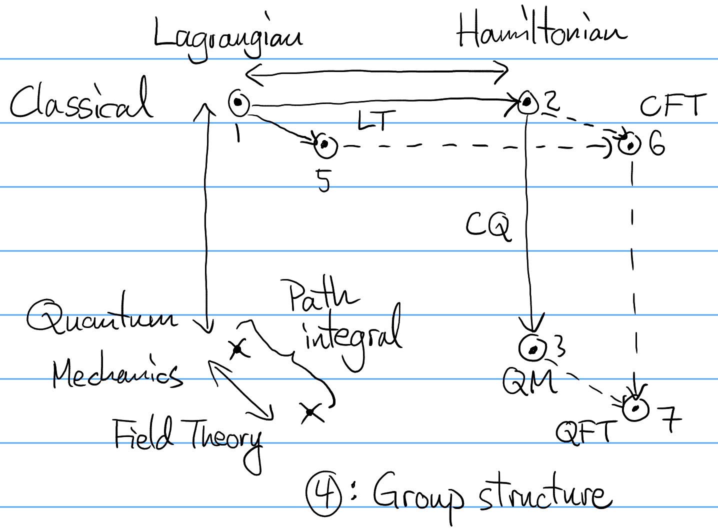
Symmetry made coherent

Outline



1 Lagrangian classical mechanics

Coords $q(t)$ (or $q^i(t)$, suppress)

Action $S=\int L(q(t),\dot q(t))\,dt$

Hamilton's principle $\Rightarrow$ EL eqs $\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}=0$

Transformation (flow) $q(t)\mapsto q(t,\varepsilon)$ with $q(t,0)=q(t)$ (write $q_\varepsilon(t)$).

Then $\dot q(t)\mapsto \dot q_\varepsilon(t)$ and $F(q,\dot q)\mapsto F_\varepsilon(q,\dot q)=F(q_\varepsilon,\dot q_\varepsilon)$ for any F.

Symmetry if $S_\varepsilon=S$, i.e. $L'=\dot X$ for some X ($'=\frac{\partial}{\partial\varepsilon}$, $\cdot=\frac{\partial}{\partial t}$).

Noether's theorem $\dot Q=0$ with $Q=\frac{\partial L}{\partial \dot q}q'-X$ (easy proof)

2 Hamiltonian classical mechanics

$p=\frac{\partial L}{\partial \dot q}$. $H=p\dot q-L$.

Poisson bracket $\{F,G\}=\sum_i\left(\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}\right)\Rightarrow [q^i,p_j]=\delta^i_j$

Generators Solve $q'=\frac{\partial G}{\partial p}$, $p'=-\frac{\partial G}{\partial q}$. Then $F'=\{F,G\}$ for all F

$\Rightarrow F_\varepsilon = e^{\varepsilon\{\cdot,G\}}F = F+\varepsilon\{F,G\}+\frac{\varepsilon^2}{2}\{\{F,G\},G\}+\cdots$.

Hamilton's eqns $\varepsilon=t \iff G=H$.

Noether's theorem $0=H'=\{H,G\}\Rightarrow 0=\{G,H\}=\dot G$.

Nontrivial: $G=Q$ from earlier.

3 ## Hamiltonian quantum mechanics

Observables: $F \to \hat{F}$

States: $(p,q) \to |\psi\rangle$ (mysterious)

Canonical quantization: $\{F,G\} \to -i[\hat{F},\hat{G}]$

$$F \mapsto F_\varepsilon = e^{\varepsilon\{\cdot,Q\}} F$$
$$\downarrow$$
$$\hat{F} \mapsto \hat{F}_\varepsilon = e^{-i\varepsilon[\cdot,\hat{Q}]}\hat{F} = e^{i\varepsilon[\hat{Q},\cdot]}\hat{F} = \{e^{[\hat{A},\cdot]}\hat{B} = e^{\hat{A}}\hat{B}e^{-\hat{A}}\} = e^{i\varepsilon\hat{Q}}\hat{F}e^{-i\varepsilon\hat{Q}}.$$

So $\langle\chi|F_\varepsilon|\psi\rangle = \langle e^{-i\varepsilon\hat{Q}}\chi|\hat{F}|e^{-i\varepsilon\hat{Q}}\psi\rangle \equiv \langle\chi_\varepsilon|\hat{F}|\psi_\varepsilon\rangle.$

4 ## Group structure

Lagr. mech: $q_\varepsilon(t) = \underbrace{U_\varepsilon[q]}(t)$; in general $F_\varepsilon(t) = U_\varepsilon[F](t)$

Lie group action; $U_\varepsilon \in G$ Lie algebra generators $t^a \in \mathfrak{g}$

1-param group $U_{\varepsilon_1}U_{\varepsilon_2} = U_{\varepsilon_1+\varepsilon_2} \leadsto U_\varepsilon = e^{i\varepsilon t^a\theta^a}$

$\downarrow$

Ham. mech: $e^{i\varepsilon t^a\theta^a}[F] = e^{\varepsilon\{\cdot,Q\}}F$

"Adjoint" Poisson bracket "representation" of $\mathfrak{g}$ on the observables:

$(it^a\theta^a)[F] = \{F,Q\}$

$\downarrow$

Quantum mech: $e^{i\varepsilon t^a\theta^a}[\hat{F}] = e^{i\varepsilon[\hat{Q},\cdot]}\hat{F} = e^{i\varepsilon\hat{Q}}\hat{F}e^{-i\varepsilon\hat{Q}}$

"Adjoint" commutator representation of $\mathfrak{g}$ on the quantum observables:

$(it^a\theta^a)[\hat{F}] = i[\hat{Q},\hat{F}]$

(or: isomorphism from $\mathfrak{g}$ into algebra $\mathfrak{q}$ of conserved quantities; $t^a\theta^a \mapsto \hat{Q}$, then adjoint rep of $\mathfrak{q}$ on itself)

Also: "Fundamental" rep of $\mathfrak{g}$ on states: $e^{i\varepsilon t^a\theta^a}[|\psi\rangle] = e^{-i\varepsilon\hat{Q}}|\psi\rangle.$

Particles and representations

$|\psi\rangle \in \mathcal{H}$ (Hilbert space).

Decompose $\mathcal{H}$ into energy eigenspaces: $\mathcal{H} = \bigoplus_i \mathcal{H}_i$.

If $\hat{Q} \in \mathfrak{q}$ (i.e. $[\hat{Q},\hat{H}]=0$) then $\hat{H}e^{-i\varepsilon\hat{Q}}|\psi_i\rangle = E_i e^{-i\varepsilon\hat{Q}}|\psi_i\rangle$;

$e^{-i\varepsilon\hat{Q}}$ doesn't change the energy. So $\mathcal{H}_i$ are invariant subspaces of the fundamental rep $e^{i\varepsilon t^a\theta^a}[|\psi\rangle] = e^{-i\varepsilon\hat{Q}}|\psi\rangle$. So this rep is <u>reducible</u>.

Decompose it as a direct sum of irreps:

$\rho_{fund} = \bigoplus_k \rho_k$, $\mathcal{H} = \bigoplus_k \mathcal{H}_k$ with $\rho_k : \mathfrak{g} \to \mathfrak{gl}(\mathcal{H}_k)$.

Then the symmetry mixes states only within the same irrep.

Cartan subalgebra of $\mathfrak{g}$: Maximal set of simultaneously observable conserved quantities. Weight vectors of ρ_k: their simultaneous eigenstates (depending on G, different orbits, spins, particles...).

like in hydrogen atom

5 Lagrangian classical field theory

Classical mechanics with uncountably many variables:

$$q^1, q^2, \dots \longrightarrow \phi(x_1), \phi(x_2), \phi(x_3), \dots$$

$$L(t) = \int d^3x\, \mathcal{L}(\phi(x,t), \partial_i\phi(x,t), \dot{\phi}(x,t))$$

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0 \longrightarrow \frac{\delta L}{\delta \phi(x)} - \frac{d}{dt}\frac{\delta L}{\delta \dot{\phi}(x)} = 0.$$

Functional derivatives: $\frac{\partial q^i}{\partial q^j} = \delta^i_j \longrightarrow \frac{\delta \phi(y)}{\delta \phi(x)} = \delta^{(3)}(x-y)$, so

$$\frac{\delta L}{\delta \phi(x)} = \int d^3y \frac{\delta \mathcal{L}(y)}{\delta \phi(x)} = \int d^3y \Big[\underbrace{\frac{\delta \phi(y)}{\delta \phi(x)}}_{\delta^{(3)}(x-y)} \frac{\partial \mathcal{L}(y)}{\partial \phi} + \underbrace{\frac{\delta(\partial_i \phi)}{\delta \phi(x)}}_{\partial_i \delta^{(3)}(x-y)} \frac{\partial \mathcal{L}(y)}{\partial(\partial_i \phi)} \Big]$$

$$= \int d^3y\, \delta^{(3)}(x-y) \Big[\frac{\partial \mathcal{L}(y)}{\partial \phi} - \partial_i \Big(\frac{\partial \mathcal{L}(y)}{\partial(\partial_i \phi)} \Big) \Big] = \underline{\frac{\partial \mathcal{L}(x)}{\partial \phi} - \partial_i \Big(\frac{\partial \mathcal{L}(x)}{\partial(\partial_i \phi)} \Big)}$$

and $$\frac{\delta L}{\delta \dot{\phi}(x)} = \int d^3y \frac{\delta \mathcal{L}(y)}{\delta \dot{\phi}(x)} = \int d^3y \underbrace{\frac{\delta \dot{\phi}(y)}{\delta \dot{\phi}(x)}}_{\delta^{(3)}(x-y)} \frac{\partial \mathcal{L}(y)}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}}$$

$$\Longrightarrow \frac{\partial \mathcal{L}(x)}{\partial \phi} - \partial_i \frac{\partial \mathcal{L}(x)}{\partial(\partial_i \phi)} - \frac{d}{dt}\frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}} = 0$$

$$\Longrightarrow \underline{\frac{\partial \mathcal{L}(x)}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \phi)} = 0}$$

6 Hamiltonian classical field theory

$$p = \frac{\partial L}{\partial \dot{q}} \longrightarrow \pi(x) = \frac{\delta L}{\delta \dot{\phi}(x)} = \{\dots\} = \frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}}.$$

$$\{F,G\} = \sum_i \Big(\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} \Big) \longrightarrow \{F,G\} = \int d^3x \Big(\frac{\delta F}{\delta \phi(x)}\frac{\delta G}{\delta \pi(x)} - \frac{\delta F}{\delta \pi(x)}\frac{\delta G}{\delta \phi(x)} \Big)$$

$$\Longrightarrow \{\phi(x), \pi(y)\} = \delta^{(3)}(x-y)$$

Noether's theorem: If $L' = \dot{X}$,

$$Q = \left(\int d^3x \frac{\delta L}{\delta \dot{\phi}(x)} \phi'(x)\right) - X \text{ is conserved (charge).}$$

More powerful: If $\mathcal{L}' = \partial_\mu \xi^\mu$,

$$j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \phi' - \xi^\mu \text{ has } \partial_\mu j^\mu = 0 \text{ (current).}$$

7 <u>Hamiltonian quantum field theory</u>

Canonical quantization like for mechanics: $\{F, G\} \to -i[\hat{F}, \hat{G}]$

Eg. $\{\phi(x), \pi(y)\} = \delta^{(3)}(x-y)$

$$\downarrow$$

$$\underline{[\phi(x), \pi(y)] = i\delta^{(3)}(x-y)}$$