Derivation of the uncertainty relation

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Lemma (Cauchy–Schwarz inequality)**.** *If ψ and χ are any states (vectors in a Hilbert space),*

$$|\psi||\chi| \geq \frac{1}{2}\Big|\langle\psi|\chi\rangle + \langle\chi|\psi\rangle\Big|. \tag{1}$$

Proof.

$$\begin{aligned} 0 &\leq \Big||\psi|\chi - |\chi|\psi\Big|^2 \\ &= |\psi|^2|\chi|^2 - |\psi||\chi|\langle\psi|\chi\rangle - |\psi||\chi|\langle\chi|\psi\rangle + |\psi|^2|\chi|^2 \\ &= |\psi||\chi|\Big[2|\psi||\chi| - \Big(\langle\psi|\chi\rangle + \langle\chi|\psi\rangle\Big)\Big], \end{aligned}$$

so

$$|\psi||\chi| \geq \frac{1}{2}\Big(\langle\psi|\chi\rangle + \langle\chi|\psi\rangle\Big).$$

Change $\psi \mapsto -\psi$ to find (1). ∎

Proposition. *If A and B are hermitian operators and $|\psi\rangle$ is any normalized state, then*

$$\sigma_A\sigma_B \geq \frac{1}{2}\Big|\langle\psi|[A,B]|\psi\rangle\Big|. \tag{2}$$

Proof. Equation (2) is easily seen to be invariant to constant shifts in A and B, so we may for simplicity assume that $\langle\psi|A|\psi\rangle = \langle\psi|B|\psi\rangle = 0$. Then

$$\sigma_A^2\sigma_B^2 = \langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle = |A\psi|^2|iB\psi|^2$$

so

$$\begin{aligned} \sigma_A\sigma_B &= |A\psi||iB\psi| \\ &\geq \frac{1}{2}\Big|\langle A\psi|iB\psi\rangle + \langle iB\psi|A\psi\rangle\Big| && \text{by (1)} \\ &= \frac{1}{2}\Big|i\langle A\psi|B\psi\rangle - i\langle B\psi|A\psi\rangle\Big| \\ &= \frac{1}{2}\Big|\langle\psi|AB\psi\rangle - \langle\psi|BA\psi\rangle\Big| && A \text{ and } B \text{ are hermitian} \\ &= \frac{1}{2}\Big|\langle\psi|[A,B]|\psi\rangle\Big|. \end{aligned}$$

∎

Corollary. *If A and B are conjugate observables ($[A,B] = i\hbar$), then*

$$\sigma_A\sigma_B \geq \frac{\hbar}{2}.$$

Remark. If we write $\sigma_A\sigma_B = |A\psi||B\psi|$ instead, we get

$$\sigma_A\sigma_B \geq \frac{1}{2}\Big|\langle\psi|\{A,B\}|\psi\rangle\Big| \tag{3}$$

with the *anticommutator* instead. However, this equation is not invariant to shifts, so it may not hold unless $\langle\psi|A|\psi\rangle = \langle\psi|B|\psi\rangle = 0$.