Geometric (Clifford) algebra

$\boxed{\text{Associative product between vectors s.t. } vv = v\cdot v}$

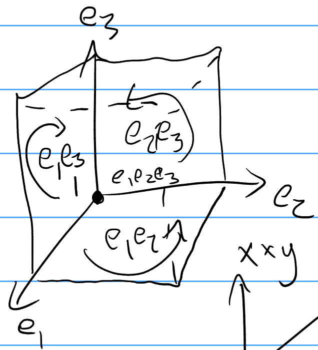
Formally, take ON basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$,
form tensor algebra (eg $7 + 4e_2e_1e_3 - 2\pi e_4$)
(contains vectors: $v_1e_1 + \dots + v_ne_n$),
impose $vv = v\cdot v$ ($v^2 = |v|^2$).

$$\left.\begin{array}{l}(x+y)^2 = (x+y)\cdot(x+y) = \overbrace{x\cdot x}^{x^2} + 2x\cdot y + \overbrace{y\cdot y}^{y^2} \\ (x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2\end{array}\right\} \Rightarrow x\cdot y = \frac{xy+yx}{2}$$

For basis vectors: $\{e_i, e_j\} = 2g_{ij}$ (Clifford algebra)
Familiar: $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}_2$; $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{1}_4$.
But no point in a matrix rep!

$x \parallel y \iff xy = yx$; $x \perp y \iff xy = -yx$
Take $g_{ij} = \delta_{ij}$: What is e_1e_2? $(e_1e_2)^2 = e_1e_2e_1e_2 = -\underbrace{e_1e_1}_{1}\underbrace{e_2e_2}_{1} = -1$

In 3D: $\mathcal{G}_3 = \mathrm{Span}_{\mathbb{R}}\{\underbrace{1}_{\text{scalars}}, \underbrace{e_1, e_2, e_3}_{\text{vectors}}, \underbrace{e_1e_2, e_1e_3, e_2e_3}_{\text{bivectors}}, \underbrace{\overbrace{e_1e_2e_3}^{I}}_{\text{pseudoscalar}}\}$



Outer product $x \wedge y = \frac{xy - yx}{2}$

$\Rightarrow \boxed{xy = x\cdot y + x\wedge y}$ ("Fundamental identity")

In 3D: $x \times y = (x\wedge y)I^{-1}$
$(I^2 = \pm 1 \Rightarrow I^{-1} = I^3)$

"Axial vectors": $(n-1)$-vectors

Vector derivative $\nabla = e^i\partial_i \to$ Higher-dim complex analysis (Geometric Calculus)

Eg In 2D: $F = (u + Iv)$: $\nabla F = (e_1\partial_1 + e_2\partial_2)(u + e_1e_2v)$
$= (\partial_1 u - \partial_2 v)e_1 + (\partial_1 v + \partial_2 u)e_2$

$\boxed{\nabla F = 0} \iff$ Cauchy-Riemann

Eg EM: Vector fields e, b. Bivector field $B = -Ib$.
EM field $e + B$.

$$\nabla F = (-\nabla \cdot e) + (\partial_t e - \nabla \times b) + (-\partial_t b - \nabla \times e) I + (\nabla \cdot b) I.$$

All 4 Maxwell eqns.

time permitting

SU(2)

"$R = a_0 + i a_i \sigma_i$" $\longrightarrow R = a_0 + Ia$

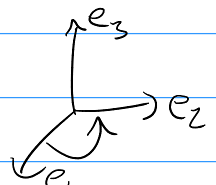
$$1 = a_0^2 + a^2 = (a_0 - Ia)(a_0 - Ia) = RR^\dagger = R^\dagger R.$$

Rotates vectors by $x \mapsto RxR^\dagger$ ($= (-R)x(-R)^\dagger$; double cover of $SO(3)$)

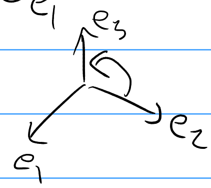
Composition: $x \mapsto R_2(R_1 x R_1^\dagger)R_2^\dagger = RxR^\dagger$ with $R = R_1 R_2$

Half-angle formula: $R = \cos(\theta/2) - i\sin(\theta/2)$; i rotation plane.

Eg $R_1 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} e_1 e_2 = \frac{1 - e_1 e_2}{\sqrt{2}}$



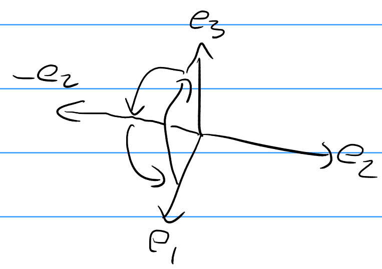
$R_2 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} e_2 e_3 = \frac{1 - e_2 e_3}{\sqrt{2}}$



$$R_2 R_1 = \frac{1}{2}\left(1 - e_1 e_2 - e_2 e_3 + e_2 e_3 e_1 e_2\right)$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2} \underbrace{\frac{e_1 e_2 + e_2 e_3 + e_1 e_3}{\sqrt{3}}}_{i \ (i^2 = -1)} = \cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)$$

$\Rightarrow$ Angle $\frac{2\pi}{3}$; axis $iI^{-1} = \frac{e_1 - e_2 + e_3}{\sqrt{3}}$



(Quaternions)

Spacetime algebra

Basis vectors $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$; $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

$\not{p} = p_\mu \gamma^\mu = p.$

$$A = \underbrace{\langle A \rangle}_{\text{scalar part}} + \langle A \rangle_1 + \langle A \rangle_2 + \langle A \rangle_3 + \langle A \rangle_4$$

time permitting

Spacetime split: $x = x\gamma_0\gamma_0 = (x\cdot\gamma_0 + x\wedge\gamma_0)\gamma_0$

$= (t + x^i\gamma_i\gamma_0)\gamma_0 = \{\text{identify } \gamma_i\gamma_0 = \vec{\sigma}_i\} = (t + \vec{x})\gamma_0.$

Different split xvv for any frame 4-velocity v $(v^2=1)$.

Dirac traces

Claim: $\mathrm{Tr}(A) = 4\langle A \rangle$ scalar part of A

Eg $\mathrm{Tr}(\not{p}_1\not{p}_2\not{p}_3) = 4\langle p_1p_2p_3\rangle = 4\langle \text{vector} + \text{trivector}\rangle = 0$

Eg $\mathrm{Tr}(\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4\langle p_1p_2p_3p_4\rangle$

$$= 4\langle (p_1\cdot p_2 + \underbrace{p_1\wedge p_2)(p_3\cdot p_4 + p_3\wedge p_4}_{\text{bivectors}})\rangle$$

$$= \underbrace{4(p_1\cdot p_2)(p_3\cdot p_4)}_{X} + 4(p_1\wedge p_2)\llcorner(p_3\wedge p_4)$$

$$= \{(A\wedge B)\llcorner C = A\llcorner(B\llcorner C)\} =$$

$$= X + 4p_1\llcorner\left(p_2\llcorner(p_3\wedge p_4)\right) = \{u\llcorner(v\wedge A) = (u\cdot v)A - v\wedge(u\llcorner A)\} =$$

$$= X + 4p_1\llcorner\left((p_2\cdot p_3)p_4 - p_3(p_2\cdot p_4)\right)$$

$$= 4\left((p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)(p_2\cdot p_4)\right)$$

time permitting

Eg $\gamma^\mu \overbrace{p_1 p_2 p_3}^{A} \gamma_\mu = \gamma^\mu(\langle A\rangle_1 + \langle A\rangle_3)\gamma_\mu$

$= \gamma^\mu(u + vI^{-1})\gamma_\mu.$

$\gamma^\mu u \gamma_\mu = \gamma^\mu(2u\cdot\gamma_\mu - \gamma_\mu u) = 2u - 4u = -2u.$

$\gamma^\mu v I^{-1}\gamma_\mu = \{ I^{-1}\gamma_\mu = -\gamma_\mu I^{-1}\} = -\gamma^\mu v \gamma_\mu I^{-1} = 2vI^{-1}$

$\Rightarrow \gamma^\mu(\langle A\rangle_1+\langle A\rangle_3)\gamma_\mu = -2(\langle A\rangle_1 - \langle A\rangle_3)$

Reverse: $(abc\ldots)^\dagger = cba$

$\gamma^{\mu\dagger}=\gamma^\mu$; For $\mu \neq \nu \neq \rho$: $\gamma^\mu\gamma^\nu\gamma^\rho = -\gamma^\rho\gamma^\nu\gamma^\mu$.

So $-2(\langle A\rangle_1 - \langle A\rangle_3) = -2A^\dagger = -2p_3p_2p_1$.

Further reading

Alan Macdonald: "A Survey of Geometric Algebra and Geometric Calculus"

Doran & Lasenby: "Geometric Algebra for Physicists"

(My BA thesis)