A NOTE ON H-CONVERGENCE

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Abstract. We give a criterion for H-convergence of conductivity matrices in terms of ordinary weak convergence of the factors in certain quotient representations of the matrices.

1. Introduction

Questions of homogenization of rapidly varying coefficients in elliptic partial differential equations have been considered by mathematicians at least since the 1970s, and by physicists and engineers much longer (cf. the references in [12], [1], [11]). A typical example is when the conductivity matrix $A^\varepsilon = A^\varepsilon(x)$ in an equation

$$-\mathrm{div}\,(A^\varepsilon \nabla u^\varepsilon) = f \tag{1.1}$$

in some (bounded) domain $\Omega \subset \mathbb{R}^n$ oscillates rapidly at a length scale $\varepsilon > 0$ and one wants to identify a limiting matrix $A = A(x)$ (presumably less oscillating than the A^ε) such that, as $\varepsilon \to 0$, the solutions u^ε converge in some weak sense to the solution u of the corresponding homogenized equation:

$$-\mathrm{div}\,(A\nabla u) = f.$$

In the 1970s and 80s, F. Murat, L. Tartar identified the appropriate type of convergence, H-convergence, for the above type of problems and started developing general theories for it. Earlier work was much concerned with special cases, like strictly periodic structures. In this little note we give an equivalent condition for general H-convergence in terms of ordinary weak convergence for the factors in certain quotient representations of the conductivity matrices, namely for matrices M^ε and P^ε appearing when writing $M^\varepsilon A^\varepsilon = P^\varepsilon$. In the special case of stratified media (A^ε depending on only one of the coordinates) and certain generalizations thereof, explicit decompositions of this type have been constructed and used for proving H-convergence in a series of papers [3], [5], [2], [6]. The purpose of this note is to point out that the existence of such quotient representations is a completely general fact in connection with H-convergence $A^\varepsilon \to A$.

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The proof consists of an adaptation of methods developed by F. Murat and L. Tartar, e.g., in [8], [9], [13], [14], [15], [10]. In fact, even the result can be said to be implicit in their work, but perhaps not explicit.

For an extension of the results in this note to H-convergence in linear elasticity, see [7].

2. H-convergence

Definition 2.1. Let $0 < \alpha \leq \beta < \infty$, $\Omega \subset \mathbb{R}^n$ (a bounded domain). Then $M(\alpha, \beta; \Omega)$ denotes the set of invertible real-valued $n \times n$ matrices $A = A(x)$ with entries in $L^\infty(\Omega)$ and satisfying almost everywhere in Ω the estimates

$$(A\xi, \xi) \geq \alpha|\xi|^2,$$
$$(A^{-1}\xi, \xi) \geq \beta^{-1}|\xi|^2$$

for $\xi \in \mathbb{R}^n$.

Above the bracket $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^n$: $(\xi, \eta) = \sum_j \xi_j \eta_j$. Divergence and curl of vector fields are defined as usual: $\operatorname{div} D$ is the scalar

$$\operatorname{div} D = \sum_j \frac{\partial D_j}{\partial x_j},$$

and $\operatorname{curl} E$ is the antisymmetric tensor with components

$$(\operatorname{curl} E)_{ij} = \frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i}.$$

We shall also need to take div and curl of matrices, and then the above definitions apply to the row vectors, i.e., to the last index. Thus, with $M = (M_{ij})$, $P = (P_{ij})$,

$$(\operatorname{div} P)_i = \sum_j \frac{\partial P_{ij}}{\partial x_j}$$
$$(\operatorname{curl} M)_{ijk} = \frac{\partial M_{ij}}{\partial x_k} - \frac{\partial M_{ik}}{\partial x_j}.$$

The parameter $\varepsilon > 0$ to be used from now on is by convention restricted to take values only in a sequence tending to zero (e.g., $\varepsilon \in \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$).

Definition 2.2. (Tartar [14]) Let $A, A^\varepsilon \in M(\alpha, \beta; \Omega)$ for some $0 < \alpha \leq \beta < \infty$ and all $\varepsilon > 0$. Then A^ε is said to H-converge to A,

$$A^\varepsilon \xrightarrow{H} A,$$

as $\varepsilon \to 0$ if the following holds. Whenever vector fields D^ε, D, E^ε, $E \in L^2(\Omega)^n$ satisfy

$$D^\varepsilon = A^\varepsilon E^\varepsilon, \tag{2.1}$$

$$D^\varepsilon \rightharpoonup D \text{ weakly in } L^2(\Omega)^n, \tag{2.2}$$

$$E^\varepsilon \rightharpoonup E \text{ weakly in } L^2(\Omega)^n \tag{2.3}$$

with

$$\{\operatorname{div} D^\varepsilon\}_{\varepsilon>0} \text{ relatively compact in } H^{-1}(\Omega), \tag{2.4}$$

$$\{\operatorname{curl} E^\varepsilon\}_{\varepsilon>0} \text{ relatively compact in } H^{-1}(\Omega)^{n\times n} \tag{2.5}$$

then

$$D = AE. \tag{2.6}$$

It is well-known [8] that the H-limit is unique, that the set $M(\alpha, \beta; \Omega)$ is sequentially compact for H-convergence, and also that H-convergence is stable under transposition of the matrices: if $A^\varepsilon \xrightarrow{H} A$ then ${}^tA^\varepsilon \xrightarrow{H} {}^tA$.

3. The result

Theorem 3.1. *Let $A, A^\varepsilon \in M(\alpha, \beta; \Omega)$ for some $0 < \alpha \le \beta < \infty$. Then $A^\varepsilon \xrightarrow{H} A$ as $\varepsilon \to 0$ if and only if there exist $n \times n$ matrices M^ε, M, P^ε, P with entries in $L^2(\Omega)$ and with M (and hence P) invertible, such that*

$$M^\varepsilon A^\varepsilon = P^\varepsilon, \tag{3.1}$$

$$MA = P, \tag{3.2}$$

$$M^\varepsilon \rightharpoonup M \text{ weakly in } L^2(\Omega)^{n\times n}, \tag{3.3}$$

$$P^\varepsilon \rightharpoonup P \text{ weakly in } L^2(\Omega)^{n\times n}. \tag{3.4}$$

with

$$\{\operatorname{curl} M^\varepsilon\}_{\varepsilon>0} \text{ relatively compact in } H^{-1}(\Omega)^{n\times n\times n}, \tag{3.5}$$

$$\{\operatorname{div} P^\varepsilon\}_{\varepsilon>0} \text{ relatively compact in } H^{-1}(\Omega)^{n}. \tag{3.6}$$

When this is the case M can be chosen to be the identity matrix I and M^ε so that $\operatorname{curl} M^\varepsilon = 0$.

Proof. The proof is based on the "div-curl lemma" of compensated compactness [8], [14], [4]. We recall that this lemma in general says that if $f^\varepsilon, g^\varepsilon, f, g \in L^2(\Omega)^n$ are vector fields such that $f^\varepsilon \rightharpoonup f$, $g^\varepsilon \rightharpoonup g$ weakly in $L^2(\Omega)^n$ and such that $\operatorname{div} f^\varepsilon$ and the components of $\operatorname{curl} g^\varepsilon$ are all contained in a compact subset of $H^{-1}(\Omega)$, then $(f^\varepsilon, g^\varepsilon) \rightharpoonup (f, g)$ weakly as distributions.

First we prove the "if"-part of the theorem, which is very easy. So assume we have the decompositions (3.1) and (3.2) with weak convergences $M^\varepsilon \rightharpoonup M$ and $P^\varepsilon \rightharpoonup P$ as in the statement. We consider E^ε and $D^\varepsilon = A^\varepsilon E^\varepsilon$ satisfying (2.2), (2.3), (2.4), (2.5). Then (3.1) acting on E^ε gives

$$M^\varepsilon D^\varepsilon = P^\varepsilon E^\varepsilon,$$

or, in components,

$$\sum_j M^\varepsilon_{ij} D^\varepsilon_j = \sum_j P^\varepsilon_{ij} E^\varepsilon_j .$$

Here the div-curl lemma applies for each i and it follows that each of the members converge in the sense of distribution, to MD and PE respectively. Thus we get

$$MD = PE, \tag{3.7}$$

which, since M is invertible, is the same as (2.6).

Now we prove the "only if" part. First we have to construct the matrices $M^\varepsilon, M, P^\varepsilon, P$. We may take $M = I$, $P = A$. Let ${}^tA$ denote the transpose of A and let e_i be the i:th unit column vector. Thus $e_i = \nabla u_i$, where u_i is the i:th coordinate function:

$$u_i(x) = x_i.$$

Setting also $f_i = \operatorname{div}({}^tAe_i)$ the equation

$$\operatorname{div}({}^tA\nabla u) = f_i$$

is trivially solved by $u = u_i$.

Now, with f_i, u_i as above ($1 \le i \le n$), there is for each $\varepsilon > 0$ a unique solution u^ε_i of the elliptic boundary value problem

$$\begin{cases} \operatorname{div}({}^tA^\varepsilon \nabla u^\varepsilon_i) = f_i, \\ u^\varepsilon_i - u_i \in H^1_0(\Omega). \end{cases} \tag{3.8}$$

Using it we define M^ε to be the matrix whose i:th row is ${}^t\nabla u^\varepsilon_i$. In other words, M^ε is the matrix with entries

$$M^\varepsilon_{ij} = \frac{\partial u^\varepsilon_i}{\partial x_j}.$$

Then we take P^ε to be

$$P^\varepsilon = M^\varepsilon A^\varepsilon,$$

so that ${}^tP^\varepsilon ={}^t A^\varepsilon(\nabla u^\varepsilon_1, \dots, \nabla u^\varepsilon_n)$. The so defined matrices M^ε, P^ε satisfy

$$\begin{cases} \operatorname{curl} M^\varepsilon = 0, \\ \operatorname{div} P^\varepsilon = f, \end{cases}$$

where f is the vector with components $f_i \in H^{-1}(\Omega)$. In particular, the components of $\operatorname{curl} M^\varepsilon$ and $\operatorname{div} P^\varepsilon$ stay within a compact subset of $H^{-1}(\Omega)$.

From (3.8) we get for each $1 \le i \le n$ the elliptic estimates

$$\|u^\varepsilon_i\|_{H^1(\Omega)} \le C < \infty,$$

$$\|{}^tA^\varepsilon \nabla u^\varepsilon_i\|_{L^2(\Omega)^n} \le C < \infty.$$

Thus for some subsequence of $\{\varepsilon\}$ and some limit fields v_i and σ_i we have convergences

$$u^\varepsilon_i \rightharpoonup v_i, \text{ weakly in } H^1(\Omega), \tag{3.9}$$

$$\nabla u^\varepsilon_i \rightharpoonup \nabla v_i, \text{ weakly in } L^2(\Omega)^n, \tag{3.10}$$

$$ {}^tA^\varepsilon\nabla u_i^\varepsilon \rightharpoonup \sigma_i,\ \text{weakly in } L^2(\Omega)^n. \tag{3.11}$$

The latter convergence together with (3.8) shows that

$$\operatorname{div}\sigma_i = f_i.$$

At this point we use the mentioned fact that H-convergence carries over to the transposed matricies. Thus ${}^tA^\varepsilon \xrightarrow{H} {}^tA$, and since $\operatorname{curl}\nabla u_i^\varepsilon = 0$ and $\operatorname{div}({}^tA^\varepsilon) = f_i$ are compact in $H^{-1}(\Omega)$ it follows from the definition of this H-convergence that

$$\sigma_i =^t A\nabla v_i.$$

Therefore v_i solves the boundary value problem

$$\begin{cases}\operatorname{div}({}^tA\nabla v_i) = f_i,\\ v_i - u_i \in H_0^1(\Omega).\end{cases} \tag{3.12}$$

But this problem has the unique solution u_i. Thus we conclude that $v_i = u_i$ and that $\sigma_i =^t A\nabla u_i$. It also follows that in (3.9)–(3.11) we have convergence for the full sequence ε (because otherwise one could extract a subsequence producing a different solution of (3.12)). With this in mind, the convergences (3.10), (3.11) state exactly that

$$M^\varepsilon \rightharpoonup M \text{ weakly in } L^2(\Omega)^{n\times n},$$
$$P^\varepsilon \rightharpoonup P \text{ weakly in } L^2(\Omega)^{n\times n}.$$

This proves the theorem.

□

Remark 3.2. It is clear that the matrices M^ε, M, P^ε, P appearing in the decompositions (3.1), (3.2) are far from being uniquely determined by A^ε, A, even when all the conditions (3.3)–(3.6) are satisfied. For example, none of the conditions (3.1)–(3.6) are affected if M^ε, M, P^ε, P are multiplied from the left by one and the same invertible matrix $R = R(x)$ with bounded Lipschitz coefficients,

For the above reasons one cannot formulate the theorem as saying that if (3.1), (3.2), (3.5), (3.6) hold with M invertible, then $A^\varepsilon \xrightarrow{H} A$ if and only if (3.3), (3.4) hold. However, such a statement is true if an appropriate normalization is imposed. Examples of such normalizations are that $M = I$ or that $P = I$. One could also move one of the conclusions to be an assumption instead. For example, the following statement is correct (and easy to deduce from the theorem): if (3.1), (3.2), (3.5), (3.6), (3.3) hold, then $A^\varepsilon \xrightarrow{H} A$ if and only if (3.4) holds.

Remark 3.3. From the formulation of the theorem one can easily pass to construction of "correctors" (cf. [14], [15]). Indeed, in order to construct correctors for A^ε one applies the theorem to the transposed matrices: if $A^\varepsilon \xrightarrow{H} A$ then ${}^tA^\varepsilon \xrightarrow{H} {}^tA$. Thus there are matrices ${}^tN^\varepsilon$, ${}^tN$, ${}^tQ^\varepsilon$, ${}^tQ$ such that ${}^tN^\varepsilon\,{}^tA^\varepsilon =^t Q^\varepsilon$, ${}^tN\,{}^tA =^t Q$, ${}^tN^\varepsilon \rightharpoonup^t N$, ${}^tQ^\varepsilon \rightharpoonup^t Q$ with $\operatorname{curl}{}^tN^\varepsilon$ and $\operatorname{div}{}^tQ^\varepsilon$

relatively compact in $H^{-1}(\Omega)$. Here we choose the normalization ${}^tN = I$, ${}^tQ = A$. If now we have vector fields D^ε, E^ε as in the definition of the H-convergence (for A^ε) and if an additional weak condition is satisfied, e.g., that N^ε, Q^ε are bounded in $L^\infty(\Omega)^{n\times n}$, then the assertion is that

$$E^\varepsilon - N^\varepsilon E \to 0 \quad \text{strongly in } L^2_{\text{loc}}(\Omega)^n,$$
$$D^\varepsilon - Q^\varepsilon E \to 0 \quad \text{strongly in } L^2_{\text{loc}}(\Omega)^n.$$

This means that $N^\varepsilon E$ and $Q^\varepsilon E$ are good approximations (correctors) of E^ε and D^ε respectively.

To prove the assertion, first notice that $D^\varepsilon - Q^\varepsilon E = A^\varepsilon(E^\varepsilon - N^\varepsilon E)$ and that therefore, for every $\omega \subset \Omega$, the $L^2(\omega)^n$-norms of both $E^\varepsilon - N^\varepsilon E$ and $D^\varepsilon - Q^\varepsilon E$ can be estimated from above and below by $\int_\omega (D^\varepsilon - Q^\varepsilon E, E^\varepsilon - N^\varepsilon E)\,dx$. But the div-curl lemma gives that

$$(D^\varepsilon - Q^\varepsilon E, E^\varepsilon - N^\varepsilon E) = (D^\varepsilon, E^\varepsilon) - (D^\varepsilon, N^\varepsilon E) - (Q^\varepsilon E, E^\varepsilon) + (Q^\varepsilon E, N^\varepsilon E)$$
$$\rightharpoonup (D, E) - (D, NE) - (QE, E) + (QE, NE) = (D - QE, E - NE) = 0$$

in the sense of distributions (see [7] for further details). From this the assertion follows.

4. Example

The criterion of H-convergence in the theorem above is particularly useful in cases where it is possible to find the matrices M^ε and P^ε *a priori* (without solving any Dirichlet problem, e.g.). The main example for which this occurs is the case of stratified media, i.e., when A^ε depends on only one of the coordinates, say x_1:

$$A^\varepsilon = A^\varepsilon(x_1).$$

Then the classical philosophy [8], [15] is that one should write the relation $D^\varepsilon = A^\varepsilon E^\varepsilon$ in such a way that the "bad" components of D^ε and E^ε are expressed in terms of the "good" ones. The good components are those for which one has control over the oscillations via the differential equation (1.1) or via compactness assumptions (2.4), (2.5).

In the stratified case, D_1^ε and $E_2^\varepsilon, \ldots, E_n^\varepsilon$ are good and the rest are bad (see [15] for explanations), and one thus writes (2.1) as (suppressing ε for a moment)

$$\begin{cases} E_1 = \frac{1}{A_{11}} D_1 - \sum_{j\ge 2} \frac{A_{1j}}{A_{11}} E_j, \\ D_i = \frac{A_{i1}}{A_{11}} D_1 + \sum_{j\ge 2} (A_{ij} - \frac{A_{i1}A_{1j}}{A_{11}}) E_j \quad (i \ge 2). \end{cases}$$

In order to write this as $MD = PE$ one should take care to multiply the bad quantities E_1 and $D_2 \ldots D_n$ only by good coefficients, for example constants. The simplest and most natural choice of matrices M and P then is

$$M = \begin{pmatrix} \frac{1}{A_{11}} & 0 \\ & \\ -\frac{A_{i1}}{A_{11}} & \delta_{ij} \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & \frac{A_{1j}}{A_{11}} \\ & \\ 0 & A_{ij} - \frac{A_{i1}A_{1j}}{A_{11}} \end{pmatrix},$$

where $i \geq 2$ is the row index, $j \geq 2$ the column index and δ_{ij} the Kronecker delta. It is immediate that $\operatorname{curl} M = 0$, $\operatorname{div} P = 0$, and, restoring ε again, we have that $A^\varepsilon \xrightarrow{H} A$ if and only if $M^\varepsilon \rightharpoonup M$, $P^\varepsilon \rightharpoonup P$ weakly in $L^2(\Omega)^{n\times n}$.

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