_\Omega |D_1^\varepsilon|^2 b_{11}^\varepsilon\,(\mathrm{d}x) \leqslant \int_\Omega |D_1|^2\, b_{11}(\mathrm{d}x).$$

So far, $\overline{\varepsilon}(p)$, and hence u^ε, depends also on $\Omega \in \mathcal{F}$, and we really have to define u^ε independently of Ω. Since the family $\mathcal{F}$ is countable this can however easily be achieved by passing to a diagonal sequence for $\overline{\varepsilon}(p)$. (For the first Ω in an enumeration of $\mathcal{F}$ we get a certain sequence $\overline{\varepsilon}(p)$, for the second Ω we choose a subsequence of this, for the third we take a further subsequence, etc. The final sequence will be the diagonal sequence of the family of subsequences selected.) This kind of comment will be in force, though not repeated, for any redefinition of $\overline{\varepsilon}(p)$. Thus we have now proved (5.8)

d) More estimates

In order to prove (5.7) and (5.9) we need some more estimates. First (5.10) gives

$$\begin{aligned} b_{11}(I_k)\left|\psi_k(x')\right|^2 &\leqslant \int_{I_k} \left|D_1^\eta(x_1,x')\right|^2 b_{11}(\mathrm{d}x_1) \leqslant \sup_{x'\in\Omega'} \int_{I_k} \left|D_1^\eta(x_1,x')\right|^2 b_{11}(\mathrm{d}x_1) \\ &\leqslant \int_{I_k} \sup_{x'\in\Omega'} \left|D_1^\eta(x_1,x')\right|^2 b_{11}(\mathrm{d}x_1), \end{aligned}$$

$$\int_{I_k} \sup_{x'\in\Omega'} \left|D_1^{n,\eta}(x_1,x')\right|^2 b_{11}(\mathrm{d}x_1) = b_{11}(I_k)\|\psi_k\|_\infty^2 \leqslant \int_{I_k} \sup_{x'\in\Omega'} \left|D_1^\eta(x_1,x')\right|^2 b_{11}(\mathrm{d}x_1)$$

and for $x_1 \in I,\ x_1 \geqslant 0$,

$$\int_{(0,x_1]} \sup_{x'\in\Omega'} \left|D_1^{n,\eta}(x_1,x')\right|^2 b_{11}(\mathrm{d}x_1) \leqslant \int_{-1}^{\sup I+1} \sup_{x'\in\Omega'} \left|D_1^\eta(x_1,x')\right|^2 b_{11}(\mathrm{d}x_1) = C(\eta) < \infty$$

(by (3.2)). Therefore, using (5.12)

$$\int_{(0,x_1]} \sup_{x'\in\Omega'} \left|D_1^{n,\eta}(x_1,x')\right|^2 b_{11}^\varepsilon(\mathrm{d}x_1) \leqslant C(\eta), \tag{5.14}$$

for $\varepsilon \leqslant \overline{\varepsilon}(n,\eta)$ (up to a change of $C(\eta)$ and $\overline{\varepsilon}(n,\eta)$ defined after (5.13)). In particular,

$$\int_{(0,x_1]} \left|D_1^{n,\eta}(t, x' - (b^\varepsilon)'(t,x_1])\right|^2 b_{11}^\varepsilon(\mathrm{d}t) \leqslant C(\eta) \tag{5.15}$$

for $\varepsilon \leqslant \overline{\varepsilon}(n,\eta)$ and for a constant $C(\eta)$ which corresponds to a domain $\Omega_1 = I \times \Omega_1'$ with Ω_1' sufficiently enlarged (compared to Ω') in all directions. Since also

$$\left|u^\eta(0, x' - (b^\varepsilon)'(0,x_1])\right| \leqslant \sup_{y'\in\Omega_1{}'} \left|u^\eta(0,y')\right| < \infty$$

by (3.1), we immediately deduce from (5.15) and the definition of $u^{\varepsilon,n,\eta}$ that $\|u^{\varepsilon,n,\eta}\|_{L^\infty(\Omega\cap\{x_1\geqslant 0\})} \leqslant C(\eta)$. As one obtains similar estimates for $x_1 < 0$, it follows that

$$\|u^{\varepsilon,n,\eta}\|_{L^\infty(\Omega)} \leqslant C(\eta) < \infty. \tag{5.16}$$

Similarly by using analogues of (5.10) with x' derivatives and by using (3.3) or (3.4) instead of (3.2), we get

$$\left\|\frac{\partial u^{\varepsilon,n,\eta}}{\partial x_i}\right\|_{L^\infty(\Omega)} \leqslant C(\eta) < \infty, \tag{5.17}$$

$$\left\|\frac{\partial^2 u^{\varepsilon,n,\eta}}{\partial x_i \partial x_j}\right\|_{L^\infty(\Omega)} \leqslant C(\eta) < \infty, \tag{5.18}$$

for $i, j \geqslant 2$ and $0 < \varepsilon < \overline{\varepsilon}(n,\eta)$, $\overline{\varepsilon}(n,\eta)$ possibly redefined.

As a final preparatory step we differentiate (5.6) with respect to x_i, $i \geqslant 2$, to obtain

$$\frac{\partial^2 u^{\varepsilon,n,\eta}}{\partial x_1 \partial x_i} + (b^\varepsilon)' \cdot \nabla' \frac{\partial u^{\varepsilon,n,\eta}}{\partial x_i} = b^\varepsilon_{11} \frac{\partial D_1^{n,\eta}}{\partial x_i}. \tag{5.19}$$

Here all terms are still in $L^\infty(\Omega)$ and are smooth in x'.

e) Reduction of (5.7) and (5.9) to estimates (5.21)–(5.24).

In order to show (5.7) and (5.9), we introduce the intermediate function

$$\begin{aligned} u^{n,\eta}(x) &= u^\eta(0, x' - b'(0, x_1]) + \int_{(0,x_1]} D_1^{n,\eta}(t, x' - b'(t, x_1]) b_{11}(\mathrm{d}t) \\ &= u^\eta(0, x' - b'(0, x_1]) + \sum_{k\in\mathbb{Z}} \int_{I_k \cap (0,x_1]} \psi_k(x' - b'(t, x_1]) b_{11}(\mathrm{d}t) \end{aligned} \tag{5.20}$$

for $x \in \mathbb{R}^N$ with $x_1 \geqslant 0$ and similarly for $x_1 < 0$. We also set

$$f_{ij}^{\varepsilon,n,\eta}(x_1) = \left(\frac{\partial u^{\varepsilon,n,\eta}}{\partial x_i}(x_1,\cdot), \frac{\partial u^{\varepsilon,n,\eta}}{\partial x_j}(x_1,\cdot)\right)_{L^2(\Omega')},$$

$$f_{ij}^{n,\eta}(x_1) = \left(\frac{\partial u^{n,\eta}}{\partial x_i}(x_1,\cdot), \frac{\partial u^{n,\eta}}{\partial x_j}(x_1,\cdot)\right)_{L^2(\Omega')},$$

$$f_{ij}^{\eta}(x_1) = \left(\frac{\partial u^{\eta}}{\partial x_i}(x_1,\cdot), \frac{\partial u^{\eta}}{\partial x_j}(x_1,\cdot)\right)_{L^2(\Omega')}.$$

Then

$$\|\nabla' u^{\varepsilon,n,\eta}\|^2_{L^2(B^{\varepsilon\prime};\Omega)} = \sum_{i,j\geqslant 2} \int_I f_{ij}^{\varepsilon,n,\eta}(x_1) b_{ij}^\varepsilon(x_1)\,\mathrm{d}x_1,$$

$$\|\nabla' u^{n,\eta}\|^2_{L^2(B';\Omega)} = \sum_{i,j\geqslant 2} \int_I f_{ij}^{n,\eta}(x_1) b_{ij}(\mathrm{d}x_1),$$

$$\|E'^{\eta}\|^2_{L^2(B';\Omega)} = \sum_{i,j\geqslant 2} \int_I f^{\eta}_{ij}(x_1) b_{ij}(\mathrm{d}x_1).$$

To show (5.7), (5.9), we will prove that

$$\|u^{\varepsilon,n,\eta} - u^{n,\eta}\|_{L^2(\Omega)} \leqslant \eta, \tag{5.21}$$

$$\big|\|\nabla' u^{\varepsilon,n,\eta}\|^2_{L^2(B^{\varepsilon\prime};\Omega)} - \|\nabla' u^{n,\eta}\|^2_{L^2(B';\Omega)}\big| \leqslant \eta, \tag{5.22}$$

for $\varepsilon \leqslant \overline{\varepsilon}(n,\eta)$ and that

$$\|u^{n,\eta} - u^{\eta}\|_{L^\infty(\Omega)} \leqslant \eta, \tag{5.23}$$

$$\big|\|\nabla' u^{n,\eta}\|^2_{L^2(B';\Omega)} - \|\nabla' u^{\eta}\|^2_{L^2(B';\Omega)}\big| \leqslant \eta, \tag{5.24}$$

for $n \geqslant \overline{n}(\eta)$. (Here again $\overline{\varepsilon}$ and $\overline{n}$ are possibly redefined.)

Assuming (5.21) to (5.24) are already proved we will have

$$\begin{aligned}\|u^{\varepsilon,n,\eta} - u\|_{L^2(\Omega)} &\leqslant \|u^{\varepsilon,n,\eta} - u^{n,\eta}\|_{L^2(\Omega)} + \|u^{n,\eta} - u^{\eta}\|_{L^2(\Omega)} + \|u^{\eta} - u\|_{L^2(\Omega)} \\ &\leqslant C\eta + \|u^{\eta} - u\|_{L^2(\Omega)}\end{aligned}$$

for $n \geqslant \overline{n}(\eta)$ and $\varepsilon \leqslant \overline{\varepsilon}(n,\eta)$ or

$$\|u^{\varepsilon} - u\|_{L^2(\Omega)} \leqslant \frac{C}{p} + \|u^{1/p} - u\|_{L^2(\Omega)}$$

for $\overline{\varepsilon}(p+1) < \varepsilon \leqslant \overline{\varepsilon}(p)$. By letting p tend to infinity and by using Lemma 3.1 we get (5.7). Similarly we will have

$$\begin{aligned}&\big|\|\nabla' u^{\varepsilon}\|^2_{L^2(B^{\varepsilon\prime};\Omega)} - \|E'\|^2_{L^2(B';\Omega)}\big| = \big|\|\nabla' u^{\varepsilon,\overline{n}(p),1/p}\|^2_{L^2(B^{\varepsilon\prime};\Omega)} - \|E'\|^2_{L^2(B';\Omega)}\big| \\ &\quad\leqslant \big|\|\nabla' u^{\varepsilon,\overline{n}(p),1/p}\|^2_{L^2(B^{\varepsilon\prime};\Omega)} - \|(E')^{1/p}\|^2_{L^2(B';\Omega)}\big| + \big|\|(E')^{1/p}\|^2_{L^2(B';\Omega)} - \|E'\|^2_{L^2(B';\Omega)}\big| \\ &\quad= \big|\|\nabla' u^{\varepsilon,\overline{n}(p),1/p}\|^2_{L^2(B^{\varepsilon\prime};\Omega)} - \|\nabla' u^{1/p}\|^2_{L^2(B';\Omega)}\big| + \big|\|(E')^{1/p}\|^2_{L^2(B';\Omega)} - \|E'\|^2_{L^2(B';\Omega)}\big| \\ &\quad\leqslant \big|\|\nabla' u^{\varepsilon,\overline{n}(p),1/p}\|^2_{L^2(B^{\varepsilon\prime};\Omega)} - \|\nabla' u^{\overline{n}(p),1/p}\|^2_{L^2(B';\Omega)}\big| \\ &\qquad + \big|\|\nabla' u^{\overline{n}(p),1/p}\|^2_{L^2(B';\Omega)} - \|\nabla' u^{1/p}\|^2_{L^2(B';\Omega)}\big| + \big|\|(E')^{1/p}\|^2_{L^2(B';\Omega)} - \|E'\|^2_{L^2(B';\Omega)}\big| \\ &\quad\leqslant \frac{2}{p} + \big|\|(E')^{1/p}\|^2_{L^2(B';\Omega)} - \|E'\|^2_{L^2(B';\Omega)}\big|,\end{aligned}$$

by using (5.22), (5.24). Then one gets (5.9) by letting ε and p tend to infinity and by using Lemma 3.1.

Thus it remains to prove (5.21) to (5.24).

f) Proof of (5.23), (5.24)

As for (5.23) we estimate

$$\begin{aligned}
&\left|\int_{I_k} D_1^{n,\eta}(t,x'-b'(t,x_1])b_{11}(\mathrm{d}t) - \int_{I_k} D_1^{\eta}(t,x'-b'(t,x_1])b_{11}(\mathrm{d}t)\right| \\
&= \frac{1}{b_{11}(I_k)}\left|\int_{I_k}\int_{I_k}(D_1^{\eta}(s,x'-b'(t,x_1]) - D_1^{\eta}(t,x'-b'(t,x_1]))b_{11}(\mathrm{d}s)b_{11}(\mathrm{d}t)\right| \\
&= \frac{1}{b_{11}(I_k)}\left|\int_{I_k}\int_{I_k}(D_1^{\eta}(t,x'-b'(s,x_1]) - D_1^{\eta}(t,x'-b'(t,x_1]))b_{11}(\mathrm{d}s)b_{11}(\mathrm{d}t)\right| \\
&\leqslant \frac{1}{b_{11}(I_k)}\int_{I_k}\int_{I_k}\sup_{y'\in\Omega_1'}|\nabla' D_1^{\eta}(t,y')|\,|b'(I_k)|b_{11}(\mathrm{d}s)b_{11}(\mathrm{d}t) \\
&= |b'(I_k)|\int_{I_k}\sup_{y'\in\Omega_1'}|\nabla' D_1^{\eta}(t,y')|b_{11}(\mathrm{d}t) \\
&\leqslant |b'(I_k)|\,b_{11}(I_k)^{1/2}\left(\int_{I_k}\sup_{y'\in\Omega_1'}|\nabla' D_1^{\eta}(t,y')|^2 b_{11}(\mathrm{d}t)\right)^{1/2},
\end{aligned}$$

with Ω_1' large enough. Summing over the finite set of values of k such that $I_k\cap(0,x_1]\neq\emptyset$, denoting by C a constant which may depend on η and using (3.3) and (2.9) we then find that for $x_1\geqslant 0,\ x_1\in I$,

$$\begin{aligned}
|u^{n,\eta}(x)-u^{\eta}(x)| &= \left|\int_{(0,x_1]} D_1^{n,\eta}(t,x'-b'(t,x_1])b_{11}(\mathrm{d}t) - \int_{(0,x_1]} D_1^{\eta}(t,x'-b'(t,x_1])b_{11}(\mathrm{d}t)\right| \\
&\leqslant C\sum_k |b'(I_k)|b_{11}(I_k)^{1/2} \leqslant C\sum_k m_{11}(I_k)^{1/2}\left(\sum_{j\geqslant 2} b_{jj}(I_k)\right)^{1/2} b_{11}(I_k)^{1/2} \\
&\leqslant C\left(\sum_k b_{11}(I_k)\right)^{1/2}\left(\sum_{j\geqslant 2}\sum_k m_{11}(I_k)b_{jj}(I_k)\right)^{1/2} \\
&\leqslant C\left(\sum_{j\geqslant 2}\sum_k\int_{I_k}\int_{I_k} m_{11}(\mathrm{d}s)b_{jj}(\mathrm{d}t)\right)^{1/2} \\
&\leqslant C\left(\sum_{j\geqslant 2}\iint_{(-1,\sup I+1)^2}\chi_{|s-t|<\frac{1}{n}} m_{11}(\mathrm{d}s)b_{jj}(\mathrm{d}t)\right)^{1/2}
\end{aligned}$$

which is independent of x and tends to zero as $n\to\infty$ by Assumption 4. A similar estimate is obtained for $x_1<0, x_1\in I$. This gives (5.23). By using (3.4) instead of (3.3) one can prove in the same manner that for fixed η, $\nabla' u^{n,\eta}\to\nabla' u^{\eta}$ uniformly as n tends to infinity. Hence also (5.24) is proved.

g) Proof of (5.21)

As for (5.21), recalling (5.16) it is enough to prove that

$$u^{\varepsilon,n,\eta}\to u^{n,\eta}\quad\text{pointwise a.e.} \tag{5.25}$$

We look at the definition (5.5) of $u^{\varepsilon,n,\eta}$ and (5.20) of $u^{n,\eta}$. Here we have

$$u^\eta(0, x' - (b^\varepsilon)'(0, x_1]) \to u^\eta(0, x' - b'(0, x_1]),$$

as $\varepsilon \to 0$, for every x (recall (2.18)). Utilizing the second forms of (5.5) and (5.20) we see that (5.25) follows if we know that

$$\int_{I_k \cap (0,x_1]} \psi_k(x' - (b^\varepsilon)'(t, x_1]) b_{11}^\varepsilon(\mathrm{d}t) \to \int_{I_k \cap (0,x_1]} \psi_k(x' - b'(t, x_1]) b_{11}(\mathrm{d}t) \tag{5.26}$$

for each k.

To show that (5.26) holds we notice that for fixed x

$$\psi_k(x' - (b^\varepsilon)'(t, x_1]) \rightharpoonup \psi_k(x' - b'(t, x_1]) \quad \text{weakly* in } BV(I_k),$$

both members regarded as functions of t. Indeed, by (2.18) we have pointwise convergence for every t, and we have uniform boundedness of the total variations:

$$\begin{aligned}\int_{I_k} \left| \frac{\mathrm{d}}{\mathrm{d}t} \psi_k(x' - (b^\varepsilon)'(t, x_1]) \right| \mathrm{d}t &\leqslant \int_{I_k} \left|\nabla' \psi_k(x' - (b^\varepsilon)'(t, x_1])\right| \left|(b^\varepsilon)'(\mathrm{d}t)\right| \\ &\leqslant C\left|(b^\varepsilon)'(I_k)\right| \leqslant C < \infty.\end{aligned}$$

Now (5.26) follows immediately from Lemma 6.2 together with Assumptions 2 and 4.

h) Proof of (5.22)

Finally it remains to prove (5.22). Note that

$$\|\nabla' u^{\varepsilon,n,\eta}\|^2_{L^2(B^{\varepsilon\prime};\Omega)} = \sum_{i,j \geqslant 2} \int_I f_{ij}^{\varepsilon,n,\eta}(x_1) b_{ij}^\varepsilon(x_1)\, \mathrm{d}x_1,$$

$$\|\nabla' u^{n,\eta}\|^2_{L^2(B';\Omega)} = \sum_{i,j \geqslant 2} \int_I f_{ij}^{n,\eta}(x_1) b_{ij}(\mathrm{d}x_1).$$

To show (5.22) we shall apply Lemma 6.2 to each term above. As we know that $b_{ij}^\varepsilon \to b_{ij}$ weakly* in $M(I)$ we then have to verify the following, for each $i, j \geqslant 2$.

$$f_{ij}^{\varepsilon,n,\eta} \rightharpoonup f_{ij}^{n,\eta} \quad \text{weakly* in } BV(I), \tag{5.27}$$

$$\int_I \int_I \chi_{|x_1 - s| < \delta} \left| \frac{\mathrm{d} f_{ij}^{\varepsilon,n,\eta}(x_1)}{\mathrm{d}x_1} \right| \left|b_{ij}^\varepsilon(s)\right| \mathrm{d}s\, \mathrm{d}x_1 \to 0, \tag{5.28}$$

$$\sum_{x_1 \in \partial I} \int_I \chi_{|x_1 - s| < \delta}(s) \left|b_{ij}^\varepsilon(s)\right| \mathrm{d}s \to 0 \tag{5.29}$$

as $\varepsilon, \delta \to 0$.

Here (5.29) is nothing but (4.17) (which we already have proved). In order to prove (5.27) it is enough (see Appendix) to prove that

$$f_{ij}^{\varepsilon,n,\eta}(x_1) \to f_{ij}^{n,\eta}(x_1) \tag{5.30}$$

for a dense set of values of x_1, together with proving that

$$\int_I \left| \frac{\mathrm{d}f_{ij}^{\varepsilon,n,\eta}}{\mathrm{d}x_1}(x_1) \right| \mathrm{d}x_1 \leqslant C(\eta) < \infty. \tag{5.31}$$

By (5.17) and the definitions of $f_{ij}^{\varepsilon,n,\eta}$ and $f_{ij}^{n,\eta}$, (5.30) follows from

$$\nabla' u^{\varepsilon,n,\eta} \to \nabla' u^{n,\eta} \quad \text{pointwise a.e.}$$

which is proved in the same way as (5.25) above.

To summarize, in order to prove (5.22) it remains to prove (5.28) and (5.31). Since

$$\frac{\partial^2 u^{\varepsilon,n,\eta}}{\partial x_1 \partial x_i} \in L^\infty(\Omega)$$

by (5.19) we can differentiate $f_{ij}^{\varepsilon,n,\eta}(x_1)$ under the integral sign, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}x_1} f_{ij}^{\varepsilon,n,\eta}(x_1) = \int_{\Omega'} \frac{\partial^2 u^{\varepsilon,n,\eta}}{\partial x_1 \partial x_i} \frac{\partial u^{\varepsilon,n,\eta}}{\partial x_j} \,\mathrm{d}x' + \int_{\Omega'} \frac{\partial u^{\varepsilon,n,\eta}}{\partial x_i} \frac{\partial^2 u^{\varepsilon,n,\eta}}{\partial x_1 \partial x_j} \,\mathrm{d}x'.$$

Using the Cauchy–Schwarz inequality several times and (5.17), (5.19), (5.18), (2.9), (2.8), (5.14) and (2.12), and denoting by C various constants which may change from step to step we estimate, for $\delta > 0$,

$$\begin{aligned}
&\int_I \int_I \chi_{|s-x_1|<\delta} \left| \frac{\mathrm{d}f_{ij}^{\varepsilon,n,\eta}(x_1)}{\mathrm{d}x_1} \right| \mathrm{d}x_1 \left|b_{ij}^\varepsilon(s)\right| \mathrm{d}s \\
&\quad \leqslant \int_I \int_I \chi_{|s-x_1|<\delta} \left| \int_{\Omega'} \frac{\partial^2 u^{\varepsilon,n,\eta}}{\partial x_1 \partial x_i} \frac{\partial u^{\varepsilon,n,\eta}}{\partial x_j} \,\mathrm{d}x' \right| \mathrm{d}x_1 \left|b_{ij}^\varepsilon(s)\right| \mathrm{d}s + \text{a similar term} \\
&\quad \leqslant C \int_I \int_I \chi_{|s-x_1|<\delta} \int_{\Omega'} \left| \frac{\partial^2 u^{\varepsilon,n,\eta}}{\partial x_1 \partial x_i} \right| \mathrm{d}x' \,\mathrm{d}x_1 \left|b_{ij}^\varepsilon(s)\right| \mathrm{d}s + \text{a similar term} \\
&\quad \leqslant C \int_I \int_I \chi_{|s-x_1|<\delta} \Bigg[b_{11}^\varepsilon(x_1) \int_{\Omega'} \left| \frac{\partial D_1^{n,\eta}}{\partial x_i} \right| \mathrm{d}x' \\
&\qquad + \sum_{\ell \geqslant 2} \left|b_{1\ell}^\varepsilon(x_1)\right| \int_{\Omega'} \left| \frac{\partial^2 u^{\varepsilon,n,\eta}}{\partial x_i \partial x_\ell} \right| \mathrm{d}x' \Bigg] \mathrm{d}x_1 \left|b_{ij}^\varepsilon(s)\right| \mathrm{d}s + \text{similar terms} \\
&\quad \leqslant C \int_I \left(\int_I \int_{\Omega'} \chi_{|s-x_1|<\delta} \left| \frac{\partial D_1^{n,\eta}}{\partial x_i} \right| \left|b_{11}^\varepsilon(x_1) \,\mathrm{d}x_1 \mathrm{d}x' \right) \left|b_{ij}^\varepsilon(s)\right| \mathrm{d}s \\
&\qquad + C \sum_{\ell \geqslant 2} \int_I \int_I \chi_{|s-x_1|<\delta} \left|b_{1\ell}^\varepsilon(x_1)\right| \mathrm{d}x_1 \left|b_{ij}^\varepsilon(s)\right| \mathrm{d}s + \text{similar terms}
\end{aligned}$$

$$\leqslant C\int_I\Big[\int_\Omega(\chi_{|s-x_1|<\delta})^2 b_{11}^\varepsilon(x_1)\,\mathrm{d}x\int_\Omega\Big|\frac{\partial D_1^{n,\eta}}{\partial x_i}\Big|^2 b_{11}^\varepsilon(x_1)\,\mathrm{d}x\Big]^{1/2}|b_{ij}^\varepsilon(s)|\,\mathrm{d}s$$

$$+C\sum_{\ell\geqslant 2}\int_I\int_I\chi_{|s-x_1|<\delta}(m_{11}^\varepsilon(x_1)\,b_{\ell\ell}^\varepsilon(x_1))^{1/2}\mathrm{d}x_1|b_{ij}^\varepsilon(s)|\,\mathrm{d}s+\text{similar terms}$$

$$\leqslant C\int_I\Big[\int_I\chi_{|s-x_1|<\delta}b_{11}^\varepsilon(x_1)\,\mathrm{d}x_1\Big]^{1/2}|b_{ij}^\varepsilon(s)|\,\mathrm{d}s\Big\|\frac{\partial D_1^{n,\eta}}{\partial x_i}\Big\|_{L^2(b_{11}^\varepsilon;\Omega)}$$

$$+C\sum_{\ell\geqslant 2}\int_I\Big[\int_I\chi_{|s-x_1|<\delta}m_{11}^\varepsilon(x_1)\,\mathrm{d}x_1\Big]^{1/2}$$

$$\times\Big[\int_I\chi_{|s-x_1|<\delta}b_{\ell\ell}^\varepsilon(x_1)\,\mathrm{d}x_1\Big]^{1/2}|b_{ij}^\varepsilon(s)|\,\mathrm{d}s+\text{ similar terms}$$

$$\leqslant C\Big[\int_I\int_I\chi_{|s-x_1|<\delta}m_{11}^\varepsilon(x_1)|b_{ij}^\varepsilon(s)|\,\mathrm{d}x_1\,\mathrm{d}s\Big]^{1/2}.$$

By Assumption 4′ this tends to zero as $\varepsilon,\delta\to 0$, proving (5.28).

The proof of (5.31) is obtained by a similar estimate as the above. Indeed, it is just to remove the factors $\chi_{|s-x_1|<\delta}$ and b_{ij}^ε from the above computations and then at the end invoke Assumption 3 instead of Assumption 4′ to reach the conclusion that the final member is bounded from above independently of $\varepsilon>0$ (and n).

This completes the proof of Lemma 5.1.

6. Appendix

This Appendix contains two technical tools of more general nature, but which we have not found readily available in the literature.

a) Absolute continuity of limit measures

The first result concerns matrices whose entries are measures or, equivalently, matrix-valued measures. In the scalar case the lemma says that if $\{\mu^\varepsilon\}_{\varepsilon>0}$ are positive measures, $\mu^\varepsilon\rightharpoonup\mu$ weakly* as measures for some measure μ and we have functions $f^\varepsilon\in L^2(\mu^\varepsilon)$ which satisfy $\|f^\varepsilon\|_{L^2(\mu^\varepsilon)}\leqslant C<\infty$, then there exists $f\in L^2(\mu)$ such that, for a subsequence, $\mu^\varepsilon f^\varepsilon\rightharpoonup\mu f$ weakly* as measures and $\|f\|_{L^2(\mu)}\leqslant\underline{\lim}\,\|f^\varepsilon\|_{L^2(\mu^\varepsilon)}$. (The same is true with L^p in place of L^2 for $1<p<\infty$, but not for $p=1$.)

To set up some notation in the matrix context, let $K\subset\mathbb{R}^N$ be compact, $\mu\in M(K)$, $f\in C(K)$. We write

$$\langle\mu,f\rangle=\int_K f(x)\mu(\mathrm{d}x)$$

and denote by $\mu f\in M(K)$ the measure defined by

$$\langle\mu f,\varphi\rangle=\int_K f(x)\varphi(x)\mu(\mathrm{d}x)$$

for any $\varphi \in C(K)$.

Let now $\mu_{ij} \in M(K)$ $(1 \leqslant i, j \leqslant N)$, $\mu = (\mu_{ij})_{i,j=1}^N \in M(K)^{N\times N}$. Let also $f = (f_i)_{i=1}^N$, $\varphi = (\varphi_i)_{i=1}^N$ with $f_i, \varphi_i \in C(K)$. We think of f and φ as column vector fields and define

$$\mu f = \left(\sum_{j=1}^N \mu_{ij} f_j\right)_{i=1}^N,$$

regarded as a column vector with entries in $M(K)$,

$$(f,\varphi)_{L^2(\mu)} = \langle \mu f, \varphi\rangle = \sum_{i=1}^N \langle (\mu f)_i, \varphi_i\rangle = \sum_{i,j=1}^N \langle \mu_{ij} f_j, \varphi_i\rangle = \sum_{i,j=1}^N \int_K f_j(x)\varphi_i(x)\mu_{ij}(\mathrm{d}x).$$

We call μ positive semidefinite if $(f,f)_{L^2(\mu)} \geqslant 0$ for all $f \in C(K)^N$. In this case one can, by completion and taking quotients, make a Hilbert space $L^2(\mu) = L^2(\mu; K)$ with norm $\|f\|^2_{L^2(\mu)} = (f,f)_{L^2(\mu)}$ out of $C(K)^N$. For an open set Ω, $L^2(\mu) = L^2(\mu;\Omega)$ is defined similarly, replacing $C(K)$ with, for example, $C_0(\Omega)$.

Lemma 6.1. *Let $K \subset \mathbb{R}^N$ be compact, μ^ε_{ij}, $\mu_{ij} \in M(K)$ $(\varepsilon > 0,\ 1 \leqslant i, j \leqslant N)$ with $\mu^\varepsilon = (\mu^\varepsilon_{ij})$ positive semidefinite, and let $f^\varepsilon = (f^\varepsilon_i)_{i=1}^N \in L^2(\mu^\varepsilon)$. If*

$$\mu^\varepsilon \rightharpoonup \mu \quad \textit{weakly* in } M(K)^{N\times N}, \tag{6.1}$$

$$\|f^\varepsilon\|_{L^2(\mu^\varepsilon)} \leqslant C < \infty, \tag{6.2}$$

then there exists $f \in L^2(\mu)$ such that, for a subsequence,

$$\mu^\varepsilon f^\varepsilon \rightharpoonup \mu f \quad \textit{weakly* in } M(K)^N, \tag{6.3}$$

$$\|f\|_{L^2(\mu)} \leqslant \underline{\lim}\, \|f^\varepsilon\|_{L^2(\mu^\varepsilon)}. \tag{6.4}$$

Remark. In general there is no convergence $f^\varepsilon \to f$.

Proof. Using (6.2) and the consequence of (6.1) that the total masses of $\{\mu^\varepsilon_{ij}\}_{\varepsilon>0}$ are uniformly bounded, we get, for any $\varphi \in C(K)^N$, $|\langle \mu^\varepsilon f^\varepsilon, \varphi\rangle| = |(f^\varepsilon,\varphi)_{L^2(\mu^\varepsilon)}| \leqslant \|f^\varepsilon\|_{L^2(\mu^\varepsilon)}\|\varphi\|_{L^2(\mu^\varepsilon)} \leqslant C\|\varphi\|_{L^\infty}$. It follows that $\{\mu^\varepsilon f^\varepsilon\}_{\varepsilon>0}$ is bounded in $M(K)^N$, hence that there exists $\nu \in M(K)^N$ such that, for a subsequence,

$$\mu^\varepsilon f^\varepsilon \rightharpoonup \nu \quad \text{weakly* in } M(K)^N.$$

For $\varphi \in C(K)^N$ we then have

$$\begin{aligned} |\langle \nu,\varphi\rangle| &= \lim_{\varepsilon\to 0} |\langle \mu^\varepsilon f^\varepsilon, \varphi\rangle| = \lim_{\varepsilon\to 0} |(f^\varepsilon,\varphi)_{L^2(\mu^\varepsilon)}| \\ &\leqslant \underline{\lim}_{\varepsilon\to 0} \|f^\varepsilon\|_{L^2(\mu^\varepsilon)} \|\varphi\|_{L^2(\mu^\varepsilon)} = \lambda \|\varphi\|_{L^2(\mu)}, \end{aligned} \tag{6.5}$$

where $\lambda = \underline{\lim}\,\|f^\varepsilon\|_{L^2(\mu^\varepsilon)}$. It follows that the functional $\varphi \mapsto \langle \nu, \varphi \rangle$ is continuous in the $L^2(\mu)$-norm, hence that there exists $f \in L^2(\mu)$ with

$$\langle \nu, \varphi \rangle = (f, \varphi)_{L^2(\mu)} = \langle \mu f, \varphi \rangle.$$

This implies that

$$\nu = \mu f \tag{6.6}$$

as measures, proving (6.3).

Inserting next (6.6) into (6.5) and thereafter letting $\varphi \to f$ in $L^2(\mu)$ gives (6.4).

b) Convergence of integrals of BV-functions

Let $f \in L^1(I)$, where $I \subset \mathbb{R}$ is a bounded open interval, and extend f by zero outside I. We say that f is of bounded variation on I, and write $f \in BV(I)$, if an estimate $|\int_I f(t)\varphi'(t)\,\mathrm{d}t| \leqslant C\|\varphi\|_\infty$ holds for all $\varphi \in C_0^\infty(I)$ or, equivalently, if the distributional derivative of f on $\mathbb{R}$ is a signed Radon measure, call it μ. Then $\operatorname{supp}\mu \subset \overline{I}$ and the cumulative distribution function of μ,

$$\tilde{f}(t) = \int_{(-\infty,t]} \mu(\mathrm{d}s) = \int_{(-\infty,t]} \mathrm{d}\mu = \mu(-\infty, t], \tag{6.7}$$

also has distributional derivative equal to μ. Thus, since $f(t) = \tilde{f}(t) = 0$ for t to the left of I, it follows that $f = \tilde{f}$ a.e. The function $\tilde{f}$ is the unique representative of f which is continuous from the right (i.e. $\tilde{f}(t) = \lim_{h\searrow 0} \tilde{f}(t+h)$ for every t). We shall often write $\mathrm{d}f$ for $\mathrm{d}\mu$, and similarly $|\mathrm{d}f|$ for the total variation measure $|\mathrm{d}\mu|$ (or $\mathrm{d}|\mu|$).

Some expressions for the total variation of f on I are

$$\begin{aligned} TV(f;I) &= |\mu|(I) \\ &= \sup\left\{\sum_{j=1}^{n-1} |\tilde{f}(t_{j+1}) - \tilde{f}(t_j)|\colon \text{ all } n \geqslant 2 \text{ and all partitions } a < t_1 < t_2 < \cdots < t_n < b\right\} \\ &= \sup\left\{\left|\int_I f(t)\varphi'(t)\,\mathrm{d}t\right|\colon \varphi \in C_0^\infty(I),\ |\varphi| \leqslant 1\right\}, \end{aligned}$$

where the last one for our purposes may be taken as the definition of $TV(f;I)$. Since constant functions have vanishing total variation, $TV(f;I)$ is only a seminorm on $BV(I)$. An appropriate norm is

$$\|f\|_{BV} = \|f\|_{L^1(I)} + TV(f;I).$$

As to topology on $BV(I)$ we only need the concept of weak* convergence. We say that $f^\varepsilon \rightharpoonup f$ weakly* in $BV(I)$ as $\varepsilon \to 0$ if and only if $\mu^\varepsilon \rightharpoonup \mu$ weakly* in $M(\mathbb{R})$, where μ^ε, μ are the distributional derivatives of f^ε, f extended by zero outside I as above, i.e., if and only if

$$\int_{\mathbb{R}} \varphi\,\mathrm{d}\mu^\varepsilon \to \int_{\mathbb{R}} \varphi\,\mathrm{d}\mu \tag{6.8}$$

for every $\varphi \in C_0(\mathbb{R})$.

The following criterion for weak* convergence in $BV(I)$ is useful: If

$$TV(f^\varepsilon; I) \leqslant C < \infty \quad \text{and}$$
$$\tilde{f}^\varepsilon(t) \to \tilde{f}(t) \quad \text{for each } t \text{ in a dense subset of } I,$$

then

$$f^\varepsilon \rightharpoonup f \quad \text{weakly* in } BV(I).$$

To prove the criterion we simply note that the two conditions are equivalent to $\{\mu^\varepsilon\}$ being bounded in norm together with (6.8) holding for every left continuous step function φ with jumps allowed only at the dense set in question. Since any function in $C(\overline{I})$ can be uniformly approximated by such step functions, the weak* star convergence $f^\varepsilon \rightharpoonup f$ follows.

Now, to prepare for the main statement, note that any $f \in BV(I)$ is bounded and Borel measurable. Therefore, for any pair $f, g \in BV(I)$ the integral $\int_S f\, dg$ over any subinterval $S \subset I$ makes sense once we have made a choice of a pointwise representative of f (necessary if f and g have common jump points). As a matter of normalization we shall always in integrals $\int_S f\, dg$ understand that f and g are extended by zero outside I and are continuous from the right. Denoting by μ, ν the signed measures corresponding to f, g as above we thus define

$$\int_S f\, \mathrm{d}g = \int_S \tilde{f}\, \mathrm{d}\nu.$$

Setting

$$R = \{(s,t) \in \mathbb{R}^2\colon\ s \leqslant t,\ t \in S\} \tag{6.9}$$

we then have, using Fubini's theorem,

$$\int_S f\, \mathrm{d}g = \int_S \mu((-\infty, t])\, \nu(\mathrm{d}t) = \iint_R \mu(\mathrm{d}s)\, \nu(\mathrm{d}t) = (\mu \otimes \nu)(R).$$

Here $\mu \otimes \nu$ denotes the product measure of μ and ν. We write the integral above also as $\int_R \mathrm{d}f \otimes \mathrm{d}g$.

The following useful lemma is similar to, and inspired by, Lemma 3.1 in [2].

Lemma 6.2. *Let f^ε, f, g^ε, g belong to $BV(I)$ and let $S \subset I$ be a subinterval (not necessarily open). Then if*

$$f^\varepsilon \rightharpoonup f \quad \textit{weakly* in } BV(I), \tag{6.10}$$
$$g^\varepsilon \rightharpoonup g \quad \textit{weakly* in } BV(I) \tag{6.11}$$

as $\varepsilon \to 0$ it follows that

$$\int_S f^\varepsilon\, \mathrm{d}g^\varepsilon \to \int_S f\, \mathrm{d}g \tag{6.12}$$

provided any one of the following two equivalent conditions is satisfied.

(i) *For every pair* μ, ν *of weak* cluster points of the total variation measures* $|\mathrm{d}f^\varepsilon|$ *and* $|\mathrm{d}g^\varepsilon|$ *there holds*

$$\nu(\{t\}) = 0 \quad \textit{for each } t \in \partial S \tag{6.13}$$

and

$$\mu(\{t\})\nu(\{t\}) = 0 \quad \textit{for each } t \in \operatorname{int} S. \tag{6.14}$$

(ii) *For every* $\eta > 0$ *there exists* $\delta > 0$ *such that, for* $\varepsilon > 0$ *small enough,*

$$\int_{s-\delta}^{s+\delta} |\mathrm{d}g^\varepsilon(t)| < \eta \quad \textit{for each } s \in \partial S \tag{6.15}$$

and

$$\int_{\operatorname{int} S}\int_{\operatorname{int} S} \chi_{|t-s|<\delta} |\mathrm{d}f^\varepsilon(s)||\mathrm{d}g^\varepsilon(t)| < \eta \tag{6.16}$$

Proof. We first notice that (6.14) is equivalent to (6.16) and (6.13) to (6.15). Indeed, let

$$\Delta_{\operatorname{int} S} = \{(s, s) \in \mathbb{R}^2\colon\ s \in \operatorname{int} S\}$$

denote the diagonal of int S in $\mathbb{R}^2$. Since $\mu(\{t\}) > 0$ for at most countably many points t we have

$$\begin{aligned}(\mu \otimes \nu)(\Delta_{\operatorname{int} S}) &= \int_{\Delta_{\operatorname{int} S}} \mathrm{d}\mu \otimes \mathrm{d}\nu = \int_{\operatorname{int} S} \left(\int_{s=t} \mathrm{d}\mu(s)\right) \mathrm{d}\nu(t) \\ &= \int_{\operatorname{int} S} \mu(\{t\})\,\mathrm{d}\nu(t) = \sum_{t\in \operatorname{int} S} \mu(\{t\})\nu(\{t\}).\end{aligned}$$

Thus (6.14) is equivalent to

$$(\mu \otimes \nu)(\Delta_{\operatorname{int} S}) = 0. \tag{6.17}$$

Now (6.17) holding for arbitrary weak* cluster points μ, ν of $|\mathrm{d}f^\varepsilon|$, $|\mathrm{d}g^\varepsilon|$ is equivalent to having, for any given $\eta > 0$,

$$(|\mathrm{d}f^\varepsilon| \otimes |\mathrm{d}g^\varepsilon|)(N_\delta) < \eta$$

for $\varepsilon > 0$, $\delta > 0$ small enough, where the N_δ denote neighbourhoods of $\Delta_{\operatorname{int} S}$ in $\mathbb{R}^2$ shrinking down to $\Delta_{\operatorname{int} S}$ as $\delta \to 0$, e.g., the ones given by

$$N_\delta = \{(s, t) \in (\operatorname{int} S)^2\colon\ |s - t| < \delta\}.$$

But this is exactly what (6.16) says. Thus (6.14) and (6.16) are equivalent. Similarly, but easier, one sees that (6.15) is equivalent to (6.13).

Next we observe that $\partial R \subset \Delta_{\mathrm{int}\, S} \cup (\mathbb{R} \times \partial S)$. Clearly (6.13) implies $(\mu \otimes \nu)(\mathbb{R} \times \partial S) = 0$. Combining with (6.17) it follows that (6.13), (6.14) together imply

$$(\mu \otimes \nu)(\partial R) = 0. \tag{6.18}$$

Now, to prove (6.12) we write

$$\int_S f^\varepsilon \, \mathrm{d}g^\varepsilon = \iint_{\mathbb{R}^2} \chi_R \, \mathrm{d}f^\varepsilon \mathrm{d}g^\varepsilon = \int_R \mathrm{d}f^\varepsilon \otimes \mathrm{d}g^\varepsilon,$$
$$\int_S f \, \mathrm{d}g = \iint_{\mathbb{R}^2} \chi_R \, \mathrm{d}f \, \mathrm{d}g = \int_R \mathrm{d}f \otimes \mathrm{d}g.$$

Thus we need to prove that

$$\int_R \mathrm{d}f^\varepsilon \otimes \mathrm{d}g^\varepsilon \to \int_R \mathrm{d}f \otimes \mathrm{d}g. \tag{6.19}$$

It is easy to see from (6.10), (6.11) that $\mathrm{d}f^\varepsilon \otimes \mathrm{d}g^\varepsilon \rightharpoonup \mathrm{d}f \otimes \mathrm{d}g$ weakly* in $M(\mathbb{R}^2)$, i.e., that

$$\int \varphi \, \mathrm{d}f^\varepsilon \otimes \mathrm{d}g^\varepsilon \to \int \varphi \, \mathrm{d}f \otimes \mathrm{d}g \tag{6.20}$$

for all $\varphi \in C_0(\mathbb{R}^2)$ (or even $\varphi \in C(\mathbb{R}^2)$). Roughly speaking, this implies $\mathrm{d}f^\varepsilon \otimes \mathrm{d}g^\varepsilon(R) \to \mathrm{d}f \otimes \mathrm{d}g(R)$, i.e., (6.19), provided ∂R has $|\mathrm{d}f| \otimes |\mathrm{d}g|$-measure zero (see [6], Theorem 1.9.1 for the case of positive measures). To be precise, ∂R actually has to have measure zero for all weak* cluster points of $|\mathrm{d}f^\varepsilon| \otimes |\mathrm{d}g^\varepsilon|$, and this is exactly what (6.18) says.

As to the details, choose a family of smooth functions $0 \leqslant \psi_\delta \leqslant 1$ such that $\psi_\delta = 1$ in a δ-neighbourhood of ∂R, $\psi_\delta = 0$ outside a 2δ-neighbourhood ($\delta > 0$). Then it follows from (6.18), or directly from (6.15), (6.16), that for every $\eta > 0$

$$\int \psi_\delta |\mathrm{d}f^\varepsilon| \otimes |\mathrm{d}g^\varepsilon| < \eta \tag{6.21}$$

for $\delta > 0$ and $\varepsilon > 0$ small enough.

Using (6.21) we estimate

$$\left| \int_R (1 - \psi_\delta) \, \mathrm{d}f^\varepsilon \otimes \mathrm{d}g^\varepsilon - \int_R \mathrm{d}f^\varepsilon \otimes \mathrm{d}g^\varepsilon \right| \leqslant \int_R \psi_\delta |\mathrm{d}f^\varepsilon| \otimes |\mathrm{d}g^\varepsilon| < \eta,$$
$$\left| \int_R (1 - \psi_\delta) \, \mathrm{d}f \otimes \mathrm{d}g - \int_R \mathrm{d}f \otimes \mathrm{d}g \right| \leqslant \int_R \psi_\delta |\mathrm{d}f| \otimes |\mathrm{d}g| \leqslant \varliminf_{\varepsilon \to 0} \int_R \psi_\delta |\mathrm{d}f^\varepsilon| \otimes |\mathrm{d}g^\varepsilon| < \eta.$$

In the second last inequality we used that, generally speaking, $\varphi \geqslant 0$ continuous and $\mu_n \rightharpoonup \mu$ weakly* imply that $\int \varphi \, \mathrm{d}|\mu| \leqslant \varliminf_{n \to \infty} \int \varphi \, \mathrm{d}|\mu_n|$, as follows from $\int \varphi \, \mathrm{d}|\mu| = \sup \{ \int \psi \, \mathrm{d}\mu \colon \psi$ continuous and $|\psi| \leqslant \varphi \}$.

Applying (6.20) with $\varphi = (1 - \psi_\delta)\chi_R$ gives

$$\int_R (1 - \psi_\delta)\, \mathrm{d}f^\varepsilon \otimes \mathrm{d}g^\varepsilon \to \int_R (1 - \psi_\delta)\, \mathrm{d}f \otimes \mathrm{d}g$$

as $\varepsilon \to 0$. Thus the above estimates show that

$$\left| \int_R \mathrm{d}f^\varepsilon \otimes \mathrm{d}g^\varepsilon - \int_R \mathrm{d}f \otimes \mathrm{d}g \right| < 3\eta$$

for $\varepsilon > 0$ small enough. Since $\eta > 0$ was arbitrary (6.19) follows.

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