Percolation of the vacant set of the Brownian excursions process

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Abstract

We prove that the critical value for percolation in the vacant set of the Brownian excursions process in the unit disc equals $\pi/3$. The proof uses the restriction property and the link to the Schramm-Loewner evolution. We also discuss some connections with the Brownian interlacements process.

1 Introduction

In this note, we study percolation in the vacant set of a Poisson process of Brownian excursions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We will give the precise definitions in Section 2, but first we describe the model informally. The Brownian excursion measure μ on $\mathbb{D}$ is supported on Brownian paths starting and ending on $\partial\mathbb{D}$ conditioned on staying inside $\mathbb{D}$ in between. On the space of continuous curves starting and ending on $\partial\mathbb{D}$ we then consider a Poisson point process ω_β with intensity measure $\beta\mu$ where $\beta > 0$. The union of the trajectories in ω_β is a random subset of $\mathbb{D}$, the law of which is conformally invariant. The complement of the union of the trajectories in ω_β is an open random subset of $\mathbb{D}$ which we denote by $\mathcal{V}_\beta$. For $z \in \mathbb{D}$, let $\mathrm{Perc}(\mathcal{V}_\beta,z)$ be the event that the component of $\mathcal{V}_\beta$ containing z is unbounded in the hyperbolic metric on $\mathbb{D}$. Define the percolation probability as $\theta(\beta) = \mathbb{P}(\mathrm{Perc}(\mathcal{V}_\beta,z))$. This probability is independent of the choice of z by conformal invariance. Standard arguments show that there is a critical value $\beta_c \in [0,\infty]$ such that $\theta(\beta) = 0$ if $\beta > \beta_c$ and $\theta(\beta) > 0$ if $\beta < \beta_c$. To be precise, we define

$$\beta_c = \sup\{\beta \geqslant 0 : \theta(\beta) > 0\}. \tag{1}$$

This note computes the value of β_c, see Theorem 1.1 below. But first, we discuss what was previously proved about β_c.

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In [ET19], a different property of $\mathcal{V}_\beta$ was studied, namely that of visibility to infinity. If there is some $\theta \in [0,2\pi)$ such that the line-segment $[o,e^{\mathrm{i}\theta})$ (which is an infinite half-line in the hyperbolic metric) is contained in $\mathcal{V}_\beta$, then we say that we have visibility to infinity from the origin. In [ET19], it was shown that with positive probability, there is visibility to infinity from the origin if and only if $\beta < \pi/4$. Since visibility to infinity implies percolation, this gives the bound $\beta_c \geqslant \pi/4$. Moreover, in [Eli18] it was shown that $\beta_c \leqslant \pi/2$. Roughly speaking, this was done by comparing the process with a (conformally invariant) hyperbolic Poisson line process induced by the end-points of the Brownian excursions, and referring to results in [BJST09].

It turns out that β_c lies in the interior of the above mentioned interval $[\pi/4,\pi/2]$. We have the following theorem[1].

Theorem 1.1. *The critical value for percolation in $\mathcal{V}_\beta$ satisfies*

$$\beta_c = \pi/3.$$

Moreover, $\theta(\beta_c) = 0$.

A consequence of Theorem 1.1 and the above mentioned result from [ET19] is that for $\beta \in [\pi/4,\pi/3)$, we have $\theta(\beta) > 0$, but a.s. no visibility to infinity from the origin.

In [ET19], questions regarding visibility were also studied for the Brownian interlacements model. Informally, this is a model created by a Poissonian cloud of double-sided, infinite Brownian motion paths in $\mathbb{R}^d$. Moreover, it was briefly discussed why the Brownian excursions process in the unit disc can be viewed as the hyperbolic plane analogue of the Brownian interlacements process. In Section 2.1 we take the opportunity to make this discussion a bit more detailed. We do this by describing the local picture of the Brownian excursions process.

The rest of this note is organized as follows. The proof of Theorem 1.1 uses connections between the Brownian excursions process and the $\mathrm{SLE}_\kappa(\rho)$ process. Background on these processes is given in Section 2. Section 2.1 also contains our discussion about the local picture of the Brownian excursions process. Finally, in Section 3.1 we give the short proof of Theorem 1.1.

2 Preliminaries

2.1 Brownian excursions

The Brownian excursion measure μ in the open unit disc $\mathbb{D}$ is a σ-finite measure on trajectories that spend their life time in the unit disk with endpoints on the boundary $\partial\mathbb{D}$, and we now briefly recall one way to construct this measure. See also for instance [LW00], [Vir03], [Law05] and [LW04].

Let

[1] An essentially equivalent statement is contained in the heuristic discussion of [QW19], Section 5, but no proof is given there.

$$W_{\mathbb{D}} := \left\{ w \in C([0,T_w],\bar{\mathbb{D}}) : w(0),w(T_w) \in \partial\mathbb{D},\ w(t) \in \mathbb{D}, \forall t \in (0,T_w) \right\}, \tag{2}$$

and for $K \Subset \mathbb{D}$ we let W_K be the set of trajectories in $W_{\mathbb{D}}$ that hit K. Define $\tilde{W}_{\mathbb{D}}$ and $\tilde{W}_K$ in the same way but with the condition $w(0) \in \partial\mathbb{D}$ replaced with the condition that $w(0) \in \mathbb{D}$. For w in $W_{\mathbb{D}}$ or $\tilde{W}_{\mathbb{D}}$ we write $X_t(w) = w(t)$.

For $x \in \mathbb{D}$, let $\mathbb{P}_x$ denote the law of complex Brownian motion started at x killed upon hitting $\partial\mathbb{D}$. Let σ_r denote the uniform probability measure on $B(o,r)$ and let $\mathbb{P}_{\sigma_r} = \int_{x \in \partial B(o,r)} \mathbb{P}_x \sigma_r(dx)$. The Brownian excursion measure μ can be defined as the weak limit

$$\mu = \lim_{\varepsilon \to 0} \frac{2\pi}{\varepsilon} \mathbb{P}_{\sigma_{1-\varepsilon}}, \tag{3}$$

as explained on p. 127-128 in [Law08].

Write δ for a Dirac mass and define

$$\Omega = \left\{ \omega = \sum_{i \geqslant 0} \delta(w_i, \beta_i) : (w_i,\beta_i) \in W_{\mathbb{D}} \times [0,\infty),\ \omega(W_K \times [0,\beta]) < \infty, \forall K \Subset \mathbb{D}, \beta \geqslant 0 \right\}.$$

Then let $\mathbb{P}$ denote the law of a Poisson point process on $W_{\mathbb{D}} \times \mathbb{R}_+$ with intensity measure $\mu \otimes \mathrm{d}\beta$. For $\beta > 0$ and $\omega = \sum_{i \geqslant 0} \delta(w_i, \beta_i) \in \Omega$ we write

$$\omega_\beta := \sum_{i \geqslant 0} \delta(w_i,\beta_i) 1\{\beta_i \leqslant \beta\}, \tag{4}$$

and note that under $\mathbb{P}$ the process ω_β is a Poisson process with intensity measure $\beta\mu$. We refer to ω_β as the Brownian excursions process at level β. For $\beta > 0$, the Brownian excursion set at level β is then defined as

$$\mathsf{BE}_\beta(\omega) := \bigcup_{\beta_i \leqslant \beta} \bigcup_{s \geqslant 0} w_i(s),\ \omega = \sum_{i \geqslant 0} \delta(w_i, \beta_i) \in \Omega, \tag{5}$$

and we let $\mathcal{V}_\beta = \mathbb{D} \setminus \mathsf{BE}_\beta$ denote the vacant set. Proposition 5.8 in [Law05] says that μ, and consequently $\mathbb{P}$, are invariant under the conformal automorphisms of $\mathbb{D}$.

We now discuss how the random set $\mathsf{BE}_\beta \cap K$ can be generated for a compact K in $\mathbb{D}$. This is what we refer to as "local picture".

We first introduce some additional notation. For $K \Subset \mathbb{D}$, let $L_K(w) = \sup\{0 < t \leqslant H_{\partial\mathbb{D}} : X_t(w) \in K\}$ denote the last exit time, with the convention that $L_K(w) = 0$ if w never hits K. Let $H_K(w) = \inf\{t > 0;\ X_t(w) \in K\}$ be the hitting time of K. A point x is said to be regular for K if $\mathbb{P}_x(H_K = 0) = 1$. We will assume that all compact sets K appearing below satisfy the condition that all $x \in K$ are regular.

For $K \Subset \mathbb{D}$ let $e_K(dy)$ denote the equilibrium measure (for Brownian motion in $\mathbb{D}$) of K, see e.g. Theorem 24.14 in [Kal02]. It is the finite measure supported on ∂K satisfying

$$\mathbb{P}_x \left(X(L_K) \in dy, 0 < L_K \right) = G(x,y) e_K(dy), \tag{6}$$

where $G(x,y)$ is the Green's function for Brownian motion in $\mathbb{D}$ stopped upon hitting $\partial\mathbb{D}$. We recall that

$$G(w,z) = \frac{1}{\pi} \log \frac{|1 - \bar{w}z|}{|w - z|} \text{ for } w,z \in \mathbb{D},$$

so that in particular

$$G(o,re^{i\theta}) = \frac{\log(1/r)}{\pi} \text{ for } 0 < r < 1 \text{ and } 0 \leqslant \theta < 2\pi.$$

Furthermore, the capacity (relative to $\mathbb{D}$) of $K \Subset \mathbb{D}$ is denoted by $\text{cap}(K)$ and is defined as the total mass of e_K.

The expression for $e_{B(o,r)}$ for $0 < r < 1$ is known, but we include a proof for convenience. Using the above and (6), we have

$$\mathbb{P}_o \left(X(L_{B(o,r)}) \in dy, 0 < L_{B(o,r)} \right) = \frac{\log(1/r)}{\pi} e_{B(o,r)}(dy). \tag{7}$$

On the other hand, by rotational invariance, we have that

$$\mathbb{P}_o \left(X(L_{B(o,r)}) \in dy, 0 < L_{B(o,r)} \right) = \sigma_r(dy), \tag{8}$$

where σ_r is the uniform probability measure on $\partial B(o,r)$. From (7) and (8) we get that

$$e_{B(o,r)}(dy) = \frac{\pi}{\log(1/r)} \sigma_r(dy). \tag{9}$$

The capacity of $B(o,r)$ is therefore given by

$$\text{cap}(B(o,r)) = \int_{y \in \partial B(o,r)} e_{B(o,r)}(dy) = \frac{\pi}{\log(1/r)}. \tag{10}$$

For $K \Subset \mathbb{D}$, the hitting kernel is defined as

$$h_K(x,dy) = \mathbb{P}_x(X(H_K) \in dy, H_K < \infty).$$

The equilibrium measure satisfies the following consistency property, see Proposition 24.15 in [Kal02]: If $K_1 \Subset K_2 \Subset \mathbb{D}$, then

$$e_{K_1}(dy) = \int_{x \in \partial K_2} h_{K_1}(x,dy) e_{K_2}(dx). \tag{11}$$

We now see that for a measurable set of trajectories A in $\tilde{W}_{\mathbb{D}}$ and $K \Subset \mathbb{D}$, we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{2\pi}{\varepsilon}\mathbb{P}_{\sigma_{1-\varepsilon}}\left(\{(X_{t+H_K})_{t\geqslant 0}\in A\}\cap\{H_K<\infty\}\right)\\
&=\lim_{\varepsilon\to 0}\frac{2\pi}{\varepsilon}E_{\sigma_{1-\varepsilon}}\left[\mathbb{P}_{X(H_K)}(A)1\{H_K<\infty\}\right]\\
&\overset{(9)}{=}\lim_{\varepsilon\to 0}\frac{-2\log(1-\varepsilon)}{\varepsilon}E_{e_{B(o,1-\varepsilon)}}\left[\mathbb{P}_{X(H_K)}(A)1\{H_K<\infty\}\right]\\
&=\lim_{\varepsilon\to 0}\frac{-2\log(1-\varepsilon)}{\varepsilon}\int_{\partial B(0,1-\varepsilon)}\int_{\partial K}\mathbb{P}_y(A)h_K(x,dy)e_{B(o,1-\varepsilon)}(dx)\\
&=\lim_{\varepsilon\to 0}\frac{-2\log(1-\varepsilon)}{\varepsilon}\int_{\partial K}\mathbb{P}_y(A)\int_{\partial B(0,1-\varepsilon)}h_K(x,dy)e_{B(o,1-\varepsilon)}(dx)\\
&\overset{(11)}{=}\lim_{\varepsilon\to 0}\frac{-2\log(1-\varepsilon)}{\varepsilon}\int_{\partial K}\mathbb{P}_y(A)e_K(dy)\\
&=2\int_{\partial K}\mathbb{P}_y(A)e_K(dy)=2\mathbb{P}_{e_K}(A).
\end{aligned}
\tag{12}
$$

We have proved the following result.

Proposition 2.1. *For $K \Subset \mathbb{D}$, let $N_K \sim Po(2\beta\mathrm{cap}(K))$ and let $\tilde{e}_K$ be the normalized equilibrium measure on K. Conditionally on N_K, let $(B_i)_{i=1}^{N_K}$ be a collection of independent Brownian motions in the unit disk with initial distribution $B_i(0) \sim \tilde{e}_K$. Then the following distributional equality holds*

$$
\mathsf{BE}_\beta \cap K \overset{d}{=} \bigcup_{i=1}^{N_K}[B_i]\cap K \tag{13}
$$

where $[B_i]$ denotes the trace of the Brownian motion.

There is a similar local description for Brownian interlacements in $\mathbb{R}^d$ for $d \geqslant 3$, see Remark 2.1.3 on p.568 of [Szn13].

2.2 Conformal restriction and SLE$_\kappa(\rho)$

We recall some facts on restriction measures. See Section 8 of [LSW03] for proofs and further discussion. Let $\mathbb{H} = \{z : \Im z > 0\}$ be the complex upper half-plane. Let X_- be the set of closed connected sets $K \subset \overline{\mathbb{H}}$ with the property that $K \cap \mathbb{R} \subset (-\infty,0]$. Suppose we have a probability measure $\mathbb{P}$ on X_-. We say that a probability measure $\mathbb{P} = \mathbb{P}_{\mathbb{H},0,\infty}$ (on X_-) satisfies one-sided conformal restriction with exponent $\alpha > 0$ if for any (relatively) compact A such that $\mathbb{H} \setminus A$ is simply connected and $\overline{A} \cap \mathbb{R} \subset (0,\infty)$,

$$
\mathbb{P}\left(K \cap A = \emptyset\right) = \varphi_A'(0)^\alpha.
$$

Here $\varphi_A : \mathbb{H} \setminus A \to \mathbb{H}$ is the conformal map fixing 0 and satisfying $\varphi_A(z) = z + o(z)$ as $z \to \infty$. A conformally invariant measure defined in some other simply connected domain with two marked boundary points is said to satisfy one-sided conformal restriction if its image in $\mathbb{H}$ does so. An important property of restriction measures is that if A is as above then the law $\mathbb{P}_{\mathbb{H},0,\infty}$ conditioned on $K \cap A = \emptyset$ is the same as $\mathbb{P}_{\mathbb{H}\setminus A,0,\infty}$, the latter defined by push-forward via conformal transformation.

Let B_t standard Brownian motion and for $\kappa > 0$, set $U_t = B_{\kappa t}$. The SLE_κ Loewner chain is defined by

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad 0 \leqslant t < T_z, \qquad g_0(z) = z,$$

where $T_z = \inf\{t \geqslant 0 : \operatorname{Im} g_t(z) = 0\}$. The SLE curve can then defined by $\gamma(t) = \lim_{y\downarrow 0} g_t^{-1}(U_t + iy)$ and the corresponding relatively compact hull $K_t = \{z \in \mathbb{H} : T_z \leqslant t\}$ is the complement of the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$. We retain this parametrization also when considering SLE in other domains.

Choosing $\kappa = 8/3$ gives the only SLE_κ with the restriction property, which in this case is two-sided. We then have $\alpha = 5/8$.

Next, consider the SDE

$$dW_t = \sqrt{\kappa} dB_t - \frac{\rho\, dt}{V_t - W_t}, \quad dV_t = \frac{2\, dt}{V_t - W_t}, \quad (W_0, V_0) = (w,v) \in \mathbb{H} \times \mathbb{R}.$$

The $\mathrm{SLE}_\kappa(\rho)$ Loewner chain started from (w,v) is the Loewner chain driven by W_t as above. The special v is called force point and we will usually take $(w,v) = (0,0+)$. Care is needed if ρ is too large and negative, though this is not an issue in the cases we will consider.

Define

$$\rho(\alpha) = \frac{-8 + 2\sqrt{1 + 24\alpha}}{3}, \qquad \alpha > 0.$$

We will need the following two well-known lemmas which link restriction measures with $\mathrm{SLE}_{8/3}(\rho)$ processes.

Lemma 2.2. *Let $\alpha > 0$ and let K_t be the hulls of an $SLE_{8/3}(\rho)$ process with $\rho = \rho(\alpha)$ and force point $0-$. Set $K = \cup_{t\geqslant 0} K_t$. Then K satisfies one-sided restriction with exponent α. In particular, the law of the right-boundary of a restriction measure of exponent α is that of $SLE_{8/3}(\rho)$, $\rho = \rho(\alpha)$, from 0 to ∞ with force point $0-$.*

Proof. See, e.g., Theorem 8.4 of [LSW03]. □

Lemma 2.3. *For $\alpha > 0$, let $\gamma = \gamma[0,\infty)$ be the trace of the $SLE_{8/3}(\rho(\alpha))$ path from 0 to ∞ in $\mathbb{H}$ with force point $0-$. If $0 < \alpha < 1/3$ then almost surely γ intersects $(-\infty, 0)$ in a set of dimension $d = -(3/8)\rho^2 - (7/4)\rho - 1$. If $\alpha \geqslant 1/3$ then almost surely γ does not intersect $(-\infty, 0)$.*

Proof. See Lemma 8.3 of [LSW03] for the statement about intersection. The dimension is computed in several places, e.g., in [Sch19]. □

Lemma 2.4. *The left-filling of a Poissonian process of Brownian excursions with intensity measure $\pi\alpha$ of the Brownian excursion measure restricted to excursions starting and ending on the upper semicircle satisfies one-sided conformal restriction with exponent α.*

Proof. See, e.g., Theorem 8 of [Wer05] (for multiplicative constant $c\alpha$) or Theorem 2.12 of [Wu15] which also computes of the constant $\pi\alpha$. $\square$

Remark 2.5. *If $\rho < 0$ is fixed, then Lemma 2.4 allows us to couple an $SLE_{8/3}(\rho)$, γ_1, with an $SLE_{8/3}$, γ_2, (both in $\mathbb{H}$ from 0 to ∞, say) so that γ_1 is to the left of γ_2 a.s. That is, γ_2 separates γ_1 from the positive real line $\mathbb{R}_+$ in $\mathbb{H}$. By topology, this implies that for any fixed crosscut η of $\mathbb{H}$ starting and ending on $\mathbb{R}_+$, $\mathbb{P}(\gamma_1 \cap \eta \neq \emptyset) \leqslant \mathbb{P}(\gamma_2 \cap \eta \neq \emptyset)$.*

3 Percolation in the vacant set

3.1 Determining the critical value: Proof of Theorem 1.1

The idea for the lower bound is very simple: consider independent excursion clouds starting and ending on the upper and lower parts of the unit circle, $\mathbb{T}_\pm$. By Lemma 2.2 and Lemma 2.4 the boundaries of the excursion clouds are independent $\mathrm{SLE}_\kappa(\rho)$ curves from -1 to 1 in $\mathbb{D}$ with force points at $e^{\pi i \mp 0}$. The $\rho(\beta)$ relation is such that the SLEs intersect the boundary away from -1 and 1 if and only if $\beta < \pi/3$ (Lemma 2.3) and do not separate o from $\partial\mathbb{D}$ with positive probability. The remaining excursions stay with positive probability in small neighborhoods of -1 and 1. In order to make this argument rigorous, we need to check a few details, e.g., the intuitive fact that with positive probability at least one path to the boundary stays open when adding on the additional excursions after the first excursion cloud has been sampled. This is done in Lemma 3.1 which provides the lower bound on the critical value. The upper bound is not implied directly by the fact that the SLEs do not intersect the boundary when $\beta \geqslant \pi/3$, but it does follow easily from this combined with monotonicity and the restriction property, see Lemma 3.2 below.

Lemma 3.1. *Suppose the intensity of the Brownian excursion Poisson process ω_β satisfies $\beta < \pi/3$. Then the probability that o is in an infinite component of the vacant set is strictly positive.*

Proof. Let $I = \{e^{i\theta} : \theta \in [\pi/4, 3\pi/4]\}$ and set $B_{-1} = B(-1, 1/10)$, $B_1 = B(1, 1/10)$. Then define

$$A = \{z \in \mathbb{D} : \Im z > 1/20 \text{ or } z \in B_{-1} \cup B_1\}.$$

Then A is a simply connected domain with -1,1 on its boundary. Write γ^+ for the $\mathrm{SLE}_\kappa(\rho), \rho = \rho(\beta/\pi)$, in $\mathbb{D}$ from -1 to 1 corresponding to the excursion cloud starting and ending on $\mathbb{T}_+$. By the choice of ρ the probability that γ^+ hits I is strictly positive. Now, by the restriction property and conformal invariance we have

$$\mathbb{P}_{\mathbb{D},-1,1}(\gamma^+ \cap I \neq \emptyset,\, \gamma^+ \subset \overline{A}) = \mathbb{P}_{A,-1,1}(\gamma^+ \cap I \neq \emptyset)\mathbb{P}_{\mathbb{D},-1,1}(\gamma^+ \subset \overline{A}) > 0.$$

(Here $\mathbb{P}_{D,-1,1}$ denotes the law of $\mathrm{SLE}_{8/3}(\rho)$ in D from -1 to 1.) Let $\varepsilon > 0$ be very small and set $S = S_\varepsilon = \overline{\{z \in \mathbb{D} : \mathrm{dist}(z, \mathbb{T}_-) \leqslant \varepsilon\}}$. Let $\tau = \inf\{t \geqslant 0 : \gamma^+[0,t] \cap I \neq \emptyset\}$. We claim that there is a constant $c < \infty$ such that conditioned on the event $\{\gamma^+_\tau \subset \overline{A}\}$, the probability that γ^+ returns to $S_\varepsilon \cap B_{-1}$ after τ is bounded by $c\varepsilon$. On the event that $\tau = \infty$ there is nothing to prove, so we assume $\tau < \infty$. Consider an arbitrary curve η starting at -1 and ending at a point $\zeta \in I$ and contained in $\overline{A}$. Let D_η be the component of 0 of $\mathbb{D} \setminus \eta$. Then we claim that the probability that an $\mathrm{SLE}_{8/3}$ curve from ζ to 1 in D_η hits a ball of radius ε centered on $\mathbb{T}_- \cap B_{-1}$ is at most $c\varepsilon^2$, where c does not depend on η, and $O(\varepsilon^{-1})$ such balls cover $S_\varepsilon \cap B_{-1}$. Indeed, this follows, e.g., from a slight variation of Proposition 3.2 and Lemma 3.3 in [FL15]. The same bound holds for γ^+ by Remark 2.5.

Also, by reversibility, the analogous estimate holds for the time reversal of γ^+ running from 1 to -1.

Now let E be the event that γ^+ stays in A, intersects I and neither γ^+ nor its time reversal returns to $S_\varepsilon \cap B_{\pm 1}$ after first hitting I. For all sufficiently small $\varepsilon > 0$, this event has positive probability and moreover, on the event E there is a path from 0 to $\zeta \in I$ that only intersects γ^+ at I and this path does not intersect S_ε.

The last observation we need to make is that with positive probability (depending on ε) the cloud of excursions that either both start and end on $\mathbb{T}_-$ or start from $\mathbb{T}_-$ and end on $\mathbb{T}_+$ is contained in S_ε. The lemma now follows from independence. $\square$

Lemma 3.2. *Suppose the intensity of the Brownian excursion Poisson process ω_β satisfies $\beta \geqslant \pi/3$. Then the probability that o is in an infinite component of the vacant set is* 0.

Proof. We consider again the cloud of excursions starting and ending on $\mathbb{T}_+$. By monotonitcity, the probability that these excursions fail to separate 0 from $\mathbb{T}_+$ is smaller than the probability that those excursions that stay in $D := \mathbb{D} \setminus [0,1)$ fail to separate 0 from $\mathbb{T}_+$. (Here we mean failure to separete as the event that there is a path from 0 to $\mathbb{T}_+$ that stays in D except for the start and end points and that does not intersect an excursion in D.) The latter probability is bounded above by the probability of the same event conditioned on the excursions staying in D. But the latter probability is 0 by the restriction property and the fact that $\mathrm{SLE}_\kappa(\rho), \rho \geqslant 1/3$, in D from -1 to 1 (understood as a prime end) a.s. does not intersect $\mathbb{T}_+$ except at -1 and 1. Similarly, we can consider excursions starting and ending on $\mathbb{T}_-$ to see that almost surely, 0 is separated from $\mathbb{T}_-$ in $\mathbb{D}$. We still need to check that -1 and 1 are not on the boundary of the component of 0 in the vacant set: this follows for instance since the two independent $\mathrm{SLE}_{8/3}(\rho)$ processes will intersect a.s. in any neighborhood of -1 (and 1). We can use the excursions to see this: in half-plane coordinates, using scale invariance, there is a constant $0 < c < 1$ such that for each dyadic half-annulus $A_n = \{2^{-n} \leqslant |z| < 2^{-n+1}\} \cap \mathbb{H}$ around 0, the probability that the union of the two sets of excursions starting and ending at $\mathbb{R}_\pm \cap A_n$ and staying in A_n separate the boundary components of A_n is at least c. The claim follows from independence. $\square$

The last two lemmas immediately imply Theorem 1.1.

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