Royal Institute of Technology

Bachelor's thesis

Geometric algebra, conformal geometry and the common curves problem

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Abstract

This bachelor's thesis gives a thorough introduction to geometric algebra (GA), an overview of conformal geometric algebra (CGA) and an application to the processing of single particle data from cryo-electron microscopy (cryo-EM).

The geometric algebra over the vector space $\mathbb{R}^{p,q}$, i.e. the Clifford algebra over an orthogonal basis of the space, is a strikingly simple algebraic construction built from the *geometric product*, which generalizes the scalar and cross products between vectors. In terms of this product, a host of algebraically and geometrically meaningful operations can be defined. These encode linear subspaces, incidence relations, direct sums, intersections and orthogonal complements, as well as reflections and rotations. It is with good reason that geometric algebra is often referred to as a universal language of geometry.

Conformal geometric algebra is the application of geometric algebra in the context of the *conformal embedding* of $\mathbb{R}^3$ into the Minkowski space $\mathbb{R}^{4,1}$. By way of this embedding, linear subspaces of $\mathbb{R}^{4,1}$ represent arbitrary points, lines, planes, point pairs, circles and spheres in $\mathbb{R}^3$. Reflections and rotations in $\mathbb{R}^{4,1}$ become conformal transformations in $\mathbb{R}^3$: reflections, rotations, translations, dilations and inversions.

The analysis of single-particle cryo-electron microscopy data leads to the *common curves* problem. By a variant of the Fourier slice theorem, this problem involves hemispheres and their intersections. This thesis presents a rewriting, inspired by CGA, into a problem of planes and lines. Concretely, an image in the Fourier domain is transformed by mapping points according to

$$\mathbf{x} \mapsto \frac{\mathbf{x}}{1 - \sqrt{1 - \mathbf{x}^2}}$$

in suitable units. The inversive nature of this transformation causes certain issues that render its usage a trade-off rather than an unconditional advantage.

Keywords: Geometric algebra, Clifford algebra, conformal geometric algebra, single particle analysis, cryo-electron microscopy, common curves, stereographic projection, inversion

Sammanfattning

Detta kandidatexamensarbete ger en grundlig introduktion till geometrisk algebra (GA), en översiktlig redogörelse för konform geometrisk algebra (CGA) samt en tillämpning på behandlingen av enpartikeldata från kryo-elektronmikroskopi (kryo-EM).

Den geometriska algebran över vektorrummet $\mathbb{R}^{p,q}$, eller Cliffordalgebran över en ortogonal bas till rummet, är en slående enkel algebraisk konstruktion som bygger på den *geometriska produkten*, vilken generaliserar skalär- och kryssprodukterna mellan vektorer. Med utgångspunkt i denna produkt kan en rad algebraiskt och geometriskt meningsfulla operationer definieras. Dessa representerar linjära delrum, incidensrelationer, direkta summor, snitt och ortogonala komplement, såväl som speglingar och rotationer. Det är av goda skäl som geometrisk algebra ofta beskrivs som ett universellt språk för geometri.

Konform geometrisk algebra är tillämpningen av geometrisk algebra i anslutning till den *konforma inbäddningen* av $\mathbb{R}^3$ i Minkowskirummet $\mathbb{R}^{4,1}$. Genom denna inbäddning representerar linjära delrum av $\mathbb{R}^{4,1}$ godtyckliga punkter, linjer, plan, punktpar, cirklar och sfärer i $\mathbb{R}^3$. Speglingar och rotationer i $\mathbb{R}^{4,1}$ blir konforma avbildningar i $\mathbb{R}^3$: speglingar, rotationer, translationer, dilationer och inversioner.

Analysen av enpartikeldata från kryo-elektronmikroskopi leder till problemet med *gemensamma kurvor*. Enligt en variant av projektionssatsen ("the Fourier slice theorem") inbegriper detta problem halvsfärer och deras skärningkurvor. I detta arbete presenteras en omskrivning, inspirerad av CGA, till ett problem rörande plan och linjer. Konkret transformeras en bild i Fourierrymden genom att punkterna skickas enligt

$$\mathbf{x} \mapsto \frac{\mathbf{x}}{1 - \sqrt{1 - \mathbf{x}^2}}$$

i lämpliga enheter. Denna avbildnings inversiva natur orsakar vissa problem som gör dess användning till en kompromiss snarare än en otvetydig fördel.

Contents

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1 Introduction

This thesis consists of three parts. The first part is a detailed introduction to the fundamentals of geometric algebra, aimed at those without prior knowledge of the subject. The second part is an overview of conformal geometric algebra (CGA), which is an interesting subject in its own right, and also serves as a source of examples of computation with geometric algebra. In the third part, inspiration and ideas from CGA are applied to the *common curves* problem from single particle cryo-electron microscopy (cryo-EM).

1.1 Geometric algebra

Vectors are highly useful mathematical objects that can be viewed in a multitude of different ways. In particular, they are *algebraic* objects that describe *geometry*. The usual vector space $\mathbb{R}^n$ supports scalar multiplication $\alpha\mathbf{x}$ and vector addition $\mathbf{x}+\mathbf{y}$, as well as the inner or scalar product $\mathbf{x}\cdot\mathbf{y}$. In three dimensions, one can also define the cross product $\mathbf{x}\times\mathbf{y}$. The usefulness of all these operations is confirmed by their abundance throughout mathematics and physics.

However, in the view that vectors are a kind of generalized numbers, there seems to be something missing: A product between vectors that behaves similarly to the ordinary product between numbers. The scalar product of two vectors always results in a scalar and not a vector (it might be better called a *bilinear form* rather than a product). The cross product is only uniquely defined in three dimensions; moreover, it is not associative ($(\mathbf{a}\times\mathbf{b})\times\mathbf{c} \neq \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$), which seems to be the least that can be required of an algebraically familiar product.

Geometric algebra is based around the *geometric product*, written simply $\mathbf{xy}$, which is associative and even invertible in many cases, and combines the scalar and cross products. The definition as a quotient algebra "requiring as little as possible" reveals a unified structure of scalars, vectors and higher-dimensional objects called multivectors. Elements in the new algebra have geometrical significance that allows linear subspaces as well as common geometrical operations to be compactly expressed. The intersection and direct sum of linear subspaces can under many conditions be expressed using the new meet and wedge products.

1.2 Conformal geometric algebra

The ability to express linear subspaces as algebraic objects is useful when dealing with $\mathbb{R}^n$ on its own, yet may be further exploited by viewing $\mathbb{R}^n$ not on its own, but embedded in an ambient space X via a map $\mathcal{X}\colon \mathbb{R}^n \to X$. Linear subspaces of X may, through their intersections with $\mathcal{X}(\mathbb{R}^n)$, represent *nonlinear* subspaces of $\mathbb{R}^n$. The simplest example is *projective geometry*, in which $X = \mathbb{R}^{n+1}$ and $\mathcal{X}(\mathbf{x}) = \mathbf{x} + \mathbf{e}_{n+1}$. Here, arbitrary points, lines and planes become linear subspaces. A more powerful construction is *conformal geometry*, where $X = \mathbb{R}^{n+1,1}$ and $\mathcal{X}$ is based on stereographic projection. In conformal geometry, linear subspaces of $\mathbb{R}^{4,1}$ represent subsets of $\mathbb{R}^3$ such as arbitrary planes, lines, points, spheres, circles and point pairs.

1.3 Cryo-electron microscopy and the common curves problem

Single particle cryo-electron microscopy (cryo-EM) is an imaging technique used to determine the three-dimensional structures of identical, isolated molecules that have been rapidly frozen in ice. The physics and practice behind the method are largely beyond the scope of this thesis, but one mathematical problem that arises in the processing of the data involves spheres and their intersections. In Section 4, I will present this problem and present a rewriting of it based on ideas from conformal geometry.

2 Geometric algebra

In this section, I will give an introduction to geometric algebra in the context of a general space $\mathbb{R}^{p,q}$ and some specific examples, without special focus on conformal geometry or the common curves problem.

The presentation is heavily based on that found in [1, Chapters 1–3] (an older version is published as [2]), which develops the subject in a powerful way that allows the proofs of many formulae to be reduced to elementary set logic. However, the treatment is quite concise and is aimed at mathematically rather experienced readers. Here, I have prioritized ease of understanding over rigour and compactness, and have tried to provide concrete examples to make the text quickly digestible. I have written it with an audience of my fellow undergraduate students in mind, trying to produce an introduction that I would have liked to read myself. The reader is only assumed to be familiar with standard linear algebra, in particular vectors, the dot product and to a lesser extent the cross product.

If the reader still wonders why one should be excited about geometric algebra, consult [3] (but beware of notational differences; see Section 2.4.1). It gives a brief introduction to the subject, followed by a long list of very elegant motivating examples. Another good introductory treatment, focusing on geometric algebra in the context of physics (for which it is very well suited) is given in [4].

2.1 Constructing the geometric algebra

Here, I will construct the geometric algebra in a way that is informal in order to convey the essential points more clearly, but that can in principle be made totally rigorous (see for example [1, Section 2.1]).

2.1.1 Vector spaces with signature

We are used to thinking of vectors as members of the Euclidean vector space $\mathbb{R}^n$ with a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, equipped with the ordinary scalar product defined by

$$\mathbf{e}_i \cdot \mathbf{e}_j \coloneqq \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases}$$

and extending bilinearly (requiring that it is a linear function in both arguments):

$$\forall \alpha, \beta \in \mathbb{R} \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n \quad \begin{aligned} (\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} &= \alpha\,(\mathbf{x} \cdot \mathbf{z}) + \beta\,(\mathbf{y} \cdot \mathbf{z}), \\ \mathbf{x} \cdot (\alpha\mathbf{y} + \beta\mathbf{z}) &= \alpha\,(\mathbf{x} \cdot \mathbf{y}) + \beta\,(\mathbf{x} \cdot \mathbf{z}). \end{aligned} \tag{2.1}$$

We will generalize this definition slightly by allowing some of the basis vectors to square to -1. More precisely, the first p vectors square to 1 and the remaining q ones (where $n = p + q$) square to -1:

$$\mathbf{e}_i \cdot \mathbf{e}_j \coloneqq \begin{cases} 1 & i = j \quad \text{and} \quad i \leq p \\ -1 & i = j \quad \text{and} \quad i > p \\ 0 & i \neq j. \end{cases} \tag{2.2}$$

This new vector space is called $\mathbb{R}^{p,q}$ and is said to have the *signature* (p, q). It is a theorem, *Sylvester's Law of Inertia* [1, Theorem 2.4], that the signature is invariant under changes of orthogonal basis.

Note that the new scalar product violates one of the conventional axioms of inner products: that $\mathbf{x} \cdot \mathbf{x} = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$. For example, in $\mathbb{R}^{1,1}$ (which has a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ with $\mathbf{e}_1 \cdot \mathbf{e}_1 = 1$ and $\mathbf{e}_2 \cdot \mathbf{e}_2 = -1$), consider the vector $\mathbf{e}_1 + \mathbf{e}_2$:

$$\begin{aligned}(\mathbf{e}_1 + \mathbf{e}_2) \cdot (\mathbf{e}_1 + \mathbf{e}_2) &= \mathbf{e}_1 \cdot \mathbf{e}_1 + 2\,\mathbf{e}_1 \cdot \mathbf{e}_2 + \mathbf{e}_2 \cdot \mathbf{e}_2 \\ &= 1 + 2 \cdot 0 - 1 \\ &= 0.\end{aligned}$$

We say that $\mathbf{e}_1 + \mathbf{e}_2$ is an example of a *null vector*. More generally, in $\mathbb{R}^{n,1}$ with the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_-\}$ (incidentally, the setting of special relativity is the *Minkowski space* $\mathbb{R}^{3,1}$, or more often $\mathbb{R}^{1,3}$, with $\mathbf{e}_-$ representing the time dimension), any vector $\mathbf{X}$ can be written

$$\mathbf{X} = \mathbf{x} + t\mathbf{e}_-$$

with $\mathbf{x} \in \operatorname{Span}\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then $\mathbf{X} \cdot \mathbf{X}$ will be 0 whenever $\mathbf{x} \cdot \mathbf{x} = t^2$. The set of null vectors thus forms a cone, known as the *null cone* (or in the terminology of relativity, the *light cone*). This will become important when we deal with conformal geometry.

2.1.2 The tensor algebra

The first step towards constructing the geometric algebra involves creating the most general conceivable structure that fulfils our expectations of a product similar to the ordinary multiplication of real numbers.

We start with only the real numbers $\mathbb{R}$ and the basis $\mathfrak{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, which we for the moment think of as nothing more than a collection of n different symbols. In this section, the scalar product will not be used at all; hence the results apply regardless of the signature of $\mathfrak{B}$ (the signature will become important in the next section). In principle, we could use any finite set of our choosing as the basis; the resulting algebra is then usually referred to as a *Clifford algebra* rather than a geometric algebra, since there may be no geometric interpretation. This concept will be alluded to in Section 2.2.2.

Let us call the new structure $\mathcal{T}$, and begin to write down a wish list for its properties (we denote elements of $\mathcal{T}$ by $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, and real numbers by α and β):

- $\mathcal{T}$ should contain all the basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$.
- $\mathcal{T}$ should be a vector space, i.e. it should be an abelian group under addition $\mathbf{A} + \mathbf{B}$ with a zero element $\mathbf{0}$ and support scalar multiplication satisfying the usual rules

$$\begin{aligned}&0\mathbf{A} = \mathbf{0} \\ &1\mathbf{A} = \mathbf{A} \\ &(\alpha + \beta)\,\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A} \\ &\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B} \\ &\alpha(\beta\mathbf{A}) = (\alpha\beta)\,\mathbf{A}\end{aligned} \tag{2.3}$$

So far, we have described the defining properties of $\mathbb{R}^n$. This means that $\mathbb{R}^n$ must be a subset of $\mathcal{T}$. Let us now add a product to our list:

- There should be a product, written simply $\mathbf{AB}$ (and $\mathbf{A}^k$ for repeated multiplication), such that $\mathcal{T}$ is closed under it, that is, for any $\mathbf{A}, \mathbf{B} \in \mathcal{T}$, $\mathbf{AB}$ is a new element of $\mathcal{T}$. This property is where the scalar product on $\mathbb{R}^n$ fails.

- The product should be distributive: $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$ and $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$.
- The product should be associative: $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$. This rules out the cross product and allows the parentheses to be omitted: $\mathbf{ABC}$.

The first rule means that one can form elements such as $\mathbf{e}_1\mathbf{e}_2$ and $2\mathbf{e}_2 + 3(\mathbf{e}_3\mathbf{e}_1)$. Notice that the last two items are very similar to the last three of the axioms (2.3) for a vector space. It also seems natural that $(3\mathbf{e}_3)\mathbf{e}_1$ should be the same thing as $3(\mathbf{e}_3\mathbf{e}_1)$. This motivates combining these two different sets of criteria into one:

- The scalars should be elements of $\mathcal{T}$, and multiplication with a scalar should be the same as the vector space scalar multiplication. That is, an expression such as $\alpha\mathbf{A}$ is unambiguous.
- Multiplication with a scalar is always commutative: $\mathbf{A}\alpha = \alpha\mathbf{A}$.

Remark. *These requirements ensure that the new multiplication is bilinear:*

$$\mathbf{A}(\alpha\mathbf{B} + \beta\mathbf{C}) = \alpha(\mathbf{AB}) + \beta(\mathbf{AC})$$
$$(\alpha\mathbf{A} + \beta\mathbf{B})\mathbf{C} = \alpha(\mathbf{AC}) + \beta(\mathbf{BC})$$

by distributivity, associativity, and the commutativity of scalars.

Notice that these last criteria were both quite natural and introduced something very new to the ordinary notion of a vector space. In the new algebra, scalars and vectors are the same kind of thing, so there must exist *mixed-grade* elements such as, for example, $1 + \mathbf{e}_1$ and $7 + 3\mathbf{e}_1\mathbf{e}_2$. This hardly appears useful at first, but such objects will in fact sometimes have geometric significance (see Examples 2.1.4 and 2.1.5). Another unconventional consequence is that there is no difference between the zero scalar and the zero vector: $0 = \mathbf{0}$.

Intuitively, we can define a structure $\mathcal{T}$ that is *generated* by the above requirements, in the sense that an element $\mathbf{A}$ is by definition in $\mathcal{T}$ if and only if it can be proven to be in $\mathcal{T}$ using the above rules. In the field of abstract algebra, there is a large machinery to give an explicit construction of such a structure, which avoids logical issues of provability and gives a simple characterization of the elements thus obtained. Suffice it to say that the resulting structure is called the *tensor algebra* or *free associative algebra* over $\mathfrak{B}$ (the algebra of non-commuting polynomials with variables in $\mathfrak{B}$ and real coefficients), denoted $\mathcal{T}(\mathbb{R}^n)$, and that it consists of finite linear combinations of items of the form $\mathbf{e}_{i_1}\mathbf{e}_{i_2}\dots\mathbf{e}_{i_k}$, where k is finite but arbitrarily large. Concretely, an element might look like

$$\mathbf{A} = 2 + \mathbf{e}_1 - \pi\,\mathbf{e}_1\mathbf{e}_1 + 8\,\mathbf{e}_1\mathbf{e}_2 + \frac{5}{2}\,\mathbf{e}_2\mathbf{e}_1 - 4\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1,$$

and so forth. The product is called the *tensor product* and is commonly written $\otimes$, but here we have omitted the symbol for compatibility with the geometric product.

Example 2.1.1. *Let us evaluate a product in* $\mathcal{T}(\mathbb{R}^2)$*:*

$$\begin{aligned}(2 + \mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1)(5 + 3\mathbf{e}_1) &= 10 + 5\mathbf{e}_2 + 5\mathbf{e}_2\mathbf{e}_1 \\ &\quad + 6\mathbf{e}_1 + 3\mathbf{e}_2\mathbf{e}_1 + 3\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1 && \textit{expanding parentheses} \\ &= 10 + 6\mathbf{e}_1 + 5\mathbf{e}_2 + 8\mathbf{e}_2\mathbf{e}_1 + 3\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1 && \textit{collecting like terms.}\end{aligned}$$

In both steps of the above example, we have made use of associativity. Notice that in a sense, the product doesn't *do* anything; it is simply juxtaposition of elements. To create a more interesting product, we need to impose an additional relation. This is the subject of the next section.

2.1.3 The geometric product

The leap from the unwieldy tensor algebra to the geometric algebra is remarkably simple. Take the vector space $\mathbb{R}^{p,q}$ with signature (p,q) and $n = p+q$ as described in Section 2.1.1 and form the tensor algebra $\mathcal{T}(\mathbb{R}^{p,q})$. Impose the following additional constraint, called the *defining equation* of the geometric algebra, connecting the tensor product to the scalar product defined in (2.2) and completing the wish list:

- For every *vector* $\mathbf{x} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$,

$$\mathbf{xx} = \mathbf{x} \cdot \mathbf{x}. \tag{2.4}$$

  (I have written $\mathbf{xx}$ instead of $\mathbf{x}^2$ for complete clarity.)

The algebra thus created is called the *geometric algebra* over $\mathbb{R}^{p,q}$ and is denoted $\mathcal{G}(\mathbb{R}^{p,q})$ (formally, it is a *quotient algebra* of $\mathcal{T}$). Its elements are called *multivectors.* The new product is called the *geometric product* and has a rich structure compared to the tensor product.

Example 2.1.2. *The defining equation* (2.4) *can be directly applied to further simplify the result of Example 2.1.1:*

$$\begin{aligned} 10 + 6\mathbf{e}_1 + 5\mathbf{e}_2 + 8\mathbf{e}_2\mathbf{e}_1 + 3\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1 &= 10 + 6\mathbf{e}_1 + 5\mathbf{e}_2 + 8\mathbf{e}_2\mathbf{e}_1 + 3\mathbf{e}_2 && \mathbf{e}_1^2 = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1 \textit{ in } \mathbb{R}^2 \\ &= 10 + 6\mathbf{e}_1 + 8\mathbf{e}_2 + 8\mathbf{e}_2\mathbf{e}_1 && \textit{collecting like terms.} \end{aligned}$$

There is also a less direct identity that expresses the scalar product of two vectors in terms of their geometric product:

Theorem 2.1. *For any two* vectors $\mathbf{x}$ *and* $\mathbf{y}$*,*

$$\mathbf{x} \cdot \mathbf{y} = \frac{\mathbf{xy} + \mathbf{yx}}{2}. \tag{2.5}$$

Proof. Expand the expression $(\mathbf{x}+\mathbf{y})^2$ in two different ways:

$$\begin{aligned} (\mathbf{x}+\mathbf{y})^2 &= (\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y}) = \mathbf{x}^2 + \mathbf{xy} + \mathbf{yx} + \mathbf{y}^2, \\ (\mathbf{x}+\mathbf{y})^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} \\ &= \mathbf{x}^2 + 2\,\mathbf{x}\cdot\mathbf{y} + \mathbf{y}^2. \end{aligned}$$

Comparing the two expressions gives (2.5). ■

Corollary 2.1.1. $\mathbf{x}$ *and* $\mathbf{y}$ *anticommute* ($\mathbf{xy} = -\mathbf{yx}$) *if and only if they are orthogonal (that is,* $\mathbf{x}\cdot\mathbf{y} = 0$*).*

Proof. Immediate from (2.5). ■

By Corollary 2.1.1, the geometric product can be specified for basis vectors similarly to (2.2):

$$\mathbf{e}_i\mathbf{e}_j = \begin{cases} 1 & i = j \text{ and } i \le p \\ -1 & i = j \text{ and } i > p \\ -\mathbf{e}_j\mathbf{e}_i & i \ne j. \end{cases} \tag{2.6}$$

By linearity (as detailed in Section 2.2.1), this completely specifies the geometric product for all multivectors, and can therefore be used as an alternative defining equation to (2.4). It is easier to compute with, but less illuminating.

Equation (2.6) can also be used to put a multivector into a standard form where every term has a strictly increasing sequence of basis vector indices.

Example 2.1.3. *Equation* (2.6) *can be used to write the result of Example 2.1.2 in a canonical form:*

$$10 + 6\mathbf{e}_1 + 8\mathbf{e}_2 + 8\mathbf{e}_2\mathbf{e}_1 = 10 + 6\mathbf{e}_1 + 8\mathbf{e}_2 - 8\mathbf{e}_1\mathbf{e}_2.$$

Rewriting a general multivector in this way, it is apparent that there are exactly 2^n distinct combinations of basis vectors. For example, in $\mathcal{G}(\mathbb{R}^3)$, every multivector can be written as a linear combination of the eight elements

$$1,\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_1\mathbf{e}_2,\ \mathbf{e}_1\mathbf{e}_3,\ \mathbf{e}_2\mathbf{e}_3,\ \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3.$$

These elements are called the *canonical basis blades.* In summary, $\mathcal{G}(\mathbb{R}^{p,q})$ is a 2^{p+q}-dimensional vector space with a basis of 2^{p+q} canonical basis blades.

Here are two more rather exciting examples of concrete computation with the geometric product. They should give a hint of the geometric significance of the newly defined product and of mixed-grade multivectors.

Example 2.1.4. *Consider* $\mathcal{G}(\mathbb{R}^2)$*, the geometric algebra of the plane, and let* $\hat{\mathbf{n}} \coloneqq \frac{1}{\sqrt{5}}(2\mathbf{e}_1 + \mathbf{e}_2)$*. Take an arbitrary vector* $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2$ *and evaluate* $\hat{\mathbf{n}}\mathbf{x}\hat{\mathbf{n}}$*:*

$$\begin{aligned}
\hat{\mathbf{n}}\mathbf{x}\hat{\mathbf{n}} &= \frac{1}{5}(2\mathbf{e}_1 + \mathbf{e}_2)(x\mathbf{e}_1 + y\mathbf{e}_2)(2\mathbf{e}_1 + \mathbf{e}_2) \\
&= \frac{1}{5}(2\mathbf{e}_1 + \mathbf{e}_2)(2x\mathbf{e}_1\mathbf{e}_1 + x\mathbf{e}_1\mathbf{e}_2 + 2y\mathbf{e}_2\mathbf{e}_1 + y\mathbf{e}_2\mathbf{e}_2) && \textit{expanding the product} \\
&= \frac{1}{5}(2\mathbf{e}_1 + \mathbf{e}_2)(2x + y + (x - 2y)\mathbf{e}_1\mathbf{e}_2) && \textit{simplifying using (2.6)} \\
&= \frac{1}{5}\big((4x + 2y)\mathbf{e}_1 + (2x - 4y)\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 \\
&\qquad + (2x + y)\mathbf{e}_2 + (x - 2y)\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\big) && \textit{expanding the product} \\
&= \frac{1}{5}\big((4x + 2y)\mathbf{e}_1 + (2x - 4y)\mathbf{e}_2 \\
&\qquad + (2x + y)\mathbf{e}_2 - (x - 2y)\mathbf{e}_1\big) && \textit{simplifying using (2.6)} \\
&= \frac{1}{5}\big((3x + 4y)\mathbf{e}_1 + (4x - 3y)\mathbf{e}_2\big) && \textit{collecting like terms.}
\end{aligned}$$

The result is the reflection in the line spanned by $\hat{\mathbf{n}}$*, as can be seen by comparing with the formula for the reflection matrix in matrix algebra:*

$$\mathsf{M} = 2\mathsf{n}\mathsf{n}^T - \mathsf{I} = \frac{2}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \quad \textit{where} \quad \mathsf{n} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We will see in Section 2.3.2 why this is the case.

Example 2.1.5. *Consider* $\mathcal{G}(\mathbb{R}^3)$*, the geometric algebra of three-dimensional space, and let* $\mathbf{R} \coloneqq \cos\varphi - \mathbf{e}_1\mathbf{e}_2\sin\varphi$ *and* $\mathbf{R}^\dagger \coloneqq \cos\varphi + \mathbf{e}_1\mathbf{e}_2\sin\varphi$*. Evaluate* $\mathbf{R}\mathbf{x}\mathbf{R}^\dagger$ *when* $\mathbf{x}$ *varies over the basis vectors. Start with* $\mathbf{e}_1$*:*

$$\begin{aligned}
\mathbf{R}\mathbf{e}_1\mathbf{R}^\dagger &= (\cos\varphi - \mathbf{e}_1\mathbf{e}_2\sin\varphi)\mathbf{e}_1(\cos\varphi + \mathbf{e}_1\mathbf{e}_2\sin\varphi) \\
&= \cos^2\varphi\,\mathbf{e}_1 - \cos\varphi\sin\varphi\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 \\
&\qquad + \cos\varphi\sin\varphi\,\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 - \sin^2\varphi\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 && \textit{expanding the product} \\
&= \cos^2\varphi\,\mathbf{e}_1 - \sin^2\varphi\,\mathbf{e}_1 \\
&\quad + \cos\varphi\sin\varphi\,\mathbf{e}_2 + \cos\varphi\sin\varphi\,\mathbf{e}_2 && \textit{simplifying using (2.6)} \\
&= (\cos^2\varphi - \sin^2\varphi)\mathbf{e}_1 + 2\cos\varphi\sin\varphi\mathbf{e}_2 && \textit{collecting like terms} \\
&= \cos(2\varphi)\mathbf{e}_1 + \sin(2\varphi)\mathbf{e}_2 && \textit{double-angle identities.}
\end{aligned}$$

In the same way, it can be shown that $\mathbf{R}\mathbf{e}_2\mathbf{R}^\dagger = -\sin(2\varphi)\mathbf{e}_1 + \cos(2\varphi)\mathbf{e}_2$. *As for* $\mathbf{e}_3$,

$$\begin{aligned}
\mathbf{R}\mathbf{e}_3\mathbf{R}^\dagger &= (\cos\varphi - \mathbf{e}_1\mathbf{e}_2\sin\varphi)\mathbf{e}_3(\cos\varphi + \mathbf{e}_1\mathbf{e}_2\sin\varphi) \\
&= (\cos\varphi - \mathbf{e}_1\mathbf{e}_2\sin\varphi)(\cos\varphi + \mathbf{e}_1\mathbf{e}_2\sin\varphi)\mathbf{e}_3 && \textit{anticommutation by } (2.6) \\
&= \big(\cos^2\varphi - \cos\varphi\sin\varphi\,\mathbf{e}_1\mathbf{e}_2 \\
&\quad + \cos\varphi\sin\varphi\,\mathbf{e}_1\mathbf{e}_2 - \sin^2\varphi\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\big)\mathbf{e}_3 && \textit{expanding the product} \\
&= \big(\cos^2\varphi + \sin^2\varphi\big)\mathbf{e}_3 && \textit{simplifying using } (2.6) \\
&= \mathbf{e}_3 && \textit{by the Pythagorean theorem.}
\end{aligned}$$

For a general $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ *we therefore find that*

$$\mathbf{R}\mathbf{x}\mathbf{R}^\dagger = \big(x\cos(2\varphi) - y\sin(2\varphi)\big)\mathbf{e}_1 + \big(x\sin(2\varphi) + y\cos(2\varphi)\big)\mathbf{e}_2 + z\mathbf{e}_3,$$

which describes a rotation by the angle 2φ *in the xy-plane. The details of rotations will be described in Section 2.3.3.*

This could be a good point to take a moment and reflect (no pun intended) on the simplicity and generality of the geometric product. The simple geometric condition of the defining equation (2.4) gives rise to the rules in (2.6), which are not obviously geometric in nature and introduce the concept of anticommutation, which does not have a foundational role in standard linear algebra.

Example 2.1.6. *It is difficult to see how the geometric algebra could be any simpler than it already is. For example, suppose that you decide to require commutativity (*$\mathbf{AB} = \mathbf{BA}$*) in addition to the defining equation* (2.4)*. Then, all higher-grade elements* $\mathbf{e}_1\mathbf{e}_2$ *etc., will be zero since, for example,* $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2$*. This leads to a contradiction (assuming that the dimension of the underlying space is at least* 2*), since for example*

$$\mathbf{e}_2 = 1\,\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1\,0 = 0,$$

but $\mathbf{e}_2 \neq 0$.

2.2 Structure of the geometric algebra

Having completed the construction of $\mathcal{G}(\mathbb{R}^{p,q})$ and given some examples of concrete computation in the previous section, this section is dedicated to defining a collection of new operations in terms of the geometric product and proving their general properties in a more abstract manner. Most proofs have been written out in considerable detail for the benefit of the reader. However, attempting the proofs oneself can be recommended as an excellent way of gaining familiarity with the algebra.

2.2.1 Proofs and linear extension

As the multivectors are constructed with explicit reference to a standard basis, a common theme will be linear extension: defining something for the basis elements only, and then extending the definition to the entire algebra by requiring it to be a linear function. This is the basic premise of linear algebra and, though no doubt familiar, is worth dwelling on for a moment because of its importance:

Lemma 2.2. *For any vector space* V*, (for example,* $V = \mathcal{G}(\mathbb{R}^{p,q})$*), a function* $f\colon \mathcal{G}(\mathbb{R}^{p,q}) \to V$ *can be uniquely specified by giving its values on the* 2^{p+q} *basis blades and requiring that it be a linear function.*

Proof. Take $\mathcal{G}(\mathbb{R}^2)$ as an example (the general case is analogous). f is specified by

$$f(a + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_{12}\mathbf{e}_1\mathbf{e}_2) = af(1) + a_1 f(\mathbf{e}_1) + a_2 f(\mathbf{e}_2) + a_{12} f(\mathbf{e}_1\mathbf{e}_2). \qquad \blacksquare$$

In the same spirit, linear equalities can be economically proven:

Lemma 2.3. *Suppose that* $f\colon \mathcal{G}(\mathbb{R}^{p,q}) \to V$ *and* $g\colon \mathcal{G}(\mathbb{R}^{p,q}) \to V$ *are linear functions that agree on the basis blades. Then* $f(\mathbf{A}) = g(\mathbf{A})$ *for any multivector* $\mathbf{A}$.

Proof. Again, take $\mathcal{G}(\mathbb{R}^2)$ as an example (the general case is again analogous).

$$\begin{aligned} f(a + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_{12}\mathbf{e}_1\mathbf{e}_2) &= af(1) + a_1 f(\mathbf{e}_1) + a_2 f(\mathbf{e}_2) + a_{12} f(\mathbf{e}_1\mathbf{e}_2) \\ &= ag(1) + a_1 g(\mathbf{e}_1) + a_2 g(\mathbf{e}_2) + a_{12} g(\mathbf{e}_1\mathbf{e}_2) \\ &= g(a + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_{12}\mathbf{e}_1\mathbf{e}_2). \qquad \blacksquare \end{aligned}$$

Lemmas 2.2 and 2.3 will be the basic tools of this section. They will also be used in variants where the domain is a different vector space than $\mathcal{G}(\mathbb{R}^{p,q})$, for example $\mathbb{R}^{p,q}$.

2.2.2 The combinatorial view

To develop a concise method for defining operators and proving formulae, it will be useful to think of a canonical basis blade $\mathbf{X} = \mathbf{e}_{i_1} \ldots \mathbf{e}_{i_k}$, with $i_1 < \cdots < i_k$, as being identical to the set of its indices $\{i_1, \ldots, i_k\}$. For example, we will write $\mathbf{e}_1\mathbf{e}_2 = \{1, 2\}$ and $\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -\{1, 2, 3\}$. In this view, it can be seen by rearranging into canonical form as in Example 2.1.3 that, for two canonical basis blades $\mathbf{X}$ and $\mathbf{Y}$,

$$\mathbf{X}\mathbf{Y} = \pm \mathbf{X} \bigtriangleup \mathbf{Y}. \tag{2.7}$$

where $\bigtriangleup$ denotes the *symmetric difference* (in computer terminology, exclusive OR) of two sets:

$$\mathbf{X} \bigtriangleup \mathbf{Y} \coloneqq (\mathbf{X} \cup \mathbf{Y}) \smallsetminus (\mathbf{X} \cap \mathbf{Y}). \tag{2.8}$$

Example 2.2.1. *Let* $\mathbf{A} = \mathbf{e}_1\mathbf{e}_4\mathbf{e}_3 = -\{1, 3, 4\}$ *and* $\mathbf{B} = \mathbf{e}_2\mathbf{e}_1 = -\{1, 2\}$ *(they are not canonical basis blades because of the minus signs). By repeated application of* (2.6) *we find* $\mathbf{AB} = -\mathbf{e}_1\mathbf{e}_4\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4 = -\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4 = -\{2, 3, 4\}$*. Indeed,* $\{1, 3, 4\} \bigtriangleup \{1, 2\} = \{2, 3, 4\}$.

2.2.3 The outer product

Look back to Equation (2.5). It states that the inner product can be thought of as a symmetrization of the geometric product. It is then natural to define a kind of dual to the inner product, called the *outer*, *exterior* or *wedge product* (all synonyms), that is the corresponding anti-symmetrization:

$$\mathbf{x} \wedge \mathbf{y} \coloneqq \frac{\mathbf{xy} - \mathbf{yx}}{2}. \tag{2.9}$$

Clearly then,

$$\mathbf{xy} = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \wedge \mathbf{y}. \tag{2.10}$$

One could extend the definition (2.9) to arbitrary multivectors rather than only vectors, but it turns out that the following definition, the first of many similar definitions, is more fruitful. We use the "Boolean" notation

$$(P) \coloneqq \begin{cases} 1 & P \text{ is true} \\ 0 & P \text{ is false} \end{cases} \tag{2.11}$$

in combination with the combinatorial view of Section 2.2.2.

Definition 2.1. *For any two canonical basis blades* $\mathbf{X}$ *and* $\mathbf{Y}$*,*

$$\mathbf{X} \wedge \mathbf{Y} \coloneqq (\mathbf{X} \cap \mathbf{Y} = \varnothing)\, \mathbf{XY}. \tag{2.12}$$

Moreover, $\wedge$ *is required to be bilinear, which extends the definition to the whole of* $\mathcal{G}(\mathbb{R}^{p,q})$*.*

Example 2.2.2. *Let us compute two outer products. The second example hints at the connection to the cross product, which is described in Example 2.2.8.*

$$\begin{aligned}(\mathbf{e}_1 + 2\mathbf{e}_2) \wedge (\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_3) &= \mathbf{e}_1 \wedge \mathbf{e}_1 + 2\mathbf{e}_2 \wedge \mathbf{e}_1 + \mathbf{e}_1 \wedge (\mathbf{e}_2\mathbf{e}_3) + 2\mathbf{e}_2 \wedge (\mathbf{e}_2\mathbf{e}_3)\\ &= 2\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -2\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3.\\ (x_1\mathbf{e}_1 + y_1\mathbf{e}_2) \wedge (x_2\mathbf{e}_1 + y_2\mathbf{e}_2) &= x_1x_2\mathbf{e}_1 \wedge \mathbf{e}_1 + y_1x_2\mathbf{e}_2 \wedge \mathbf{e}_1 + x_1y_2\mathbf{e}_1 \wedge \mathbf{e}_2 + y_1y_2\mathbf{e}_2 \wedge \mathbf{e}_2\\ &= y_1x_2\mathbf{e}_2\mathbf{e}_1 + x_1y_2\mathbf{e}_1\mathbf{e}_2 = (x_1y_2 - y_1x_2)\mathbf{e}_1\mathbf{e}_2.\end{aligned}$$

The next theorem shows how this type of definition can be manipulated when proving formulae:

Theorem 2.4. *The wedge product is associative: For any multivectors* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{G}(\mathbb{R}^{p,q})$*,*

$$(\mathbf{A} \wedge \mathbf{B}) \wedge \mathbf{C} = \mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}). \tag{2.13}$$

Proof. First, note that both sides are linear functions of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Therefore, by Lemma 2.3, it is sufficient to prove the formula for arbitrary canonical basis blades. This reduces the proof to set logic. Compare the left and right hand sides:

$$\begin{aligned}(\mathbf{X} \wedge \mathbf{Y}) \wedge \mathbf{Z} &= \big((\mathbf{X} \cap \mathbf{Y} = \varnothing)\, \mathbf{XY}\big) \wedge \mathbf{Z} && \text{applying (2.12)}\\ &= (\mathbf{X} \cap \mathbf{Y} = \varnothing)\, (\mathbf{XY}) \wedge \mathbf{Z} &&\\ &= (\mathbf{X} \cap \mathbf{Y} = \varnothing)\big((\mathbf{X} \triangle \mathbf{Y}) \cap \mathbf{Z} = \varnothing\big)\, \mathbf{XYZ} && \text{applying (2.12) and (2.7)}\\ &= (\mathbf{X} \cap \mathbf{Y} = \varnothing)\big((\mathbf{X} \cup \mathbf{Y}) \cap \mathbf{Z} = \varnothing\big)\, \mathbf{XYZ} && \text{set logic}\\ &= (\mathbf{X} \cap \mathbf{Y} = \varnothing)(\mathbf{X} \cap \mathbf{Z} = \varnothing)(\mathbf{Y} \cap \mathbf{Z} = \varnothing)\, \mathbf{XYZ} && \text{set logic,}\end{aligned}$$

$$\begin{aligned}\mathbf{X} \wedge (\mathbf{Y} \wedge \mathbf{Z}) &= \mathbf{X} \wedge \big((\mathbf{Y} \cap \mathbf{Z} = \varnothing)\, \mathbf{YZ}\big) && \text{applying (2.12)}\\ &= (\mathbf{Y} \cap \mathbf{Z} = \varnothing)\, \mathbf{X} \wedge (\mathbf{YZ}) &&\\ &= \big(\mathbf{X} \cap (\mathbf{Y} \triangle \mathbf{Z}) = \varnothing\big)(\mathbf{Y} \cap \mathbf{Z} = \varnothing)\, \mathbf{XYZ} && \text{applying (2.12) and (2.7)}\\ &= \big(\mathbf{X} \cap (\mathbf{Y} \cup \mathbf{Z}) = \varnothing\big)(\mathbf{Y} \cap \mathbf{Z} = \varnothing)\, \mathbf{XYZ} && \text{set logic}\\ &= (\mathbf{X} \cap \mathbf{Y} = \varnothing)(\mathbf{X} \cap \mathbf{Z} = \varnothing)(\mathbf{Y} \cap \mathbf{Z} = \varnothing)\, \mathbf{XYZ} && \text{set logic.}\end{aligned}$$

The expressions are identical. ∎

Theorem 2.5. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors, the definitions* (2.9) *and* (2.12) *are equivalent.*

Proof. Both definitions are clearly bilinear, so use a variant of Lemma 2.3 where the domain is $\mathbb{R}^{p,q}$. It then remains to prove that the two expressions are the same for arbitrary basis vectors $\mathbf{x} = \mathbf{e}_i = \{i\}$ and $\mathbf{y} = \mathbf{e}_j = \{j\}$. The definition simplifies:

$$\begin{aligned}\mathbf{e}_i \wedge \mathbf{e}_j &= (\{i\} \cap \{j\} = \varnothing)\, \mathbf{e}_i\mathbf{e}_j &&\\ &= (i \neq j)\, \mathbf{e}_i\mathbf{e}_j &&\\ &= (i \neq j)\, \frac{\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_i\mathbf{e}_j}{2} &&\\ &= (i \neq j)\, \frac{\mathbf{e}_i\mathbf{e}_j - \mathbf{e}_j\mathbf{e}_i}{2} && \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \text{ when } i \neq j\\ &= \frac{\mathbf{e}_i\mathbf{e}_j - \mathbf{e}_j\mathbf{e}_i}{2} && \mathbf{e}_i\mathbf{e}_j - \mathbf{e}_j\mathbf{e}_i = 0 \text{ when } i = j\\ &= \frac{\mathbf{xy} - \mathbf{yx}}{2}.\end{aligned}$$

∎

Corollary 2.5.1. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors, then* $\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}$.

Corollary 2.5.2. *If* $\mathbf{x}$ *is a vector, then* $\mathbf{x} \wedge \mathbf{x} = 0$.

2.2.4 Exterior algebra

In the previous section, the exterior product was defined in terms of the geometric product. However, the exterior product can also stand on its own and was in fact historically invented before the geometric product. Instead of the geometric algebra, another algebra can be constructed by enforcing a defining equation different from (2.4). Start with the tensor algebra $\mathcal{T}(\mathbb{R}^{p,q})$ as described in Section 2.1.2, but to avoid confusion, write the tensor product as $\mathbf{A} \wedge \mathbf{B}$ rather than $\mathbf{AB}$. Then, add the defining equation (require that Corollary 2.5.2 holds):

For any vector $\mathbf{x}$,

$$\mathbf{x} \wedge \mathbf{x} = 0. \tag{2.14}$$

The algebra thus obtained is called the *exterior algebra* over $\mathbb{R}^{p,q}$ and is denoted $\wedge^*(\mathbb{R}^{p,q})$. A short introductory treatment of exterior algebra (albeit single-grade, $\wedge^k(\mathbb{R}^n)$) is given in [5]. The product is called the exterior product because it will turn out to have precisely the same properties as the product of Definition 2.1. In particular, it is automatically associative since it inherits that property from the tensor product. That $\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}$ for any vectors $\mathbf{x}$ and $\mathbf{y}$ (the statement of Corollary 2.5.1) can be seen by expanding $(\mathbf{x}+\mathbf{y}) \wedge (\mathbf{x}+\mathbf{y})$:

$$\begin{aligned}
0 &= (\mathbf{x}+\mathbf{y}) \wedge (\mathbf{x}+\mathbf{y}) && \text{by the defining equation (2.14)} \\
&= \mathbf{x} \wedge \mathbf{x} + \mathbf{x} \wedge \mathbf{y} + \mathbf{y} \wedge \mathbf{x} + \mathbf{y} \wedge \mathbf{y} && \text{expanding the product} \\
&= \mathbf{x} \wedge \mathbf{y} + \mathbf{y} \wedge \mathbf{x} && \text{by (2.14) again.}
\end{aligned}$$

An attractive feature of viewing the outer product this way is that the definition of the exterior algebra makes no mention of the scalar product, the geometric product or the signature of $\mathbb{R}^{p,q}$. This means that, as far as the wedge product is concerned, $\mathbb{R}^n = \mathbb{R}^{p+q}$ and $\mathbb{R}^{p,q}$ are the same space. This fact will be used to simplify the proof of Theorem 2.8 in Section 2.2.6.

2.2.5 Generalizations of the inner product

In Section 2.2.3 the outer product was defined, first for vectors and then for arbitrary multivectors, but the inner product has not yet been defined for arbitrary multivectors. There are many similar ways to define inner products, but the following are perhaps the most elegant:

Definition 2.2. *Given two canonical basis blades* $\mathbf{X}$ *and* $\mathbf{Y}$*, the* ***scalar product*** $\mathbf{X} * \mathbf{Y}$*, the* ***left inner product*** $\mathbf{X} \llcorner \mathbf{Y}$ *and the* ***right inner product*** $\mathbf{X} \lrcorner \mathbf{Y}$ *are defined by*

$$\mathbf{X} * \mathbf{Y} \coloneqq (\mathbf{X} = \mathbf{Y})\, \mathbf{XY} \tag{2.15}$$

$$\mathbf{X} \llcorner \mathbf{Y} \coloneqq (\mathbf{X} \subseteq \mathbf{Y})\, \mathbf{XY} \tag{2.16}$$

$$\mathbf{X} \lrcorner \mathbf{Y} \coloneqq (\mathbf{X} \supseteq \mathbf{Y})\, \mathbf{XY} \tag{2.17}$$

The definitions are extended linearly to the whole of $\mathcal{G}(\mathbb{R}^{p,q})$.

Of these, the left inner product will be the most used. Note that in the case of two vectors, the products are all equal to each other and to the usual scalar product:

$$\mathbf{x} * \mathbf{y} = \mathbf{x} \llcorner \mathbf{y} = \mathbf{x} \lrcorner \mathbf{y} = \mathbf{x} \cdot \mathbf{y} \tag{2.18}$$

and are given by linearly extending

$$\mathbf{e}_i * \mathbf{e}_j = \mathbf{e}_i \llcorner \mathbf{e}_j = \mathbf{e}_i \lrcorner \mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_j = (i = j)\, \mathbf{e}_i \mathbf{e}_j. \tag{2.19}$$

Keep in mind that the notation for these products varies; see Section 2.4.1.

A few comments on the scalar product $\mathbf{A} * \mathbf{B}$. It is immediate from its definition (by linearity) that it is commutative;

$$\mathbf{A} * \mathbf{B} = \mathbf{B} * \mathbf{A} \tag{2.20}$$

for all multivectors $\mathbf{A}$ and $\mathbf{B}$. It can also be shown using straightforward set logic that

$$\mathbf{A} * \mathbf{B} = \langle \mathbf{AB} \rangle_0 \tag{2.21}$$

where $\langle \mathbf{X} \rangle_0 \coloneqq (\mathbf{X} = \varnothing)\, \mathbf{X}$ (extended linearly) is the **scalar part** of $\mathbf{X}$ (see also (2.28)). It is perhaps a little surprising that $\langle \mathbf{AB} \rangle_0 = \langle \mathbf{BA} \rangle_0$ for arbitrary multivectors.

Turning now to the left inner product, the following formula is taken from [1, Proposition 2.9];

Theorem 2.6. *For any multivectors* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$,

$$(\mathbf{A} \wedge \mathbf{B}) \llcorner \mathbf{C} = \mathbf{A} \llcorner (\mathbf{B} \llcorner \mathbf{C}). \tag{2.22}$$

Proof. Both sides are linear in $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, so by Lemma 2.3, it is sufficient to prove the formula for arbitrary canonical basis blades, which takes nothing but set logic:

$$\begin{aligned}
(\mathbf{A} \wedge \mathbf{B}) \llcorner \mathbf{C} &= (\mathbf{A} \cap \mathbf{B} = \varnothing)\, (\mathbf{AB}) \llcorner \mathbf{C} \\
&= (\mathbf{A} \cap \mathbf{B} = \varnothing)(\mathbf{A} \vartriangle \mathbf{B} \subseteq \mathbf{C})\, \mathbf{ABC} \\
&= (\mathbf{A} \cap \mathbf{B} = \varnothing)(\mathbf{A} \cup \mathbf{B} \subseteq \mathbf{C})\, \mathbf{ABC} \\
&= (\mathbf{A} \cap \mathbf{B} = \varnothing)(\mathbf{A} \subseteq \mathbf{C})(\mathbf{B} \subseteq \mathbf{C})\, \mathbf{ABC} \\
&= (\mathbf{A} \cap \mathbf{B} = \varnothing)(\mathbf{A} \subseteq \mathbf{C})\, \mathbf{A}(\mathbf{B} \llcorner \mathbf{C}) \\
&= (\mathbf{A} \subseteq \mathbf{C} \setminus \mathbf{B})\, \mathbf{A}(\mathbf{B} \llcorner \mathbf{C}) \\
&= (\mathbf{A} \subseteq \mathbf{B} \vartriangle \mathbf{C})\, \mathbf{A}(\mathbf{B} \llcorner \mathbf{C}) \\
&= \mathbf{A} \llcorner (\mathbf{B} \llcorner \mathbf{C}).
\end{aligned}$$

∎

The following formula, adapted from [6, Equation (3.70)], is also quite powerful:

Theorem 2.7. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors and* $\mathbf{A}$ *is any multivector, then*

$$\mathbf{x} \llcorner (\mathbf{y} \wedge \mathbf{A}) = (\mathbf{x} \cdot \mathbf{y})\, \mathbf{A} - \mathbf{y} \wedge (\mathbf{x} \llcorner \mathbf{A}). \tag{2.23}$$

Proof. Again by linearity, consider only $\mathbf{x} = \mathbf{e}_i$, $\mathbf{y} = \mathbf{e}_j$ and $\mathbf{A} = \mathbf{X}$, an arbitrary canonical basis blade.

$$\begin{aligned}
\mathbf{e}_i \llcorner (\mathbf{e}_j \wedge \mathbf{X}) &= (j \notin \mathbf{X})\, \mathbf{e}_i \llcorner (\mathbf{e}_j \mathbf{X}) && \text{by (2.12)} \\
&= (j \notin \mathbf{X})(i \in \{j\} \cup \mathbf{X})\, \mathbf{e}_i \mathbf{e}_j \mathbf{X} && \text{by (2.16)} \\
&= (j \notin \mathbf{X})[(i = j) + (i \in \mathbf{X})]\, \mathbf{e}_i \mathbf{e}_j \mathbf{X} && i = j \text{ and } i \in \mathbf{X} \text{ disjoint} \\
&= (j \notin \mathbf{X})[(\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{X} + (i \in \mathbf{X})\mathbf{e}_i \mathbf{e}_j \mathbf{X}] && \text{by (2.19)} \\
&= (j \notin \mathbf{X})[(\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{X} - (i \in \mathbf{X})\mathbf{e}_j \mathbf{e}_i \mathbf{X}] && \mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i \text{ if } i \neq j \\
&= (j \notin \mathbf{X})[(\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{X} - \mathbf{e}_j (\mathbf{e}_i \llcorner \mathbf{X})] && \text{by (2.16)} \\
&= (j \notin \mathbf{X})[(\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{X} - \mathbf{e}_j \wedge (\mathbf{e}_i \llcorner \mathbf{X})] && \text{by (2.12); } j \notin \mathbf{X} \Rightarrow j \notin \mathbf{X} \setminus \{i\}.
\end{aligned}$$

The formula is proven if it can be shown that $(\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{X} - \mathbf{e}_j \wedge (\mathbf{e}_i \llcorner \mathbf{X})$ is zero whenever $j \in \mathbf{X}$; then the factor of $(j \notin \mathbf{X})$ is superfluous. Assume therefore that $j \in \mathbf{X}$. If $i \notin \mathbf{X}$, both terms are zero. If $i \in \mathbf{X}$ and $i \neq j$, then $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ and $\mathbf{e}_j \wedge (\mathbf{e}_i \llcorner \mathbf{X}) = 0$ because $j \in \mathbf{X} \setminus \{i\}$. Finally, if $i \in \mathbf{X}$ and $i = j$, then $(\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{X} = \mathbf{X}$ and $\mathbf{e}_j \wedge (\mathbf{e}_i \llcorner \mathbf{X}) = \mathbf{X}$, so the difference is 0. ■

The formulae (2.22) and (2.23) can be used to give expanded expressions for the inner products between blades (blades will be introduced in Section 2.2.6), as the following examples show. See also Example 2.2.7.

Example 2.2.3. *Setting* $\mathbf{A}$ *to a vector* $\mathbf{z}$ *in* (2.23)*, the formula simplifies to*

$$\mathbf{x} \llcorner (\mathbf{y} \wedge \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y})\, \mathbf{z} - (\mathbf{x} \cdot \mathbf{z})\, \mathbf{y}. \tag{2.24}$$

Notice that this is the negation of the well-known expression for the three-dimensional triple cross product: $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z})\, \mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\, \mathbf{z}$. *See Example 2.2.8 for an elaboration on this.*

Example 2.2.4. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$ *and* $\mathbf{d}$ *be vectors. Let us evaluate* $(\mathbf{a} \wedge \mathbf{b}) \llcorner (\mathbf{c} \wedge \mathbf{d})$.

$$\begin{aligned} (\mathbf{a} \wedge \mathbf{b}) \llcorner (\mathbf{c} \wedge \mathbf{d}) &= \mathbf{a} \llcorner \big(\mathbf{b} \llcorner (\mathbf{c} \wedge \mathbf{d})\big) && \text{by (2.22)} \\ &= \mathbf{a} \llcorner \big((\mathbf{b} \cdot \mathbf{c})\, \mathbf{d} - (\mathbf{b} \cdot \mathbf{d})\, \mathbf{c}\big) && \text{by (2.24)} \\ &= (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \llcorner \mathbf{d}) - (\mathbf{b} \cdot \mathbf{d})(\mathbf{a} \llcorner \mathbf{c}) \\ &= (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{d})(\mathbf{a} \cdot \mathbf{c}) && \text{by (2.18).} \end{aligned}$$

We have thus proven the Binet–Cauchy identity*,*

$$(\mathbf{a} \wedge \mathbf{b}) \llcorner (\mathbf{c} \wedge \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{d})(\mathbf{a} \cdot \mathbf{c}). \tag{2.25}$$

2.2.6 Blades and linear subspaces

In contrast to the geometric product, the wedge product has a clear geometrical interpretation: If $\mathbf{x}$ and $\mathbf{y}$ are vectors, $\mathbf{x} \wedge \mathbf{y}$ should be thought of as the plane spanned by $\mathbf{x}$ and $\mathbf{y}$. In general, any linear subspace of $\mathbb{R}^{p,q}$ can be represented in by a multivector in the following way:

Definition 2.3. *Any multivector* $\mathbf{B}$ *that can be written in the form*

$$\mathbf{B} = \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k, \tag{2.26}$$

where the $\mathbf{x}_i$ *are vectors, is called a* ***blade*** *(more specifically, a* k***-blade****).*

Definition 2.4. *Let* $\mathbf{B} \in \mathcal{G}(\mathbb{R}^{p,q})$ *be a blade. The* ***outer null space*** *of* $\mathbf{B}$ *is*

$$\overline{\mathbf{B}} \coloneqq \{\mathbf{x} \in \mathbb{R}^{p,q} \mid \mathbf{B} \wedge \mathbf{x} = 0\}. \tag{2.27}$$

From the definition, it is clear that $\overline{\mathbf{B}}$ is a linear subspace of $\mathbb{R}^{p,q}$. The next theorem shows which subspace it is.

Theorem 2.8. *Let* $\mathbf{x}_1, \ldots, \mathbf{x}_k$ *be* k *linearly independent vectors and let* $\mathbf{B} \coloneqq \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k$. *Then* $\overline{\mathbf{B}} = \operatorname{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.

Proof sketch. The following is a rather high-level argument, and would require some work in filling out the details to make a formally complete proof.

Notice that the theorem is formulated exclusively in terms of the wedge product. We are therefore examining the exterior algebra. As indicated in Section 2.2.4, any result in the exterior algebra cannot depend on the signature. Therefore, let us assume, *without loss of generality*, that the space is Euclidean ($\mathbb{R}^n = \mathbb{R}^{n,0}$). This manoeuvre makes all the tools of standard linear algebra available.

In particular, $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ can be orthonormalized (e.g. by the Gram–Schmidt method) to form an orthonormal basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_k\}$ of $\mathrm{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ such that

$$\begin{aligned}
\mathbf{x}_1 &= a_{11}\mathbf{e}'_1 \\
\mathbf{x}_2 &= a_{21}\mathbf{e}'_1 + a_{22}\mathbf{e}'_2 \\
&\vdots \\
\mathbf{x}_k &= a_{k1}\mathbf{e}'_1 + a_{k2}\mathbf{e}'_2 + \dots + a_{kk}\mathbf{e}'_k.
\end{aligned}$$

Then, $\mathbf{x}_1 \wedge \mathbf{x}_2 = a_{11}\mathbf{e}'_1 \wedge (a_{21}\mathbf{e}'_1 + a_{22}\mathbf{e}'_2) = a_{11}a_{22}\,\mathbf{e}'_1 \wedge \mathbf{e}'_2$, and in general,

$$\mathbf{B} = a_{11}a_{22}\dots a_{kk}\ \mathbf{e}'_1 \wedge \mathbf{e}'_2 \wedge \dots \wedge \mathbf{e}'_k.$$

The set $\{\mathbf{e}'_1, \dots, \mathbf{e}'_k\}$ can be extended to an orthogonal basis $\mathfrak{B}' = \{\mathbf{e}'_1, \dots, \mathbf{e}'_k, \mathbf{e}'_{k+1}, \dots, \mathbf{e}'_n\}$ of the whole of $\mathbb{R}^n$. Now (this is the informal part), the new basis vectors have all the algebraic properties of the standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$. Therefore, the exterior algebras (and indeed the geometric algebras) generated by the two bases are isomorphic. This means that

$$\begin{aligned}
(\mathbf{e}'_1 \wedge \mathbf{e}'_2 \wedge \dots \wedge \mathbf{e}'_k) \wedge \mathbf{e}'_j &= (\{1\}, \{2\}, \dots, \{k\}, \{j\} \text{ all disjoint})\, \mathbf{e}'_1\mathbf{e}'_2 \dots \mathbf{e}'_k\mathbf{e}'_j && \text{by (2.12)} \\
&= (1, 2, \dots, k, j \text{ all different})\, \mathbf{e}'_1\mathbf{e}'_2 \dots \mathbf{e}'_k\mathbf{e}'_j \\
&= (j \notin \{1, 2, \dots, k\})\, \mathbf{e}'_1\mathbf{e}'_2 \dots \mathbf{e}'_k\mathbf{e}'_j.
\end{aligned}$$

For an arbitrary vector $\mathbf{x} = x_1\mathbf{e}'_1 + \dots + x_n\mathbf{e}'_n$, since the elements

$$(\mathbf{e}'_1\mathbf{e}'_2 \dots \mathbf{e}'_k\mathbf{e}'_{k+1}),\ \dots,\ (\mathbf{e}'_1\mathbf{e}'_2 \dots \mathbf{e}'_k\mathbf{e}'_n)$$

are all linearly independent, $\mathbf{B} \wedge \mathbf{x} = 0$ must be equivalent to that $x_{k+1} = \dots = x_n = 0$ which means precisely that $\mathbf{x} \in \mathrm{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$. ■

This theorem may be phrased in another way:

Corollary 2.8.1. *Let $\mathbf{x}_1, \dots, \mathbf{x}_k$ be vectors. Then $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k = 0$ if and only if they are linearly dependent.*

We now turn to the algebraic properties of blades. For a general multivector $\mathbf{A}$, not much can be said about $\mathbf{A}^2$ or even $\mathbf{A} \wedge \mathbf{A}$. For blades, the situation is more orderly.

Lemma 2.9. *For any blade $\mathbf{B}$, $\mathbf{B} \wedge \mathbf{B} = 0$.*

Proof. Since $\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}$,

$$\begin{aligned}
&\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k\ \wedge\ \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k \\
&= \pm\ \mathbf{x}_k \wedge \dots \wedge \underbrace{\mathbf{x}_1\ \wedge\ \mathbf{x}_1}_{0} \wedge \dots \wedge \mathbf{x}_k = 0.
\end{aligned}$$ ■

Theorem 2.10 (Proposition 3.2 in [1]). *Every blade* $\mathbf{B} = \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k$ *can be written as a* geometric *product* $\mathbf{B} = \mathbf{y}_1 \ldots \mathbf{y}_k$ *where* $\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$ *is an orthogonal set.*

Proof. By Theorem A.2, there exists an orthogonal basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ of $\overline{\mathbf{B}}$. Writing

$$\mathbf{x}_i = \sum_{j=1}^{k} a_{ij} \mathbf{b}_j,$$

we find that

$$\mathbf{B} = \lambda \; \mathbf{b}_1 \ldots \mathbf{b}_k$$

where $\lambda \in \mathbb{R}$ is some function of the a_{ij} (in fact, $\lambda = \det[a_{ij}]$, as is argued in [1] by noting that λ is multilinear and alternating in the $\mathbf{b}_j$). ∎

Corollary 2.10.1. *If* $\mathbf{B}$ *is a blade,* $\mathbf{B}^2$ *is a scalar.*

Proof. By Theorem 2.10, write $\mathbf{B}$ as a geometric product $\mathbf{B} = \mathbf{y}_1 \ldots \mathbf{y}_k$ where the $\mathbf{y}_i$ are orthogonal to each other. Then,

$$\begin{aligned}
\mathbf{y}_1 \ldots \mathbf{y}_k \; \mathbf{y}_1 \ldots \mathbf{y}_k &= \pm \; \mathbf{y}_k \ldots \mathbf{y}_2 \mathbf{y}_1^2 \mathbf{y}_2 \ldots \mathbf{y}_k && \text{orthogonal vectors anticommute} \\
&= \pm \; \mathbf{y}_1^2 \; \mathbf{y}_k \ldots \mathbf{y}_3 \mathbf{y}_2^2 \mathbf{y}_3 \ldots \mathbf{y}_k && \mathbf{y}_1^2 \in \mathbb{R} \text{ and scalars commute} \\
&\;\;\vdots \\
&= \pm \; \mathbf{y}_1^2 \mathbf{y}_2^2 \ldots \mathbf{y}_k^2.
\end{aligned}$$

∎

Remark. *It can also be shown (though it is not as easy), that any multivector* $\mathbf{A}$ *with* $\mathbf{A}^2 \in \mathbb{R}$ *must be a blade (see [1, Example 3.1 and Exercise 6.12]).*

Corollary 2.10.2. *Any blade* $\mathbf{B}$ *with* $\mathbf{B}^2 \neq 0$ *(a* non-null *blade) is invertible with respect to the geometric product.*

Proof. The inverse is $\mathbf{B}^{-1} = \frac{\mathbf{B}}{\mathbf{B}^2}$. ∎

Example 2.2.5. *A standard problem in linear algebra is solving systems of linear equations. For example, a three-dimensional system can be written* $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{r}$*, where* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{r} \in \mathbb{R}^3$ *are given and* $x, y, z \in \mathbb{R}$ *are unknown. The system can be solved for e.g.* y *using the exterior product: Left-multiplying by* $\mathbf{a}$ *and right-multiplying by* $\mathbf{c}$ *gives* $\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c} = y \; \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$*. Therefore, since blades are invertible,* $y = (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c})(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}$*, and similarly for* x *and* z*.*

The **grade** of a nonzero blade $\mathbf{B}$ can be defined as $\operatorname{gr} \mathbf{B} \coloneqq \dim \overline{\mathbf{B}}$. A blade of grade k is called a *k-blade*. The zero element is a k-blade for every k. Following the notation of [6], we will sometimes write $\mathbf{A}_{\langle k \rangle}$, $\mathbf{B}_{\langle k \rangle}$ etc. for blades of grade k. A *k-vector* or *multivector of grade k* is a linear combination of k-blades. The set of all k-vectors is a linear subspace of $\mathcal{G}(\mathbb{R}^{p,q})$ denoted $\mathcal{G}^k(\mathbb{R}^{p,q})$.

Example 2.2.6. *In three dimensions, we have*

$$\begin{aligned}
\mathcal{G}^0(\mathbb{R}^3) &= \operatorname{Span}\{1\}, & \mathcal{G}^2(\mathbb{R}^3) &= \operatorname{Span}\{\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3\}, \\
\mathcal{G}^1(\mathbb{R}^3) &= \operatorname{Span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}, & \mathcal{G}^3(\mathbb{R}^3) &= \operatorname{Span}\{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}.
\end{aligned}$$

The **grade projection** onto grade k is defined for all canonical basis blades $\mathbf{X}$ by

$$\langle \mathbf{X} \rangle_k \coloneqq (|\mathbf{X}| = k)\, \mathbf{X} \tag{2.28}$$

where $|\mathbf{X}|$ is the cardinality of $\mathbf{X}$, and extended linearly to the whole of $\mathcal{G}(\mathbb{R}^{p,q})$. It is then readily seen that a number of definitions can be specialized for k-vectors:

$$\mathbf{A}_{\langle k \rangle} \wedge \mathbf{B}_{\langle l \rangle} = \left\langle \mathbf{A}_{\langle k \rangle} \mathbf{B}_{\langle l \rangle} \right\rangle_{k+l} \tag{2.29}$$

$$\mathbf{A}_{\langle k \rangle} \llcorner \mathbf{B}_{\langle l \rangle} = \left\langle \mathbf{A}_{\langle k \rangle} \mathbf{B}_{\langle l \rangle} \right\rangle_{l-k} \tag{2.30}$$

$$\mathbf{A}_{\langle k \rangle} \lrcorner \mathbf{B}_{\langle l \rangle} = \left\langle \mathbf{A}_{\langle k \rangle} \mathbf{B}_{\langle l \rangle} \right\rangle_{k-l}. \tag{2.31}$$

Proof. We prove (2.30). The others can be proven very similarly. Since the k-vectors form a vector space (and so do the l-vectors) and both sides are bilinear, it suffices to consider canonical basis k- and l-blades. Compare the expressions:

$$\mathbf{X}_{\langle k \rangle} \llcorner \mathbf{Y}_{\langle l \rangle} = (\mathbf{X}_{\langle k \rangle} \subseteq \mathbf{Y}_{\langle l \rangle})\, \mathbf{X}_{\langle k \rangle} \mathbf{Y}_{\langle l \rangle},$$

$$\left\langle \mathbf{X}_{\langle k \rangle} \mathbf{Y}_{\langle l \rangle} \right\rangle_{l-k} = \left(\left| \mathbf{X}_{\langle k \rangle} \triangle \mathbf{Y}_{\langle l \rangle} \right| = l - k \right) \mathbf{X}_{\langle k \rangle} \mathbf{Y}_{\langle l \rangle}.$$

We must show that $\mathbf{X}_{\langle k \rangle} \subseteq \mathbf{Y}_{\langle l \rangle}$ is equivalent to $\left| \mathbf{X}_{\langle k \rangle} \triangle \mathbf{Y}_{\langle l \rangle} \right| = l - k$. We have

$$\begin{aligned}
\left| \mathbf{X}_{\langle k \rangle} \triangle \mathbf{Y}_{\langle l \rangle} \right| &= \left| \mathbf{X}_{\langle k \rangle} \cup \mathbf{Y}_{\langle l \rangle} \right| - \left| \mathbf{X}_{\langle k \rangle} \cap \mathbf{Y}_{\langle l \rangle} \right| \\
&= \left| \mathbf{X}_{\langle k \rangle} \right| + \left| \mathbf{Y}_{\langle l \rangle} \right| - 2 \left| \mathbf{X}_{\langle k \rangle} \cap \mathbf{Y}_{\langle l \rangle} \right| \\
&= k + l - 2 \left| \mathbf{X}_{\langle k \rangle} \cap \mathbf{Y}_{\langle l \rangle} \right| \\
&= l - k \iff \left| \mathbf{X}_{\langle k \rangle} \cap \mathbf{Y}_{\langle l \rangle} \right| = k,
\end{aligned}$$

and this is certainly equivalent to $\mathbf{X}_{\langle k \rangle} \subseteq \mathbf{Y}_{\langle l \rangle}$. ∎

A large enough number of results have now been collected that many useful formulae can be quickly derived. The sequence of steps in the following derivation is quite interesting:

Example 2.2.7. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two vectors. We wish to evaluate* $(\mathbf{x} \wedge \mathbf{y})^2$. *Set* $\mathbf{B} \coloneqq \mathbf{B}_{\langle 2 \rangle} \coloneqq \mathbf{x} \wedge \mathbf{y}$. *By Corollary 2.10.1, we know that* $\mathbf{B}^2$ *is a scalar, so* $\mathbf{B}^2 = \langle \mathbf{B}^2 \rangle_0 = \langle \mathbf{B}^2 \rangle_{2-2} = \mathbf{B} \llcorner \mathbf{B}$ *by* (2.30). *That is,* $(\mathbf{x} \wedge \mathbf{y})^2 = (\mathbf{x} \wedge \mathbf{y}) \llcorner (\mathbf{x} \wedge \mathbf{y})$. *Applying the Binet–Cauchy identity* (2.25) *then results in the so-called* Lagrange identity

$$(\mathbf{x} \wedge \mathbf{y})^2 = (\mathbf{x} \cdot \mathbf{y})^2 - \mathbf{x}^2 \mathbf{y}^2. \tag{2.32}$$

In particular, if the space is Euclidean, this quantity is always non-positive by the Cauchy-Schwarz inequality. The reader may verify the identity directly using (2.5) *and* (2.9).

2.2.7 The dual and orthogonal complements

Given a linear subspace V of $\mathbb{R}^{p,q}$, the *orthogonal complement* of V is defined to be $V^\perp \coloneqq \{ \mathbf{x} \in \mathbb{R}^{p,q} \mid \forall \mathbf{y} \in V \ \mathbf{x} \cdot \mathbf{y} = 0 \}$. We have seen in Section 2.2.6 that a linear subspace can be thought of as the outer null space $\overline{\mathbf{B}}$ of a blade $\mathbf{B}$. The orthogonal complement of $\overline{\mathbf{B}}$ can be represented algebraically by the following construction.

Definition 2.5. *The* ***pseudoscalar*** *in* $\mathbb{R}^{p,q}$ *is*

$$\mathbf{I} \coloneqq \mathbf{e}_1 \mathbf{e}_2 \ldots \mathbf{e}_{p+q}. \tag{2.33}$$

Notice that $\mathbf{I}^2 = \pm 1$, so $1/\mathbf{I}^2 = \mathbf{I}^2$ and $\mathbf{I}^{-1} = \mathbf{I}^3$. The pseudoscalar has a characteristic algebraic property:

Lemma 2.11. *For any multivector* $\mathbf{A}$*,*

$$\mathbf{AI} = \mathbf{A} \llcorner \mathbf{I}. \tag{2.34}$$

Proof. This follows from linearity because all canonical basis blades are subsets of $\mathbf{I} = \{1, \dots, n\}$. ∎

Definition 2.6. *Given a multivector* $\mathbf{A}$*, the* ***dual*** *or* ***complement*** *of* $\mathbf{A}$ *is*

$$\mathbf{A}^{\mathbf{c}} \coloneqq \mathbf{AI}^{-1} = \mathbf{A} \llcorner \mathbf{I}^{-1}. \tag{2.35}$$

Lemma 2.12. *For any multivectors* $\mathbf{A}$ *and* $\mathbf{B}$*,*

$$\mathbf{A} \llcorner \mathbf{B}^{\mathbf{c}} = (\mathbf{A} \wedge \mathbf{B})^{\mathbf{c}}, \tag{2.36}$$
$$\mathbf{A} \wedge \mathbf{B}^{\mathbf{c}} = (\mathbf{A} \llcorner \mathbf{B})^{\mathbf{c}}. \tag{2.37}$$

Proof. Equation (2.36) is simply (2.22) in disguise:

$$\mathbf{A} \llcorner \mathbf{B}^{\mathbf{c}} = \mathbf{A} \llcorner (\mathbf{B} \llcorner \mathbf{I}^{-1}) \overset{(2.22)}{=} (\mathbf{A} \wedge \mathbf{B}) \llcorner \mathbf{I}^{-1} = (\mathbf{A} \wedge \mathbf{B})^{\mathbf{c}}.$$

From this, (2.37) follows:

$$\begin{aligned} \mathbf{A} \wedge \mathbf{B}^{\mathbf{c}} &= (\mathbf{A} \wedge \mathbf{B}^{\mathbf{c}})\,\mathbf{I}^{-1}\mathbf{I} = (\mathbf{A} \wedge \mathbf{B}^{\mathbf{c}})^{\mathbf{c}}\,\mathbf{I} \\ &= (\mathbf{A} \llcorner (\mathbf{B}^{\mathbf{c}})^{\mathbf{c}})\,\mathbf{I} && \text{by (2.36)} \\ &= (\mathbf{A} \llcorner (\mathbf{BI}^{-2}))\,\mathbf{I} = (\mathbf{A} \llcorner \mathbf{B})\,\mathbf{I}^{-1} = (\mathbf{A} \llcorner \mathbf{B})^{\mathbf{c}}. \end{aligned}$$ ∎

It is a theorem (refer to [1, Theorem 3.3] for a proof) that if $\mathbf{B}$ is a blade then so is $\mathbf{B}^{\mathbf{c}}$. Its outer null space is the orthogonal complement of that of $\mathbf{B}$:

Theorem 2.13. *If* $\mathbf{B}$ *is a nonzero blade, then* $\overline{\mathbf{B}}^{\perp} = \overline{\mathbf{B}^{\mathbf{c}}}$*.*

Proof. Let $\mathbf{x}$ be an arbitrary vector. First note that $\mathbf{B}^{\mathbf{c}} \wedge \mathbf{x} = \pm \mathbf{x} \wedge \mathbf{B}^{\mathbf{c}}$ because $\mathbf{B}^{\mathbf{c}}$ is a blade. By (2.37), $\mathbf{x} \wedge \mathbf{B}^{\mathbf{c}} = 0$ if and only if $\mathbf{x} \llcorner \mathbf{B} = 0$. Writing $\mathbf{B} = \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k$, repeated application of (2.23) gives

$$\begin{aligned} \mathbf{x} \llcorner \mathbf{B} = \quad & (\mathbf{x} \cdot \mathbf{x}_1)(\mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4 \wedge \dots \wedge \mathbf{x}_k) \\ & - (\mathbf{x} \cdot \mathbf{x}_2)(\mathbf{x}_1 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4 \wedge \dots \wedge \mathbf{x}_k) \\ & + (\mathbf{x} \cdot \mathbf{x}_3)(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_4 \wedge \dots \wedge \mathbf{x}_k) \\ & \vdots \\ & \pm (\mathbf{x} \cdot \mathbf{x}_k)(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4 \wedge \dots \wedge \mathbf{x}_{k-1}). \end{aligned} \tag{2.38}$$

Since the above is a linear combination of linearly independent blades, it is zero if and only if all the coefficients $(\mathbf{x} \cdot \mathbf{x}_i)$ are zero. Therefore, $\mathbf{x} \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in \operatorname{Span}\{\mathbf{x}_1, \dots \mathbf{x}_k\}$, which by Theorem 2.8 is equal to $\overline{\mathbf{B}}^{\perp}$. ∎

Example 2.2.8. *In* $\mathbb{R}^3$*, the familiar* cross product *can be defined for two vectors:*

$$\mathbf{x} \times \mathbf{y} \coloneqq (\mathbf{x} \wedge \mathbf{y})^{\mathbf{c}}. \tag{2.39}$$

In Example 2.2.3 we mentioned the triple cross product $\mathbf{x} \times (\mathbf{y} \times \mathbf{z})$*, which can now be expressed in terms of geometric algebra:*

$$\begin{aligned}
\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) &= (\mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z})^{\mathbf{c}})^{\mathbf{c}} && \textit{by } (2.39) \\
&= \left((\mathbf{x} \llcorner (\mathbf{y} \wedge \mathbf{z}))^{\mathbf{c}}\right)^{\mathbf{c}} && \textit{by } (2.37) \\
&= (\mathbf{x} \llcorner (\mathbf{y} \wedge \mathbf{z}))\, \mathbf{I}^2 && \textit{by } (2.35) \\
&= -\mathbf{x} \llcorner (\mathbf{y} \wedge \mathbf{z}) && \mathbf{I}^2 = -1 \textit{ in } \mathbb{R}^3 \\
&= (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z} && \textit{by } (2.24)
\end{aligned}$$

as was noted in Example 2.2.3.

In two dimensions, the same expression (2.39) *gives a scalar rather than a vector (cf. Example 2.2.2):*

$$(x_1\mathbf{e}_1 + y_1\mathbf{e}_2) \times (x_2\mathbf{e}_1 + y_2\mathbf{e}_2) = x_1 y_2 - y_1 x_2.$$

This is indeed a common definition of a two-dimensional "cross product".

Example 2.2.9. *The* determinant *or* triple product *of three vectors in* $\mathbb{R}^3$ *is often defined as* $\det(\mathbf{x}, \mathbf{y}, \mathbf{z}) \coloneqq \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$*. In the language of duals, we find*

$$\begin{aligned}
\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) &= \mathbf{x} \llcorner (\mathbf{y} \times \mathbf{z}) && \textit{by } (2.18) \\
&= \mathbf{x} \llcorner (\mathbf{y} \wedge \mathbf{z})^{\mathbf{c}} && \textit{by } (2.39) \\
&= (\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z})^{\mathbf{c}} && \textit{by } (2.36).
\end{aligned}$$

Notice that this expression is more symmetric and can be easily generalized to n *vectors in* $\mathbb{R}^n$ *for any* n*.*

2.2.8 Sums and intersections of subspaces

In Section 2.2.6, the outer product was shown to be connected to linear subspaces. A dual operation to the outer product, the **meet** product, is defined for two multivectors by

$$(\mathbf{A} \vee \mathbf{B})^{\mathbf{c}} = \mathbf{A}^{\mathbf{c}} \wedge \mathbf{B}^{\mathbf{c}}. \tag{2.40}$$

By (2.37), we find the explicit formula

$$\mathbf{A} \vee \mathbf{B} = (\mathbf{A}^{\mathbf{c}} \llcorner \mathbf{B})\, \mathbf{I}^2. \tag{2.41}$$

The following theorem is quoted from [1, Proposition 3.4 and Corollary 3.3]. For two linear spaces V and W, define their **direct sum** $V + W \coloneqq \{\mathbf{v} + \mathbf{w} \mid \mathbf{v} \in V, \mathbf{w} \in W\}$.

Theorem 2.14. *For two nonzero blades* $\mathbf{A}$ *and* $\mathbf{B}$*,* $\mathbf{A} \wedge \mathbf{B}$ *and* $\mathbf{A} \vee \mathbf{B}$ *are blades and*

$$\overline{\mathbf{A} \wedge \mathbf{B}} = \overline{\mathbf{A}} + \overline{\mathbf{B}} \quad \textit{if} \quad \overline{\mathbf{A}} \cap \overline{\mathbf{B}} = \{0\}, \tag{2.42}$$

$$\overline{\mathbf{A} \vee \mathbf{B}} = \overline{\mathbf{A}} \cap \overline{\mathbf{B}} \quad \textit{if} \quad \overline{\mathbf{A}} + \overline{\mathbf{B}} = \mathbb{R}^{p,q}. \tag{2.43}$$

2.3 Linear transformations

In the previous section, we explored some of the algebraic and geometric properties of vectors, blades and general multivectors. This section is dedicated to examining how specific and general linear transformations can be represented, and how they act on the new objects in the algebra.

2.3.1 Projections

The **projection** of a multivector $\mathbf{A}$ onto a non-null blade $\mathbf{B}$ is defined

$$\mathcal{P}_{\mathbf{B}}(\mathbf{A}) = (\mathbf{A} \llcorner \mathbf{B})\mathbf{B}^{-1}. \tag{2.44}$$

To support the claim that it is a projection, note that for canonical basis blades,

$$\mathcal{P}_{\mathbf{Y}}(\mathbf{X}) = (\mathbf{X} \subseteq \mathbf{Y})\mathbf{X} \tag{2.45}$$

and that for two vectors, it reduces to the familiar formula

$$\mathcal{P}_{\mathbf{n}}(\mathbf{x}) = \frac{\mathbf{n} \cdot \mathbf{x}}{\mathbf{n}^2}\mathbf{n}.$$

A full proof is given in [1, Section 3.3].

2.3.2 Reflections

In Example 2.1.4, we saw the first example of the geometric significance of the geometric product. An explanation will now be given. Consider two vectors $\mathbf{n}$ and $\mathbf{x}$. By (2.5),

$$\mathbf{n}\mathbf{x} = 2(\mathbf{n} \cdot \mathbf{x}) - \mathbf{x}\mathbf{n}. \tag{2.46}$$

We therefore find that the expression $\mathbf{n}\mathbf{x}\mathbf{n}^{-1}$ simplifies as

$$\begin{aligned} \mathbf{n}\mathbf{x}\mathbf{n}^{-1} &= \frac{1}{\mathbf{n}^2}\mathbf{n}\mathbf{x}\mathbf{n} \\ &= \frac{1}{\mathbf{n}^2}(2(\mathbf{n} \cdot \mathbf{x}) - \mathbf{x}\mathbf{n})\mathbf{n} && \text{by (2.46)} \\ &= \frac{2(\mathbf{n} \cdot \mathbf{x})}{\mathbf{n}^2}\mathbf{n} - \mathbf{x} && \tfrac{\mathbf{n}^2}{\mathbf{n}^2} = 1. \end{aligned} \tag{2.47}$$

This is recognized as the formula for reflection perpendicular to $\mathbf{n}$. The reflection *along* $\mathbf{n}$ is given by $-\mathbf{n}\mathbf{x}\mathbf{n}^{-1}$.

2.3.3 Rotations

Having found a succinct formula for reflection, it can be put to good use in deriving a general formula for rotations. Let $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ be two *unit* vectors in $\mathbb{R}^3$. We shall construct the rotation that takes $\hat{\mathbf{u}}$ to $\hat{\mathbf{v}}$ and has the plane of rotation $\overline{\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}}$.

The angular bisector unit vector between $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ is

$$\hat{\mathbf{n}} := \frac{\hat{\mathbf{u}} + \hat{\mathbf{v}}}{\|\hat{\mathbf{u}} + \hat{\mathbf{v}}\|}.$$

The reflection $\mathbf{x} \mapsto \hat{\mathbf{n}}\mathbf{x}\hat{\mathbf{n}}$ must then take $\hat{\mathbf{u}}$ to $\hat{\mathbf{v}}$, which can also be shown algebraically:

$$\begin{aligned} \hat{\mathbf{n}}\hat{\mathbf{u}}\hat{\mathbf{n}} &= \frac{(\hat{\mathbf{u}} + \hat{\mathbf{v}})\hat{\mathbf{u}}(\hat{\mathbf{u}} + \hat{\mathbf{v}})}{(\hat{\mathbf{u}} + \hat{\mathbf{v}})^2} = \frac{\hat{\mathbf{u}}\hat{\mathbf{u}}\hat{\mathbf{u}} + \hat{\mathbf{v}}\hat{\mathbf{u}}\hat{\mathbf{u}} + \hat{\mathbf{u}}\hat{\mathbf{u}}\hat{\mathbf{v}} + \hat{\mathbf{v}}\hat{\mathbf{u}}\hat{\mathbf{v}}}{(\hat{\mathbf{u}} + \hat{\mathbf{v}})^2} \\ &= \frac{\hat{\mathbf{u}} + 2\hat{\mathbf{v}} + \hat{\mathbf{v}}\hat{\mathbf{u}}\hat{\mathbf{v}}}{(\hat{\mathbf{u}} + \hat{\mathbf{v}})^2} = \frac{\overbrace{(\hat{\mathbf{u}}\hat{\mathbf{v}} + 2 + \hat{\mathbf{v}}\hat{\mathbf{u}})}^{(\hat{\mathbf{u}}+\hat{\mathbf{v}})^2}\hat{\mathbf{v}}}{(\hat{\mathbf{u}} + \hat{\mathbf{v}})^2} = \hat{\mathbf{v}}. \end{aligned}$$

Performing a second reflection in $\hat{\mathbf{v}}$ leaves $\hat{\mathbf{v}}$ unchanged ($\hat{\mathbf{v}}\hat{\mathbf{v}}\hat{\mathbf{v}} = \hat{\mathbf{v}}$), so because the composition of two reflections is a rotation (both reflection and rotation are orthogonal transformations, reflections have determinant -1 and rotations have determinant 1), the desired rotation must be given by $\mathbf{x} \mapsto \hat{\mathbf{v}}\hat{\mathbf{n}}\mathbf{x}\hat{\mathbf{n}}\hat{\mathbf{v}} =: \mathbf{R}\mathbf{x}\mathbf{R}^\dagger$, where we have defined the *rotor* $\mathbf{R}$ and its *reverse* $\mathbf{R}^\dagger$ (the reverse will be properly defined in Section 2.3.5)

$$\mathbf{R} := \hat{\mathbf{v}}\hat{\mathbf{n}} = \frac{1 + \hat{\mathbf{v}}\hat{\mathbf{u}}}{\|\hat{\mathbf{u}} + \hat{\mathbf{v}}\|} \quad \text{and} \quad \mathbf{R}^\dagger := \hat{\mathbf{n}}\hat{\mathbf{v}} = \frac{1 + \hat{\mathbf{u}}\hat{\mathbf{v}}}{\|\hat{\mathbf{u}} + \hat{\mathbf{v}}\|}. \tag{2.48}$$

There is another very elegant expression for rotors. Let θ be the angle between $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, i.e. $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \cos\theta$ (and $0 \leq \theta < \pi$). By (2.32) we then find that $(\hat{\mathbf{u}} \wedge \hat{\mathbf{v}})^2 = -\sin^2\theta$. Defining the **magnitude** of a blade $\mathbf{B}$ by

$$\|\mathbf{B}\| := \sqrt{|\mathbf{B}^2|}, \tag{2.49}$$

we can construct the *unit pseudoscalar* in $\overline{\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}}$

$$\mathbf{i} := \frac{\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}}{\|\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}\|},$$

so that $\mathbf{i}^2 = -1$ (hence the suggestive name) and $\hat{\mathbf{u}} \wedge \hat{\mathbf{v}} = \mathbf{i}\sin\theta$. Then, the rotor $\mathbf{R}$ can be rewritten (using $\hat{\mathbf{v}}\hat{\mathbf{u}} = \hat{\mathbf{v}} \cdot \hat{\mathbf{u}} + \hat{\mathbf{v}} \wedge \hat{\mathbf{u}}$ by (2.10)) into the not so attractive form

$$\mathbf{R} = \frac{1 + \cos\theta - \mathbf{i}\sin\theta}{\sqrt{2(1 + \cos\theta)}}.$$

The aesthetics of this expression can be rescued by the change of variables $\varphi := \theta/2$ and the double-angle formulae $\cos\theta = \cos^2\varphi - \sin^2\varphi$ and $\sin\theta = 2\sin\varphi\cos\varphi$. From these identities follows that $1 + \cos\theta = 2\cos^2\varphi$, so

$$\mathbf{R} = \frac{2\cos^2\varphi - 2\mathbf{i}\sin\varphi\cos\varphi}{\sqrt{4\cos^2\varphi}} = \cos\varphi - \mathbf{i}\sin\varphi.$$

Extending the standard definition of the exponential function to arbitrary multivectors,

$$e^{\mathbf{A}} := \exp(\mathbf{A}) := \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}, \tag{2.50}$$

gives as usual that $\exp(\mathbf{i}\phi) = \cos\phi + \mathbf{i}\sin\phi$, and so leads to the conclusion that the rotation by θ in the plane with unit pseudoscalar $\mathbf{i}$ is given by the rotor

$$\mathbf{R} = \exp\left(-\frac{\mathbf{i}\theta}{2}\right). \tag{2.51}$$

2.3.4 Outermorphisms and linear algebra

Definition 2.7. *A function $F\colon \mathcal{G}(\mathbb{R}^{p,q}) \to \mathcal{G}(\mathbb{R}^{p,q})$ is an **outermorphism** if*

- *F is linear.*
- *$F(1) = 1$.*
- *For any k-vector $\mathbf{B}$, $F(\mathbf{B})$ is also a k-vector (F is* grade-preserving*).*
- *For all multivectors $\mathbf{A}$ and $\mathbf{B}$, $F(\mathbf{A} \wedge \mathbf{B}) = F(\mathbf{A}) \wedge F(\mathbf{B})$.*

The second condition of the definition is there only to exclude the trivial map $\forall \mathbf{A}\ F(\mathbf{A}) = 0$, since if there is any $\mathbf{A}$ for which $F(\mathbf{A}) \neq 0$, then $F(\mathbf{A}) = F(1 \wedge \mathbf{A}) = F(1) \wedge F(\mathbf{A}) = F(1)\, F(\mathbf{A})$ implies $F(1) = 1$.

Given any linear map $f\colon \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$, it can be *uniquely* extended to an outermorphism $f_\wedge\colon \mathcal{G}(\mathbb{R}^{p,q}) \to \mathcal{G}(\mathbb{R}^{p,q})$ by simply defining $f_\wedge(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k) \coloneqq f(\mathbf{x}_1) \wedge \cdots \wedge f(\mathbf{x}_k)$. As an aside, the determinant of f can then be computed via

$$\det f = f_\wedge(\mathbf{I})^{\mathbf{c}}. \tag{2.52}$$

Because of uniqueness, to prove that two outermorphisms are equal it suffices to prove that they agree on a set of basis *vectors* (as opposed to basis *blades*).

2.3.5 Versors

One of the advantages of geometric algebra is that and many important linear transformations are represented by elements of the same algebra as the objects that they act on. The objects are linear subspaces, represented by blades. The transformations in question will be represented by *versors*, of which vectors and rotors are examples:

Definition 2.8. *A* ***versor*** *is a multivector that can be written as the geometric product of a number of non-null vectors:*

$$\mathbf{V} = \mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_k. \tag{2.53}$$

The so-called *reverse* will be used for constructing the inverse of a versor.

Definition 2.9. *For a canonical basis blade $\mathbf{X} = \mathbf{e}_{i_1} \ldots \mathbf{e}_{i_k}$, the* ***reverse*** *is defined by*

$$\mathbf{X}^\dagger \coloneqq \mathbf{e}_{i_k} \ldots \mathbf{e}_{i_1} \tag{2.54}$$

or equivalently

$$\mathbf{X}^\dagger \coloneqq (-1)^{\binom{|\mathbf{X}|}{2}}\,\mathbf{X}, \tag{2.55}$$

and the definition is extended linearly to the whole of $\mathcal{G}(\mathbb{R}^{p,q})$.

That (2.54) and (2.55) are equivalent is seen by noting that the number of swaps needed to reverse a list of k elements is $\sum_{i=1}^{k}(i-1) = \frac{k(k-1)}{2} = \binom{k}{2}$.

Lemma 2.15. *For any multivectors $\mathbf{A}$ and $\mathbf{B}$,*

$$(\mathbf{A}\mathbf{B})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger, \tag{2.56}$$

$$(\mathbf{A} \wedge \mathbf{B})^\dagger = \mathbf{B}^\dagger \wedge \mathbf{A}^\dagger. \tag{2.57}$$

Proof. Both sides are bilinear, and (2.54) shows that both (2.56) and (2.57) hold for canonical basis blades. ∎

Since $\mathbf{x}^\dagger = \mathbf{x}$ for any vector $\mathbf{x}$, we find that for a versor according to (2.53),

$$\mathbf{V}^\dagger = \mathbf{x}_k \ldots \mathbf{x}_2\mathbf{x}_1. \tag{2.58}$$

This leads to the following:

Lemma 2.16. *For any versor $\mathbf{V}$, $\mathbf{V}\mathbf{V}^\dagger = \mathbf{V}^\dagger\mathbf{V}$ is a nonzero scalar.*

Proof. It is immediate from (2.53) and (2.58) that $\mathbf{V}\mathbf{V}^\dagger = \mathbf{V}^\dagger\mathbf{V} = \mathbf{x}_1^2 \ldots \mathbf{x}_k^2$. ∎

Corollary 2.16.1. *Any versor* $\mathbf{V}$ *has an inverse*

$$\mathbf{V}^{-1} = \frac{\mathbf{V}^\dagger}{\mathbf{V}\mathbf{V}^\dagger}. \tag{2.59}$$

A versor $\mathbf{V}$ encodes a linear transformation, known as the *adjoint action* of $\mathbf{V}$:

$$\begin{aligned} \mathrm{Ad}_{\mathbf{V}} \colon \mathcal{G}(\mathbb{R}^{p,q}) &\to \mathcal{G}(\mathbb{R}^{p,q}) \\ \mathbf{A} &\mapsto \mathbf{V}\mathbf{A}\mathbf{V}^{-1}. \end{aligned} \tag{2.60}$$

Comparing with (2.47) shows that the adjoint action is the composition of the reflections perpendicular to the constituent vectors.

$$\mathrm{Ad}_{\mathbf{V}}(\mathbf{A}) = \mathbf{x}_1 \dots \mathbf{x}_k\, \mathbf{A}\, \mathbf{x}_k^{-1} \dots \mathbf{x}_1^{-1}.$$

The proof of the following important theorem is given in Appendix B.

Theorem 2.17. *For any versor* $\mathbf{V}$*,* $\mathrm{Ad}_{\mathbf{V}}$ *is an outermorphism.*

This means that, for example, rotating the constituent vectors of a blade is the same as rotating the blade as a whole: $\mathbf{R}(\mathbf{x} \wedge \mathbf{y})\mathbf{R}^\dagger = (\mathbf{R}\mathbf{x}\mathbf{R}^\dagger) \wedge (\mathbf{R}\mathbf{y}\mathbf{R}^\dagger)$.

The following property is also important:

Theorem 2.18. *For any versor* $\mathbf{V}$*,* $\mathrm{Ad}_{\mathbf{V}}$ *is an* isometry*. That is, for any vectors* $\mathbf{x}$ *and* $\mathbf{y}$*,*

$$(\mathbf{V}\mathbf{x}\mathbf{V}^{-1}) \cdot (\mathbf{V}\mathbf{y}\mathbf{V}^{-1}) = \mathbf{x} \cdot \mathbf{y}. \tag{2.61}$$

Proof. Because scalars commute, for all vectors $\mathbf{x}$,

$$(\mathbf{V}\mathbf{x}\mathbf{V}^{-1})^2 = \mathbf{V}\mathbf{x}\mathbf{V}^{-1}\mathbf{V}\mathbf{x}\mathbf{V}^{-1} = \mathbf{V}\mathbf{x}\mathbf{x}\mathbf{V}^{-1} = \mathbf{x}^2\mathbf{V}\mathbf{V}^{-1} = \mathbf{x}^2.$$

Therefore,

$$\begin{aligned} (\mathbf{V}\mathbf{x}\mathbf{V}^{-1}) \cdot (\mathbf{V}\mathbf{y}\mathbf{V}^{-1}) &= \frac{(\mathbf{V}\mathbf{x}\mathbf{V}^{-1} + \mathbf{V}\mathbf{y}\mathbf{V}^{-1})^2 - (\mathbf{V}\mathbf{x}\mathbf{V}^{-1})^2 - (\mathbf{V}\mathbf{y}\mathbf{V}^{-1})^2}{2} \\ &= \frac{(\mathbf{V}(\mathbf{x}+\mathbf{y})\mathbf{V}^{-1})^2 - (\mathbf{V}\mathbf{x}\mathbf{V}^{-1})^2 - (\mathbf{V}\mathbf{y}\mathbf{V}^{-1})^2}{2} \\ &= \frac{(\mathbf{x}+\mathbf{y})^2 - \mathbf{x}^2 - \mathbf{y}^2}{2} = \mathbf{x} \cdot \mathbf{y}. \end{aligned}$$

■

Of some special concern will be versors for which $\mathbf{V}^\dagger\mathbf{V} = 1$ (equivalently, $\mathbf{V}^{-1} = \mathbf{V}^\dagger$), called **unitary** versors. The rotor constructed in Section 2.3.3 is an example of a unitary versor.

A variant of the adjoint action, the *twisted adjoint action*, turns out to be, in a sense, more fundamental. We need the following new operation:

Definition 2.10. *The* ***grade involution*** *is defined for canonical basis blades by*

$$\mathbf{X}^* \coloneqq (-1)^{|\mathbf{X}|}\, \mathbf{X} \tag{2.62}$$

and extended linearly to the whole of $\mathcal{G}(\mathbb{R}^{p,q})$*.*

Clearly $\mathbf{x}^* = -\mathbf{x}$ for any vector $\mathbf{x}$. It can also be proven by the usual methods that for all multivectors $\mathbf{A}$ and $\mathbf{B}$,

$$(\mathbf{A}\mathbf{B})^* = \mathbf{A}^*\mathbf{B}^*. \tag{2.63}$$

Now, the **twisted adjoint action** is defined by

$$\widetilde{\mathrm{Ad}}_{\mathbf{V}}(\mathbf{A}) \coloneqq \mathbf{V}^*\mathbf{A}\mathbf{V}^{-1}. \tag{2.64}$$

This is the composition of reflections *along* the constituent vectors (refer back to Section 2.3.2):

$$\widetilde{\mathrm{Ad}}_{\mathbf{V}}(\mathbf{A}) = (-\mathbf{x}_1) \dots (-\mathbf{x}_k)\, \mathbf{A}\, \mathbf{x}_k^{-1} \dots \mathbf{x}_1^{-1}.$$

2.4 Miscellaneous comments

2.4.1 Notation and conventions

The definitions and notation used in this thesis are consistent with [1]. However, it is important to be aware that many texts use different conventions.

Regarding inner products, sometimes the symbols $\llcorner$ and $\lrcorner$ are interchanged compared to our usage, or written $\lfloor$ and $\rfloor$. The symbol $\bullet$ is also sometimes used in place of $\llcorner$. More often, however, this symbol is used for yet another inner product (known simply as the *inner product*), defined by $\mathbf{X} \bullet \mathbf{Y} \coloneqq (\mathbf{X} \subseteq \mathbf{Y} \text{ or } \mathbf{Y} \subseteq \mathbf{X})\,\mathbf{X}\mathbf{Y}$ for canonical basis blades, or in general $\mathbf{A} \bullet \mathbf{B} = \mathbf{A} \llcorner \mathbf{B} + \mathbf{A} \lrcorner \mathbf{B} - \mathbf{A} * \mathbf{B}$. In this thesis, $\bullet$ is used exclusively between *vectors*, where there is no distinction between the products.

Regarding the grade involution $\mathbf{A}^*$, the reverse $\mathbf{A}^\dagger$ and the dual $\mathbf{A}^\mathbf{c}$, the notation varies greatly. $\mathbf{A}^\dagger$ is often written $\widetilde{\mathbf{A}}$ or $\mathbf{A}^\sim$, while $\mathbf{A}^*$ is commonly used in place of $\mathbf{A}^\mathbf{c}$. There seems to be no standard notation for the grade involution, but similar operations (like the "conjugate" in [6, Section 3.1.4]) are sometimes denoted $\mathbf{A}^\dagger$.

2.4.2 Inverses

We have discussed the inverses of blades and versors, for which we have seen that the left and right inverses are the same. In fact, this is true in general, and the proof (adapted from [1, Example 6.2]) is simple yet subtle:

Theorem 2.19. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are multivectors such that* $\mathbf{AB} = 1$*, then* $\mathbf{BA} = 1$.

Proof. First, note that if $\mathbf{B}$ has a right inverse at all, it must be $\mathbf{A}$, since if $\mathbf{BC} = 1$,

$$\mathbf{C} = 1\mathbf{C} = \mathbf{ABC} = \mathbf{A}1 = \mathbf{A}.$$

It is less obvious that there does in fact exist such a $\mathbf{C}$. To show this, note that the map defined by $T(\mathbf{X}) \coloneqq \mathbf{BX}$ (for general multivectors $\mathbf{X}$) is linear, and that if $T(\mathbf{X}) = 0$, then

$$0 = \mathbf{A}0 = \mathbf{A}T(\mathbf{X}) = \mathbf{ABX} = 1\mathbf{X} = \mathbf{X}.$$

In other words, T is a linear map with kernel $\{0\}$ from $\mathcal{G}(\mathbb{R}^{p,q})$ to itself. By standard linear algebra (the rank–nullity theorem), it must therefore be surjective, which means that there is some $\mathbf{C}$ such that $T(\mathbf{C}) = 1$, i.e. $\mathbf{C}$ is a right inverse of $\mathbf{B}$. By the above, we then find that $\mathbf{C} = \mathbf{A}$, so $\mathbf{BA} = 1$. ∎

This reasoning is not peculiar to geometric algebra; the exact same proof also holds for e.g. (finite-dimensional!) matrices. Not all multivectors are invertible; simple examples are 0 and any blade $\mathbf{B}$ with $\mathbf{B}^2 = 0$. A more interesting example is the multivector

$$\mathbf{A} \coloneqq \frac{1 + \mathbf{e}_1}{2}.$$

It is simple to check that $\mathbf{A}^2 = \mathbf{A}$, i.e. $\mathbf{A}$ is *idempotent*. Any such element cannot be invertible (unless it is equal to 1); if $\mathbf{AB} = 1$ then

$$1 = \mathbf{AB} = \mathbf{AAB} = \mathbf{A}.$$

3 Conformal geometric algebra

In this section, conformal geometry is presented using the tools of geometric algebra. The term “conformal geometry” has different connotations in different contexts; the usage here is in the context of *conformal geometric algebra* (CGA). For a fuller treatment of CGA, refer to for example [10], [9] (these use the notation $n = \mathbf{e}_\infty$, $\bar{n} = 2\mathbf{e}_o$) or [7].

In Section 2.2.6, it was demonstrated that linear subspaces of $\mathbb{R}^{p,q}$ correspond to blades in $\mathcal{G}(\mathbb{R}^{p,q})$. However, not all of geometry is linear algebra. There are many interesting geometric entities that are not linear subspaces. The approach of this section is to map points in $\mathbb{R}^n$ into a higher-dimensional space through a nonlinear embedding so as to render a larger set of shapes representable as linear subspaces.

3.1 Projective geometry

Before we dive into conformal geometry, we shall consider a simpler embedding of the same sort, that has the added advantage of being easily visualized. Consider the plane $\mathbb{R}^2$. The interesting linear subspaces are simply lines through the origin and can be represented as the outer null spaces of vectors. The idea of projective geometry is to view $\mathbb{R}^2$ as embedded in $\mathbb{R}^3$ with z-coordinate 1. More precisely, define the *projective* or *homogeneous embedding*

$$\begin{aligned} &\mathcal{H}\colon \mathbb{R}^2 \to \mathbb{R}^3 \\ &\mathcal{H}(\mathbf{x}) \coloneqq \mathbf{x} + \mathbf{e}_3. \end{aligned} \tag{3.1}$$

The reason for the name “homogeneous” is that we shall consider $\mathbf{x} \in \mathbb{R}^2$ to be represented by $\mathbf{X} = \lambda\mathcal{H}(\mathbf{x}) = \lambda(\mathbf{x} + \mathbf{e}_3)$ for any λ. A point $\mathbf{x}$ is therefore represented by the *linear subspace*

$$\overline{\mathcal{H}}(\mathbf{x}) \coloneqq \overline{\mathcal{H}(\mathbf{x})} = \{\lambda\mathcal{H}(\mathbf{x}) \mid \lambda \in \mathbb{R}\}. \tag{3.2}$$

We may therefore in a slight abuse of notation define an inverse mapping

$$\begin{aligned} &\mathcal{H}^{-1}\colon \mathbb{R}^3 \to \mathbb{R}^2 \\ &\mathcal{H}^{-1}(\mathbf{X}) \coloneqq \mathcal{P}_{\mathbf{e}_1\mathbf{e}_2}\left(\frac{\mathbf{X}}{\mathbf{e}_3 \cdot \mathbf{X}}\right). \end{aligned} \tag{3.3}$$

The line through the two points $\mathbf{x}_1$ and $\mathbf{x}_2$ can be represented by the plane spanned by $\mathcal{H}(\mathbf{x}_1) \wedge \mathcal{H}(\mathbf{x}_2)$. This is used in the following computational example.

Example 3.1.1. *Let us algebraically find the intersection of two lines in the plane. Take the four points*

$$\mathbf{x}_1 \coloneqq 0 \quad \mathbf{x}_2 \coloneqq \mathbf{e}_1 \quad \mathbf{x}_3 \coloneqq \mathbf{e}_1 + \mathbf{e}_2 \quad \mathbf{x}_4 \coloneqq 3\mathbf{e}_2.$$

These map to the vectors

$$\mathbf{X}_1 \coloneqq \mathbf{e}_3 \quad \mathbf{X}_2 \coloneqq \mathbf{e}_1 + \mathbf{e}_3 \quad \mathbf{X}_3 \coloneqq \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \quad \mathbf{X}_4 \coloneqq 3\mathbf{e}_2 + \mathbf{e}_3.$$

The line through $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *and the line through* $\mathbf{x}_3$ *and* $\mathbf{x}_4$ *are represented by the planes spanned by*

$$\begin{aligned} \mathbf{L}_1 &\coloneqq \mathbf{X}_1 \wedge \mathbf{X}_2 = -\mathbf{e}_1\mathbf{e}_3 \\ \mathbf{L}_2 &\coloneqq \mathbf{X}_3 \wedge \mathbf{X}_4 = 3\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_3 - 2\mathbf{e}_2\mathbf{e}_3. \end{aligned}$$

These are linearly independent and therefore together span $\mathbb{R}^3$*, so by* (2.43) *and* (2.41) *their intersection is given by*

$$\begin{aligned}\mathbf{P} &\coloneqq \mathbf{L}_1 \vee \mathbf{L}_2 = (\mathbf{L}_1^{\mathrm{c}} \llcorner \mathbf{L}_2)\mathbf{I}^2 \\ &= (-\mathbf{e}_1\mathbf{e}_3\ \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1) \llcorner (3\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_3 - 2\mathbf{e}_2\mathbf{e}_3)\,\overbrace{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^2}^{-1} \\ &= -\mathbf{e}_2 \llcorner (3\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_3 - 2\mathbf{e}_2\mathbf{e}_3) \\ &= -\mathbf{e}_2(3\mathbf{e}_1\mathbf{e}_2 - 2\mathbf{e}_2\mathbf{e}_3) && \textit{using } (2.16) \\ &= 3\mathbf{e}_1 + 2\mathbf{e}_3.\end{aligned}$$

Mapping back to $\mathbb{R}^2$ *via* (3.3) *we find the intersection point*

$$\mathcal{H}^{-1}(\mathbf{P}) = \frac{3}{2}\mathbf{e}_1.$$

3.2 Stereographic projection

The construction of the conformal geometry consists of two steps, the first of which is the inverse stereographic projection. Consider first the two-dimensional case: take the unit circle $\mathbb{S}^1$ as a subset of $\mathbb{R}^2 = \mathrm{Span}\{\mathbf{e}_1, \mathbf{e}_+\}$ (the reason for this notation will become clear in the general case) and the line $\mathbb{R}^1 \coloneqq \mathrm{Span}\{\mathbf{e}_1\}$ (clearly $\mathbb{R}^1$ is isomorphic to $\mathbb{R}$, but for generality it is best to think of it as the vector space $\mathbb{R}^1$). Given a unit vector $\mathbf{X} = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_+ \in \mathbb{S}^1$, the *stereographic projection*, which shall be denoted $\mathbf{x} = \mathcal{S}^{-1}(\mathbf{X})$, is defined as the intersection with $\mathbb{R}^1$ of the line through the points $\mathbf{e}_+$ and $\mathbf{X}$ (see Figure 1).

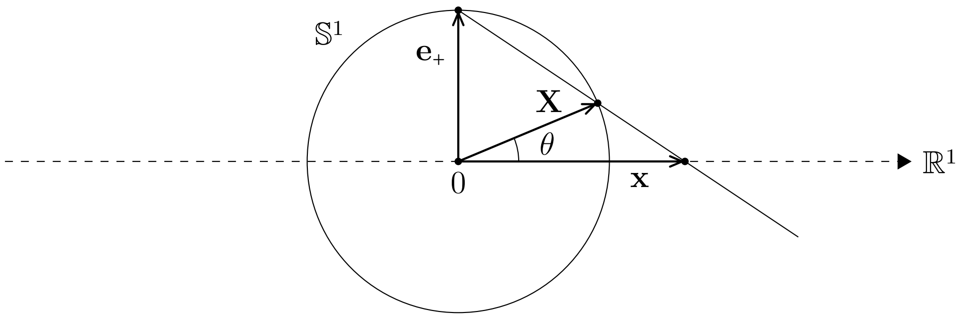


Figure 1: Stereographic projection $\mathcal{S}^{-1}\colon \mathbb{S}^1 \to \mathbb{R}^1$.

By a little trigonometry, it can be found that

$$\mathcal{S}^{-1}(\cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_+) = \frac{\cos\theta}{1-\sin\theta}\mathbf{e}_1. \tag{3.4}$$

In fact, Equation (3.4) generalizes straightforwardly to the case of arbitrary dimension:

$$\begin{aligned}&\mathcal{S}^{-1}\colon \mathbb{S}^n \to \mathbb{R}^n \\ &\mathcal{S}^{-1}(\mathbf{X}) \coloneqq \frac{\mathcal{P}_{\mathbf{e}_1\ldots\mathbf{e}_n}(\mathbf{X})}{1-\mathbf{e}_+\cdot\mathbf{X}}\end{aligned} \tag{3.5}$$

(in two dimensions, we have $\mathcal{P}_{\mathbf{e}_1}(\mathbf{X}) = \cos\theta\,\mathbf{e}_1$ and $\mathbf{e}_+\cdot\mathbf{X} = \sin\theta$). This equation can also be inverted to give the *stereographic embedding*, which shall be the more interesting map for our purposes:

$$\begin{aligned}&\mathcal{S}\colon \mathbb{R}^n \to \mathbb{S}^n \subset \mathbb{R}^{n+1} \\ &\mathcal{S}(\mathbf{x}) \coloneqq \frac{2}{\mathbf{x}^2+1}\mathbf{x} + \frac{\mathbf{x}^2-1}{\mathbf{x}^2+1}\mathbf{e}_+.\end{aligned} \tag{3.6}$$

3.3 Conformal embedding

The second step needed complete the construction of the conformal geometry is *homogenization*; making it so that $\mathbf{X}$ and $\lambda\mathbf{X}$ map back to the same point, just as for projective geometry. This is achieved in a rather spectacular way, as follows. Add a new basis vector $\mathbf{e}_-$ such that $\mathbf{e}_-^2 = -1$ and consider the map $\mathbb{R}^n \to \mathbb{R}^{n+1,1}$ given by $\mathbf{x} \mapsto \mathcal{S}(\mathbf{x}) + \mathbf{e}_-$ (essentially $\mathcal{H} \circ \mathcal{S}$ where $\mathcal{H}$ is the projective embedding (3.1)). Since $\mathcal{S}(\mathbf{x})$ is a unit vector, it will be the case that $(\mathcal{S}(\mathbf{x}) + \mathbf{e}_-)^2 = 0$ (a null vector; refer back to Section 2.1.1), which will turn out to be a highly useful algebraic property.

We are now ready to define the conformal embedding in its standard form. For convenience, the expression $\mathcal{S}(\mathbf{x}) + \mathbf{e}_-$ is multiplied by $(\mathbf{x}^2 + 1)/2$, and a slight change of basis is made:

Definition 3.1 (Conformal geometry)**.** *Given any Euclidean space* $\mathbb{R}^n = \mathrm{Span}\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, *construct the space* $\mathbb{R}^{n+1,1}$ *by adding two basis vectors* $\mathbf{e}_+$ *and* $\mathbf{e}_-$*, both orthogonal to* $\mathbb{R}^n$ *and to each other, such that*

$$\mathbf{e}_+^2 = 1, \qquad \mathbf{e}_-^2 = -1. \tag{3.7}$$

Define two new vectors

$$\mathbf{e}_\infty \coloneqq \mathbf{e}_- + \mathbf{e}_+ \qquad \textit{and} \qquad \mathbf{e}_\mathrm{o} \coloneqq \frac{\mathbf{e}_- - \mathbf{e}_+}{2}. \tag{3.8}$$

The ***null cone*** *in* $\mathbb{R}^{n+1,1}$ *is*

$$\mathbb{K}^{n+1} \coloneqq \left\{\mathbf{X} \in \mathbb{R}^{n+1,1} \;\middle|\; \mathbf{X}^2 = 0\right\}. \tag{3.9}$$

The ***conformal embedding*** *is defined by*

$$\begin{aligned}
\mathcal{C}\colon \mathbb{R}^n &\to \mathbb{K}^{n+1} \subset \mathbb{R}^{n+1,1}\\
\mathcal{C}(\mathbf{x}) &\coloneqq \frac{\mathbf{x}^2+1}{2}(\mathcal{S}(\mathbf{x}) + \mathbf{e}_-)\\
&= \mathbf{x} + \frac{\mathbf{x}^2-1}{2}\mathbf{e}_+ + \frac{\mathbf{x}^2+1}{2}\mathbf{e}_-\\
&= \mathbf{x} + \frac{\mathbf{x}^2}{2}\mathbf{e}_\infty + \mathbf{e}_\mathrm{o}.
\end{aligned} \tag{3.10}$$

Also define, for any subset $S \subseteq \mathbb{R}^n$,

$$\overline{\mathcal{C}}(S) \coloneqq \{\lambda\, \mathcal{C}(\mathbf{x}) \mid \mathbf{x} \in S,\ \lambda \in \mathbb{R}\}. \tag{3.11}$$

The last of the expressions (3.10) for $\mathcal{C}(\mathbf{x})$ is the most convenient one. Importantly, the two new vectors $\mathbf{e}_\infty$ and $\mathbf{e}_\mathrm{o}$ satisfy

$$\mathbf{e}_\infty^2 = 0, \qquad \mathbf{e}_\mathrm{o}^2 = 0, \qquad \mathbf{e}_\infty \cdot \mathbf{e}_\mathrm{o} = -1. \tag{3.12}$$

The reason for their names is that $\mathcal{C}(0) = \mathbf{e}_\mathrm{o}$ (o for "origin") and that, in a sense, $\mathcal{C}(\infty) \propto \mathbf{e}_\infty$. Precisely speaking,

$$\lim_{|\mathbf{x}|\to\infty} \frac{2}{\mathbf{x}^2}\mathcal{C}(\mathbf{x}) = \mathbf{e}_\infty.$$

The fact that the point at infinity is an element in the algebra means that the set of points that can be represented is not $\mathbb{R}^n$, but its *one-point compactification* $\mathbb{R}^n \cup \{\infty\}$.

In what follows, the case $n = 3$ shall be of chief interest. Most of the properties discussed extend straightforwardly to $\mathbb{R}^n$ for any n. Moreover, the stereographic embedding can also be generalized to be of type $\mathbb{R}^{p,q} \to \mathbb{R}^{p+1,q+1}$, but we will not go down that path here.

3.4 Geometric entities

Here, a subset of the geometric entities in $\mathbb{R}^3$ that can be represented as linear subspaces of $\mathbb{R}^{4,1}$ are presented. These are in the simplest view arbitrary points, point pairs, circles and spheres. A number of other objects can be viewed as special cases of these: Planes and lines are spheres and circles with infinite radius, spheres and circles may have zero or imaginary radius and lines at infinity may be represented. The treatment is largely based on [7, Section 4.3], from which a number of results are quoted without proof. The focus will be mainly on those aspects that will be directly applied to the common curves problem in Section 4.

A blade $\mathbf{B} \in \mathcal{G}(\mathbb{R}^{4,1})$ represents a geometric entity in $\mathbb{R}^3$, called its **geometric outer null space**:

$$\mathcal{C}^{-1}(\overline{\mathbf{B}}) = \left\{\mathbf{x} \in \mathbb{R}^3 \;\middle|\; \mathcal{C}(\mathbf{x}) \wedge \mathbf{B} = 0\right\}. \tag{3.13}$$

It will often be useful to think of a geometric entity as having an outer-product representation $\mathbf{B}$ and a corresponding inner-product representation $\mathbf{B}^{\mathbf{c}}$ (since $\mathbf{X} \llcorner \mathbf{B}^{\mathbf{c}} = 0 \iff \mathbf{X} \wedge \mathbf{B} = 0$ by (2.36)).

Clearly, $\mathcal{C}^{-1}(\overline{\lambda\mathbf{B}}) = \mathcal{C}^{-1}(\overline{\mathbf{B}})$ for nonzero λ, so we will often ignore scalar multipliers. In that spirit, we shall use the notation $\mathbf{A} \propto \mathbf{B}$ to signify "$\mathbf{A} = \lambda\mathbf{B}$ for a nonzero $\lambda \in \mathbb{R}$".

3.4.1 Points

The properties of conformal embeddings of points deserve some more attention. We have the following coordinate systems:

$$\mathbf{X} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + s\mathbf{e}_+ + t\mathbf{e}_- \tag{3.14}$$

$$= x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + u\mathbf{e}_\infty + v\mathbf{e}_\text{o}, \tag{3.15}$$

and the following coordinate transformations:

$$\begin{aligned} s &= u - v/2 & u &= \frac{t+s}{2} \\ t &= u + v/2 & v &= t - s. \end{aligned} \tag{3.16}$$

The coordinates can be found by inner products with the basis vectors; the formulae for u and v deserve special attention:

$$\begin{aligned} x &= \mathbf{e}_1 \cdot \mathbf{X} \\ y &= \mathbf{e}_2 \cdot \mathbf{X} \\ z &= \mathbf{e}_3 \cdot \mathbf{X} \end{aligned} \qquad \begin{aligned} s &= \mathbf{e}_+ \cdot \mathbf{X} \\ t &= -\mathbf{e}_- \cdot \mathbf{X} \end{aligned} \qquad \begin{aligned} u &= -\mathbf{e}_\text{o} \cdot \mathbf{X} \\ v &= -\mathbf{e}_\infty \cdot \mathbf{X}. \end{aligned} \tag{3.17}$$

Given any conformal point $\mathbf{X} = \lambda\,\mathcal{C}(\mathbf{x})$, (3.10) shows that $\lambda = v = -\mathbf{e}_\infty \cdot \mathbf{X}$. Hence, $\mathcal{C}(\mathbf{x})$ and the Euclidean point $\mathbf{x}$ can be recovered using

$$\mathcal{C}(\mathbf{x}) = \frac{\mathbf{X}}{-\mathbf{X} \cdot \mathbf{e}_\infty} \tag{3.18}$$

$$\text{and} \quad \mathbf{x} = \mathcal{P}_{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}(\mathcal{C}(\mathbf{x})). \tag{3.19}$$

It is therefore warranted, though a slight abuse of notation, to define an inverse mapping $\mathcal{C}^{-1}\colon \mathbb{K}^{n+1} \to \mathbb{R}^n$

$$\mathcal{C}^{-1}(\mathbf{X}) \coloneqq \mathcal{P}_{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}\left(\frac{\mathbf{X}}{-\mathbf{e}_\infty \cdot \mathbf{X}}\right) \tag{3.20}$$

such that $\mathcal{C}^{-1}(\lambda\,\mathcal{C}(\mathbf{x})) = \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$.

Consider now the scalar product $\mathcal{C}(\mathbf{x}) \cdot \mathcal{C}(\mathbf{y})$:

$$\begin{aligned}
\mathcal{C}(\mathbf{x}) \cdot \mathcal{C}(\mathbf{y}) &= \left(\mathbf{x} + \frac{\mathbf{x}^2}{2}\mathbf{e}_\infty + \mathbf{e}_o\right) \cdot \left(\mathbf{y} + \frac{\mathbf{y}^2}{2}\mathbf{e}_\infty + \mathbf{e}_o\right) && \text{by (3.10)} \\
&= \mathbf{x} \cdot \mathbf{y} - \frac{\mathbf{x}^2}{2} - \frac{\mathbf{y}^2}{2} && \text{by (3.12)} \\
&= -\frac{(\mathbf{x} - \mathbf{y})^2}{2},
\end{aligned}$$

which encodes the distance between $\mathbf{x}$ and $\mathbf{y}$. The homogeneous version of this formula (that is, for $\mathbf{X} = \lambda_1\,\mathcal{C}(\mathbf{x})$ and $\mathbf{Y} = \lambda_2\,\mathcal{C}(\mathbf{y})$) is obtained using (3.18):

$$-\frac{(\mathbf{x} - \mathbf{y})^2}{2} = \frac{\mathbf{X} \cdot \mathbf{Y}}{(-\mathbf{e}_\infty \cdot \mathbf{X})(-\mathbf{e}_\infty \cdot \mathbf{Y})}. \tag{3.21}$$

Note also that since $\mathbf{X}^2 = \mathbf{Y}^2 = 0$, it is the case that $(\mathbf{X} + \mathbf{Y})^2 = -(\mathbf{X} - \mathbf{Y})^2 = 2\mathbf{X} \cdot \mathbf{Y}$, so a better-looking expression can be obtained:

$$(\mathbf{x} - \mathbf{y})^2 = \frac{(\mathbf{X} - \mathbf{Y})^2}{(-\mathbf{e}_\infty \cdot \mathbf{X})(-\mathbf{e}_\infty \cdot \mathbf{Y})}. \tag{3.22}$$

3.4.2 Spheres

Theorem 3.1. *A sphere $\Sigma \subset \mathbb{R}^3$ of radius $\rho \geq 0$ centred on $\mathbf{m} \in \mathbb{R}^3$ can be written*

$$\Sigma = \mathcal{C}^{-1}(\overline{\mathbf{S}}) \quad \textit{where } \mathbf{S} \textit{ is the 4-blade such that} \quad \mathbf{S}^\mathbf{c} = \mathcal{C}(\mathbf{m}) - \frac{\rho^2}{2}\mathbf{e}_\infty. \tag{3.23}$$

Proof. That $\mathbf{x} \in \mathcal{C}^{-1}(\overline{\mathbf{S}})$ is equivalent to that

$$\begin{aligned}
0 &= \mathcal{C}(\mathbf{x}) \llcorner \mathbf{S}^\mathbf{c} = \mathcal{C}(\mathbf{x}) \cdot \mathbf{S}^\mathbf{c} && \text{by (2.18)} \\
&= \mathcal{C}(\mathbf{x}) \cdot \mathcal{C}(\mathbf{m}) - \frac{\rho^2}{2}\,\mathcal{C}(\mathbf{x}) \cdot \mathbf{e}_\infty \\
&= -\frac{(\mathbf{x} - \mathbf{m})^2}{2} + \frac{\rho^2}{2} && \text{by (3.21), (3.10) and (3.12)}
\end{aligned}$$

or $(\mathbf{x} - \mathbf{m})^2 = \rho^2$, which describes the sphere. ■

It is also shown in [7] that an outer-product representation is

$$\Sigma = \mathcal{C}^{-1}(\overline{\mathcal{C}(\mathbf{x}_1) \wedge \mathcal{C}(\mathbf{x}_2) \wedge \mathcal{C}(\mathbf{x}_3) \wedge \mathcal{C}(\mathbf{x}_4)}) \tag{3.24}$$

where $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$ are four distinct points on the sphere.

Given any representation $\mathbf{S}$ such that $\Sigma = \mathcal{C}^{-1}(\overline{\mathbf{S}})$, information about Σ can be extracted as follows: We know from (3.23) that $\mathbf{S}^\mathbf{c}$ is of the form $\lambda\left(\mathcal{C}(\mathbf{m}) - \frac{\rho^2}{2}\mathbf{e}_\infty\right)$. By (3.10) and (3.12) we find that $\lambda = -\mathbf{e}_\infty \cdot \mathbf{S}^\mathbf{c}$ and that $(\mathbf{S}^\mathbf{c})^2 = \lambda\rho^2$, which leads to an expression for the radius:

$$\rho^2 = \frac{(\mathbf{S}^\mathbf{c})^2}{(-\mathbf{e}_\infty \cdot \mathbf{S}^\mathbf{c})^2} = \frac{\mathbf{S}^2}{(\mathbf{S} \wedge \mathbf{e}_\infty)^2} \tag{3.25}$$

where the last equality follows from (2.36), that $\mathbf{e}_\infty \wedge \mathbf{S} = \mathbf{S} \wedge \mathbf{e}_\infty$ (because $\mathbf{S}$ is a 4-blade; four anticommuting swaps), and the fact that the pseudoscalar $\mathbf{I} \coloneqq \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_+\mathbf{e}_-$ commutes with everything (for vectors: four anticommuting swaps and one commuting one). By (3.23), (3.10) and (3.12), the centre can be extracted via the same formula as for points (which can in fact be viewed as spheres of radius 0):

$$\mathbf{m} = \mathcal{P}_{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}\left(\frac{\mathbf{S}^\mathbf{c}}{-\mathbf{e}_\infty \cdot \mathbf{S}^\mathbf{c}}\right). \tag{3.26}$$

There is another interesting formula for the centre of a sphere, with the connotation of reflecting the point at infinity $\mathbf{e}_\infty$ in the sphere:

Theorem 3.2. *If* $\Sigma = \mathcal{C}^{-1}(\overline{\mathbf{S}})$ *is a sphere with centre* $\mathbf{m}$*, then*

$$\mathcal{C}(\mathbf{m}) \propto \mathbf{S}\mathbf{e}_\infty\mathbf{S}. \tag{3.27}$$

Proof. First recall that the pseudoscalar $\mathbf{I}$ commutes with everything. Therefore,

$$\begin{aligned}
\mathbf{S}\mathbf{e}_\infty\mathbf{S} &= \mathbf{S}^\mathbf{c}\mathbf{I}\mathbf{e}_\infty\mathbf{S}^\mathbf{c}\mathbf{I} && \text{by (2.35)}\\
&= \mathbf{I}^2\mathbf{S}^\mathbf{c}\mathbf{e}_\infty\mathbf{S}^\mathbf{c}\\
&= -\mathbf{S}^\mathbf{c}\mathbf{e}_\infty\mathbf{S}^\mathbf{c} && \mathbf{I}^2 = -1\\
&\propto \left(\mathcal{C}(\mathbf{m}) - \frac{\rho^2}{2}\mathbf{e}_\infty\right)\mathbf{e}_\infty\left(\mathcal{C}(\mathbf{m}) - \frac{\rho^2}{2}\mathbf{e}_\infty\right) && \text{by (3.23)}\\
&= \mathcal{C}(\mathbf{m})\mathbf{e}_\infty\,\mathcal{C}(\mathbf{m}) && \mathbf{e}_\infty^2 = 0\\
&= \left(\mathbf{m} + \frac{\mathbf{m}^2}{2}\mathbf{e}_\infty + \mathbf{e}_\mathrm{o}\right)\mathbf{e}_\infty\left(\mathbf{m} + \frac{\mathbf{m}^2}{2}\mathbf{e}_\infty + \mathbf{e}_\mathrm{o}\right) && \text{by (3.10)}\\
&= (\mathbf{m} + \mathbf{e}_\mathrm{o})\mathbf{e}_\infty(\mathbf{m} + \mathbf{e}_\mathrm{o}) && \mathbf{e}_\infty^2 = 0\\
&= \mathbf{m}\mathbf{e}_\infty\mathbf{m} + \mathbf{m}\mathbf{e}_\infty\mathbf{e}_\mathrm{o} + \mathbf{e}_\mathrm{o}\mathbf{e}_\infty\mathbf{m} + \mathbf{e}_\mathrm{o}\mathbf{e}_\infty\mathbf{e}_\mathrm{o}\\
&= -\mathbf{m}^2\mathbf{e}_\infty + (\mathbf{e}_\infty\mathbf{e}_\mathrm{o} + \mathbf{e}_\mathrm{o}\mathbf{e}_\infty)\mathbf{m} + \mathbf{e}_\mathrm{o}\mathbf{e}_\infty\mathbf{e}_\mathrm{o} && \text{anticommutation}\\
&= -\mathbf{m}^2\mathbf{e}_\infty - 2\mathbf{m} - 2\mathbf{e}_\mathrm{o} && 2(\mathbf{e}_\infty \cdot \mathbf{e}_\mathrm{o}) = -2 \text{ and } \mathbf{e}_\mathrm{o}\mathbf{e}_\infty\mathbf{e}_\mathrm{o} = -2\mathbf{e}_\mathrm{o}\\
&= -2\,\mathcal{C}(\mathbf{m}) && \text{by (3.10).}
\end{aligned}$$

■

3.4.3 Planes

Theorem 3.3. *A plane* $\Pi \subset \mathbb{R}^3$ *with unit normal* $\hat{\mathbf{n}}$ *and orthogonal distance* d *from the origin can be written*

$$\Pi = \mathcal{C}^{-1}(\overline{\mathbf{P}}) \quad \textit{where } \mathbf{P} \textit{ is the 4-blade such that} \quad \mathbf{P}^\mathbf{c} = \hat{\mathbf{n}} + d\,\mathbf{e}_\infty. \tag{3.28}$$

Proof. That $\mathbf{x} \in \mathcal{C}^{-1}(\overline{\mathbf{P}})$ is equivalent to that

$$\begin{aligned}
0 &= \mathcal{C}(\mathbf{x}) \cdot \mathbf{P}^\mathbf{c}\\
&= \mathcal{C}(\mathbf{x}) \cdot \hat{\mathbf{n}} + d\,\mathcal{C}(\mathbf{x}) \cdot \mathbf{e}_\infty\\
&= \mathbf{x} \cdot \hat{\mathbf{n}} - d && \text{by (3.10) and (3.12),}
\end{aligned}$$

or $\mathbf{x} \cdot \hat{\mathbf{n}} = d$, which describes the plane. ■

It is also shown in [7] that an outer-product representation is

$$\Pi = \mathcal{C}^{-1}(\overline{\mathcal{C}(\mathbf{x}_1) \wedge \mathcal{C}(\mathbf{x}_2) \wedge \mathcal{C}(\mathbf{x}_3) \wedge \mathbf{e}_\infty}) \tag{3.29}$$

where $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are three distinct points in the plane.

A plane can alternatively be specified by a unit normal vector $\hat{\mathbf{n}}$ and a point $\mathbf{x}$ in the plane. The proper choice of d for equation (3.28) can, following [8], be found from any $\mathbf{X} = \lambda\,\mathcal{C}(\mathbf{x})$:

$$(\hat{\mathbf{n}} + d\mathbf{e}_\infty)\cdot\mathbf{X} = 0 \implies d = -\frac{\mathbf{X}\cdot\hat{\mathbf{n}}}{\mathbf{X}\cdot\mathbf{e}_\infty}. \tag{3.30}$$

Substituting back into (3.28) gives an inner-product representation

$$\mathbf{P}^\mathbf{c} = \hat{\mathbf{n}} - \frac{\mathbf{X}\cdot\hat{\mathbf{n}}}{\mathbf{X}\cdot\mathbf{e}_\infty}\mathbf{e}_\infty \propto -(\mathbf{X}\cdot\mathbf{e}_\infty)\hat{\mathbf{n}} + (\mathbf{X}\cdot\hat{\mathbf{n}})\mathbf{e}_\infty = \mathbf{X}\llcorner(\hat{\mathbf{n}}\wedge\mathbf{e}_\infty) \tag{3.31}$$

where the last step follows from (2.24).

For a vector $\mathbf{x}\in\mathbb{R}^3$ and an inner-product representation $\mathbf{P}^\mathbf{c} = \lambda(\hat{\mathbf{n}} + d\mathbf{e}_\infty)$ of a plane $\Pi = \mathcal{C}^{-1}(\overline{\mathbf{P}})$, we find as above that the signed distance from the plane is encoded by the scalar product:

$$\mathbf{P}^\mathbf{c}\cdot\mathcal{C}(\mathbf{x}) = \lambda(\hat{\mathbf{n}}\cdot\mathbf{x} - d). \tag{3.32}$$

We will be interested in knowing on which side of the plane $\mathbf{x}$ is, that is, we wish to know the sign of $\hat{\mathbf{n}}\cdot\mathbf{x} - d$. If d is taken to be positive, we could evaluate $|\lambda| = \|\mathcal{P}_{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}(\mathbf{P}^\mathbf{c})\|$ and then normalize $\mathbf{P}^\mathbf{c}$, making sure that the $\mathbf{e}_\infty$ component is positive. However, it is simpler to note that $\mathbf{e}_o\cdot\mathbf{P}^\mathbf{c} = -\lambda d$, and so it is sufficient to check the sign of

$$\frac{\mathcal{C}(\mathbf{x})\cdot\mathbf{P}^\mathbf{c}}{-\mathbf{e}_o\cdot\mathbf{P}^\mathbf{c}} = \frac{\hat{\mathbf{n}}\cdot\mathbf{x} - d}{d}. \tag{3.33}$$

3.4.4 Circles

Suppose that a circle is described as the intersection of two spheres: $\Gamma = \Sigma_1\cap\Sigma_2$, where $\Sigma_1 = \mathcal{C}^{-1}(\overline{\mathbf{S}_1})$ and $\Sigma_2 = \mathcal{C}^{-1}(\overline{\mathbf{S}_2})$. Since $\mathbf{S}_1$ and $\mathbf{S}_2$ are linearly independent (as Σ_1 and Σ_2 are distinct) and both are 4-blades, they must together span $\mathbb{R}^{4,1}$. Therefore, by (2.43), their intersection must be represented by their meet product:

$$\Gamma = \mathcal{C}^{-1}(\overline{\mathbf{C}}) \quad \text{where} \quad \mathbf{C} = \mathbf{S}_1\vee\mathbf{S}_2. \tag{3.34}$$

Alternatively, they can be represented as the intersection of a sphere and a plane: $\mathbf{C} = \mathbf{S}\vee\mathbf{P}$. It is also shown in [7] that

$$\Gamma = \mathcal{C}^{-1}(\overline{\mathcal{C}(\mathbf{x}_1)\wedge\mathcal{C}(\mathbf{x}_2)\wedge\mathcal{C}(\mathbf{x}_3)}) \tag{3.35}$$

where $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are three distinct points on the circle.

Given any outer-product representation $\mathbf{C}$ of a circle Γ, the radius can be found by an interesting conjuring trick, outlined in [7]:

Theorem 3.4. *If* $\Gamma = \mathcal{C}^{-1}(\overline{\mathbf{C}})$ *is a circle, its radius* ρ *can be found by*

$$\rho^2 = -\frac{\mathbf{C}^2}{(\mathbf{C}\wedge\mathbf{e}_\infty)^2}. \tag{3.36}$$

Note the sign difference from the corresponding formula (3.25) *for spheres.*

Proof. First, an outer-product representation of the circle's plane Π is obtained simply by $\mathbf{P} = \mathbf{C}\wedge\mathbf{e}_\infty$ (just compare (3.35) to (3.29)). Now, let $\mathbf{P}_n$ and $\mathbf{S}_n$ be *normalized* representation of Π and the sphere Σ with the same centre $\mathbf{m}$ and radius ρ as Γ respectively:

$$\mathbf{P}_n^\mathbf{c} = -(\mathbf{M}\cdot\mathbf{e}_\infty)\hat{\mathbf{n}} + (\mathbf{M}\cdot\hat{\mathbf{n}})\mathbf{e}_\infty \qquad \text{by (3.31)}$$

$$\mathbf{S}_n^\mathbf{c} = \mathbf{M} - \frac{\rho^2}{2}\mathbf{e}_\infty \qquad \text{by (3.23)}$$

where $\mathbf{M} \coloneqq \mathcal{C}(\mathbf{m})$ for brevity. Then we know that $\mathbf{P}^\mathbf{c} = \lambda \mathbf{P}_n^\mathbf{c}$ for some nonzero λ, so $\mathbf{C}^\mathbf{c} = \lambda(\mathbf{S}_n \vee \mathbf{P}_n)^\mathbf{c} = \lambda(\mathbf{S}_n^\mathbf{c} \wedge \mathbf{P}_n^\mathbf{c})$. Using (3.10), the expression for $\mathbf{P}_n^\mathbf{c}$ simplifies further: $\mathbf{P}_n^\mathbf{c} = \hat{\mathbf{n}} + (\mathbf{m} \cdot \hat{\mathbf{n}})\mathbf{e}_\infty$. Now, apply the Lagrange identity (2.32) to find that $(\mathbf{C}^\mathbf{c})^2 = \lambda^2((\mathbf{S}_n^\mathbf{c} \cdot \mathbf{P}_n^\mathbf{c})^2 - (\mathbf{S}_n^\mathbf{c})^2(\mathbf{P}_n^\mathbf{c})^2)$. It is straightforward to find

$$\begin{aligned} \mathbf{S}_n^\mathbf{c} \cdot \mathbf{P}_n^\mathbf{c} &= 0 \\ (\mathbf{S}_n^\mathbf{c})^2 &= \rho^2 \\ (\mathbf{P}_n^\mathbf{c})^2 &= 1. \end{aligned}$$

Therefore, $(\mathbf{C}^\mathbf{c})^2 = -\lambda^2\rho^2$ and $(\mathbf{P}^\mathbf{c})^2 = \lambda^2$, so we find the formula

$$\rho^2 = -\frac{(\mathbf{C}^\mathbf{c})^2}{(\mathbf{P}^\mathbf{c})^2} = -\frac{\mathbf{C}^2}{\mathbf{P}^2}. \qquad \blacksquare$$

The centre of a circle can be extracted in the same way as for a sphere:

Theorem 3.5. *If* $\Gamma = \mathcal{C}^{-1}(\overline{\mathbf{C}})$ *is a circle with centre* $\mathbf{m}$*, then*

$$\mathcal{C}(\mathbf{m}) \propto \mathbf{C}\mathbf{e}_\infty\mathbf{C}. \tag{3.37}$$

A proof similar to that of Theorem 3.2 seem intractable. The proof is deferred to Section 3.6, where it becomes easy.

3.4.5 Point pairs

The outer product of two conformal points can be rather simply analysed. Let $\mathbf{A} \coloneqq \mathcal{C}(\mathbf{a})$ and $\mathbf{B} \coloneqq \mathcal{C}(\mathbf{b})$ with $\mathbf{a} \neq \mathbf{b}$, Then $\mathbf{X} \wedge \mathbf{A} \wedge \mathbf{B} = 0$ if and only if $\mathbf{X} = \alpha\mathbf{A} + \beta\mathbf{B}$ for scalars α and β. If $\mathbf{X}$ is a conformal point then $\mathbf{X}^2 = 0$, which means

$$\begin{aligned} 0 = \mathbf{X}^2 &= \alpha^2 \underbrace{\mathbf{A}^2}_{0} + 2\alpha\beta\mathbf{A} \cdot \mathbf{B} + \beta^2 \underbrace{\mathbf{B}^2}_{0} \\ &= -\alpha\beta(\mathbf{a} - \mathbf{b})^2 \qquad \text{by (3.21).} \end{aligned}$$

Since $\mathbf{a} \neq \mathbf{b}$ (and they live in an Euclidean space), this is true only if $\alpha = 0$ or $\beta = 0$, that is $\mathbf{x} = \mathbf{b}$ or $\mathbf{x} = \mathbf{a}$. In summary, we have the point pair

$$\mathcal{C}^{-1}(\overline{\mathcal{C}(\mathbf{a}) \wedge \mathcal{C}(\mathbf{b})}) = \{\mathbf{a}, \mathbf{b}\}. \tag{3.38}$$

This is the zero-dimensional analogue of circles (1D) and spheres (2D).

3.5 Conformal transformations

An essential feature of CGA is that *conformal transformations* in $\mathbb{R}^3$ become versor transformations in $\mathbb{R}^{4,1}$. A conformal transformation is a function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ that preserves the (oriented) angles between curves. Formally (a similar definition is given in [9, Section 10.3]): For all points $\mathbf{x} \in \mathbb{R}^3$ and all tangent vectors $\mathbf{a}$ and $\mathbf{b}$ at $\mathbf{x}$, the cosine of the angle θ between $\mathbf{a}$ and $\mathbf{b}$ is locally preserved:

$$\frac{(\mathbf{a} \cdot \nabla f(\mathbf{x})) \cdot (\mathbf{b} \cdot \nabla f(\mathbf{x}))}{\|\mathbf{a} \cdot \nabla f(\mathbf{x})\| \|\mathbf{b} \cdot \nabla f(\mathbf{x})\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \tag{3.39}$$

The simplest conformal transformations to represent in conformal geometric algebra are the versor transformations laid out in Section 2.3. In particular, the twisted adjoint action (2.64) commutes with the conformal embedding ($\widetilde{\mathrm{Ad}}_{\mathbf{V}} \circ \mathcal{C} = \mathcal{C} \circ \widetilde{\mathrm{Ad}}_{\mathbf{V}}$) because for $\mathbf{n} \in \mathbb{R}^3$,

$$\begin{aligned} -\mathbf{n}\,\mathcal{C}(\mathbf{x})\mathbf{n}^{-1} &= -\mathbf{n}\mathbf{x}\mathbf{n}^{-1} - \frac{\mathbf{x}^2}{2}\mathbf{n}\mathbf{e}_\infty\mathbf{n}^{-1} - \mathbf{n}\mathbf{e}_\mathrm{o}\mathbf{n}^{-1} \\ &= -\mathbf{n}\mathbf{x}\mathbf{n}^{-1} + \frac{\mathbf{x}^2}{2}\mathbf{e}_\infty + \mathbf{e}_\mathrm{o} \\ &= \mathcal{C}(-\mathbf{n}\mathbf{x}\mathbf{n}^{-1}), \end{aligned}$$

i.e. reflections along vectors are the same in Euclidean and conformal space. The most important case is that of rotations. For a rotor, $\mathbf{R}^* = \mathbf{R}$ (it has only even-grade components) and $\mathbf{R}^{-1} = \mathbf{R}^\dagger$, so

$$\mathcal{C}(\mathbf{R}\mathbf{x}\mathbf{R}^\dagger) = \mathbf{R}\,\mathcal{C}(\mathbf{x})\mathbf{R}^\dagger. \tag{3.40}$$

Translations in $\mathbb{R}^3$ are represented as follows. Given a vector $\mathbf{a} \in \mathbb{R}^3$, define the rotor

$$\mathbf{T}_\mathbf{a} \coloneqq \exp\left(\frac{\mathbf{e}_\infty\mathbf{a}}{2}\right) = 1 + \frac{\mathbf{e}_\infty\mathbf{a}}{2} \tag{3.41}$$

(the series expansion of the exponential terminates as $(\mathbf{e}_\infty\mathbf{a})^2 = -\mathbf{e}_\infty^2\mathbf{a}^2 = 0$).

Theorem 3.6. *For all* $\mathbf{x} \in \mathbb{R}^3$*,*

$$\mathbf{T}_\mathbf{a}\,\mathcal{C}(\mathbf{x})\mathbf{T}_\mathbf{a}^\dagger = \mathcal{C}(\mathbf{x} + \mathbf{a}). \tag{3.42}$$

Proof. Since $\mathbf{X} \mapsto \mathbf{T}_\mathbf{a}\mathbf{X}\mathbf{T}_\mathbf{a}^\dagger$ is linear, evaluate it for the three terms in $\mathcal{C}(\mathbf{x})$ separately:

$$\begin{aligned} \mathbf{T}_\mathbf{a}\mathbf{x}\mathbf{T}_\mathbf{a}^\dagger &= \left(1 + \frac{\mathbf{e}_\infty\mathbf{a}}{2}\right)\mathbf{x}\left(1 + \frac{\mathbf{a}\mathbf{e}_\infty}{2}\right) \\ &= \mathbf{x} + \frac{\mathbf{e}_\infty\mathbf{a}\mathbf{x} + \mathbf{x}\mathbf{a}\mathbf{e}_\infty}{2} + \frac{\mathbf{e}_\infty\mathbf{a}\mathbf{x}\mathbf{a}\mathbf{e}_\infty}{4} \\ &= \mathbf{x} + \frac{\mathbf{a}\mathbf{x} + \mathbf{x}\mathbf{a}}{2}\mathbf{e}_\infty && \text{anticommutation, } \mathbf{e}_\infty^2 = 0 \\ &= \mathbf{x} + (\mathbf{a}\cdot\mathbf{x})\,\mathbf{e}_\infty && \text{by (2.5),} \\ \mathbf{T}_\mathbf{a}\mathbf{e}_\infty\mathbf{T}_\mathbf{a}^\dagger &= \left(1 + \frac{\mathbf{e}_\infty\mathbf{a}}{2}\right)\mathbf{e}_\infty\left(1 + \frac{\mathbf{a}\mathbf{e}_\infty}{2}\right) = \mathbf{e}_\infty && \text{anticommutation, } \mathbf{e}_\infty^2 = 0, \\ \mathbf{T}_\mathbf{a}\mathbf{e}_\mathrm{o}\mathbf{T}_\mathbf{a}^\dagger &= \left(1 + \frac{\mathbf{e}_\infty\mathbf{a}}{2}\right)\mathbf{e}_\mathrm{o}\left(1 + \frac{\mathbf{a}\mathbf{e}_\infty}{2}\right) \\ &= \mathbf{e}_\mathrm{o} + \frac{\mathbf{e}_\infty\mathbf{a}\mathbf{e}_\mathrm{o} + \mathbf{e}_\mathrm{o}\mathbf{a}\mathbf{e}_\infty}{2} + \frac{\mathbf{e}_\infty\mathbf{a}\mathbf{e}_\mathrm{o}\mathbf{a}\mathbf{e}_\infty}{4} \\ &= \mathbf{e}_\mathrm{o} - \frac{\mathbf{e}_\infty\mathbf{e}_\mathrm{o} + \mathbf{e}_\mathrm{o}\mathbf{e}_\infty}{2}\,\mathbf{a} - \frac{\mathbf{e}_\infty\mathbf{e}_\mathrm{o}\mathbf{e}_\infty}{4}\,\mathbf{a}^2 && \text{anticommutation} \\ &= \mathbf{e}_\mathrm{o} + \mathbf{a} + \frac{\mathbf{e}_\infty}{2}\,\mathbf{a}^2 && \text{by (3.12) and } \mathbf{e}_\infty\mathbf{e}_\mathrm{o}\mathbf{e}_\infty = -2\mathbf{e}_\infty \\ &= \mathcal{C}(\mathbf{a}) && \text{by (3.10).} \end{aligned}$$

Then

$$\begin{aligned} \mathbf{T}_\mathbf{a}\,\mathcal{C}(\mathbf{x})\mathbf{T}_\mathbf{a}^\dagger &= \mathbf{T}_\mathbf{a}\mathbf{x}\mathbf{T}_\mathbf{a}^\dagger + \frac{\mathbf{x}^2}{2}\,\mathbf{T}_\mathbf{a}\mathbf{e}_\infty\mathbf{T}_\mathbf{a}^\dagger + \mathbf{T}_\mathbf{a}\mathbf{e}_\mathrm{o}\mathbf{T}_\mathbf{a}^\dagger \\ &= \Big[\mathbf{x} + (\mathbf{a}\cdot\mathbf{x})\,\mathbf{e}_\infty\Big] + \left[\frac{\mathbf{x}^2}{2}\,\mathbf{e}_\infty\right] + \left[\mathbf{a} + \frac{\mathbf{a}^2}{2}\,\mathbf{e}_\infty + \mathbf{e}_\mathrm{o}\right] \\ &= (\mathbf{x} + \mathbf{a}) + \frac{\mathbf{x}^2 + 2(\mathbf{x}\cdot\mathbf{a}) + \mathbf{a}^2}{2}\,\mathbf{e}_\infty + \mathbf{e}_\mathrm{o} \\ &= \mathcal{C}(\mathbf{x} + \mathbf{a}). \end{aligned}$$

■

By similar methods, it can be shown that the reflection in a plane given by $\mathbf{P^c} = \hat{\mathbf{n}} + d\mathbf{e}_\infty$ is given by

$$\mathbf{P}\,\mathcal{C}(\mathbf{x})\mathbf{P} = \mathcal{C}(\mathrm{refl}_{\mathbf{P}}(\mathbf{x})). \tag{3.43}$$

and that a dilation (scaling) $\mathbf{x} \mapsto s\mathbf{x}$ is given by

$$\mathcal{C}(s\mathbf{x}) = s\,\mathbf{D}_s\,\mathcal{C}(\mathbf{x})\mathbf{D}_s^\dagger \quad \text{where} \quad \mathbf{D}_s \coloneqq \frac{(1+s) + (1-s)\,\mathbf{e}_\infty \wedge \mathbf{e}_\mathrm{o}}{2\sqrt{s}} \tag{3.44}$$

(note that $\mathbf{e}_\infty \wedge \mathbf{e}_\mathrm{o} = \mathbf{e}_+\mathbf{e}_-$).

Perhaps less familiar is the *inversion* $\mathbf{x} \mapsto \mathbf{x}^{-1} = {}^{\mathbf{x}}/_{\mathbf{x}^2}$. This inversion can be shown, by similarly straightforward methods to those above, to correspond to a reflection along $\mathbf{e}_+$:

$$-\,\mathbf{e}_+\,\mathcal{C}(\mathbf{x})\mathbf{e}_+ = \mathbf{x}^2\,\mathcal{C}(\mathbf{x}^{-1}) \propto \mathcal{C}(\mathbf{x}^{-1}). \tag{3.45}$$

3.6 Transforming geometric entities

A great advantage of expressing conformal transformations as versors is that it allows them to act not only on points, but on all the geometric entities of Section 3.4:

Theorem 3.7. *If* $\mathbf{V}$ *is a versor and* $\mathbf{A}$ *is a blade, then* $\mathrm{Ad}_{\mathbf{V}}(\overline{\mathbf{A}}) = \overline{\mathrm{Ad}_{\mathbf{V}}(\mathbf{A})}$. *Concretely, if* $\mathbf{X} \in \mathbb{R}^{4,1}$, *then* $\mathbf{X} \wedge \mathbf{A} = 0$ *if and only if* $(\mathbf{V}\mathbf{X}\mathbf{V}^{-1}) \wedge (\mathbf{V}\mathbf{A}\mathbf{V}^{-1}) = 0$.

Proof. By Theorem 2.17, $(\mathbf{V}\mathbf{X}\mathbf{V}^{-1}) \wedge (\mathbf{V}\mathbf{A}\mathbf{V}^{-1}) = \mathbf{V}(\mathbf{X} \wedge \mathbf{A})\mathbf{V}^{-1}$. Since $\mathbf{V}$ is invertible this is zero if and only if $\mathbf{X} \wedge \mathbf{A}$ is. ■

This means that, for example, if $\mathcal{C}^{-1}(\overline{\mathbf{C}})$ is a circle and $\mathbf{T_a}$ represents a translation, then $\mathcal{C}^{-1}(\overline{\mathbf{T_a}\mathbf{C}\mathbf{T_a^\dagger}})$ is the translated circle. This fact allows for a simple and interesting proof of the formula (3.37) for the centre of a circle.

Proof of Theorem 3.5. First, we verify that the formula correctly gives the centre of the unit circle in the xy-plane, centred on the origin. The unit sphere is given by $\mathbf{S^c} = \mathbf{e}_\mathrm{o} - \frac{\mathbf{e}_\infty}{2} = -\mathbf{e}_+$ by (3.23) and the xy-plane is given by its normal $\mathbf{P^c} = \mathbf{e}_3$. The unit circle can then be computed as $\mathbf{C}_0^\mathbf{c} = \mathbf{S^c} \wedge \mathbf{P^c} = \mathbf{e}_3\mathbf{e}_+$. Therefore,

$$\mathbf{C}_0\mathbf{e}_\infty\mathbf{C}_0 = -\mathbf{C}_0^\mathbf{c}\mathbf{e}_\infty\mathbf{C}_0^\mathbf{c} = -\mathbf{e}_3\mathbf{e}_+(\mathbf{e}_- + \mathbf{e}_+)\mathbf{e}_3\mathbf{e}_+ = \mathbf{e}_- - \mathbf{e}_+ = 2\mathbf{e}_0 = 2\,\mathcal{C}(0).$$

To prove that the formula holds for any circle $\Gamma = \mathcal{C}^{-1}(\overline{\mathbf{C}})$ with centre $\mathbf{m}$, note that the unit circle can be transformed into Γ by scaling it to the same radius, rotating it into the correct plane, and then translating it to the correct position. Symbolically then, by Theorem 3.7 (assuming a normalization on $\mathbf{C}$)

$$\mathbf{C} = \mathbf{T_m}\mathbf{R}\mathbf{D}_\rho\;\mathbf{C}_0\;\mathbf{D}_\rho^\dagger\mathbf{R}^\dagger\mathbf{T}_\mathbf{m}^\dagger.$$

Then, because translations, rotations and dilations leave $\mathbf{e}_\infty$ (the point at infinity) unchanged, and because rotations and dilations leave $\mathbf{e}_\mathrm{o}$ (the origin) unchanged (for dilations, there is a scalar factor by (3.44); $\mathbf{D}_s\mathbf{e}_\infty\mathbf{D}_s^\dagger = \frac{1}{s}\mathbf{e}_\infty$ and $\mathbf{D}_s^\dagger\mathbf{e}_\infty\mathbf{D}_s = \mathbf{D}_{1/s}\mathbf{e}_\infty\mathbf{D}_{1/s}^\dagger = s\mathbf{e}_\infty$, and analogously for $\mathbf{e}_\mathrm{o}$),

$$\begin{aligned}
\mathbf{C}\mathbf{e}_\infty\mathbf{C} &= \mathbf{T_m}\mathbf{R}\mathbf{D}_\rho\;\mathbf{C}_0\;\mathbf{D}_\rho^\dagger\mathbf{R}^\dagger\mathbf{T}_\mathbf{m}^\dagger\;\mathbf{e}_\infty\;\mathbf{T_m}\mathbf{R}\mathbf{D}_\rho\;\mathbf{C}_0\;\mathbf{D}_\rho^\dagger\mathbf{R}^\dagger\mathbf{T}_\mathbf{m}^\dagger \\
&= \rho\;\mathbf{T_m}\mathbf{R}\mathbf{D}_\rho\;\mathbf{C}_0\;\mathbf{e}_\infty\;\mathbf{C}_0\;\mathbf{D}_\rho^\dagger\mathbf{R}^\dagger\mathbf{T}_\mathbf{m}^\dagger \\
&= \rho\;\mathbf{T_m}\mathbf{R}\mathbf{D}_\rho\;2\mathbf{e}_\mathrm{o}\;\mathbf{D}_\rho^\dagger\mathbf{R}^\dagger\mathbf{T}_\mathbf{m}^\dagger \\
&= 2\,\mathbf{T_m}\mathbf{e}_\mathrm{o}\mathbf{T}_\mathbf{m}^\dagger \\
&= 2\,\mathcal{C}(\mathbf{m}).
\end{aligned}$$

■

This method of proving something algebraically for a special case and then reasoning using transformations to establish the general case is referred to in [10] as using the *covariance* of the conformal model.

The transformations of geometric entities that will be significant in Section 4 are rotations and inversions of spheres, planes, circles and lines. In particular, that spheres and circles that contain the origin map to planes and lines will be the basis of Section 4.5.

4 Cryo-electron microscopy and the common curves problem

Cryo-electron microscopy is an imaging technique where samples of molecules to be studied are rapidly frozen so that they become immersed in amorphous ice, before being analysed using electron microscopy. It is an example of *single particle analysis* in that the specimens are identical and spaced so far apart that single molecules can be distinguished.

4.1 Electron microscopy imaging of thin specimens

A detailed survey of electron microscopy is given in [11]. A much simplified model is as follows. Consider a incident plane wave of electrons travelling in the positive z direction. Their stationary wave function $\psi\colon \mathbb{R}^3 \to \mathbb{C}$ obeys the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) = E\psi(\mathbf{x}) \tag{4.1}$$

where E is the electron energy and $V\colon \mathbb{R}^3 \to \mathbb{R}$ is the potential energy as function of position ($V(\mathbf{x}) = -eU(\mathbf{x})$ where $-e$ is the electron charge and U is the electrostatic potential). Assume that $V(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3)$ is zero except for in the interval $-L < z < 0$, where a molecule is present. The goal is then to determine V.

For $z \le -L$, the electrons are described by the incident plane wave $\psi(\mathbf{x}) = Ce^{ik_0(z+L)}$ where $k_0 \coloneqq \frac{\sqrt{2mE}}{\hbar}$. In the region $-L < z < 0$, let us make an ansatz of the form

$$\psi(\mathbf{x}) = Ce^{i\varphi(\mathbf{x})} \tag{4.2}$$

for a constant $C \in \mathbb{R}$ and assume that the potential varies slowly enough that the electron can be locally described as a plane wave. Informally, this means that for a small displacement $\mathrm{d}z$,

$$\psi(\mathbf{x} + \mathrm{d}z\,\mathbf{e}_3) = \psi(\mathbf{x})e^{ik(\mathbf{x})\mathrm{d}z} = Ce^{i(\varphi(\mathbf{x})+k(\mathbf{x})\mathrm{d}z)}. \tag{4.3}$$

Formally, the above amounts to approximating $\nabla^2\psi(\mathbf{x}) \approx \partial_z^2\psi(\mathbf{x})$ and $\partial_z^2\varphi(\mathbf{x}) \approx 0$ in the Schrödinger equation (4.1); then

$$-\frac{2m(E - V(\mathbf{x}))}{\hbar^2}\psi(\mathbf{x}) = \nabla^2\psi(\mathbf{x}) \approx \partial_z^2\psi(\mathbf{x}) = \Big[-\big(\partial_z\varphi(\mathbf{x})\big)^2 + i\,\overbrace{\partial_z^2\varphi(\mathbf{x})}^{\approx 0}\Big]\psi(\mathbf{x}) \tag{4.4}$$

In the standard terminology of [11, Section 4], we have made the *WKB approximation* and the *projection approximation*. Either way, we find the differential equation

$$\partial_z\varphi(\mathbf{x}) = k(\mathbf{x}) \coloneqq \frac{\sqrt{2m(E - V(\mathbf{x}))}}{\hbar}. \tag{4.5}$$

Supposing that $\varphi(\mathbf{x})\big|_{z=-L} = 0$, the solution at $z = 0$ is

$$\varphi(x, y, 0) = \int_{-L}^{0} k(x, y, z)\,\mathrm{d}z \tag{4.6}$$

If a detector is placed at the plane $z = 0$ this phase shift can be measured.

Since the potential was assumed to be zero outside $-L < z < 0$, defining

$$f(x,y,z) \coloneqq \frac{\sqrt{2mE}}{\hbar} - k(x,y,z) \tag{4.7}$$

$$\gamma(x,y) \coloneqq \frac{\sqrt{2mE}}{\hbar} L - \varphi(x,y,0) \tag{4.8}$$

lets us rewrite (4.6) into

$$\gamma(x,y) = \int_{-L}^{0} f(x,y,z)\,\mathrm{d}z = \int_{-\infty}^{\infty} f(x,y,z)\,\mathrm{d}z\,, \tag{4.9}$$

which will be useful because of the following theorem:

Theorem 4.1 (Fourier slice theorem, specialized variant). *Let* $f\colon \mathbb{R}^3 \to \mathbb{R}$ *and* $\gamma\colon \mathbb{R}^2 \to \mathbb{R}$ *be related as in* (4.9)*, and let* $\hat{f}\colon \mathbb{R}^3 \to \mathbb{C}$ *and* $\hat{\gamma}\colon \mathbb{R}^2 \to \mathbb{C}$ *be their respective Fourier transforms:*

$$\hat{f}(\mathbf{k}) \coloneqq \int_{\mathbb{R}^3} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}\,,$$

$$\hat{\gamma}(\boldsymbol{\kappa}) \coloneqq \int_{\mathbb{R}^2} \gamma(\boldsymbol{\xi}) e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{\xi}}\,\mathrm{d}\boldsymbol{\xi}\,.$$

Then

$$\hat{\gamma} = \hat{f}\big|_{\mathbb{R}^2} \tag{4.10}$$

where $\hat{f}|_{\mathbb{R}^2}$ *is the restriction of* $\hat{f}$ *to* $\mathbb{R}^2$*, that is,* $\hat{f}|_{\mathbb{R}^2}(x,y) \coloneqq \hat{f}(x,y,0)$.

Proof. Just expand definitions:

$$\hat{\gamma}(\boldsymbol{\kappa}) = \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} f(\boldsymbol{\xi} + z\mathbf{e}_3)\,\mathrm{d}z\, e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{\xi}}\,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} f(\mathbf{x}) e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{\xi}}\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3} f(\mathbf{x}) e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x} = \hat{f}(\boldsymbol{\kappa}). \quad \blacksquare$$

The theorem says that *projecting* f onto the xy-plane and then taking the Fourier transform is the same as first taking the Fourier transform and then *slicing* it (restricting it to $\mathbb{R}^2$), hence the name.

4.2 Single particle analysis

If a number of identical molecules with random orientation are imaged in the way described above (that is, functions $\hat{\gamma}_1, \ldots, \hat{\gamma}_n$ are collected), Theorem 4.1 amounts to values of $\hat{f}$ being known on n planes that go through the origin, but the orientation of the planes being unknown. If a more accurate model of the physics is used, the analysis is similar, but a generalization of Theorem 4.1 states that instead of being restricted to planes, $\hat{f}$ is restricted to *hemispheres* that are halves of *Ewald spheres* (see for example [12, Theorem 3.1] and [13]). The mathematical problem thus obtained will now be made precise.

Let us first fix some notation in order to be able to state the problem succinctly. For convenience, we shall omit the hats on $\hat{f}$ and $\hat{\gamma}$ to indicate Fourier transforms. From the point of view of the mathematical problem, it is not so important that they are in fact

Fourier transforms. Refer to the following definitions:

$$k \in \mathbb{R} \qquad \text{a positive constant} \tag{4.11}$$

$$D := \left\{ \mathbf{x} \in \mathbb{R}^3 \;\middle|\; \mathbf{e}_3 \cdot \mathbf{x} = 0,\ \|\mathbf{x}\| < k \right\} \qquad \text{the disk of radius } k \tag{4.12}$$

$$T\colon D \to \mathbb{R}^3$$
$$T(\mathbf{x}) := \left(k - \sqrt{k^2 - \mathbf{x}^2}\right)\mathbf{e}_3 + \mathbf{x} \qquad \text{the lifting map} \tag{4.13}$$

$$H := T(D) \qquad \text{the canonical hemisphere} \tag{4.14}$$

$$R_1, \ldots, R_n \in \mathrm{SO}(3) \qquad \text{unknown rotations} \tag{4.15}$$

$$H_i := R_i(H) \qquad \text{unknown hemispheres} \tag{4.16}$$

$$\alpha_{ij} := H_i \cap H_j \qquad \text{the common curves} \tag{4.17}$$

$$f\colon \mathbb{R}^3 \to \mathbb{C} \qquad \text{the target function} \tag{4.18}$$

$$\gamma_1, \ldots, \gamma_n\colon D \to \mathbb{C} \qquad \text{the data} \tag{4.19}$$

$$g_1, \ldots, g_n\colon H \to \mathbb{C}$$
$$g_i := \gamma_i \circ T^{-1} \qquad \text{the lifted data.} \tag{4.20}$$

The problem can then be stated as follows: There is a function $f\colon \mathbb{R}^3 \to \mathbb{C}$, which is related to the quantum-mechanical potential of the molecule's three-dimensional structure as indicated in the previous section. The ultimate objective is to approximate f. The data are functions $\gamma_1, \ldots, \gamma_n\colon D \to \mathbb{C}$, each corresponding to the Fourier transform of the recorded images. It is known by the generalization of the Fourier slice theorem mentioned in the previous section that for all $i \in \{1, \ldots, n\}$,

$$g_i = f \circ R_i|_H \tag{4.21}$$

where $R_i|_H\colon H \to \mathbb{R}^3$ is the restriction of $R_i\colon \mathbb{R}^3 \to \mathbb{R}^3$ to H. In other words, values of f are known on the n hemispheres $H_1, \ldots, H_n$, but the orientations of the hemispheres relative to each other are not known.

4.3 The common curves problem

Clearly, the data does not contain enough information to determine f exactly. However, if all the rotations R_i could be determined, then the values of f would be known on the subset $\bigcup_{i=1}^n H_i \subset \mathbb{R}^3$, whence values of f could be interpolated.

We obtain the *common curves problem* by the following observation: Since any two hemispheres H_i and H_j intersect in a circular arc α_{ij} (see Figure 2), the relative rotation $R_i \circ R_j^{-1}$ can be found by some numerical method that tries different rotations and maximizes the correlation between the two sets of function values on α_{ij} (that is, between $g_i \circ R_i^{-1}|_{\alpha_{ij}}$ and $g_j \circ R_j^{-1}|_{\alpha_{ij}}$).

Ideally, all rotations can be determined by fixing for example $R_1 = \mathrm{id}$ and executing the numerical method $n - 1$ times with $j = 1$. However, there is redundant information available by executing the numerical method $\binom{n}{2} = \frac{n(n-1)}{2}$ times so as to find approximations of $R_i \circ R_j^{-1}$ for all pairs (i, j) with $1 \le i < j \le n$. Then, the problem remains of how to decide on best estimates of R_i from the redundant information.

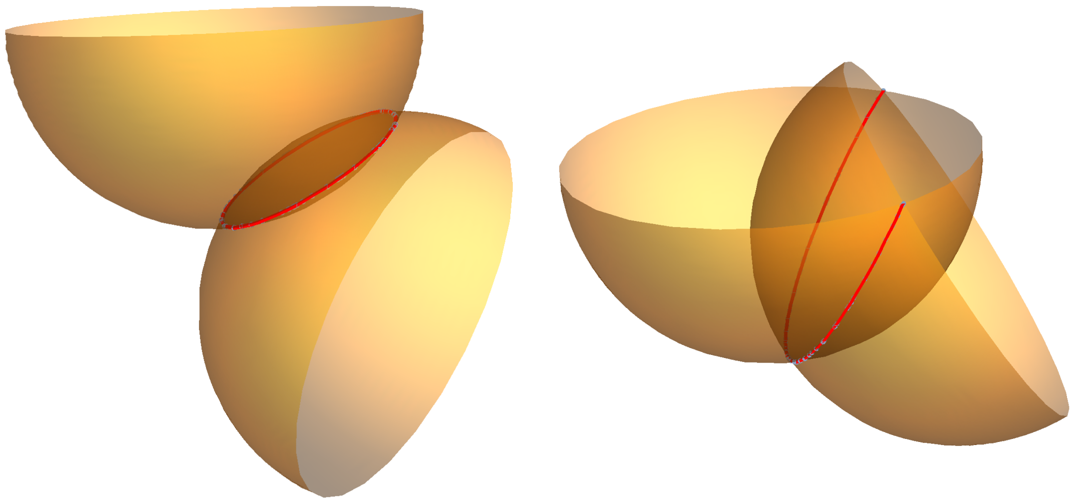


Figure 2: Hemispheres H_i and their intersections α_{ij}.

There are thus three quite separate parts to this approach: First, find the common curves. Second, use the common curves to determine the unknown rotations. Third, approximate f by suitably interpolating between the known values on $\bigcup_{i=1}^n H_i \subset \mathbb{R}^3$. In the following sections, the problem will be rewritten in a form that potentially has practical advantages when applied to the first two of these parts.

The motivation for attempting to reformulate the problem is that the limiting case $k \to \infty$, i.e. when the hemispheres become planes, has already been studied, for example in [14]. Therefore, it would be desirable to find a transformation that reduces the problem to a linear one.

4.4 Reformulation in conformal geometric algebra

In this section, the common curves problem is translated into conformal geometric algebra in as straightforward a manner as possible. In essence, all parts of the problem are transformed verbatim via the map $\overline{\mathcal{C}}$, as constructed in Definition 3.1. First, the rotations are shown to extend naturally into the conformal space by (4.23). Then, the central equation (4.21) is translated into (4.28). The remaining sections make the resulting linear subspaces explicit as blades in $\mathcal{G}(\mathbb{R}^{4,1})$.

A note on units of measurement is now in order. The conformal embedding (3.10) is, as it stands, dimensionally inconsistent. The reason for this is that the stereographic embedding (3.5) was defined in terms of the *unit* circle. Let us therefore assume that units have been chosen, so that $k = k_{\text{dim}}/k_0$ is a dimensionless quantity with k_{dim} being the corresponding dimensional quantity.

4.4.1 Expressing the unknown rotations

As discussed in Section 2.3.3, a rotation $R_i \in \text{SO}(3)$ can be expressed as a rotor $\mathbf{R}_i \in \mathcal{G}(\mathbb{R}^3)$ via

$$R_i(\mathbf{x}) = \mathbf{R}_i \mathbf{x} \mathbf{R}_i^\dagger \tag{4.22}$$

and $\mathbf{R}_i^\dagger = \mathbf{R}_i^{-1}$. The composition $R_i \circ R_j$ of two rotations is represented by the geometric product $\mathbf{R}_i \mathbf{R}_j$ of the corresponding rotors. A crucial feature of the present problem is that the same rotor as in Euclidean geometry can be used to represent the rotation in conformal geometry, as stated in (3.40). In other words,

$$\mathcal{C} \circ R_i = \tilde{R}_i \circ \mathcal{C} \tag{4.23}$$

where $\tilde{R}_i \colon \mathbb{R}^{4,1} \to \mathbb{R}^{4,1}$ is the linear extension of R_i to $\mathbb{R}^{4,1}$ that leaves $\mathbf{e}_+$ and $\mathbf{e}_-$ unchanged.

4.4.2 Translating the data maps

We now turn to the problem of converting the functions g_i into the conformal representation. Recall from (4.21) that

$$\begin{aligned} g_i &: H \to \mathbb{C} \\ g_i &= f \circ R_i|_H. \end{aligned} \tag{4.24}$$

This motivates the very natural definition

$$\begin{aligned} \tilde{g}_i &: \overline{\mathcal{C}}(H) \to \mathbb{C} \\ \tilde{g}_i &\coloneqq g_i \circ \mathcal{C}^{-1}, \end{aligned} \tag{4.25}$$

which says that to find the image value associated with $\mathbf{X} = \lambda\,\mathcal{C}(\mathbf{x})$ we simply take the image value associated with $\mathbf{x}$. We find that this definition plays nicely with the structure of the problem:

$$\begin{aligned} \tilde{g}_i &= g_i \circ \mathcal{C}^{-1} \\ &= f \circ R_i|_H \circ \mathcal{C}^{-1} && \text{by (4.24)} \\ &= f \circ \mathcal{C}^{-1} \circ \tilde{R}_i|_{\overline{\mathcal{C}}(H)} && \text{rotations commute with } \mathcal{C} \text{ by (4.23)} \end{aligned} \tag{4.26}$$

Defining $\tilde{f}$ analogously to $\tilde{g}$;

$$\tilde{f} \coloneqq f \circ \mathcal{C}^{-1}, \tag{4.27}$$

the central equation (4.21) is translated into its analogue

$$\tilde{g}_i = \tilde{f} \circ \tilde{R}_i|_{\overline{\mathcal{C}}(H)}. \tag{4.28}$$

In summary, the following diagram commutes:

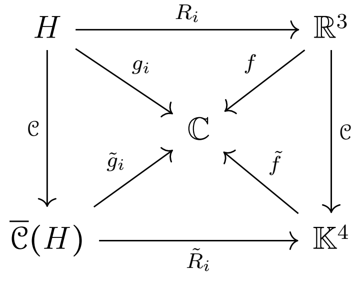


Figure 3: Commutative diagram for the transformed functions $\tilde{f}$ and $\tilde{g}_i$.

4.4.3 Expressing the hemispheres

To describe the hemisphere H and its rotated cousins $H_1, \ldots, H_n$, let us start by describing the corresponding spheres Σ (set $\Sigma_0 \coloneqq \Sigma$ and $H_0 \coloneqq H$ for compactness of notation) and $\Sigma_1, \ldots, \Sigma_n$. The centre of Σ_i is $k\hat{\mathbf{n}}_i \coloneqq k\mathbf{R}_i\mathbf{e}_3\mathbf{R}_i^\dagger$. Equation (3.23) simplifies, since $\mathbf{m}^2 = \rho^2 = k^2$, to yield the representation

$$\Sigma_i = \mathcal{C}^{-1}(\overline{\mathbf{S}_i}) \quad \text{where} \quad \mathbf{S}_i^{\mathbf{c}} \coloneqq k\hat{\mathbf{n}}_i + \mathbf{e}_o. \tag{4.29}$$

To describe the hemispheres H_i, consider the plane Π_i that cuts the sphere in half. Its unit normal is $\hat{\mathbf{n}}_i$ and its orthogonal distance from the origin is k, so by (3.28) it can be written

$$\Pi_i = \mathcal{C}^{-1}(\overline{\mathbf{P}_i}) \quad \text{where} \quad \mathbf{P}_i^{\mathbf{c}} \coloneqq \hat{\mathbf{n}}_i + k\mathbf{e}_\infty. \tag{4.30}$$

The half-space of $\mathbb{R}^3$ that is on the same side of Π_i as the origin cannot be expressed as a blade in $\mathbb{R}^{4,1}$. However, the condition $(\hat{\mathbf{n}}_i \cdot \mathbf{x} \leq k)$ can be expressed according to (3.33):

$$\frac{\mathcal{C}(\mathbf{x}) \cdot \mathbf{P}_i^{\mathbf{c}}}{-\mathbf{e}_o \cdot \mathbf{P}_i^{\mathbf{c}}} \leq 0. \tag{4.31}$$

We therefore arrive at a (necessarily slightly clumsy) expression for the hemisphere H_i:

$$H_i = \left\{ \mathbf{x} \in \mathbb{R}^3 \;\middle|\; \mathbf{S}_i^{\mathbf{c}} \cdot \mathcal{C}(\mathbf{x}) = 0,\; \frac{\mathcal{C}(\mathbf{x}) \cdot \mathbf{P}_i^{\mathbf{c}}}{-\mathbf{e}_o \cdot \mathbf{P}_i^{\mathbf{c}}} \leq 0 \right\}. \tag{4.32}$$

4.4.4 Making the hemisphere representation homogeneous

In order to completely transform the problem into a problem of linear subspaces in the new domain we would like to replace $\mathcal{C}(\mathbf{x})$ by $\mathbf{X} = \lambda\,\mathcal{C}(\mathbf{x})$ where the value of $\lambda \in \mathbb{R}$ is unimportant. In doing so, we can explicitly describe the corresponding region $\overline{\mathcal{C}}(H_i)$. The first condition of (4.32) translates directly to $\mathbf{X} \cdot \mathbf{S}_i^{\mathbf{c}} = 0$. The translation of the second criterion is complicated by the fact that λ can be negative. We can substitute using (3.18) to get

$$\frac{\mathbf{X} \cdot \mathbf{P}_i^{\mathbf{c}}}{(-\mathbf{e}_o \cdot \mathbf{P}_i^{\mathbf{c}})(-\mathbf{e}_\infty \cdot \mathbf{X})} \leq 0 \tag{4.33}$$

This gives the homogeneous representation

$$\overline{\mathcal{C}}(H_i) = \left\{ \mathbf{X} \in \mathbb{R}^{4,1} \;\middle|\; \mathbf{X}^2 = 0,\; \mathbf{X} \cdot \mathbf{S}_i^{\mathbf{c}} = 0,\; \frac{\mathbf{X} \cdot \mathbf{P}_i^{\mathbf{c}}}{(-\mathbf{e}_o \cdot \mathbf{P}_i^{\mathbf{c}})(-\mathbf{e}_\infty \cdot \mathbf{X})} \leq 0 \right\}. \tag{4.34}$$

This is the intersection with the four-dimensional null cone of a four-dimensional hyperplane $\overline{\mathbf{S}_i}$ (that contains the origin in $\mathbb{R}^{4,1}$), bounded by the four-dimensional hyperplane $\overline{\mathbf{P}_i}$ (that also contains the origin in $\mathbb{R}^{4,1}$).

4.4.5 The intersection of two hemispheres

To describe the intersection $\alpha_{ij} \coloneqq H_i \cap H_j$, consider first the intersection $\Sigma_i \cap \Sigma_j$ of the corresponding spheres. This is a circle, given by the inner-product representation

$$\begin{aligned} \mathbf{C}_{ij}^{\mathbf{c}} &\coloneqq (\mathbf{S}_i \vee \mathbf{S}_j)^{\mathbf{c}} = \mathbf{S}_i^{\mathbf{c}} \wedge \mathbf{S}_j^{\mathbf{c}} = (k\hat{\mathbf{n}}_i + \mathbf{e}_o) \wedge (k\hat{\mathbf{n}}_j + \mathbf{e}_o) \\ &= k^2(\hat{\mathbf{n}}_i \wedge \hat{\mathbf{n}}_j) + k(\hat{\mathbf{n}}_i - \hat{\mathbf{n}}_j)\mathbf{e}_o. \end{aligned} \tag{4.35}$$

We describe $\overline{\mathcal{C}}(\alpha_{ij})$ naively as $\overline{\mathcal{C}}(H_i) \cap \overline{\mathcal{C}}(H_j)$, using the above:

$$\overline{\mathcal{C}}(\alpha_{ij}) = \left\{ \mathbf{X} \in \mathbb{R}^{4,1} \;\middle|\; \mathbf{X}^2 = 0,\; \mathbf{X} \llcorner \mathbf{C}_{ij}^{\mathbf{c}} = 0,\; \frac{\mathbf{X} \cdot \mathbf{P}_i^{\mathbf{c}}}{(-\mathbf{e}_o \cdot \mathbf{P}_i^{\mathbf{c}})(-\mathbf{e}_\infty \cdot \mathbf{X})} \leq 0,\; \frac{\mathbf{X} \cdot \mathbf{P}_j^{\mathbf{c}}}{(-\mathbf{e}_o \cdot \mathbf{P}_j^{\mathbf{c}})(-\mathbf{e}_\infty \cdot \mathbf{X})} \leq 0 \right\}. \tag{4.36}$$

This is the intersection of two 4D hyperplanes (i.e. a 3D hyperplane), bounded by two 4D hyperplanes, all intersected with the null cone.

4.5 Reformulation by inversion

The reformulation given in the preceding section in a sense reduces the original common curves problem to a linear one, because the original hemispheres and circular arcs are represented by hyperplanes through the origin. In actuality, however, the intersection of these hyperplanes with the null cone are not linear spaces. Moreover, the mapping $\mathcal{C}^{-1}$ from the transformed space to the original space is not injective. From the point of view of concrete implementation, this amounts to the copying of data; the value of g at a single point $\mathbf{x} \in H$ is copied to every point on the line $\overline{\mathcal{C}}(\mathbf{x})$. It may therefore be argued that little is gained from explicitly transforming the problem into conformal geometric algebra.

However, there are a number of related concepts that can be applied to construct more practical transformations that do not increase the dimensionality of the problem. During the preparation of this thesis, three such transformations were discovered, which shall be called *stereographic projection*, *inversion* and *cutting hyperplanes*. They all share the property of sending spheres through the origin to planes and circles through the origin to lines. In fact, a main result of this section is that they are all equivalent. While their application to the problem at hand was inspired by conformal geometric algebra, they do not require geometric algebra for their formulation.

4.5.1 Stereographic projection

A natural candidate for a map that could transform circles on the hemisphere into lines in the plane is a translated and scaled version of the stereographic projection described in Section 3.2. Let

$$\Pi \coloneqq \left\{(x, y, k) \in \mathbb{R}^3 \;\middle|\; (x, y) \in \mathbb{R}^2\right\} \tag{4.37}$$

and define the projection $P\colon H \to \Pi$ where by

$$P(x, y, z) \coloneqq \mathcal{S}^{-1}(x, y, k - z) + k\mathbf{e}_3 = \frac{k}{k - (k - z)}(x, y) + k\mathbf{e}_3 = \frac{k}{z}(x, y, z). \tag{4.38}$$

where $\mathbf{e}_+$ has been replaced by $\mathbf{e}_3$ and the unit circle has been replaced by one of radius k in (3.4).

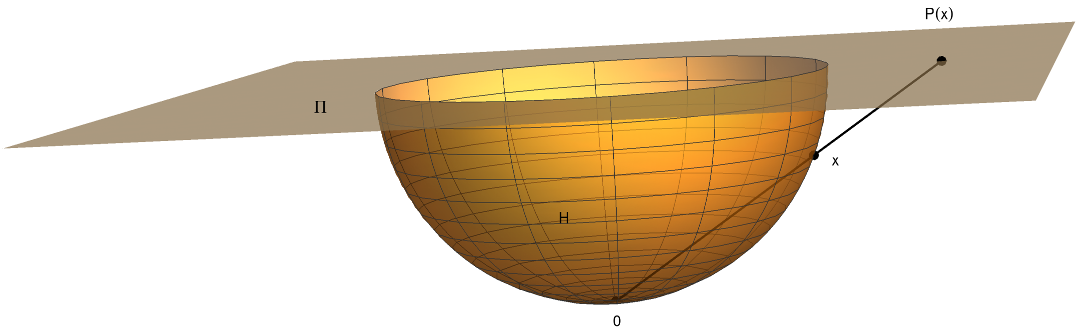


Figure 4: The modified stereographic projection (4.38).

4.5.2 Inversion

An inversion in the origin is a map

$$\mathbf{x} \mapsto \frac{R^2}{\mathbf{x}^2}\mathbf{x} \tag{4.39}$$

where R is the radius of the sphere that is invariant under the transformation. If such an inversion is applied to points on the hemisphere H, the formula simplifies: Because $x^2 + y^2 + (k - z)^2 = k^2$ on H, (4.39) becomes

$$(x, y, z) \mapsto \frac{R^2}{x^2 + y^2 + z^2}(x, y, z) = \frac{R^2}{k^2 - (k - z)^2 + z^2}(x, y, z) = \frac{R^2}{2kz}(x, y, z). \tag{4.40}$$

Comparison with (4.38) shows that the modified stereographic projection is precisely the inversion in the sphere of radius $R = \sqrt{2}k$. These considerations also give new light to the ordinary stereographic projection (3.4) as an inversion in the circle of radius $\sqrt{2}$ centred on $\mathbf{e}_+$; see Figure 5.

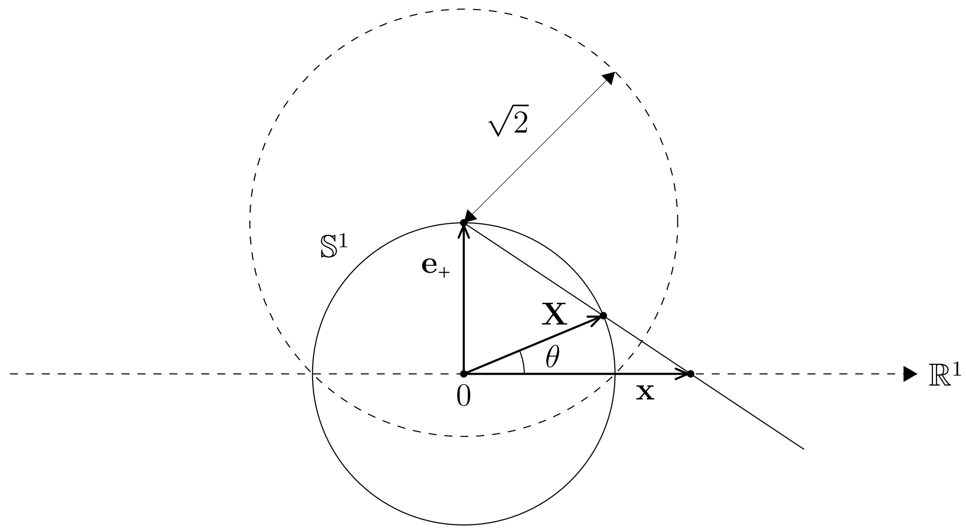


Figure 5: The common stereographic projection (3.4) as an inversion.

4.5.3 Hyperplane cutting

It was discovered empirically using Wolfram Mathematica that a transformation with similar properties to those above can be extracted from the general reformulation constructed in Section 4.4 as follows: Take the intersection of $\overline{\mathcal{C}}(H)$ with the hyperplane given by $u = u_0$ for some constant u_0 (refer back to (3.15) for the definition of the u coordinate), then project onto $\mathbb{R}^3$.

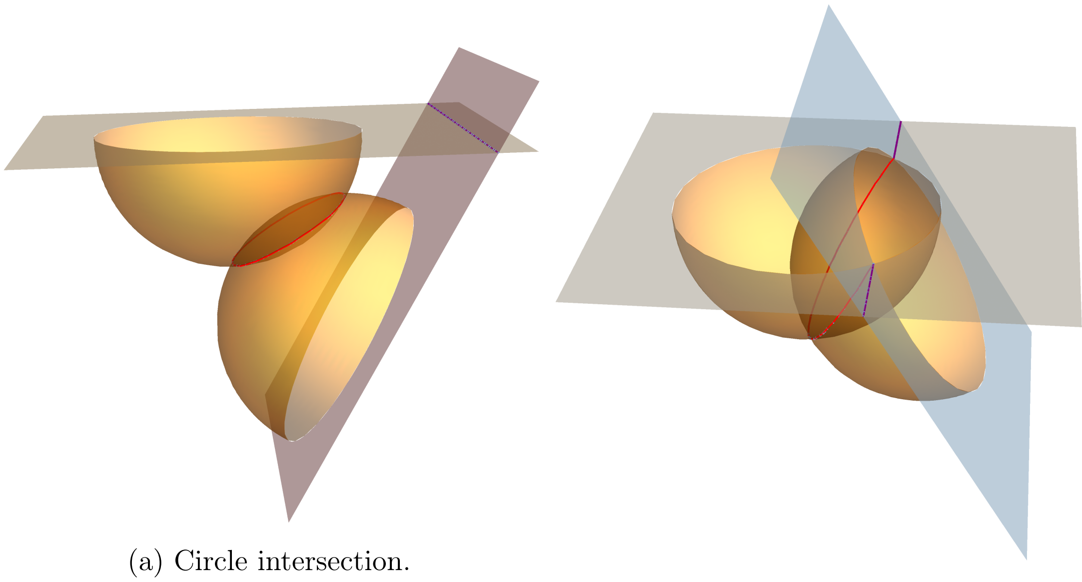
In symbols, this means that a point $\mathbf{x} \in H$ is first mapped onto $\lambda\,\mathcal{C}(\mathbf{x}) = \lambda\big(\mathbf{x} + \frac{\mathbf{x}^2}{2}\mathbf{e}_\infty + \mathbf{e}_\text{o}\big)$. Then, a particular λ is chosen (depending on $\mathbf{x}$) so that $u = u_0$. Since $u = \frac{\lambda\mathbf{x}^2}{2}$ this means $\lambda = \frac{2u_0}{\mathbf{x}^2}$. Finally, the result is projected onto $\mathbb{R}^3$. The resulting map is

$$\mathbf{x} \mapsto \frac{2u_0}{\mathbf{x}^2}\mathbf{x}, \tag{4.41}$$

which is an inversion in the sphere of radius $\sqrt{2u_0}$. If $u_0 = k^2$, it corresponds precisely to the stereographic projection (4.38).

4.5.4 The transformed problem

We have now seen that the three methods of transforming the data described above are all equivalent up to a dilation in the origin. We shall therefore henceforth focus on the stereographic projection (4.38). Figure 6 sums up the application of this transformation to the problem at hand.



(a) Circle intersection.

(b) Circular arc intersection.

Figure 6: Transformed hemispheres $P(H_i)$, $P(H_j)$ and their intersections $P(\alpha_{ij})$.

We shall now examine the image processing practicalities arising from this transformation. To this end, let us first explicitly write down the transformation $\mathcal{T}\colon D \to \mathbb{R}^2 \setminus D$ of two-dimensional images. Reusing the notation (3.1) (but including k this time);

$$\begin{aligned} \mathcal{H}\colon \mathbb{R}^2 &\to \Pi \\ \mathcal{H}(\mathbf{x}) &\coloneqq \mathbf{x} + k\mathbf{e}_3 \end{aligned} \tag{4.42}$$

we find

$$\mathcal{T} = \mathcal{H}^{-1} \circ P \circ T\colon\ \mathbf{x} \mapsto \frac{k}{k - \sqrt{k^2 - \mathbf{x}^2}}\mathbf{x}. \tag{4.43}$$

Transforming D through this map means that the data $\gamma_i\colon D \to \mathbb{C}$ are transformed into $\gamma_i \circ \mathcal{T}^{-1}\colon \mathbb{R}^2 \setminus D \to \mathbb{C}$. The inverse can be found to be

$$\mathcal{T}^{-1}(\mathbf{x}) = \frac{2k^2}{\mathbf{x}^2 + k^2}\mathbf{x}. \tag{4.44}$$

Figure 7 shows two images transformed in this way (corresponding to Figure 6a). The problem of finding the rotation that makes the data agree on the intersection corresponding hemispheres has now been reduced to that of identifying common lines in the two-dimensional transformed images. The task of a numerical method is to find a triple (d, θ, φ) (note that like SO(3), the search space is three-dimensional), which can be translated into the two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and the two direction vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, so that the values on L agree with those on L'; that is, for all $\lambda \in \mathbb{R}$,

$$(\gamma_i \circ \mathcal{T}^{-1})(\mathbf{x} + \lambda\hat{\mathbf{u}}) = (\gamma_j \circ \mathcal{T}^{-1})(\mathbf{y} + \lambda\hat{\mathbf{v}}) \tag{4.45}$$

whenever $\mathbf{x} + \lambda\hat{\mathbf{u}}$ and $\mathbf{y} + \lambda\hat{\mathbf{v}}$ lie outside of the disk D.

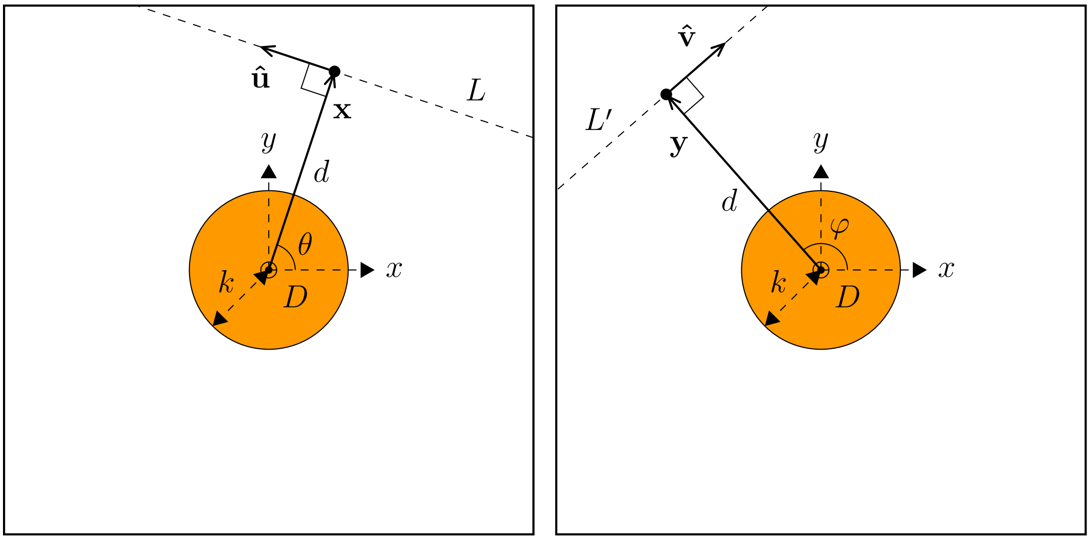


Figure 7: An identified common line between two images on $\mathbb{R}^2 \setminus D$.

4.5.5 Extracting rotations from common lines

Suppose that, given the two images shown in Figure 7, the numerical method has found $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and the direction vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$. The problem is now to map the solution back into SO(3); to find the rotation $R \in \mathrm{SO}(3)$ that transforms the line L' to the line L, when the images are viewed as lying in the plane $\Pi = \mathcal{H}(\mathbb{R}^2)$. Note that the handedness of $(\mathbf{x}, \hat{\mathbf{u}})$ is different from that of $(\mathbf{y}, \hat{\mathbf{v}})$. This can be seen by geometric intuition considering Figure 6, or by noting that if the handedness were the same, the solution would be a rotation in the xy-plane, which is evidently not the case if there is a unique common line.

To solve the problem, let us view the rotation as a rotor $\mathbf{R} \in \mathcal{G}(\mathbb{R}^3)$ (if only to emphasize the fact that rotation is a linear transformation). Defining

$$\mathbf{p} \coloneqq \mathcal{H}(\mathbf{x}) = \mathbf{x} + k\mathbf{e}_3 \tag{4.46}$$

$$\mathbf{q} \coloneqq \mathcal{H}(\mathbf{y}) = \mathbf{y} + k\mathbf{e}_3, \tag{4.47}$$

the fact that $\mathbf{R}$ transforms L into L' is algebraically

$$\forall \lambda \in \mathbb{R} \quad \mathbf{p} + \lambda\hat{\mathbf{u}} = \mathbf{R}(\mathbf{q} + \lambda\hat{\mathbf{v}})\mathbf{R}^\dagger. \tag{4.48}$$

From this we find that $\mathbf{R}\mathbf{q}\mathbf{R}^\dagger = \mathbf{p}$ and $\mathbf{R}\hat{\mathbf{v}}\mathbf{R}^\dagger = \hat{\mathbf{u}}$. Now, rotations respect cross products; this can be easily proven:

$$\begin{aligned}
\mathbf{R}(\mathbf{a} \times \mathbf{b})\mathbf{R}^\dagger &= \mathbf{R}(\mathbf{a} \wedge \mathbf{b})^{\mathbf{c}}\mathbf{R}^\dagger && \text{by (2.39)} \\
&= \left(\mathbf{R}(\mathbf{a} \wedge \mathbf{b})\mathbf{R}^\dagger\right)^{\mathbf{c}} && \mathbf{I} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \text{ commutes with everything} \\
&= \left((\mathbf{R}\mathbf{a}\mathbf{R}^\dagger) \wedge (\mathbf{R}\mathbf{b}\mathbf{R}^\dagger)\right)^{\mathbf{c}} && \text{outermorphism (Theorem 2.17)} \\
&= (\mathbf{R}\mathbf{a}\mathbf{R}^\dagger) \times (\mathbf{R}\mathbf{b}\mathbf{R}^\dagger) && \text{by (2.39).}
\end{aligned}$$

Thus, $\mathbf{R}(\mathbf{q} \times \hat{\mathbf{v}})\mathbf{R}^\dagger = \mathbf{p} \times \hat{\mathbf{u}}$. Since $\{\mathbf{q}, \hat{\mathbf{v}}, \mathbf{q} \times \hat{\mathbf{v}}\}$ is a basis of $\mathbb{R}^3$ for which the rotated basis vectors are known, the rotation is completely specified.

In fact, the rotor may be explicitly calculated by a formula from [15, Section 10.3.2]

$$\mathbf{R} \propto 1 + \sum_{i=1}^{3} \mathbf{f}^i \mathbf{b}_i \tag{4.49}$$

where $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = (\mathbf{q}, \hat{\mathbf{v}}, \mathbf{q} \times \hat{\mathbf{v}})$ and $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = (\mathbf{p}, \hat{\mathbf{u}}, \mathbf{p} \times \hat{\mathbf{u}})$ and where $\{\mathbf{f}^i\}$ denotes the *reciprocal basis* such that $\mathbf{f}^i \cdot \mathbf{f}_j = (i = j)$: $\mathbf{f}^1 = \frac{\mathbf{f}_2 \times \mathbf{f}_3}{(\mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \mathbf{f}_3)^c}$ et cetera. The rotor must be normalized so that $\mathbf{R}^\dagger \mathbf{R} = 1$.

4.5.6 Discussion

The mapping $\mathcal{T}$ developed in Section 4.5.4 has the property of mapping data close to the origin very far away. This leads to a data set that, while planar, in principle has infinite extent. In the limiting case $k \to \infty$ without applying $\mathcal{T}$, the hemispheres H_i become planes through the origin and the common curves α_{ij} become lines through the origin. In contrast, if $\mathcal{T}$ is applied, the limiting behaviour is the exact opposite; the planes and lines recede *infinitely far* from the origin.

However, it is not clear to what degree this is a problem in practice. The data γ_i are, after all, pixelated, i.e. there is a positive pixel size ε. The disk D might therefore more accurately be modelled as the annulus $\varepsilon < \|\mathbf{x}\| < k$, which by $\mathcal{T}$ is sent to another annulus $k < \|\mathbf{x}\| < \frac{k\varepsilon}{k - \sqrt{k^2 - \varepsilon^2}} \approx \frac{2k^2}{\varepsilon}$ which, if properly scaled, could conceivably be a well-behaved search space.

Furthermore, let us not forget that the solution to the common curves problem consists of more than merely identifying the common curves between two hemispheres. For example, there is the problem of finding the best fit for the orientations of *all* hemispheres given the relative orientations of pairs. Solving that problem in terms of the transformed planes could well prove to have advantages over working with the hemispheres directly.

4.6 Conclusion

We have developed two methods for transforming the common curves problem into one of planes and lines rather than spheres and circles. The first approach is to translate it directly into conformal geometric algebra. This is certainly elegant, but it requires copying the original data multiple times into the new space, which is likely to be computationally infeasible. The method of "cutting hyperplanes" was devised to solve this, but turns out to be equivalent to the simpler transformation of inversion, alternatively stereographic projection. This method appears on the surface to be a simplification of the problem, but the practical concerns described in Section 4.5.6 could limit its usefulness. Which of these approaches to select, if any, will depend on its performance on actual data, and that must be left to experts in the field.

4.7 Further work

In this thesis, three superficially different transformation methods were devised and subsequently found to be equivalent. This might lead one to ask the question: Is it at all possible to transform the problem into one of planes and lines in a way that is *not* equivalent to the inversion (4.39)? In particular, can the issue of sending data infinitely far away be avoided? Intuition seems to suggest not, but it would be worthwhile to formulate and examine this question rigorously.

It is known that the functions f and γ are Fourier transforms of real-valued functions; therefore $f(-\mathbf{x}) = \overline{f(\mathbf{x})}$ where $z \mapsto \overline{z}$ denotes complex conjugation, and similarly for γ. Thus, in fact the values of f are known on hourglass-shaped *pairs* of hemispheres. This could potentially be exploited, for example to make it possible to deal only with full circles rather than circular arcs.

This thesis did not treat the problem of reconstructing the orientations of all hemispheres given the relative orientation of all pairs of hemispheres (with some uncertainty). Problems such as this can benefit from a geometric algebra treatment. In particular, Perwass shows [7, Equation (4.15)] that a good approximation for the mean rotor of a set of n rotors is given by

$$\mathbf{R}_M \propto \sum_{i=1}^{n} \mathbf{R}_i \tag{4.50}$$

(where $\mathbf{R}_M$ must be normalized so that $\mathbf{R}_M^\dagger \mathbf{R}_M = 1$), which could indeed be directly applicable since the rotor $\mathbf{R}_{ij} \coloneqq \mathbf{R}_i \mathbf{R}_j^{-1}$ can be approximated as $\mathbf{R}_{ik}\mathbf{R}_{kj}$ for every k.

Starting from the transformed images of Figure 7, one may perform a further inversion in the circle ∂D, described by the map $\mathcal{I}\colon \mathbb{R}^2 \setminus D \to D$ with $\mathcal{I}(\mathbf{x}) \coloneqq \frac{k^2}{\mathbf{x}^2}\mathbf{x}$. This has the effect of transforming the common lines into circles through the origin, whereas they were ellipses in the original image. The compound map is given by

$$\begin{aligned} \mathcal{I} \circ \mathcal{T}\colon D &\to D \\ \mathbf{x} &\mapsto \frac{k(k - \sqrt{k^2 - \mathbf{x}^2})}{\mathbf{x}^2}\,\mathbf{x}. \end{aligned} \tag{4.51}$$

It would be worth considering whether transforming ellipses into circles in this way could simplify the problem by allowing for existing knowledge about circles to be applied.

A Basic geometry in $\mathbb{R}^{p,q}$

In this appendix, some basic concepts that are familiar in the Euclidean space $\mathbb{R}^n$ are shown to generalize straightforwardly into any $\mathbb{R}^{p,q}$.

Lemma A.1. *For any vector* $\mathbf{n} \neq 0$*, the set* $\Pi := \{\mathbf{x} \in \mathbb{R}^{p,q} \mid \mathbf{n} \cdot \mathbf{x} = 0\}$ *is a* $(p+q-1)$*-dimensional hyperplane.*

Proof. The lemma is assumed to be known to hold in $\mathbb{R}^{p+q}$ (the Euclidean space). Write $\mathbf{n}$ in the standard basis:

$$\begin{aligned}\mathbf{n} &= n_1\mathbf{e}_1 + \cdots + n_p\mathbf{e}_p \\ &\quad + n_{p+1}\mathbf{e}_{p+1} + \cdots + n_{p+q}\mathbf{e}_{p+q}.\end{aligned}$$

Define a new vector

$$\begin{aligned}\mathbf{n}' &:= n_1\mathbf{e}_1 + \cdots + n_p\mathbf{e}_p \\ &\quad - n_{p+1}\mathbf{e}_{p+1} - \cdots - n_{p+q}\mathbf{e}_{p+q}.\end{aligned} \tag{A.1}$$

Then $\mathbf{n} \cdot \mathbf{x} = \mathbf{n}' \circ \mathbf{x}$, where $\circ$ stands for the *Euclidean* scalar product. This immediately proves the lemma. ∎

Remark. *This theorem is true for the entire space* $\mathbb{R}^{p,q}$*, but may not be true for subspaces. For example, take* $V := \operatorname{Span}\{\mathbf{n}\} \subset \mathbb{R}^{1,1}$ *where* $\mathbf{n} := \mathbf{e}_1 + \mathbf{e}_2$*. Then* $\dim V = 1$ *but* $\Pi := \{\mathbf{x} \in V \mid \mathbf{n} \cdot \mathbf{x} = 0\} = V$ *since* $\mathbf{n}^2 = 0$*. Therefore* $\dim \Pi$ *is not* 0*, but* 1*.*

Theorem A.2. *Every linear subspace of* $\mathbb{R}^{p,q}$ *has an orthogonal basis.*

Proof. Let V be a linear subspace of $\mathbb{R}^{p,q}$. Proceed by induction on $\dim V$. The base case $\dim V = 0$ is trivial, so assume that every subspace W with $\dim W < \dim V$ has an orthogonal basis.

If $\forall \mathbf{x} \in V\ \mathbf{x}^2 = 0$, then any basis for V is orthogonal, since $\forall \mathbf{x}, \mathbf{y} \in V$,

$$\mathbf{x} \cdot \mathbf{y} = \frac{(\mathbf{x}+\mathbf{y})^2 - \mathbf{x}^2 - \mathbf{y}^2}{2} = 0.$$

Assume therefore that $\exists \mathbf{x} \in V\ \mathbf{x}^2 \neq 0$ and let $\mathbf{x}$ be such a vector. Define $W := \{\mathbf{y} \in V \mid \mathbf{x} \cdot \mathbf{y} = 0\}$. Clearly, $\mathbf{x} \notin W$ which means that $\dim W < \dim V$, and so W has an orthogonal basis $\mathfrak{B}$. By Lemma A.1, $\dim W$ can be no less than $\dim V - 1$. Therefore, it must be the case that $\dim W = \dim V - 1$, which shows that $\mathfrak{B} \cup \{\mathbf{x}\}$ is an orthogonal basis of V. ∎

B Proof of Theorem 2.17

The proof will be done in a series of steps. That $\mathrm{Ad}_{\mathbf{V}}$ is linear and $\mathrm{Ad}_{\mathbf{V}}(1) = 1$ are immediate. We thus need to prove that $\mathrm{Ad}_{\mathbf{V}}$ is grade-preserving and that $\mathrm{Ad}_{\mathbf{V}}(\mathbf{A} \wedge \mathbf{B}) = \mathrm{Ad}_{\mathbf{V}}(\mathbf{A}) \wedge \mathrm{Ad}_{\mathbf{V}}(\mathbf{B})$ for all $\mathbf{A}$ and $\mathbf{B}$.

First, note that if $\mathbf{x}$ is a vector, then $\mathrm{Ad}_{\mathbf{V}}(\mathbf{x})$ is a vector as well (since we saw that $\mathrm{Ad}_{\mathbf{V}}$ is a composition of reflections). We will use this to prove the wedge product property, but we need the following determinant-like expansion of a blade into geometric products:

Lemma B.1 (Equation (3.1) in [1]). *If $\mathbf{x}_1, \dots, \mathbf{x}_k$ are vectors, then*

$$\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k = \frac{1}{k!} \sum_{\pi \in S_k} \mathrm{sign}(\pi)\, \mathbf{x}_{\pi(1)} \dots \mathbf{x}_{\pi(k)} \tag{B.1}$$

where S_k is the symmetric group of order k (the set of permutations of $(1, \dots, k)$).

Proof. Both sides are clearly linear in all $\mathbf{x}_i$, and the left side is clearly alternating (that is, swapping two of the vectors multiplies it by -1). That the right side is also alternating can be seen by noting that swapping two vectors amounts to replacing π with $\pi \circ \sigma$ where $\sigma \in S_k$ is the swap, and then using $\mathrm{sign}(\pi \circ \sigma) = -\mathrm{sign}(\pi)$.

Now, because both sides are linear and alternating, it suffices to consider an ordered orthogonal basis $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ of $\mathrm{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ (such a basis exists by Theorem A.2). Then, $\mathbf{b}_1 \wedge \dots \wedge \mathbf{b}_k = \mathbf{b}_1 \dots \mathbf{b}_k$ and $\mathrm{sign}(\pi)\, \mathbf{b}_{\pi(1)} \dots \mathbf{b}_{\pi(k)} = \mathbf{b}_1 \dots \mathbf{b}_k$ for every π by anticommutation. Since there are $k!$ terms ($|S_k| = k!$), the two sides are equal. ■

Using (B.1), we now find

$$\begin{aligned}
\mathrm{Ad}_{\mathbf{V}}(\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k) &= \mathbf{V}(\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k)\mathbf{V}^{-1} \\
&= \frac{1}{k!} \sum_{\pi \in S_k} \mathbf{V}\mathbf{x}_{\pi(1)} \dots \mathbf{x}_{\pi(k)} \mathbf{V}^{-1} \\
&= \frac{1}{k!} \sum_{\pi \in S_k} \mathbf{V}\mathbf{x}_{\pi(1)}\mathbf{V}^{-1} \dots \mathbf{V}\mathbf{x}_{\pi(k)} \mathbf{V}^{-1} \\
&= (\mathbf{V}\mathbf{x}_1\mathbf{V}^{-1}) \wedge \dots \wedge (\mathbf{V}\mathbf{x}_k\mathbf{V}^{-1})
\end{aligned}$$

because $\mathbf{V}^{-1}\mathbf{V} = 1$ and all $\mathbf{V}\mathbf{x}_i\mathbf{V}^{-1}$ are vectors. This proves that $\mathrm{gr}\,\mathrm{Ad}_{\mathbf{V}}(\mathbf{A}) = \mathrm{gr}\,\mathbf{A}$ and $\mathrm{Ad}_{\mathbf{V}}(\mathbf{A} \wedge \mathbf{B}) = \mathrm{Ad}_{\mathbf{V}}(\mathbf{A}) \wedge \mathrm{Ad}_{\mathbf{V}}(\mathbf{B})$ for arbitrary *blades*. By linearity, the same results follow for k-vectors and general multivectors.

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