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Quantum structure of holographic black holes

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Abstract

We study a free quantum scalar field in the BTZ spacetime as a model of the AdS/CFT correspondence for black holes, and show the essential steps in computing Bogolyubov coefficients between modes on either side of the wormhole. As background, we review the BTZ geometry in standard, Kruskal and Poincaré coordinates, holographic renormalisation of the dual field theory and canonical quantisation in curved spacetime.

Keywords: BTZ spacetime, black hole, wormhole, AdS/CFT, holographic renormalisation, analytic continuation, Bogolyubov transformation, QFT in curved spacetime.

Sammanfattning

Vi studerar ett fritt skalärt kvantfält i BTZ-rumtiden som en modell av AdS/CFT-dualiteten för svarta hål och visar huvudstegen i beräkningen av Bogolyubov-koefficienter mellan moder på olika sidor av maskhålet. Som bakgrund redogör vi för BTZ-geometrin i standard-, Kruskal- och Poincarékoordinater, holografisk renormering av den duala fältteorin och kanonisk kvantisering i krökt rumtid.

Contents

1 Introduction

The problem of quantum gravity is one of the most fundamental difficulties in present-day physics. Hawking [1] made a decisive step with his prediction of thermal radiation from black holes, a striking example of a quantum-mechanical effect in the presence of a strong gravitational field. Another piece of the puzzle was discovered by Maldacena [2] in the form of the AdS/CFT correspondence, which provides a concrete realisation of gravity as a quantum-mechanical system via holography. Thus, the study of black holes in a holographic setting seems to carry great promise for the development of an understanding of quantum gravity. The Bañados–Teitelboim–Zanelli (BTZ) black hole [3] is an especially simple black hole spacetime amenable to a holographic treatment. This work investigates the relationship between the holographic (near-boundary) properties of the BTZ spacetime and its black hole (near-horizon) structure.

The AdS/CFT correspondence may be expressed on several levels of detail and precision. In full generality, it posits the equivalence of string theory, compactified as a theory on a $(d+1)$-dimensional asymptotically anti-de Sitter (AdS) spacetime X, called the bulk, to a supersymmetric conformal field theory (CFT) on its d-dimensional boundary $M = \partial X$. Often, the correspondence is analysed in the supergravity limit, that is, the classical limit of the particular string theory compactification. This work goes one step further and considers a simplified theory of supergravity consisting of a single dynamical free scalar field Φ on a fixed gravitational background geometry X.

Specifying the bulk theory involves choosing three things: the bulk spacetime X, the action $S[\Phi]$ and boundary conditions for Φ on M. For the action, we shall always take that of a free scalar field. The near-boundary behaviour of the bulk theory then defines the boundary CFT up to two undetermined functions on M: a source term for a specific local operator $\mathcal{O}(x)$, and the vacuum expectation value (VEV) for the same operator (more details in section 3). The choice of boundary conditions for Φ fixes the source, while the choice of X fixes or constrains the VEV in terms of the source (see section 4).

For the bulk spacetime X, we shall concentrate on the Bañados–Teitelboim–Zanelli (BTZ) black hole. This is a three-dimensional ($d = 2$) solution of the Einstein equations with negative cosmological constant, and an asymptotically AdS spacetime (defined in section 2). The standard metric for the nonrotating BTZ black hole is

$$\mathrm{d}s^2 = -\frac{\rho^2 - \rho_h^2}{\ell^2}\,\mathrm{d}t^2 + \frac{\ell^2}{\rho^2 - \rho_h^2}\,\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2 \qquad \begin{aligned} &t \in \mathbb{R} \\ &\rho_h < \rho < \infty \\ &0 \le \phi < 2\pi. \end{aligned} \tag{1.1}$$

This describes a static spacetime with Killing vector fields ∂_t and ∂_ϕ. There is an event horizon as $\rho \to \rho_h$, but the metric solves the Einstein equations equally well for $0 < \rho < \rho_h$—the interior of the black hole—and in this region there is a singularity as $\rho \to 0$; unlike in the Schwarzschild case, it is not a curvature singularity but a singularity in the causal structure [4]. As we shall see in section 5, the maximal analytic extension of the BTZ spacetime contains four regions; the original exterior region, past and future interior regions and an additional exterior region. The two exterior regions may be interpreted as the two sides of a non-traversable wormhole, and for both regions, one may define a holographic boundary.

Because the two holographic boundaries are part of the same theory, the boundary field theories are also related. Our goal is to establish the basic facts about the relationship between the two boundary theories by analysing the limits (near-boundary and near-horizon) and global properties of the bulk field, and to lay the groundwork for an understanding of the combined system in terms of the Hilbert space structure and possibly quantum correlations between the boundaries. After building up the necessary background in sections 2 to 5, we quantify the relationship in section 6. We review the methods used by Boulware [5] in the Rindler and Schwarzschild spacetimes and apply them to the BTZ spacetime, deriving equations (6.36) which relate sources and VEVs across the horizons. In section 7, we show how the results apply to normalised quantum modes.

2 Geometry of AdS and asymptotically AdS spaces

Anti-de Sitter (AdS) space is a maximally symmetric spacetime satisfying Einstein's equation

$$R_{ab} - \frac{1}{2}RG_{ab} + \Lambda G_{ab} = 8\pi T_{ab} \tag{2.1}$$

in a vacuum ($T_{ab} = 0$) and with negative cosmological constant Λ. In $d+1$ dimensions, one defines a length scale ℓ by

$$\Lambda = -\frac{d(d+1)}{2\ell^2}. \tag{2.2}$$

The space AdS_{d+1} can then be defined as a hypersurface in $\mathbb{R}^{d,2}$, endowed with the $-(+)^d-$ metric $\bar{\eta}_{MN} = \mathrm{diag}(-1, 1, \dots, 1, -1)$, by imposing $\bar{\eta}_{MN}X^M X^N = -\ell^2$ [6, §2.3.2]. Like any maximally symmetric spacetime, it can be recognised by the simple form of the Riemann curvature tensor,

$$R_{abcd} = \frac{R}{d(d+1)}(G_{ac}G_{bd} - G_{ad}G_{bc}). \tag{2.3}$$

We will not concern ourselves further with the global structure of AdS space. For our purposes, it will suffice to describe the geometry in the so-called *Poincaré patch* [6], where the metric is given by[1]

$$\mathrm{d}s^2 = G_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{\ell^2}{z^2}\tilde{\eta}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{\ell^2}{z^2}\left(\mathrm{d}z^2 + \eta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\right) \tag{2.4}$$

in coordinates $z \in (0, \infty)$ and $x^0, x^1, \dots, x^{d-1} \in \mathbb{R}$, with $\eta = \mathrm{diag}(-1, 1, \dots, 1)$ and $\tilde{\eta} = \mathrm{diag}(1, \eta)$. This coordinate patch only covers half of AdS_{d+1}, and $z \to \infty$ corresponds to a coordinate horizon in the interior of the space. Meanwhile, $z \to 0$ is the holographic boundary, on which the boundary CFT is defined.

Our aim is to study the AdS/CFT correspondence in *asymptotically* AdS spacetimes. To understand what this means, we must first define the concept of a conformally compact spacetime [7, §3; 8, §1, §2].

Definition. A manifold X with metric G_{ab} is *conformally compact* if one can find a positive function $z\colon X \to \mathbb{R}$ such that if X is instead endowed with the metric $g_{ab} = \frac{z^2}{\ell^2}G_{ab}$, then X can be equipped with a boundary $M = \partial X$ onto which g_{ab} and z extend smoothly, and on which $z = 0$ and $\mathrm{d}z \neq 0$. Such a function z is called a *defining function* for the conformal boundary.

That is, one compactifies the space X by a conformal transformation $G_{ab} \to \frac{z^2}{\ell^2}G_{ab}$, where z is chosen so that it is positive in the interior, but has a simple zero at the boundary. The AdS metric (2.4) is conformally compact; taking the z coordinate as the defining function, the conformal compactification takes

$$G \to \frac{z^2}{\ell^2}G = \mathrm{d}z^2 + \eta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{2.5}$$

[1] When multiplying differentials, we technically mean the symmetrised tensor product:

$$\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{1}{2}(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu).$$

which is a flat metric and can certainly be continued to the boundary at $z = 0$. The boundary metric becomes $g\big|_M = \eta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$, a standard Minkowski metric.

Notice that if z and z' are two defining functions, we may write $z' = z\mathrm{e}^{\omega}$ with ω regular. Switching defining functions $z \longrightarrow z'$ thus amounts to a conformal transformation $g_{ij} \longrightarrow \mathrm{e}^{2\omega} g_{ij}$ of the boundary metric. The bulk geometry X therefore determines the boundary metric up to conformal transformations, that is, the boundary conformal class is an invariant of X.

Next, we define asymptotically AdS manifolds.

Definition. An *asymptotically AdS manifold* is a conformally compact manifold satisfying the vacuum, negative-Λ Einstein equation (2.1).

This definition requires some motivation. We will show that the metric of a conformally compact Einstein manifold can be made to resemble the AdS metric (2.4) close to the boundary, with the defining function z being the same as the z coordinate. Specifically, it can be written

$$G = \frac{\ell^2}{z^2}\Big(\mathrm{d}z^2 + g_{ij}(z,\mathbf{x})\,\mathrm{d}x^i\,\mathrm{d}x^j\Big). \tag{2.6}$$

in some coordinates $(z, x^0, \dots, x^{d-1})$ where z is a defining function. Given a choice of z, the x^i may always be chosen orthogonal to z, so the substance of eq. (2.6) is the form of the first term; $G_{zz} = \frac{\ell^2}{z^2}$, which translates[2] to $|\mathrm{d}z|_g^2 = 1$.

That $|\mathrm{d}z|_g^2 = 1$ on the boundary is in fact true for *any* defining function z. It can be shown [8, eq. 2.1] that the Riemann tensor has the asymptotic form

$$R_{abcd} = -\frac{|\mathrm{d}z|_g^2}{\ell^2}(G_{ac}G_{bd} - G_{ad}G_{bc}) + O\big(z^{-3}\big). \tag{2.7}$$

Taking the trace, using $G^a{}_a = d+1$, shows that

$$R = -|\mathrm{d}z|_g^2 \frac{d(d+1)}{\ell^2} + O(z). \tag{2.8}$$

Further, taking the trace of the Einstein equation (2.1) gives

$$R = -\frac{d(d+1)}{\ell^2}. \tag{2.9}$$

For these to agree, it must be the case that $|\mathrm{d}z|_g^2\big|_{\partial X} = 1$.

Finally, it can be shown [8, Lemma 2.1] that, given a prescribed boundary metric $g_{ij}\big|_M$ in the correct conformal class, there is a *unique* defining function z with $|\mathrm{d}z|_g^2 = 1$ in a neighbourhood of ∂X. The proof is an appeal to uniqueness of the solution to a PDE. Thus, any conformally compact Einstein manifold has a metric of the form (2.6), at least in a neighbourhood of the boundary.

The BTZ black hole is conformally compact, and therefore an asymptotically AdS spacetime. A simple way of seeing this is to consider the BTZ metric (1.1) as $\rho \longrightarrow \infty$, which corresponds to $\rho_h \longrightarrow 0$. Then, the metric takes the form

$$\mathrm{d}s^2 = -\frac{\rho^2}{\ell^2}\,\mathrm{d}t^2 + \frac{\ell^2}{\rho^2}\,\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2 \tag{2.10}$$

[2] As follows: $1 = |\mathrm{d}z|_g^2 = g^{ab}(\mathrm{d}z)_a(\mathrm{d}z)_b = g^{\mu\nu}\partial_\mu z\,\partial_\nu z = g^{zz} = \frac{\ell^2}{z^2}G^{zz} = \frac{\ell^2}{z^2}\Big/G_{zz}$.

The substitution $z = \ell^2/\rho$ transforms this into

$$\mathrm{d}s^2 = G_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{\ell^2}{z^2}\left(\mathrm{d}z^2 - \mathrm{d}t^2 + \ell^2\,\mathrm{d}\phi^2\right). \tag{2.11}$$

This is the same as the AdS_3 metric (2.4) with $x^0 = t$ and $x^1 = \ell\phi$. Like in the AdS case, the conformally transformed metric $g = \mathrm{d}z^2 - \mathrm{d}t^2 + \ell^2\,\mathrm{d}\phi^2$ can be continued to $z = 0$, and the boundary has a flat metric $g\big|_M = -\,\mathrm{d}t^2 + \ell^2\,\mathrm{d}\phi^2$.

For a more careful analysis, we consider the full BTZ metric (1.1), and try to exhibit a defining function z such that $|\mathrm{d}z|_g^2 = 1$ everywhere. We choose to look for a function of ρ only. Then we must solve

$$1 = |\mathrm{d}z|_g^2 = \frac{z^2}{\ell^2}|\mathrm{d}z|_G^2 = \frac{z^2}{\ell^2}\frac{\ell^2}{\rho^2-\rho_h^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^2 \implies \frac{\mathrm{d}z}{z} = \overset{(+)}{-}\frac{\mathrm{d}\rho}{\sqrt{\rho^2-\rho_h^2}}. \tag{2.12}$$

A conveniently normalised solution is

$$z = \frac{2\ell^2}{\rho_h}\mathrm{e}^{-\operatorname{arccosh}\left(\frac{\rho}{\rho_h}\right)} = \frac{2\ell^2}{\rho + \sqrt{\rho^2-\rho_h^2}} = \frac{2\ell^2}{\rho_h^2}\left(\rho - \sqrt{\rho^2-\rho_h^2}\right). \tag{2.13}$$

The inverse transformation is

$$\rho = \rho_h\cosh\left[\ln\left(\frac{2\ell^2}{\rho_h z}\right)\right] = \frac{\rho_h}{2}\left(\frac{2\ell^2}{\rho_h z} + \frac{\rho_h z}{2\ell^2}\right) \tag{2.14}$$

$$\rho^2 - \rho_h^2 = \rho_h^2\sinh^2\left[\ln\left(\frac{2\ell^2}{\rho_h z}\right)\right] = \left[\frac{\rho_h}{2}\left(\frac{2\ell^2}{\rho_h z} - \frac{\rho_h z}{2\ell^2}\right)\right]^2. \tag{2.15}$$

Rewriting the metric using these substitutions, we eventually find the BTZ Poincaré metric

$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\left[\mathrm{d}z^2 - \left(1 - \frac{\rho_h^2 z^2}{4\ell^4}\right)^2\mathrm{d}t^2 + \left(1 + \frac{\rho_h^2 z^2}{4\ell^4}\right)^2\ell^2\,\mathrm{d}\phi^2\right]. \tag{2.16}$$

Notice that that in the limit $\rho \longrightarrow \infty$ or $\rho_h \longrightarrow 0$ or $z \longrightarrow 0$, (2.13) turns into $z = \ell^2/\rho$ and (2.16) agrees with (2.11). After multiplying by the conformal factor z^2/ℓ^2, the metric can be continued to $z = 0$, resulting in the same boundary metric $g\big|_M = -\,\mathrm{d}t^2 + \ell^2\,\mathrm{d}\phi^2$.

3 Statement of the AdS/CFT correspondence

We will now make more precise the discussion from section 1 and formulate the quantitative statement of the AdS/CFT correspondence.

Let X be a $(d+1)$-dimensional manifold with metric tensor G_{ab} of signature $-(+)^d$ satisfying the vacuum Einstein equations, and let Φ be a free scalar field of mass m. To specify the bulk theory, we must decide on boundary conditions for Φ. This means specifying the asymptotic behaviour of Φ close to the boundary ∂X. The boundary conditions may be encoded as a function $\phi_{(0)}$ on ∂X, and we may then write the partition function as

$$\mathcal{Z}_\Phi[\phi_{(0)}] = \int_{\Phi|_{\partial X} \sim \phi_{(0)}} \mathcal{D}\Phi\, \mathrm{e}^{\mathrm{i}S[\Phi]} \tag{3.1}$$

where we take for simplicity

$$S[\Phi] = -\int_X \mathrm{d}^{d+1}x\, \sqrt{|G|}\Big(\frac{1}{2}G^{ab}\nabla_a\Phi\nabla_b\Phi + \frac{1}{2}m^2\Phi^2\Big), \tag{3.2}$$

which is the action for a free scalar field in mostly-plus signature. We will elucidate the precise meaning of the boundary conditions "$\Phi\big|_{\partial X} \sim \phi_{(0)}$" in section 4.

If X is asymptotically AdS, the general statement of the AdS/CFT correspondence is

$$\mathcal{Z}_\Phi[\phi_{(0)}] = \mathcal{Z}_{\mathrm{CFT}}[\phi_{(0)}] \tag{3.3}$$

where $\mathcal{Z}_{\mathrm{CFT}}[\phi_{(0)}]$ is the partition function of some conformal field theory on ∂X subject to a source $\phi_{(0)}$. Namely,

$$\mathcal{Z}_{\mathrm{CFT}}[\phi_{(0)}] = \int \mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}\left[S_{\mathrm{CFT}}[\chi] - \int_{\partial X}\mathrm{d}^d x \sqrt{|g|}\,\mathcal{O}(\mathbf{x})\phi_{(0)}(\mathbf{x})\right]} \tag{3.4}$$

where $S_{\mathrm{CFT}}[\chi]$ is some action with conformal symmetry, g_{ab} is the boundary metric as in section 2, and $\mathcal{O}(\mathbf{x})$ is some local function of the fundamental degree(s) of freedom $\chi(\mathbf{x})$. Notice how the boundary conditions $\phi_{(0)}$ of the bulk theory enter the boundary partition function as a source term for the operator $\mathcal{O}(\mathbf{x})$. In both cases, $\phi_{(0)}$ is a nondynamical parameter.

In the supergravity limit, we make the stationary phase approximation

$$\mathcal{Z}_\Phi[\phi_{(0)}] \approx \mathrm{e}^{\mathrm{i}S[\Phi]} \tag{3.5}$$

where Φ is now the solution of the classical equations of motion, $\delta S[\Phi] = 0$, subject to the boundary conditions determined by $\phi_{(0)}$. In terms of the generating function $\mathcal{W}_{\mathrm{CFT}}[\phi_{(0)}]$ of connected correlation functions, given by $\mathcal{Z}_{\mathrm{CFT}} = \mathrm{e}^{\mathrm{i}\mathcal{W}_{\mathrm{CFT}}}$, we therefore find

$$\mathcal{W}_{\mathrm{CFT}}[\phi_{(0)}] \approx S[\Phi]. \tag{3.6}$$

This is the form of the AdS/CFT correspondence that we shall make use of. As usual, to compute correlation functions (connected and in the presence of the source) is simply to take functional derivatives of $\mathcal{W}_{\mathrm{CFT}}[\phi_{(0)}]$, or of $S[\Phi]$ by (3.6):

$$\langle\mathcal{O}(\mathbf{x}_1)\ldots\mathcal{O}(\mathbf{x}_n)\rangle_{\mathrm{conn}} = \frac{\mathrm{i}}{\sqrt{|g(\mathbf{x}_1)|}}\frac{\delta}{\delta\phi_{(0)}(\mathbf{x}_1)}\cdots\frac{\mathrm{i}}{\sqrt{|g(\mathbf{x}_n)|}}\frac{\delta}{\delta\phi_{(0)}(\mathbf{x}_n)}\,\mathrm{i}S[\Phi].$$

In particular, the one-point function, or VEV, of $\mathcal{O}(\mathbf{x})$ is

$$\langle\mathcal{O}(\mathbf{x})\rangle = -\frac{1}{\sqrt{|g(\mathbf{x})|}}\frac{\delta}{\delta\phi_{(0)}(\mathbf{x})}S[\Phi]. \tag{3.7}$$

4 Holographic renormalisation

To exploit the AdS/CFT correspondence (3.6) to get information about the boundary theory, we must evaluate the bulk action $S[\Phi]$ on the classical solution Φ. We shall see that the action as currently stated diverges, and the naive theory is thus ill-defined. The standard way to obtain a well-defined theory is through counterterm renormalisation: One parameterises the divergence by regularising the action, amends the regularised action with counterterms that cancel the divergences, and finally removes the regulariser. These are the essentials of the *holographic renormalisation* framework. Skenderis [7] gives an introduction.[3]

We start with the action (3.2),

$$S[\Phi] = -\int_X \mathrm{d}^3x\,\sqrt{|G|}\frac{1}{2}\Big(\nabla_a\Phi\nabla^a\Phi + m^2\Phi^2\Big). \tag{4.1}$$

For reasons which will become clear below, the cases where

$$-1 < m^2\ell^2 < 0 \tag{4.2}$$

are the simplest (note that we allow for negative values of m^2).[4] We will consider only these cases.

Before we can evaluate the action, we must find the classical solution Φ. The action principle is $\delta S = 0$, where the variation is taken in the space of field configurations subject to some boundary conditions $\Phi\big|_{\partial X} \sim \phi_{(0)}$, which we have yet to specify. Taking a smooth, compactly supported variation $\delta\Phi$, we find, regardless of the boundary conditions,

$$\delta S = -\int_X \mathrm{d}^3x\,\sqrt{|G|}\Big(\Box\Phi + m^2\Phi\Big)\delta\Phi. \tag{4.3}$$

where $\Box = -\nabla^2 = -G^{ab}\nabla_a\nabla_b$. This yields the Klein–Gordon equation

$$(\Box + m^2)\Phi = 0. \tag{4.4}$$

Using the Voss–Weyl formula

$$\nabla^2\Phi = \frac{1}{\sqrt{|G|}}\partial_\mu\Big(\sqrt{|G|}G^{\mu\nu}\partial_\nu\Phi\Big), \tag{4.5}$$

we can express this equation explicitly for any metric. The eventual aim is to analyse the near-boundary BTZ metric (2.16), but it is highly instructive to first consider AdS space. We shall see that the important conclusions are identical for both spaces.

4.1 Series solution of the wave equation

The Klein–Gordon equation in AdS_3, with metric (2.4), takes the form

$$-\frac{z}{\ell^2}\partial_z\Phi + \frac{z^2}{\ell^2}\partial_z^2\Phi + \frac{z^2}{\ell^2}\Box_0\Phi - m^2\Phi = 0 \tag{4.6}$$

[3]Unlike Skenderis, we work in Minkowski signature, and in the z coordinate rather than the Fefferman–Graham coordinate $\rho = z^2$.

[4]The lower part, $m^2\ell^2 > -1$, is the *Breitenlohner–Freedman bound* [9; 6, box 5.3] for $d = 2$. It cannot be violated in a consistent field theory.

where $\Box_0 = -\eta_{ij}\partial_i\partial_j = -\partial_t^2 + \partial_x^2$. Viewed as an ODE in z (properly, after a Fourier transform $\Box_0 \to -\mathbf{k}^2$), it has a regular singular point at $z = 0$. We follow the standard procedure (see e.g. [10, §5.6]) and seek series solutions of the form

$$\Phi(z, \mathbf{x}) = \sum_{n=0}^{\infty} z^{r+n}\Phi_{(r+n)}(\mathbf{x}). \tag{4.7}$$

Equation (4.6) becomes

$$\sum_{n=0}^{\infty} z^{r+n}\Big\{\big[(r+n)(r+n-2) - m^2\ell^2\big]\Phi_{(r+n)} + \Box_0\Phi_{(r+n-2)}\Big\} = 0 \tag{4.8}$$

(the second term taken as 0 when $n < 2$). The admissible values of the exponent r are given by the indicial equation (the $n = 0$ term):

$$r(r-2) - m^2\ell^2 = 0. \tag{4.9}$$

Its solutions, by convention called $\Delta^+ = \Delta$ and $\Delta^- = 2 - \Delta$ (in general $d - \Delta$), are

$$\Delta^\pm = 1 \pm \sqrt{1 + m^2\ell^2}. \tag{4.10}$$

The previously chosen bounds (4.2) for m^2 thus correspond to $1 < \Delta < 2$. Here, it is beginning to become clear why this is the simplest case: If Δ were an integer, the two roots would differ by an integer, necessitating a logarithmic term in the expansion.

Because (4.9) is fulfilled, (4.8) becomes

$$n(2\Delta^\pm - 2 + n)\Phi_{(\Delta^\pm+n)} + \Box_0\Phi_{(\Delta^\pm+n-2)} = 0. \tag{4.11}$$

The equation thus couples terms together only when their exponents differ by 2. Furthermore, for n nonzero and even, the factor $n(2\Delta^\pm - 2 + n)$ is never zero (as $\Delta^\pm$ is not an integer), so the two coefficients with $n = 0$, namely $\Phi_{(\Delta^\pm)}(\mathbf{x})$, recursively determine the coefficients with even n. For odd n, the factor is generally nonzero, and (4.11) says that $\Phi_{(\Delta^\pm+n)} = 0$ for odd n. The exception is when $n = 1$ and $\Delta = 3/2$, in which case $2\Delta^- - 2 + 1 = 0$. Then, the two roots $\Delta^\pm$ differ by 1, and $\Phi_{(\Delta^-+n)}$ for odd n coincide with $\Phi_{(\Delta^++n)}$ for even n. In all cases, however, there are two independent coefficients $\Phi_{(\Delta^\pm)}$ which, when given, fix all remaining coefficients $\Phi_{(\Delta^\pm+2n)}$. In conclusion, the complete series solution is

$$\begin{aligned}\Phi(z, \mathbf{x}) &= \sum_{n=0}^{\infty} z^{\Delta^-+2n}\Phi_{(\Delta^-+2n)}(\mathbf{x}) + \sum_{n=0}^{\infty} z^{\Delta^++2n}\Phi_{(\Delta^++2n)}(\mathbf{x}) \\ &= z^{\Delta^-}\Phi_{(\Delta^-)} + z^{\Delta^+}\Phi_{(\Delta^+)} + z^{\Delta^-+2}\Phi_{(\Delta^-+2)} + z^{\Delta^++2}\Phi_{(\Delta^++2)} + \cdots \\ &= z^{2-\Delta}\Phi_{(2-\Delta)} + z^{\Delta}\Phi_{(\Delta)} + z^{4-\Delta}\Phi_{(4-\Delta)} + z^{\Delta+2}\Phi_{(\Delta+2)} + \cdots\end{aligned} \tag{4.12}$$

where the last two expressions are written in order of increasing powers of z, and the functions $\Phi_{(2-\Delta+2n)}(\mathbf{x})$ and $\Phi_{(\Delta+2n)}(\mathbf{x})$ are all determined recursively by $\Phi_{(2-\Delta)}(\mathbf{x})$ and $\Phi_{(\Delta)}(\mathbf{x})$ via (4.11); explicitly

$$\Phi_{(\Delta^\pm+2n)} = \frac{1}{2n(2 - 2\Delta^\pm - 2n)}\Box_0\Phi_{(\Delta^\pm+2n-2)}. \tag{4.13}$$

4.2 Boundary conditions

We are now in a position to precisely define the boundary conditions that appeared in the statement of the AdS/CFT correspondence in section 3. The near-boundary expansion that we have just derived contains two undetermined functions $\Phi_{(2-\Delta)}(\mathbf{x})$ and $\Phi_{(\Delta)}(\mathbf{x})$ which, from the point of view of the boundary, are unrelated to each other. Therefore, it might seem natural to consider them both boundary conditions that need to be imposed. However, the two are generally related to each other through regularity conditions in the interior of the spacetime, far away from the boundary. Therefore, specifying $\Phi_{(2-\Delta)}$ also determines $\Phi_{(\Delta)}$, or at least restricts it to a limited set of possibilities. Indeed, $\Phi_{(\Delta)}$ is not part of the proper boundary conditions, and the source term $\phi_{(0)}(\mathbf{x})$ mentioned in section 3 is simply $\Phi_{(2-\Delta)}$ (more generally $\Phi_{(d-\Delta)}$):

$$\Phi\Big|_{\partial X} \sim \phi_{(0)} \quad \text{means} \quad \Phi_{(2-\Delta)}(\mathbf{x}) = \phi_{(0)}(\mathbf{x}) \quad \text{or} \quad \lim_{z\to 0} z^{\Delta-2}\Phi(z,\mathbf{x}) = \phi_{(0)}(\mathbf{x}). \tag{4.14}$$

The ambiguity in determining $\Phi_{(\Delta)}$ reflects the fact that the boundary CFT can be in different quantum states.

4.3 Regularising the action

Evaluating the action (4.1) on the solution just obtained leads to a divergence in the $z \to 0$ part of the integral. To parameterise the divergence, we construct a regularised action by limiting the domain of integration to $z \geq \epsilon$, for some $\epsilon > 0$:

$$S_{\text{reg}}[\Phi;\epsilon] = -\int_{z\geq\epsilon} \mathrm{d}z\,\mathrm{d}^2x\,\sqrt{|G|}\,\frac{1}{2}\Big(\nabla_a\Phi\nabla^a\Phi + m^2\Phi^2\Big). \tag{4.15}$$

Thus, we recover the original action by taking $\epsilon \to 0$. From now on, we assume that Φ satisfies the equation of motion (4.4) so that (4.12) and (4.13) hold (that is, we work "on-shell"). The action then reduces to a boundary term through partial integration:

$$\begin{aligned} S_{\text{reg}}[\Phi;\epsilon] &= -\int_{z\geq\epsilon} \mathrm{d}z\,\mathrm{d}^2x\,\sqrt{|G|}\,\frac{1}{2}\Phi\underbrace{\Big(-\nabla_a\nabla^a + m^2\Big)\Phi}_{0} - \int_{z=\epsilon}\mathrm{d}^2x\,\sqrt{|\gamma|}\,\frac{1}{2}n_a\Phi\nabla^a\Phi \\ &= -\int_{z=\epsilon}\mathrm{d}^2x\,\sqrt{|\gamma|}\,\frac{1}{2}\Phi\, n^\mu\partial_\mu\Phi \\ &= \int_{z=\epsilon}\mathrm{d}^2x\,\sqrt{|\gamma|}\sqrt{G^{zz}}\,\frac{1}{2}\Phi\partial_z\Phi. \end{aligned} \tag{4.16}$$

Here, γ_{ab} is the induced metric on the boundary ($\gamma_{\mu\nu} = G_{\mu\nu}$ with $\mu,\nu \in \{t,x\}$) and n^a is the outward-pointing unit normal, namely $n^\mu = -\sqrt{G^{zz}}\,\delta^{\mu z}$ because the metric is diagonal (this will also hold in BTZ space). Explicitly for AdS_3,

$$S_{\text{reg}}[\Phi;\epsilon] = \int_{z=\epsilon}\mathrm{d}^2x\,\frac{\ell}{2z}\Phi\partial_z\Phi. \tag{4.17}$$

Now, substitute the series solution (4.12) into the integrand:

$$\begin{aligned} \frac{\ell}{2z^2}\Phi z\partial_z\Phi\Big|_{z=\epsilon} &= \frac{\ell}{2\epsilon^2}\Big(\epsilon^{2-\Delta}\Phi_{(2-\Delta)} + \epsilon^{\Delta}\Phi_{(\Delta)} + O\big(\epsilon^{4-\Delta}\big)\Big) \\ &\qquad \Big((2-\Delta)\epsilon^{2-\Delta}\Phi_{(2-\Delta)} + \Delta\epsilon^{\Delta}\Phi_{(\Delta)} + O\big(\epsilon^{4-\Delta}\big)\Big) \\ &= \frac{\ell}{2}\Big((2-\Delta)\epsilon^{-2(\Delta-1)}\Phi^2_{(\Delta-2)} + 2\Phi_{(2-\Delta)}\Phi_{(\Delta)} \\ &\qquad + O\big(\epsilon^{2(\Delta-1)}\big) + O\big(\epsilon^{2(2-\Delta)}\big)\Big). \end{aligned} \tag{4.18}$$

As $\epsilon \to 0$, the first term diverges, the second term remains finite and all other terms vanish. Here, we see that $1 < \Delta < 2$ is indeed the simplest case; if $\Delta > 2$, some $O(\epsilon^{2(2-\Delta)})$ terms would also diverge. In the case $\Delta = 2, 3, 4\ldots$, there would in addition be logarithmically divergent terms.

4.4 Renormalisation with counterterms

The problem that renormalisation solves is that observables, specifically correlation functions, diverge in the naive theory. A renormalisation scheme must subtract the divergences of correlation functions in a consistent way. The simplest method to ensure consistency is to add counterterms to the action such that it is finite. The correlation functions—functional derivatives of the action, as in (3.7)—are then automatically finite. We need therefore only subtract a term that diverges as the first term of (4.18).

There are many possible counterterms that diverge in the required way. When choosing between counterterms, we prefer the least invasive ones—those that preserve important features of the naive theory. In particular, to preserve the bulk equations of motion, we consider only counterterms that depend on the values of Φ at the boundary $z = \epsilon$. Furthermore, in the interest of preserving as much symmetry as possible, we will choose only counterterms that are diffeomorphism invariant.[5] This is ensured if it is the integral of a scalar function times the invariant measure $\mathrm{d}^2x\,\sqrt{|\gamma|}$, where γ_{ab} is the induced metric on $z = \epsilon$. The simplest counterterm that fulfils these requirements is

$$S_{\mathrm{ct}}[\Phi;\epsilon] = \int_{z=\epsilon} \mathrm{d}^2x\,\sqrt{|\gamma|}\frac{\Delta-2}{2\ell}\Phi^2. \tag{4.19}$$

The integrand expands as

$$\begin{aligned}\sqrt{|\gamma|}\frac{\Delta-2}{2\ell}\Phi^2\bigg|_{z=\epsilon} &= \frac{\ell^2}{\epsilon^2}\frac{\Delta-2}{2\ell}\Big(\epsilon^{2-\Delta}\Phi_{(2-\Delta)} + \epsilon^{\Delta}\Phi_{(\Delta)} + O\big(\epsilon^{4-\Delta}\big)\Big)^2 \\ &= \frac{\ell(\Delta-2)}{2}\Big(\epsilon^{2(1-\Delta)}\Phi_{(2-\Delta)}^2 + 2\Phi_{(2-\Delta)}\Phi_{(\Delta)} \\ &\qquad + O\big(\epsilon^{2(\Delta-1)}\big) + O\big(\epsilon^{2(2-\Delta)}\big)\Big).\end{aligned} \tag{4.20}$$

The first term is indeed the negative of the divergent term in (4.18). Thus, the subtracted action

$$\begin{aligned}S_{\mathrm{sub}}[\Phi;\epsilon] = S_{\mathrm{reg}} + S_{\mathrm{ct}} = &-\int_{z\geq\epsilon} \mathrm{d}z\,\mathrm{d}^2x\,\sqrt{|G|}\frac{1}{2}\big(\nabla_a\Phi\nabla^a\Phi + m^2\Phi^2\big) \\ &+ \int_{z=\epsilon} \mathrm{d}^2x\,\sqrt{|\gamma|}\frac{\Delta-2}{2\ell}\Phi^2\end{aligned} \tag{4.21}$$

is finite; indeed, combining (4.18) and (4.20) yields

$$S_{\mathrm{sub}} = \int \mathrm{d}^2x\,\ell(\Delta-1)\Big(\Phi_{(2-\Delta)}\Phi_{(\Delta)} + O\big(\epsilon^{2(\Delta-1)}\big) + O\big(\epsilon^{2(2-\Delta)}\big)\Big). \tag{4.22}$$

The final, renormalised on-shell action is defined by taking $\epsilon \to 0$:

$$S_{\mathrm{ren}}[\Phi] = \lim_{\epsilon\to 0} S_{\mathrm{sub}}[\Phi;\epsilon] = \int \mathrm{d}^2x\,\ell(\Delta-1)\Phi_{(2-\Delta)}\Phi_{(\Delta)}. \tag{4.23}$$

[5] Invariant under diffeomorphisms of the bulk space X, or equivalently invariant under the induced diffeomorphisms on the boundary $z = \epsilon$.

4.5 The one-point function

To compute the one-point function (3.7), we must take the functional derivative of (4.23) with respect to $\phi_{(0)} = \Phi_{(2-\Delta)}$, which is complicated by the fact that we lack knowledge of the functional derivative $\frac{\delta\Phi_{(\Delta)}(\mathrm{y})}{\delta\Phi_{(2-\Delta)}(\mathrm{x})}$. The standard approach—and perhaps the more edifying one—is to go back to the covariant expression (4.21) and carefully take the variation using the chain rule $\frac{\delta}{\delta\phi_{(0)}(\mathrm{x})} = \int \mathrm{d}z\, \mathrm{d}^2y\, \frac{\delta\Phi(z,\mathrm{y})}{\delta\phi_{(0)}(\mathrm{x})}\frac{\delta}{\delta\Phi(z,\mathrm{y})}$. Here, however, we show how to circumvent the issue using a trick.

First, note that varying $\phi_{(0)}$ entails varying Φ while by definition keeping it within the space of on-shell field configurations. Both Φ and $\Phi+\delta\Phi$ thus satisfy the equation of motion (4.4). Since this equation is linear, $\delta\Phi$ also solves it:

$$(-\nabla^2 + m^2)\delta\Phi = 0. \tag{4.24}$$

Therefore, performing two partial integrations, recalling $n^\mu = -\sqrt{G^{zz}}\delta^{\mu z} = -\frac{z}{\ell}\delta^{\mu z}$, we have the identity

$$\begin{aligned}
&-\int_{z=\epsilon} \mathrm{d}^2x\, \sqrt{|\gamma|}\Phi\frac{z}{\ell}\partial_z\delta\Phi \\
&= -\int_{z=\epsilon} \mathrm{d}^2x\, \sqrt{|\gamma|}\Phi\frac{z}{\ell}\partial_z\delta\Phi - \int_{z\geq\epsilon} \mathrm{d}z\, \mathrm{d}^2x\, \sqrt{|G|}\Phi\overbrace{\left(-\nabla^2\delta\Phi + m^2\delta\Phi\right)}^{0} \\
&= -\int_{z\geq\epsilon} \mathrm{d}z\, \mathrm{d}^2x\, \sqrt{|G|}\left(\nabla_\mu\Phi\nabla^\mu\delta\Phi + m^2\Phi\delta\Phi\right) \\
&= -\int_{z=\epsilon} \mathrm{d}^2x\, \sqrt{|\gamma|}\delta\Phi\frac{z}{\ell}\partial_z\Phi - \int_{z\geq\epsilon} \mathrm{d}z\, \mathrm{d}^2x\, \sqrt{|G|}\delta\Phi\overbrace{\left(-\nabla^2\Phi + m^2\Phi\right)}^{0} \\
&= -\int_{z=\epsilon} \mathrm{d}^2x\, \sqrt{|\gamma|}\delta\Phi\frac{z}{\ell}\partial_z\Phi.
\end{aligned} \tag{4.25}$$

Expanding Φ and $\delta\Phi$ on both sides, we find

$$\begin{aligned}
&\int_{z=\epsilon} \mathrm{d}^2x\, \frac{\ell}{z^2}\left(z^{2-\Delta}\Phi_{(2-\Delta)} + z^\Delta\Phi_{(\Delta)} + \cdots\right)\left((2-\Delta)z^{2-\Delta}\delta\Phi_{(2-\Delta)} + \Delta z^\Delta\delta\Phi_{(\Delta)} + \cdots\right) \\
=&\int_{z=\epsilon} \mathrm{d}^2x\, \frac{\ell}{z^2}\left(z^{2-\Delta}\delta\Phi_{(2-\Delta)} + z^\Delta\delta\Phi_{(\Delta)} + \cdots\right)\left((2-\Delta)z^{2-\Delta}\Phi_{(2-\Delta)} + \Delta z^\Delta\Phi_{(\Delta)} + \cdots\right).
\end{aligned} \tag{4.26}$$

The leading-order terms cancel, and, setting $\epsilon = 0$ we obtain

$$\begin{aligned}
&\int \mathrm{d}^2x\, \ell\left(\Delta\Phi_{(2-\Delta)}\delta\Phi_{(\Delta)} + (2-\Delta)\Phi_{(\Delta)}\delta\Phi_{(2-\Delta)}\right) \\
=&\int \mathrm{d}^2x\, \ell\left(\Delta\delta\Phi_{(2-\Delta)}\Phi_{(\Delta)} + (2-\Delta)\delta\Phi_{(\Delta)}\Phi_{(2-\Delta)}\right).
\end{aligned} \tag{4.27}$$

Collecting like terms and cancelling the common factor $2\ell(1-\Delta)$ shows that

$$\int \mathrm{d}^2x\, \Phi_{(2-\Delta)}\delta\Phi_{(\Delta)} = \int \mathrm{d}^2x\, \Phi_{(\Delta)}\delta\Phi_{(2-\Delta)}. \tag{4.28}$$

With this result in hand, we find the variation of (4.23):

$$\begin{aligned}
\delta S_{\mathrm{ren}}[\Phi] &= \int \mathrm{d}^2x\, \ell(\Delta-1)\left(\Phi_{(\Delta)}\delta\Phi_{(2-\Delta)} + \Phi_{(2-\Delta)}\delta\Phi_{(\Delta)}\right) \\
&\overset{(4.28)}{=} \int \mathrm{d}^2x\, 2\ell(\Delta-1)\Phi_{(\Delta)}\delta\Phi_{(2-\Delta)}
\end{aligned}$$

and finally the one-point function

$$\langle O(\mathbf{x})\rangle = -\frac{\delta S_{\text{ren}}}{\delta\Phi_{(2-\Delta)}(\mathbf{x})} = -2\ell(\Delta-1)\Phi_{(\Delta)}(\mathbf{x}). \tag{4.29}$$

The above result highlights a general feature of holographic duality: Of the two undetermined coefficients in the expansion (4.12), one is the source for a boundary field theory operator $O(\mathbf{x})$, while the other is closely related to the VEV of the same operator. These are the coefficients of $z^{2-\Delta}$ (in general $z^{d-\Delta}$) and z^{Δ}, respectively. This is the most important conclusion of this chapter.

4.6 Generalisation to the BTZ spacetime

We shall now argue that the essential conclusions of holographic renormalisation are identical in the BTZ spacetime. For the near-boundary BTZ metric (2.16), the Klein–Gordon equation takes the form

$$-\frac{z}{\ell^2}\frac{1+3\left(\frac{\rho_h z}{2\ell^2}\right)^4}{1-\left(\frac{\rho_h z}{2\ell^2}\right)^4}\partial_z\Phi + \frac{z^2}{\ell^2}\left[\partial_z^2\Phi - \frac{1}{\left(1-\frac{\rho_h^2 z^2}{4\ell^4}\right)^2}\partial_t^2\Phi + \frac{1}{\left(1+\frac{\rho_h^2 z^2}{4\ell^4}\right)^2}\frac{1}{\ell^2}\partial_\phi^2\Phi\right] - m^2\Phi = 0. \tag{4.30}$$

This differs from the AdS equation (4.6), but only in higher powers of z. In particular, the indicial equation (4.9) is the same, and since the differential equation also couples terms together whose powers differ by an even number, the expansion (4.12) is the same, at least to the lowest orders. The regularised action is

$$S_{\text{reg}}[\Phi;\epsilon] = \int_{z=\epsilon} \mathrm{d}^2x\,\sqrt{|\gamma|}\sqrt{G^{zz}}\,\frac{1}{2}\Phi\partial_z\Phi = \int_{z=\epsilon}\mathrm{d}^2x\,\left(1+O\left(z^4\right)\right)\frac{\ell}{2z^2}\Phi z\partial_z\Phi \tag{4.31}$$

which has the same divergent and finite parts as in AdS space. The same holds for the counterterm (4.19), so the renormalised expressions for the action (4.23) and one-point functions (4.29) are exactly the same.

5 Global geometry

5.1 Rindler spacetime

We would like to construct the maximal analytic extension of the BTZ spacetime. As a warm-up, we first construct the extension of a simpler spacetime, *Rindler space*, following the steps from Wald [11, chap. 6.4]. Rindler space is given by the metric

$$\mathrm{d}s^2 = -x^2\,\mathrm{d}t^2 + \mathrm{d}x^2 \qquad \begin{array}{l} t \in \mathbb{R} \\ 0 < x < \infty. \end{array} \tag{5.1}$$

The metric is singular as $x \to 0$, but this is in fact just a coordinate singularity. To show this, we construct new coordinates by considering null geodesics. A null geodesic must fulfil $-x^2\,\mathrm{d}t^2 + \mathrm{d}x^2 = 0$, meaning $\mathrm{d}t = \pm\,\mathrm{d}x\,/x$. In terms of the "tortoise coordinate" $r_* = \ln x$ this becomes $\mathrm{d}t = \pm\,\mathrm{d}r_*$ or

$$t \mp r_* = \text{const.} \tag{5.2}$$

Calling $u = t - r_*$ and $v = t + r_*$, we find that there are two classes of null geodesics, given by $u = \text{const}$ and $v = \text{const}$ respectively. In terms of the new coordinates, the metric is

$$\mathrm{d}s^2 = -\mathrm{e}^{v-u}\,\mathrm{d}u\,\mathrm{d}v \qquad \begin{array}{l} u \in \mathbb{R} \\ v \in \mathbb{R}. \end{array} \tag{5.3}$$

Next, we define the *Kruskal coordinates* $U = -\mathrm{e}^{-u}$ and $V = \mathrm{e}^{v}$ (Wald motivates this choice by showing that they are affine parameters of the null geodesics with $v = \text{const}$ and $u = \text{const}$, respectively). The metric becomes

$$\mathrm{d}s^2 = -\,\mathrm{d}U\,\mathrm{d}V \qquad \begin{array}{l} -\infty < U < 0 \\ 0 < V < \infty. \end{array} \tag{5.4}$$

There is now no obstacle to extending the coordinate range to $U \in \mathbb{R}$ and $V \in \mathbb{R}$. This extended spacetime is simply Minkowski space; we can see this by defining new coordinates T and X by $U = T - X$ and $V = T + X$, giving

$$\mathrm{d}s^2 = -\,\mathrm{d}T^2 + \mathrm{d}X^2 \qquad \begin{array}{l} T \in \mathbb{R} \\ X \in \mathbb{R}. \end{array} \tag{5.5}$$

The results are summarised in fig. 1. The original coordinates just cover the right wedge—region I—while (U, V) and (T, X) cover the whole spacetime. In I, the relationship between (t, x) and (U, V) is

$$x = \sqrt{-UV} \qquad t = \frac{1}{2}\ln\left(\frac{V}{-U}\right). \tag{5.6}$$

These equations may, however, be generalised to define coordinates (x, t) in the entire spacetime, apart from at $UV = 0$:

$$x = \sqrt{|UV|} \qquad t = \frac{1}{2}\ln\left|\frac{V}{U}\right| \tag{5.7}$$

with inverse

$$|U| = \mathrm{e}^{-t+\ln x} \qquad |V| = \mathrm{e}^{t+\ln x}. \tag{5.8}$$

The metric then takes the form

$$\mathrm{d}s^2 = \pm(-x^2\,\mathrm{d}t^2 + \mathrm{d}x^2) \qquad \begin{array}{l} + \text{ in I and IV} \\ - \text{ in II and III.} \end{array} \tag{5.9}$$

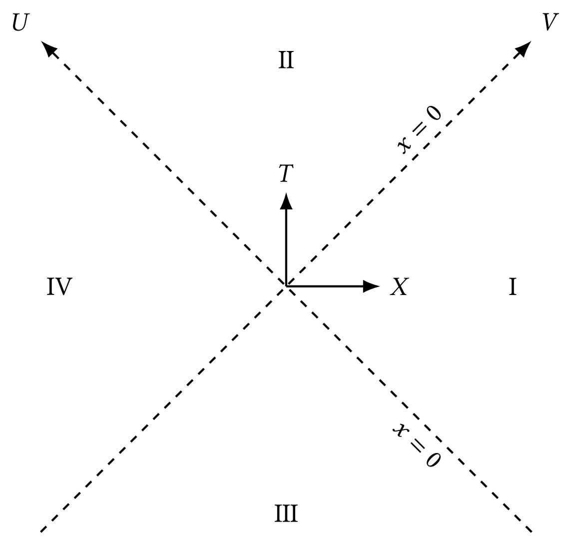

Figure 1: Spacetime diagram of the maximal analytic extension of Rindler space in global coordinates (U, V).

5.2 BTZ spacetime

We now turn to the BTZ spacetime. Consider again the metric (1.1),

$$\mathrm{d}s^2 = -\frac{\rho^2 - \rho_h^2}{\ell^2}\,\mathrm{d}t^2 + \frac{\ell^2}{\rho^2 - \rho_h^2}\,\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2 \qquad \begin{array}{l} t \in \mathbb{R} \\ \rho_h < \rho < \infty \\ 0 \le \phi < 2\pi. \end{array} \tag{5.10}$$

In analogy with the above, we seek to define new coordinates related to null geodesics with constant ϕ. Such a geodesic must fulfil

$$\frac{\ell^2}{\rho^2 - \rho_h^2}\,\mathrm{d}\rho = \pm\,\mathrm{d}t\,. \tag{5.11}$$

We define a new coordinate r_* by $\mathrm{d}r_* = \frac{\ell^2}{\rho^2 - \rho_h^2}\,\mathrm{d}\rho$, and solve the differential equation to find

$$r_* = -\frac{\ell^2}{\rho_h}\,\mathrm{arccoth}\Big(\frac{\rho}{\rho_h}\Big) \iff \rho = -\rho_h \coth\Big(\frac{\rho_h r_*}{\ell^2}\Big). \tag{5.12}$$

We have chosen a constant of integration to be zero, such that $r_* \to 0$ corresponds to $\rho \to \infty$. The metric takes the form

$$\mathrm{d}s^2 = \frac{\rho_h^2}{\ell^2 \sinh^2\Big(\frac{\rho_h r_*}{\ell^2}\Big)}\Big[-\mathrm{d}t^2 + \mathrm{d}r_*^2 + \cosh^2\Big(\frac{\rho_h r_*}{\ell^2}\Big)\,\ell^2\,\mathrm{d}\phi^2\Big]. \tag{5.13}$$

Null geodesics with constant ϕ are then given by $\mathrm{d}t = \pm\,\mathrm{d}r_*$, that is, $u = \text{const}$ or $v = \text{const}$ where $u = t - r_*$ and $v = t + r_*$. In terms of u and v, the metric is

$$\mathrm{d}s^2 = \frac{\rho_h^2}{\ell^2 \sinh^2\big(\frac{\rho_h(v-u)}{2\ell^2}\big)}\Big[-\mathrm{d}u\,\mathrm{d}v + \cosh^2\Big(\frac{\rho_h(v-u)}{2\ell^2}\Big)\,\ell^2\,\mathrm{d}\phi^2\Big]. \tag{5.14}$$

Like for Rindler space, we now compactify the coordinate range by introducing the Kruskal coordinates $U = -\mathrm{e}^{-\frac{\rho_h u}{\ell^2}}$ and $V = \mathrm{e}^{\frac{\rho_h v}{\ell^2}}$. (Unlike for Rindler space, these are not affine parameters of the corresponding null geodesics. They are nevertheless simple coordinates that can be extended past the original horizon.) The coordinate range is $U < 0,\ V > 0,\ -UV < 1$, but can be extended to $U, V \in \mathbb{R},\ |UV| < 1$. The final metric for the extended BTZ spacetime becomes

$$\mathrm{d}s^2 = \frac{4\ell^2}{(1+UV)^2}\Big(-\,\mathrm{d}U\,\mathrm{d}V + \frac{\rho_h^2}{4\ell^2}(1-UV)^2\,\mathrm{d}\phi^2\Big) \qquad \begin{array}{l} U, V \in \mathbb{R} \\ |UV| < 1 \\ 0 \leq \phi < 2\pi. \end{array} \tag{5.15}$$

The coordinates t and ρ are at present only defined in the original region, in which they are related to U and V by

$$\frac{\rho}{\rho_h} = \frac{1-UV}{1+UV} \qquad t = \frac{\ell^2}{2\rho_h}\ln\left(\frac{V}{-U}\right). \tag{5.16}$$

However, after a slight generalisation, these equations define coordinates (t, ρ) in the full extended spacetime, apart from at $UV = 0$ (which would give $\rho = \rho_h$)[6]:

$$\frac{\rho}{\rho_h} = \frac{1-UV}{1+UV} \qquad t = \frac{\ell^2}{2\rho_h}\ln\left|\frac{V}{U}\right|. \tag{5.17}$$

Using them to rewrite the metric (5.15), we find that it takes the form of the standard metric (5.10), even in regions with $\rho < \rho_h$.

In summary, we have arrived at the spacetime shown in fig. 2. We started with a metric describing only region I ($U < 0$, $V > 0$), the exterior of the black hole with $\rho > \rho_h$. The spacetime does, however, extend through the event horizon at $\rho = \rho_h$ into regions with $\rho < \rho_h$, namely the black hole and white hole regions, II ($U, V > 0$) and III ($U, V < 0$), both of which contain true singularities as $\rho \to 0$. In these regions, examining the metric (5.10) shows that t is spacelike while ρ is timelike. Furthermore, there is an additional exterior region IV, where $\rho > \rho_h$. We may call this a parallel universe, and say that the BTZ spacetime describes a wormhole between the right and left exterior regions I and IV. However, as no timelike or lightlike geodesic can pass between them, the wormhole is not traversable. This is all closely analogous to the case of the Kruskal extension of the Schwarzschild spacetime.

[6] It is also possible to take (5.16) at face value in all regions, choosing a branch of the complex logarithm function. We will work with (5.17) in order to keep both coordinates real.

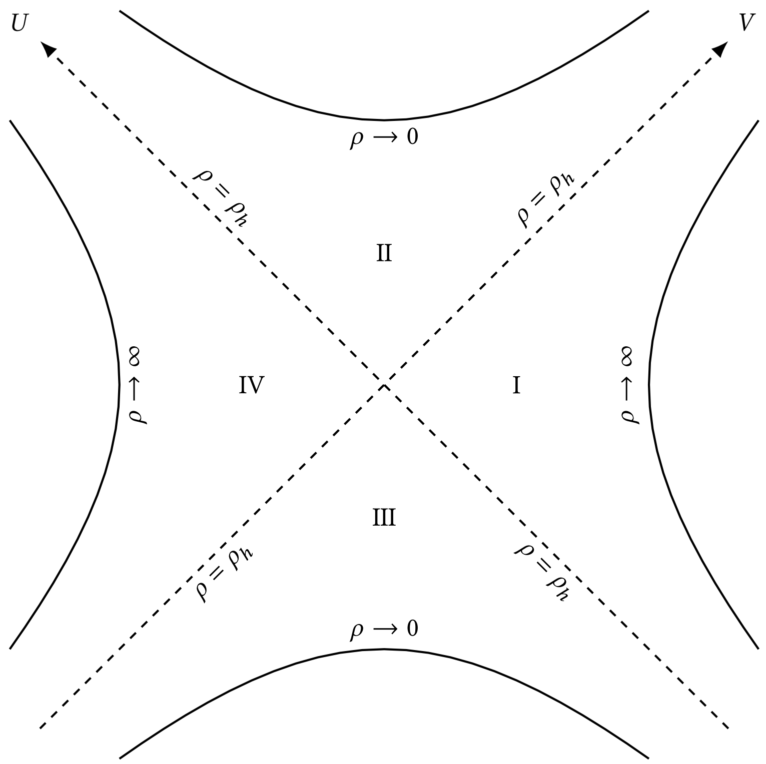

Figure 2: Spacetime diagram of the maximal analytic extension of BTZ space in global coordinates (U, V), with right exterior (I), future interior (II), past interior (III) and left exterior (IV) regions. The ϕ coordinate is suppressed, and corresponds to a circle at each point of the diagram.

6 Behaviour of a scalar field across the horizon

The holographic dual CFT is defined in terms of the behaviour of the bulk theory near the boundary. But the relationship between the source and the VEV is determined by the global properties of the bulk spacetime. We must therefore examine how the Klein–Gordon solutions in one region are related to those in other regions. Like when constructing the extended spacetime, it is instructive to first consider the analogous problem in the Rindler spacetime. Boulware [5] has shown how to solve this, and we begin by retracing his steps.[7]

6.1 Rindler spacetime

We start from the standard Rindler metric (5.9) and write out the Klein–Gordon equation $(\Box + m^2)\Phi = 0$, like in the beginning of section 4:

$$\left[\pm\left(\partial_t^2 - x^2\partial_x^2 - x\partial_x\right) + m^2x^2\right]\Phi = 0. \tag{6.1}$$

with the top sign applying in regions I and IV of fig. 1, and the bottom sign in regions II and III. Because the equation is t-translation invariant, it is convenient to express Φ as a Fourier transform in the t variable:

$$\Phi(x,t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega t}\Phi(x,\omega). \tag{6.2}$$

This transforms (6.1) into

$$\left[x^2\partial_x^2 + x\partial_x + (\omega^2 \mp x^2m^2)\right]\Phi = 0. \tag{6.3}$$

Switching variables to $z = mx$ and viewing ω^2 as $-(\mathrm{i}\omega)^2$, this becomes

$$\left[z^2\partial_z^2 + z\partial_z + (\mp z^2 - (\mathrm{i}\omega)^2)\right]\Phi = 0, \tag{6.4}$$

which is exactly Bessel's modified equation (with the top sign) or the standard Bessel equation (with the bottom sign). It has two linearly independent solutions

$$\begin{aligned} \Phi(x,\omega) &\propto I_{\pm\mathrm{i}\omega}(mx) \quad \text{in I and IV} \\ \Phi(x,\omega) &\propto J_{\pm\mathrm{i}\omega}(mx) \quad \text{in II and III} \end{aligned} \tag{6.5}$$

where $J_\nu(z)$ and $I_\nu(z)$ are the standard and modified Bessel functions, respectively (the case of $\omega = 0$ is technical and will not be dealt with here). Thus, any solution of (6.1) can be written

$$\Phi(x,t) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega t}\left(c_+^{\mathrm{I/IV}}(\omega)\, I_{\mathrm{i}\omega}(mx) + c_-^{\mathrm{I/IV}}(\omega)\, I_{-\mathrm{i}\omega}(mx)\right) & \text{in I/IV} \\ \displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega t}\left(c_+^{\mathrm{II/III}}(\omega)\, J_{\mathrm{i}\omega}(mx) + c_-^{\mathrm{II/III}}(\omega)\, J_{-\mathrm{i}\omega}(mx)\right) & \text{in II/III} \end{cases} \tag{6.6}$$

for some choice of $c_\pm^{\mathrm{I,II,III,IV}}(\omega)$.

Equation (6.6) describes the solution in each wedge of fig. 1, with apparently independent coefficients $c_\pm^{\mathrm{I,II,III,IV}}(\omega)$. However, solving the equation in global coordinates (the usual Klein–Gordon equation $(\partial_T^2 - \partial_X^2 + m^2)\Phi = 0$) would reveal that the

[7] Boulware's (τ, Z) and (t, z) are our (t, x) and $(T, X) = (\frac{U+V}{2}, \frac{V-U}{2})$.

wedges are not at all independent—the horizons are, after all, nothing but coordinate singularities. Therefore, the coefficients must be related to each other. To find this relationship, we express a solution close to the horizon in global coordinates (U, V), and see how it analytically continues into the region on the other side. We make use of the asymptotic forms [12, eqs. 10.7.3 and 10.30.1]

$$J_{\pm i\omega}(mx) \sim I_{\pm i\omega}(mx) \sim \left(\frac{mx}{2}\right)^{\pm i\omega} \Big/ \Gamma(1 \pm i\omega) \qquad \text{as } x \to 0. \tag{6.7}$$

In particular, starting with the solution in I or IV, we have

$$\begin{aligned} e^{-i\omega t} I_{\pm i\omega}(mx) &\overset{x\to 0}{\simeq} \frac{1}{\Gamma(1 \pm i\omega)} e^{-i\omega t}\left(\frac{mx}{2}\right)^{\pm i\omega} = \frac{(m/2)^{\pm i\omega}}{\Gamma(1 \pm i\omega)} e^{-i\omega(t \mp \ln x)} \\ &\overset{(5.8)}{=} \frac{(m/2)^{\pm i\omega}}{\Gamma(1 \pm i\omega)} |\{U, V\}|^{\pm i\omega} \end{aligned} \tag{6.8}$$

(where $\{U, V\}$ means U for the top sign and V for the bottom sign), and the same expansion holds for the solutions in II and III:

$$e^{-i\omega t} J_{\pm i\omega}(mx) \overset{x\to 0}{\simeq} \frac{(m/2)^{\pm i\omega}}{\Gamma(1 \pm i\omega)} |\{U, V\}|^{\pm i\omega}. \tag{6.9}$$

For example, the solution $e^{-i\omega t} I_{-i\omega}(mx)$ in region I is of the asymptotic form $\frac{(m/2)^{-i\omega}}{\Gamma(1-i\omega)} V^{-i\omega}$. This expression can be analytically continued across the $U = 0$ horizon into region II, and there it precisely agrees with the asymptotic form of $e^{-i\omega t} J_{-i\omega}(mx)$. We have thus found that $c_-^{\mathrm{II}}(\omega) = c_-^{\mathrm{I}}(\omega)$. The same solution cannot, however, be continued across the $V = 0$ horizon into region III, because $V^{-i\omega}$ is singular there, so we find no relationship between $c_-^{\mathrm{I}}(\omega)$ and $c_-^{\mathrm{III}}(\omega)$.[8]

The situation is the opposite for the other solution $e^{-i\omega t} I_{i\omega}(mx)$. It has the asymptotic form $\frac{(m/2)^{i\omega}}{\Gamma(1+i\omega)} (-U)^{i\omega}$, which continues without problem across the $V = 0$ horizon into region III, where it agrees with $e^{-i\omega t} J_{i\omega}(mx)$. This shows that $c_+^{\mathrm{III}}(\omega) = c_+^{\mathrm{I}}(\omega)$, but says nothing about $c_+^{\mathrm{II}}(\omega)$.

After going through the analogous steps for region IV, we summarise our conclusions as follows:

$$\begin{aligned} c_-^{\mathrm{I}}(\omega) &= c_-^{\mathrm{II}}(\omega) & \qquad c_+^{\mathrm{I}}(\omega) &= c_+^{\mathrm{III}}(\omega) \\ c_+^{\mathrm{IV}}(\omega) &= c_+^{\mathrm{II}}(\omega) & \qquad c_-^{\mathrm{IV}}(\omega) &= c_-^{\mathrm{III}}(\omega). \end{aligned} \tag{6.10}$$

6.2 BTZ spacetime

Like for Rindler space, we start from the standard BTZ metric (5.10) and write out the Klein–Gordon equation:

$$\left[\left(\rho^2 - \rho_h^2\right)\partial_\rho^2 + \frac{3\rho^2 - \rho_h^2}{\rho}\partial_\rho - \frac{\ell^4}{\rho^2 - \rho_h^2}\partial_t^2 + \frac{\ell^2}{\rho^2}\partial_\phi^2 - \ell^2 m^2\right]\Phi = 0. \tag{6.11}$$

After a Fourier transform

$$\Phi(\rho, t, \phi) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{k\in\mathbb{Z}} e^{-i\omega t + ik\phi} \Phi(\rho, \omega, k) \tag{6.12}$$

[8] One might first think that the solution not being analytic means that its coefficients must be zero, but the Riemann–Lebesgue lemma shows that a continuous wavepacket may have nonzero coefficients of oscillatory components like $V^{-i\omega}$, and that their contribution vanishes as the frequency approaches infinity [13, above eq. B.2].

it becomes

$$\left[(\rho^2-\rho_h^2)\partial_\rho^2+\frac{3\rho^2-\rho_h^2}{\rho}\partial_\rho+\frac{\ell^4}{\rho^2-\rho_h^2}\omega^2-\frac{\ell^2}{\rho^2}k^2-\ell^2m^2\right]\Phi(\rho,\omega,k)=0. \tag{6.13}$$

Two linearly independent solutions are (see appendix A)

$$\Phi_\pm(\rho,\omega,k)=\left(\frac{\rho^2}{\rho^2-\rho_h^2}\right)^a\left(\frac{\rho_h^2}{\rho^2-\rho_h^2}\right)^{\Delta^\pm/2}\mathbf{F}\left(\begin{matrix}\frac{\Delta^\pm}{2}+a+b,\ \frac{\Delta^\pm}{2}+a-b\\ \Delta^\pm\end{matrix};\frac{\rho_h^2}{\rho_h^2-\rho^2}\right). \tag{6.14}$$

where $\mathbf{F}\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};z\right)$ is the normalised hypergeometric function[9]

$$\mathbf{F}\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};z\right)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=0}^{\infty}\frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)s!}z^s \tag{6.15}$$

and

$$a=\frac{\mathrm{i}k\ell}{2\rho_h},\qquad b=\frac{\mathrm{i}\omega\ell^2}{2\rho_h},\qquad \Delta^+=\Delta,\qquad \Delta^-=2-\Delta \tag{6.16}$$

(here $\Delta(\Delta-2)=\ell^2m^2$ and $\Delta^+>\Delta^-$, just as in (4.10)). As $\rho\to\infty$, the solutions go as[10]

$$\Phi_\pm\overset{\rho\to\infty}{\sim}\rho^{-\Delta^\pm}\sim z^{\Delta^\pm} \tag{6.17}$$

(recall from section 2 that the defining function z for BTZ goes as $z\sim\ell^2/\rho$ close to the boundary). Thus, by our discussions in section 4, Φ_- corresponds to a boundary theory configuration with source but no VEV and Φ_+ to one with VEV but no source. To summarise, any solution of (6.11) can be written, locally in a region I/II/III/IV, as

$$\begin{aligned}\Phi(\rho,t,\phi)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\sum_{k\in\mathbb{Z}}&\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}k\phi}\\ &\left(c_+^{\mathrm{I/II/III/IV}}(\omega,k)\Phi_+(\rho,\omega,k)+c_-^{\mathrm{I/II/III/IV}}(\omega,k)\Phi_-(\rho,\omega,k)\right)\end{aligned} \tag{6.18}$$

for some choice of $c_\pm^{\mathrm{I,II,III,IV}}(\omega,k)$, with $\Phi_\pm(\rho,\omega,k)$ as in (6.14).

In the same way as in Rindler space, we must now relate the solutions in the different regions I–IV by converting to global coordinates and analytically continuing across the horizon. Consider a specific mode

$$\Phi_{\omega k\pm}(\rho,t,\phi)=\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}k\phi}\Phi_\pm(\rho,\omega,k) \tag{6.19}$$

(these are displayed in fig. 3). Close to the horizon, it behaves as

$$\Phi_{\omega k\pm}\overset{\rho\to\rho_h}{\sim}\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}k\phi}\left(\frac{\rho_h^2}{\rho^2-\rho_h^2}\right)^{\frac{\Delta^\pm}{2}+a}\mathbf{F}\left(\begin{matrix}\frac{\Delta^\pm}{2}+a+b,\ \frac{\Delta^\pm}{2}+a-b\\ \Delta^\pm\end{matrix};\frac{\rho_h^2}{\rho_h^2-\rho^2}\right). \tag{6.20}$$

[9] $\mathbf{F}\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};z\right)=\frac{1}{\Gamma(\gamma)}F\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};z\right)$, where F is the ordinary hypergeometric function ${}_2F_1$ [12, §15.1, §15.2]. We take the principal branch, which has a branch cut at $z\in\mathbb{R}, z\geq1$, on which by convention $\mathbf{F}\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};x\right)=\mathbf{F}\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};x-\mathrm{i}0\right)$.

[10] The hypergeometric function approaches a constant: $\mathbf{F}\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};0\right)=\frac{1}{\Gamma(\gamma)}$.

By the identity [12, eqs. 15.8.2 and 5.5.3]

$$\mathbf{F}\begin{pmatrix}\alpha,\ \beta \\ \gamma\end{pmatrix}; z\Bigg) = \frac{\Gamma(\beta-\alpha)\Gamma(\alpha-\beta+1)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-z)^{-\alpha}\mathbf{F}\begin{pmatrix}\alpha,\ \alpha-\gamma+1 \\ \alpha-\beta+1\end{pmatrix}; \frac{1}{z}\Bigg) + (\alpha \leftrightarrow \beta) \tag{6.21}$$

we may rewrite $\Phi_\pm$ as

$$\begin{aligned}
&\Phi_{\omega k\pm}(\rho, t, \phi) \\
&= \mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}k\phi}\left(\frac{\rho_h^2}{\rho^2-\rho_h^2}\right)^{a}\left(\frac{\rho^2}{\rho^2-\rho_h^2}\right)^{\frac{\Delta^\pm}{2}}\left(\frac{\rho_h^2}{\rho^2-\rho_h^2}\right)^{-\frac{\Delta^\pm}{2}-a-b} \\
&\qquad \frac{\Gamma(-2b)\Gamma(1+2b)}{\Gamma\left(\frac{\Delta^\pm}{2}+a-b\right)\Gamma\left(\frac{\Delta^\pm}{2}-a-b\right)}\mathbf{F}\left(\begin{matrix}\frac{\Delta^\pm}{2}+a+b,\ \frac{\Delta^\mp}{2}+a+b \\ 1+2b\end{matrix}; 1-\frac{\rho^2}{\rho_h^2}\right) \\
&\qquad + (b \rightarrow -b) \\
&= \mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}k\phi}\left(\frac{\rho^2}{\rho_h^2}\right)^{a}\left(\frac{\rho_h^2}{\rho_h^2-\rho^2}\right)^{-b} \\
&\qquad \frac{\Gamma(-2b)\Gamma(1+2b)}{\Gamma\left(\frac{\Delta^\pm}{2}+a-b\right)\Gamma\left(\frac{\Delta^\pm}{2}-a-b\right)}\mathbf{F}\left(\begin{matrix}\frac{\Delta^+}{2}+a+b,\ \frac{\Delta^-}{2}+a+b \\ 1+2b\end{matrix}; 1-\frac{\rho^2}{\rho_h^2}\right) \\
&\qquad + (b \rightarrow -b).
\end{aligned} \tag{6.22}$$

Close to the horizon at $\rho = \rho_h$, the hypergeometric functions in the above approach constants $\frac{1}{\Gamma(1\pm 2b)}$. The behaviour close to the horizon is thus determined by the factors $\mathrm{e}^{-\mathrm{i}\omega t}\left(\frac{\rho_h^2}{\rho^2-\rho_h^2}\right)^{\mp b}$. In order to express these in global coordinates, we now recall the coordinate transformations (5.17), which may be rewritten as

$$\frac{\rho_h^2}{\rho^2-\rho_h^2} = \frac{(1+UV)^2}{-4UV}, \qquad \mathrm{e}^{-\mathrm{i}\omega t} = \left|\frac{U}{V}\right|^b. \tag{6.23}$$

We find

$$\mathrm{e}^{-\mathrm{i}\omega t}\left(\frac{\rho_h^2}{\rho^2-\rho_h^2}\right)^{-b} = \left(\frac{1+UV}{2}\right)^{-2b}|U|^{2b}(-s)^{-b} \tag{6.24}$$

$$\mathrm{e}^{-\mathrm{i}\omega t}\left(\frac{\rho_h^2}{\rho^2-\rho_h^2}\right)^{b} = \left(\frac{1+UV}{2}\right)^{2b}|V|^{-2b}(-s)^{b} \tag{6.25}$$

where $s = \mathrm{sign}(UV) = \mathrm{sign}(\rho_h - \rho)$.[11] Thus, we see from (6.22), (6.24) and (6.25) that

$$\Phi_{\omega k\{U,V\}}(\rho, t, \phi) = \mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}k\phi}\left(\frac{\rho^2}{\rho_h^2}\right)^{a}\left(\frac{\rho_h^2}{\rho^2-\rho_h^2}\right)^{\mp b}\mathbf{F}\left(\begin{matrix}\frac{\Delta^+}{2}+a\pm b,\ \frac{\Delta^-}{2}+a\pm b \\ 1\pm 2b\end{matrix}; 1-\frac{\rho^2}{\rho_h^2}\right) \tag{6.26}$$

are solutions of the wave equation (6.11) with near-horizon behaviour

$$\Phi_{\omega kU} \overset{\rho\to\rho_h}{\sim} \frac{\mathrm{e}^{\mathrm{i}k\phi}}{\Gamma(1+2b)}|2U|^{2b}(-s)^{-b}, \qquad \Phi_{\omega kV} \overset{\rho\to\rho_h}{\sim} \frac{\mathrm{e}^{\mathrm{i}k\phi}}{\Gamma(1-2b)}|2V|^{-2b}(-s)^{b}, \tag{6.27}$$

[11] For the complex function $z^\alpha = \mathrm{e}^{\alpha \ln z}$, we take the principal branch with a branch cut at $z = -x < 0$, with $\ln(-x) = \ln(-x + \mathrm{i}0) = \ln(x) + \mathrm{i}\pi$ by convention. Then, $(-x)^\alpha = \mathrm{e}^{\mathrm{sign}(x)\mathrm{i}\pi\alpha}x^\alpha = (-1)^{\mathrm{sign}(x)\alpha}x^\alpha$.

and which are linear combinations of $\Phi_{\omega k\pm}$, given by

$$\Phi_{\omega k\pm} = A_{\omega k\pm}\Phi_{\omega kU} + B_{\omega k\pm}\Phi_{\omega kV} \tag{6.28}$$

where the coefficients are

$$A_{\omega k\pm} = \frac{\Gamma(-2b)\Gamma(1+2b)}{\Gamma\big(\frac{\Delta^\pm}{2}+a-b\big)\Gamma\big(\frac{\Delta^\pm}{2}-a-b\big)} \tag{6.29}$$

$$B_{\omega k\pm} = \frac{\Gamma(2b)\Gamma(1-2b)}{\Gamma\big(\frac{\Delta^\pm}{2}+a+b\big)\Gamma\big(\frac{\Delta^\pm}{2}-a+b\big)} \tag{6.30}$$

(note that $B_{\omega k\pm}$ is obtained from $A_{\omega k\pm}$ by replacing b with $-b$). Inverting (6.28), we find

$$\begin{aligned}\Phi_{\omega kU} &= \frac{B_{\omega k-}\Phi_{\omega k+} - B_{\omega k+}\Phi_{\omega k-}}{A_{\omega k+}B_{\omega k-} - A_{\omega k-}B_{\omega k+}}\\ \Phi_{\omega kV} &= \frac{-A_{\omega k-}\Phi_{\omega k+} + A_{\omega k+}\Phi_{\omega k-}}{A_{\omega k+}B_{\omega k-} - A_{\omega k-}B_{\omega k+}}.\end{aligned} \tag{6.31}$$

From (6.27), it is clear that $\Phi_{\omega kU}$ can be analytically continued across the $V=0$ horizon, and vice versa for $\Phi_{\omega kV}$. For example, starting in region I or IV, in which $s=-1$, the $\Phi_{\omega kU}$ solution goes as

$$\Phi^{\mathrm{I/IV}}_{\omega kU} \overset{\rho\to\rho_h}{\sim} \frac{\mathrm{e}^{\mathrm{i}k\phi}}{\Gamma(1+2b)}|2U|^{2b}. \tag{6.32}$$

This may be analytically continued across the $V=0$ horizon, into region III or II respectively. In this region, $s=+1$ and so

$$\Phi^{\mathrm{III/II}}_{\omega kU} \overset{\rho\to\rho_h}{\sim} \frac{\mathrm{e}^{\mathrm{i}k\phi}}{\Gamma(1+2b)}|2U|^{2b}(-1)^{-b}. \tag{6.33}$$

The analytic continuation of $\Phi^{\mathrm{I/IV}}_{\omega kU}$ thus differs from $\Phi^{\mathrm{III/II}}_{\omega kU}$ by a factor $(-1)^b = \mathrm{e}^{\mathrm{i}\pi b} = \mathrm{e}^{-\frac{\pi\omega\ell^2}{2\rho_h}}$. In other words, $\Phi^{\mathrm{I/IV}}_{\omega kU}$ analytically continues into $\mathrm{e}^{-\frac{\pi\omega\ell^2}{2\rho_h}}\Phi^{\mathrm{III/II}}_{\omega kU}$. After repeating this analysis for the other cases, the results can be summarised by stating that, upon analytic continuation from a region O across a horizon into the target region T,

$$\Phi^{\mathrm{O}}_{\omega kU} \quad \text{continues into} \quad \mathrm{e}^{-s^{\mathrm{T}}\frac{\pi\omega\ell^2}{2\rho_h}}\Phi^{\mathrm{T}}_{\omega kU} \tag{6.34}$$

$$\Phi^{\mathrm{O}}_{\omega kV} \quad \text{continues into} \quad \mathrm{e}^{s^{\mathrm{T}}\frac{\pi\omega\ell^2}{2\rho_h}}\Phi^{\mathrm{T}}_{\omega kV} \tag{6.35}$$

where s^{T} is the value of $s=\mathrm{sign}(UV)$ in T; that is, -1 in I and IV, and 1 in II and III. See fig. 4 for an example.

Referring back to (6.18), we may summarise the above as four constraints on the eight coefficients $c_\pm^{\mathrm{I,II,III,IV}}(\omega,k)$, much like in Rindler space:

$$\begin{aligned}
\mathrm{e}^{\frac{\pi\omega\ell^2}{2\rho_h}}\Big(B_{\omega k+}c_+^{\mathrm{I}}(\omega,k) + B_{\omega k-}c_-^{\mathrm{I}}(\omega,k)\Big) &= B_{\omega k+}c_+^{\mathrm{II}}(\omega,k) + B_{\omega k-}c_-^{\mathrm{II}}(\omega,k)\\
\mathrm{e}^{-\frac{\pi\omega\ell^2}{2\rho_h}}\Big(A_{\omega k+}c_+^{\mathrm{I}}(\omega,k) + A_{\omega k-}c_-^{\mathrm{I}}(\omega,k)\Big) &= A_{\omega k+}c_+^{\mathrm{III}}(\omega,k) + A_{\omega k-}c_-^{\mathrm{III}}(\omega,k)\\
\mathrm{e}^{\frac{\pi\omega\ell^2}{2\rho_h}}\Big(B_{\omega k+}c_+^{\mathrm{IV}}(\omega,k) + B_{\omega k-}c_-^{\mathrm{IV}}(\omega,k)\Big) &= B_{\omega k+}c_+^{\mathrm{III}}(\omega,k) + B_{\omega k-}c_-^{\mathrm{III}}(\omega,k)\\
\mathrm{e}^{-\frac{\pi\omega\ell^2}{2\rho_h}}\Big(A_{\omega k+}c_+^{\mathrm{IV}}(\omega,k) + A_{\omega k-}c_-^{\mathrm{IV}}(\omega,k)\Big) &= A_{\omega k+}c_+^{\mathrm{II}}(\omega,k) + A_{\omega k-}c_-^{\mathrm{II}}(\omega,k).
\end{aligned} \tag{6.36}$$

Note also that for a real solution, the coefficients are subject to additional reality conditions; more on this in section 7.3.

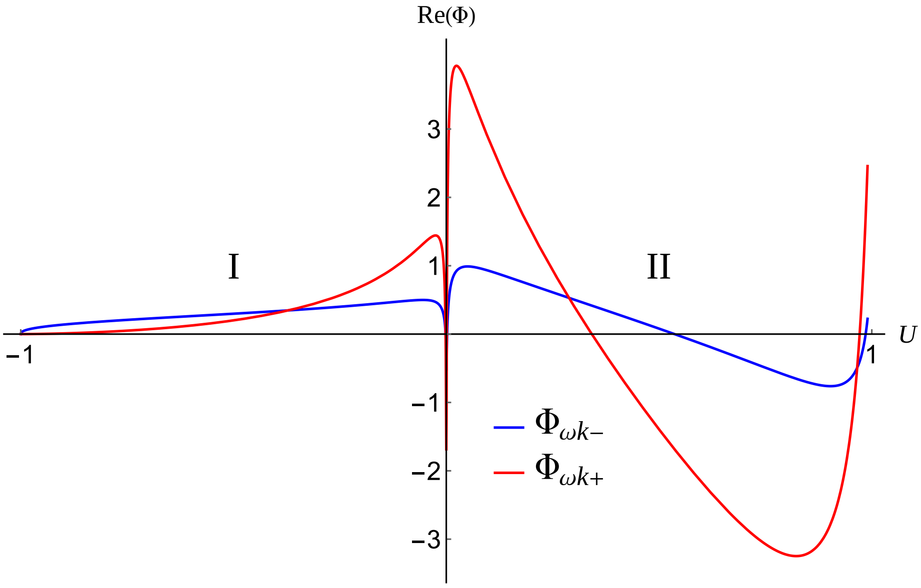

Figure 3: The solutions $\Phi_{\omega k-}$ (blue, with source) and $\Phi_{\omega k+}$ (red, with VEV) on the line $V = 1$, with parameters $\Delta = 1.6$, $k = 3$, $\omega = 2.3$, $\rho_h = 3$, $\ell = 0.8$, $\phi = 0$. Only the real part is shown. Both solutions blow up as $U \longrightarrow 1$ ($\rho \longrightarrow 0$), and are badly behaved at the event horizon $U = 0$. Note also the visible difference between $\rho^{-\Delta^-}$ and $\rho^{-\Delta^+}$ decay as $U \longrightarrow -1$ ($\rho \longrightarrow \infty$).

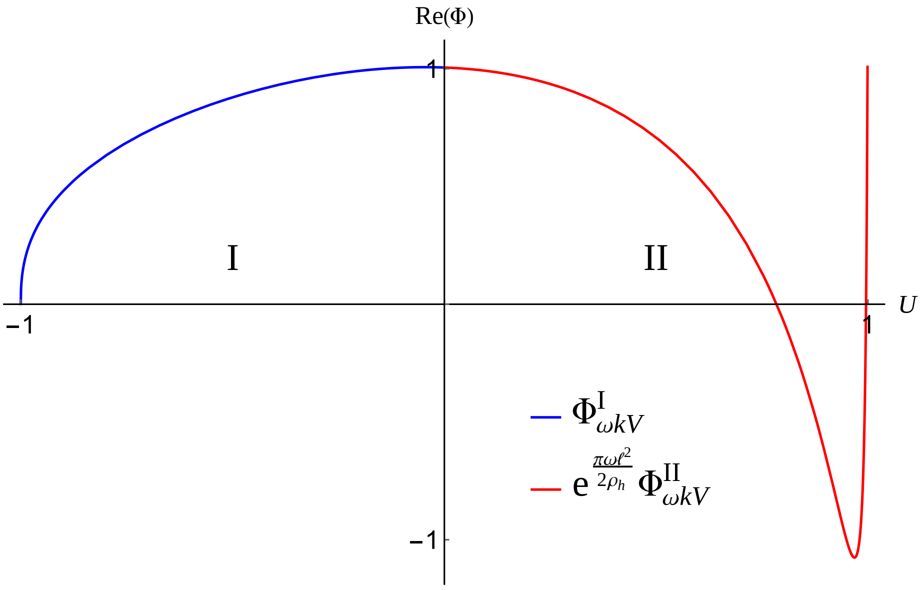

Figure 4: The solutions $\Phi_{\omega kV}$ in region I (blue) and $e^{\frac{\pi\omega\ell^2}{2\rho_h}} \Phi_{\omega kV}$ in region II (red) on the line $V = 1$, with parameters like in fig. 3. Only the real part is shown. These are the particular linear combinations of $\Phi_{\omega k\pm}$ that smoothly continue into each other across the horizon.

7 Quantum structure

7.1 Canonical quantisation in curved spacetime

We loosely follow Jacobson [14]. Starting with a $(d+1)$-dimensional spacetime X, choose a time coordinate t such that the metric can be written as

$$\mathrm{d}s^2 = G_{tt}\,\mathrm{d}t^2 + \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \tag{7.1}$$

Let Σ_t be the surface of constant value t of this coordinate, and split the action as

$$S = \int \mathrm{d}t\, L \quad \text{with} \quad L = -\int_{\Sigma_t} \mathrm{d}^d x\,\sqrt{|G|}\,\frac{1}{2}\big(\nabla_a\Phi\nabla^a\Phi + m^2\Phi^2\big). \tag{7.2}$$

Define the canonical momentum as

$$\Pi(\mathbf{x},t) = \frac{1}{\sqrt{|\gamma(\mathbf{x})|}}\frac{\delta L(t)}{\delta\Phi(\mathbf{x},t)} \tag{7.3}$$

(Jacobson does not include the factor $1/\sqrt{|\gamma(\mathbf{x})|}$). A short calculation then shows that one may write

$$\Pi = n^a\nabla_a\Phi \tag{7.4}$$

where $n^\mu = -\sqrt{-G_{tt}}\,G^{t\mu}$ is the future-pointing unit normal vector of Σ_t. We then quantise the theory by imposing the canonical equal-time commutation relations

$$\begin{aligned}[\Phi(\mathbf{x},t),\Pi(\mathbf{y},t)] &= \frac{1}{\sqrt{|\gamma(\mathbf{x})|}}\mathrm{i}\delta^{(\Sigma_t)}(\mathbf{x},\mathbf{y})\\ [\Phi(\mathbf{x},t),\Phi(\mathbf{y},t)] &= [\Pi(\mathbf{x},t),\Pi(\mathbf{y},t)] = 0\end{aligned} \tag{7.5}$$

where $\delta^{(\Sigma_t)}$ is a delta function on Σ_t, defined by the property that

$$\int_{\Sigma_t} \mathrm{d}^d x\,\delta^{(\Sigma_t)}(\mathbf{x},\mathbf{y}) f(\mathbf{x}) = f(\mathbf{y}). \tag{7.6}$$

7.2 Quantisation by mode expansion

Again, we follow the approach of Jacobson [14], with aspects from similar treatments in Wald [11, §14.2], Ford [15, §1], and Mukhanov and Winitzki [16, §6, §8]. We define the Klein–Gordon inner product, or Wronskian, of two solutions Φ and Ψ as

$$(\Phi,\Psi) = \mathrm{i}\int_\Sigma \mathrm{d}\Sigma^a\,\Phi^*\overleftrightarrow{\nabla}_a\Psi = \mathrm{i}\int_\Sigma \mathrm{d}\Sigma^a\,(\Phi^*\nabla_a\Psi - \nabla_a\Phi^*\Psi) \tag{7.7}$$

where Σ is a spacelike Cauchy surface (for example, a time slice Σ_t) and $\mathrm{d}\Sigma^a = \mathrm{d}^2x\,\sqrt{|\gamma|}\,n^a$ is its area element [11, eq. 14.2.5].[12] It is a sesquilinear form:

$$\begin{aligned}(\Phi,\alpha\Psi_1+\beta\Psi_2) &= \alpha(\Phi,\Psi_1)+\beta(\Phi,\Psi_2)\\ (\alpha\Phi_1+\beta\Phi_2,\Psi) &= \alpha^*(\Phi_1,\Psi)+\beta^*(\Phi_2,\Psi)\end{aligned} \tag{7.8}$$

[12] Note that, since Σ is spacelike, the sign of its normal vector n^a requires extra attention. For the purposes of Gauss' theorem, the normal vector should point into the volume rather than out of it [11, eq. B.2.24; 17, eq. 9.67]. Here, however, we choose n^a to be future-pointing.

and additionally satisfies

$$(\Phi, \Psi)^* = -(\Phi^*, \Psi^*) = (\Psi, \Phi) \quad \text{and} \quad (\Phi, \Phi^*) = 0. \tag{7.9}$$

To see that it is independent of the choice of Cauchy surface Σ, first use the equations of motion to show that $j_a = \mathrm{i}(\Phi^* \nabla_a \Psi - \nabla_a \Phi^* \Psi)$ is a conserved current, i.e. $\nabla^a j_a = 0$:

$$\begin{aligned} \nabla^a j_a &= \mathrm{i}(\nabla^a \Phi^* \nabla_a \Psi + \Phi^* \nabla^a \nabla_a \Psi - \nabla^a \nabla_a \Phi^* \Psi - \nabla_a \Phi^* \nabla^a \Psi) \\ &= \mathrm{i}\big(\Phi^* m^2 \Psi - m^2 \Phi^* \Psi\big) = 0. \end{aligned} \tag{7.10}$$

Then, for two Cauchy surfaces Σ and Σ', let V be the region bounded by them. By Gauss' theorem,

$$\left(\int_{\Sigma'} - \int_{\Sigma}\right) \mathrm{d}\Sigma^a \, j_a = \int_{\partial V} \mathrm{d}\Sigma^a \, j_a = \int_V \mathrm{d}V \, \nabla^a j_a = 0 \tag{7.11}$$

where $\mathrm{d}V = \mathrm{d}^3 x \sqrt{|G|}$ (if V is not a compact set, Φ and Ψ are assumed to decay sufficiently fast at infinity).

A *complete set of modes* is now defined as a set of solutions $\{u_k\}$, indexed by a discrete label k, such that $\{u_k\} \cup \{u_k^*\}$ forms a basis of all solutions, and such that

$$(u_k, u_{k'}) = \delta_{kk'} \qquad (u_k^*, u_{k'}^*) = -\delta_{kk'} \qquad (u_k, u_{k'}^*) = 0. \tag{7.12}$$

(in the continuous case, $\delta_{kk'}$ is replaced by a delta function). A general complex solution may then be expanded as

$$\Phi = \sum_k \left(a_k^- u_k + a_k^+ u_k^*\right) \tag{7.13}$$

with some coefficients $a_k^\pm$. For a real solution, we must evidently have $a_k^+ = (a_k^-)^*$. If Φ is promoted to an operator (and therefore also $a_k^\pm$, with $a_k^+ = (a_k^-)^\dagger$) and f and g are any two Klein–Gordon solutions, one may show the commutator identity [14, eq. 4.13]

$$[(f, \Phi), (g, \Phi)] = -(f, g^*) \tag{7.14}$$

by means of a short calculation, expanding the inner products, replacing $n^a \nabla_a \Phi$ by Π and using the canonical commutation relations (7.5). Since the normalisation (7.12) implies that $a_k^- = (u_k, \Phi)$ and $a_k^+ = -(u_k^*, \Phi)$, we find the commutation relations

$$\left[a_k^-, a_{k'}^+\right] = \delta_{kk'} \qquad \left[a_k^-, a_{k'}^-\right] = \left[a_k^+, a_{k'}^+\right] = 0. \tag{7.15}$$

The motivating example to keep in mind is $(d+1)$-dimensional Minkowski space $\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2$. Here, the modes

$$u_\mathbf{k} = \frac{1}{\sqrt{2\omega_\mathbf{k}}} \mathrm{e}^{-\mathrm{i}\omega_\mathbf{k} t + \mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{7.16}$$

where $\omega_\mathbf{k} = \sqrt{\mathbf{k}^2 + m^2}$, are solutions that satisfy $(u_k, u_{k'}) = (2\pi)^d \delta^{(d)}(\mathbf{k} - \mathbf{k}')$. The continuous version of (7.13) becomes the familiar free-field expansion

$$\Phi(\mathbf{x}, t) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(a_\mathbf{k}^- u_\mathbf{k} + a_\mathbf{k}^+ u_\mathbf{k}^*\right) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\sqrt{2\omega_\mathbf{k}}} \left(a_\mathbf{k}^- \mathrm{e}^{-\mathrm{i}\omega_\mathbf{k} t + \mathrm{i}\mathbf{k}\cdot\mathbf{x}} + a_\mathbf{k}^+ \mathrm{e}^{\mathrm{i}\omega_\mathbf{k} t - \mathrm{i}\mathbf{k}\cdot\mathbf{x}}\right). \tag{7.17}$$

Given a complete set of modes, the u_k^* are called *positive-frequency modes* and the u_k are called *negative-frequency modes*. Linear combinations of only positive/negative-frequency modes are called positive/negative-frequency solutions. The positive/negative frequency solutions both form a vector space, within which $(\Phi, \Phi) \gtrless 0$ respectively. Note, however, that not every positive-norm solution is a positive-frequency solution; indeed, $\{\Phi \mid (\Phi, \Phi) > 0\}$ is not a linear space. The notion of positive frequency thus depends on the chosen complete set of modes.

Consider therefore a different complete set of modes $\{\tilde{u}_\ell\}$. Expanding each $\tilde{u}_\ell$ in the $\{u_k, u_k^*\}$ basis as

$$u_\ell = \sum_k \left(\alpha_{\ell k} u_k + \beta_{\ell k} u_k^*\right) \tag{7.18}$$

and requiring that both bases are normalised by (7.12), one finds that $\alpha_{\ell k}$ and $\beta_{\ell k}$, called the *Bogolyubov coefficients*, must satisfy

$$\sum_k \left(\alpha_{\ell k}^* \alpha_{\ell' k} - \beta_{\ell k}^* \beta_{\ell' k}\right) = \delta_{\ell \ell'} \qquad \sum_k \left(\alpha_{\ell k} \beta_{\ell' k} - \beta_{\ell k} \alpha_{\ell' k}\right) = 0. \tag{7.19}$$

Expanding the field (7.13) in the new set of modes as

$$\Phi = \sum_\ell \left(b_\ell^- \tilde{u}_\ell + b_\ell^+ \tilde{u}_\ell^*\right), \tag{7.20}$$

one finds the *Bogolyubov transformations*

$$a_k^- = \sum_\ell \left(\alpha_{\ell k} b_\ell^- + \beta_{\ell k}^* b_\ell^+\right) \qquad a_k^+ = \sum_\ell \left(\alpha_{\ell k}^* b_\ell^+ + \beta_{\ell k} b_\ell^-\right) \tag{7.21}$$

$$b_\ell^- = \sum_k \left(\alpha_{\ell k}^* a_k^- - \beta_{\ell k}^* a_k^+\right) \qquad b_\ell^+ = \sum_k \left(\alpha_{\ell k} a_k^+ - \beta_{\ell k} a_k^-\right). \tag{7.22}$$

Having found a complete set of modes with associated ladder operators $a_k^\pm$, one may construct the Hilbert space of the quantum theory as a Fock space in the usual way: Postulate a (normalised) vacuum state $\left|{}_{(a)}0\right\rangle$ such that $a_k^- \left|{}_{(a)}0\right\rangle = 0$ for all k, then build one-particle states $\left|{}_{(a)}1_k\right\rangle = a_k^+ \left|{}_{(a)}0\right\rangle$ and in general N-particle states

$$\left|{}_{(a)}\ell_{k_1} m_{k_2} \dots n_{k_N}\right\rangle = \frac{1}{\sqrt{\ell! m! \dots n!}} (a_{k_1}^+)^\ell (a_{k_2}^+)^m \dots (a_{k_N}^+)^n \left|{}_{(a)}0\right\rangle. \tag{7.23}$$

The notion of particles clearly depends on the choice of mode decomposition. In particular, the vacuum state $\left|{}_{(b)}0\right\rangle$ associated with another set of ladder operators $b_\ell^\pm$ is not the same as $\left|{}_{(a)}0\right\rangle$, as the latter is annihilated by all b_ℓ^- only if all $\beta_{\ell k}$ are zero in the Bogolyubov transformation between the two bases. Thus the b-vacuum contains a-particles and vice versa.

As previously remarked, the space of positive-frequency solutions with respect to a mode decomposition is linear while the space of positive-norm solutions (independent of decomposition) is not linear. Indeed, the former corresponds to to the space of one-particle states via the isomorphism $f \leftrightarrow (f, \Phi)\left|{}_{(a)}0\right\rangle$ (Jacobson calls $(f, \Phi) = a(f)$), while the latter corresponds to all states that are one-particle with respect to *some* mode decomposition. This is another way to see that a one-particle state in one basis is generally a superposition of states with various numbers of particles in a different basis.

A concrete application of this formalism is in describing particle creation phenomena such as the Unruh effect and the Hawking effect; see for example [16, chap. 8–9; 1].

7.3 Modes in the BTZ spacetime

For the study of the quantum scalar in the BTZ spacetime, we focus on region I— naturally, the situation is completely analogous in region IV— and consider only the simplest case in which the source for $O(\mathbf{x})$ is turned off in the boundary CFT. This amounts to a boundary condition prohibiting any coefficients of $\Phi_{\omega k-}$ in the expansion (6.18) of Φ. According to Papadodimas and Raju [13, above eq. A.1], the remaining $\Phi_{\omega k+}$ are in fact orthogonal with respect to the Klein–Gordon product (7.7), and the normalisation constant is such that the following modes are orthonormal:

$$u_{\omega k}(\rho, t, \phi) = \frac{1}{\sqrt{2\omega\rho_h}}\sqrt{\frac{\Gamma\left(\frac{\Delta}{2}+a+b\right)\Gamma\left(\frac{\Delta}{2}+a-b\right)\Gamma\left(\frac{\Delta}{2}-a+b\right)\Gamma\left(\frac{\Delta}{2}-a-b\right)}{\Gamma(2b)\Gamma(-2b)}}\Phi_{\omega k+}(\rho, t, \phi) \tag{7.24}$$

(notice that the square root is real and positive; it is equal to $\left|\frac{\Gamma\left(\frac{\Delta}{2}+a+b\right)\Gamma\left(\frac{\Delta}{2}+a-b\right)}{\Gamma(2b)}\right|$ because complex conjugation takes $a \longrightarrow -a$ and $b \longrightarrow -b$). Thus, expanding

$$\Phi = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \sum_{k\in\mathbb{Z}} \left(a_{\omega k} u_{\omega k} + a^\dagger_{\omega k} u^*_{\omega k}\right), \tag{7.25}$$

the ladder operators $a_{\omega k}$, $a^\dagger_{\omega k}$ satisfy canonical commutation relations

$$\left[a_{\omega k}, a^\dagger_{\omega' k'}\right] = 2\pi\delta(\omega' - \omega)\delta_{kk'} \qquad \left[a_{\omega k}, a_{\omega' k'}\right] = \left[a^\dagger_{\omega k}, a^\dagger_{\omega' k'}\right] = 0. \tag{7.26}$$

Notice that the expansion runs over only positive ω, because $u_{\omega k}$ has positive Klein–Gordon norm only for $\omega > 0$ (this is not difficult to check, taking Σ to be the $t = 0$ surface). The solutions with negative ω are represented as $u_{-\omega,k} = u^*_{\omega,-k}$.

8 Conclusions and further work

The main result of this work is the system of constraints (6.36), which relate solutions with given near-boundary behaviour across the BTZ event horizon.

To obtain a relationship between the holographic boundaries in regions I and IV respectively, one must impose (6.36) across the four boundaries, and furthermore, impose boundary conditions at the singularities in regions II and III. It is not clear what boundary conditions are appropriate, if any, not least considering the subtle nature of the BTZ singularity, so we leave this question for further investigations. Once boundary conditions are selected, however, the connection between I and IV is established and it is easy to compute Bogolyubov coefficients between the modes (7.24) on either side. This enables studying the dependence of the theory on constraints at the holographic boundaries. For example, imposing boundary conditions—such as the absence of sources—on both sides, one expects to find a quantisation condition on the allowed frequencies and wavenumbers.

Having determined a connection between I and IV, the natural next step is to write it in terms of boundary CFT objects. In this way, we can hope to illuminate the Hilbert space structure of the composite system and even compute correlation functions between local observables on different sides.

A Solution of the BTZ wave equation

Using computer algebra, one obtains two linearly independent solutions to (6.11); one is

$$\Phi_1(\rho,\omega,k) = \left(\frac{\rho_h^2}{\rho^2-\rho_h^2}\right)^b \left(\frac{\rho^2}{\rho_h^2}\right)^a \mathbf{F}\left(\begin{matrix}\frac{\Delta^+}{2}+a-b,\ \frac{\Delta^-}{2}+a-b\\ 1+2a\end{matrix};\frac{\rho^2}{\rho_h^2}\right) \tag{A.1}$$

The other solution, $\Phi_2(\rho,\omega,k)$, is obtained by substituting $a \to -a$. Using the identity [12, eq. 15.8.3]

$$\frac{\sin\big(\pi(\beta-\alpha)\big)}{\pi}\mathbf{F}\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};z\right) = \frac{\left(\frac{1}{1-z}\right)^\alpha}{\Gamma(\beta)\Gamma(\gamma-\alpha)}\mathbf{F}\left(\begin{matrix}\alpha,\ \gamma-\beta\\ \alpha-\beta+1\end{matrix};\frac{1}{1-z}\right) - (\alpha\leftrightarrow\beta) \tag{A.2}$$

we may, disregarding the overall constant, rewrite this as

$$\begin{aligned}
&\Phi_1(\rho,\omega,k)\\
&\overset{(A.2)}{\propto} \frac{\left(\frac{\rho_h^2}{\rho^2-\rho_h^2}\right)^b\left(\frac{\rho^2}{\rho_h^2}\right)^a\left(\frac{\rho_h^2}{\rho_h^2-\rho^2}\right)^{\frac{\Delta^+}{2}+a-b}}{\Gamma\left(\frac{\Delta^-}{2}+a+b\right)\Gamma\left(\frac{\Delta^-}{2}+a-b\right)}\mathbf{F}\left(\begin{matrix}\frac{\Delta^+}{2}+a+b,\ \frac{\Delta^+}{2}+a-b\\ \Delta^+\end{matrix};\frac{\rho_h^2}{\rho_h^2-\rho^2}\right)\\
&\quad - (\Delta^+\leftrightarrow\Delta^-)\\
&= \frac{\left(\frac{\rho^2}{\rho_h^2-\rho^2}\right)^a\left(\frac{\rho_h^2}{\rho_h^2-\rho^2}\right)^{\frac{\Delta^+}{2}}}{\Gamma\left(\frac{\Delta^-}{2}+a+b\right)\Gamma\left(\frac{\Delta^-}{2}+a-b\right)}\mathbf{F}\left(\begin{matrix}\frac{\Delta^+}{2}+a+b,\ \frac{\Delta^+}{2}+a-b\\ \Delta^+\end{matrix};\frac{\rho_h^2}{\rho_h^2-\rho^2}\right)\\
&\quad - (\Delta^+\leftrightarrow\Delta^-).
\end{aligned} \tag{A.3}$$

Then, using the identity [12, eq. 15.8.1]

$$\mathbf{F}\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};z\right) = (1-z)^{\gamma-\alpha-\beta}\mathbf{F}\left(\begin{matrix}\gamma-\alpha,\ \gamma-\beta\\ \gamma\end{matrix};z\right) \tag{A.4}$$

we rewrite the expression for $\Phi_2(\rho,\omega,k)$ (obtained from (A.3) by changing $a \to -a$) as

$$\begin{aligned}
&\Phi_2(\rho,\omega,k)\\
&\propto \frac{\left(\frac{\rho^2}{\rho_h^2-\rho^2}\right)^{-a}\left(\frac{\rho_h^2}{\rho_h^2-\rho^2}\right)^{\frac{\Delta^+}{2}}}{\Gamma\left(\frac{\Delta^-}{2}-a+b\right)\Gamma\left(\frac{\Delta^-}{2}-a-b\right)}\mathbf{F}\left(\begin{matrix}\frac{\Delta^+}{2}-a+b,\ \frac{\Delta^+}{2}-a-b\\ \Delta^+\end{matrix};\frac{\rho_h^2}{\rho_h^2-\rho^2}\right)\\
&\quad - (\Delta^+\leftrightarrow\Delta^-)\\
&\overset{(A.4)}{=} \frac{\left(\frac{\rho^2}{\rho_h^2-\rho^2}\right)^{-a}\left(\frac{\rho_h^2}{\rho_h^2-\rho^2}\right)^{\frac{\Delta^+}{2}}\left(\frac{\rho^2}{\rho^2-\rho_h^2}\right)^{2a}}{\Gamma\left(\frac{\Delta^-}{2}-a+b\right)\Gamma\left(\frac{\Delta^-}{2}-a-b\right)}\mathbf{F}\left(\begin{matrix}\frac{\Delta^+}{2}+a+b,\ \frac{\Delta^+}{2}+a-b\\ \Delta^+\end{matrix};\frac{\rho_h^2}{\rho_h^2-\rho^2}\right)\\
&\quad - (\Delta^+\leftrightarrow\Delta^-)\\
&\propto \frac{\left(\frac{\rho^2}{\rho_h^2-\rho^2}\right)^{a}\left(\frac{\rho_h^2}{\rho_h^2-\rho^2}\right)^{\frac{\Delta^+}{2}}}{\Gamma\left(\frac{\Delta^-}{2}-a+b\right)\Gamma\left(\frac{\Delta^-}{2}-a-b\right)}\mathbf{F}\left(\begin{matrix}\frac{\Delta^+}{2}+a+b,\ \frac{\Delta^+}{2}+a-b\\ \Delta^+\end{matrix};\frac{\rho_h^2}{\rho_h^2-\rho^2}\right)\\
&\quad - (\Delta^+\leftrightarrow\Delta^-).
\end{aligned} \tag{A.5}$$

This is the same as (A.4), apart from the factors in the denominator. In particular, we see that Φ_1 and Φ_2 are independent linear combinations of the two solutions $\Phi_\pm$ given in (6.14).

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